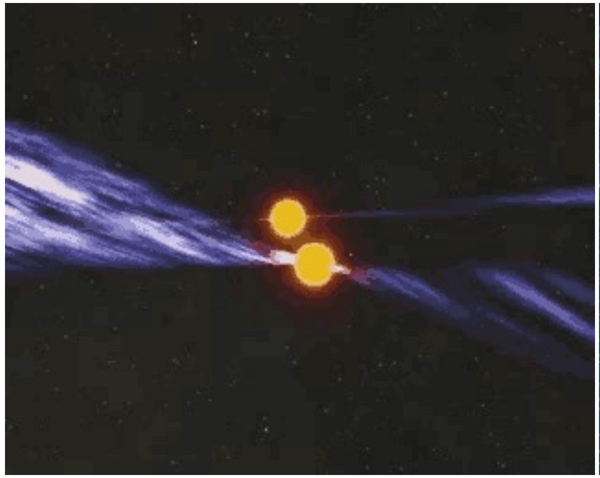
1st TEONGRAV international workshop on the theory of Gravitational Waves

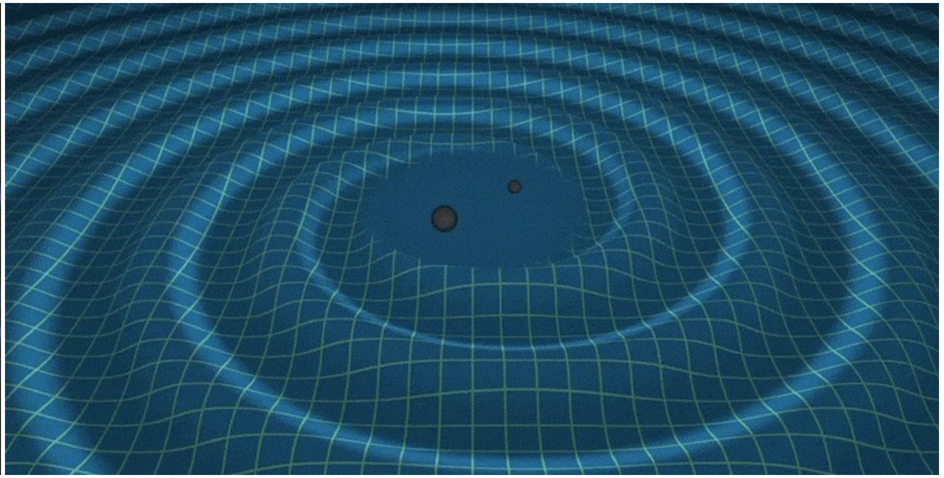
Sapienza University of Rome

16 - 20 September 2024

SAPIENZA UNIVERSITÀ DI ROMA

INFN Istituto Nazionale di Fisica Nucleare





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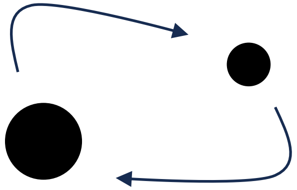
Analytical coordinate time at first post-Newtonian order

https://iopscience.iop.org/article/10.1209/0295-5075/acb07e/pdf

Astrophysical motivations of my talk

- General Relativity (GR) can be tested via ***compact binary systems***
- The non-linear structure of GR can be approximated via the *post-Newtonian (PN) method*



PN sources
(slowly moving, weakly self-gravitating, and weakly stressed)
+
BODIES WELL SEPARATED

⇒

Newtonian absolute Euclidean space

+ PN CORRECTIONS
$nPN = c^{-2n}$

LOST OF THE GR GENERAL COVARIANCE IN FAVOUR OF *SPECIAL COORDINATE SYSTEMS* (E.G., HARMONIC COORDINATES)

- The PN method is extensively used to investigate the relativistic two-body dynamics for precision tests of gravity theories and neutron star mass measurements in binary pulsars

- These assessments are based on the *quasi-elliptic 1PN-accurate GR motion of a two-body system*, being the **analytical solution** provided by Damour & Deruelle 1985 (DD85)

Aim of the talk

PREMISES: binary system's motion in the center-of-mass frame with harmonic coordinates

Use the analytical result of DD85 at 1PN order $R(\varphi)$

R = two-body seperation
φ = polar angle

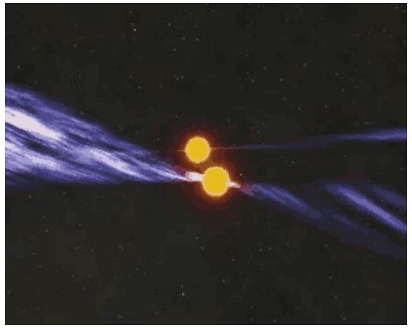
Provide the analytical expression at 1PN order $t(\varphi)$

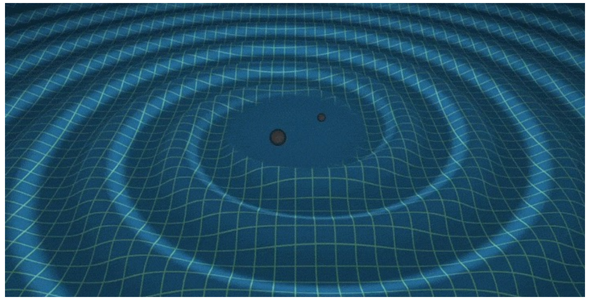
t = coordinate time

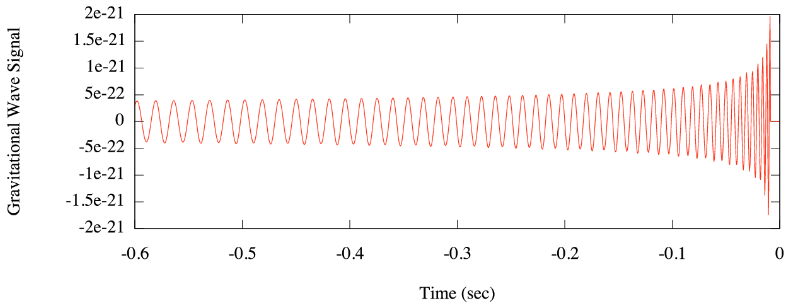
Why is it important?

Speed up the calculations for binary pulsars implemented in TEMPO, TEMPO2, PINT

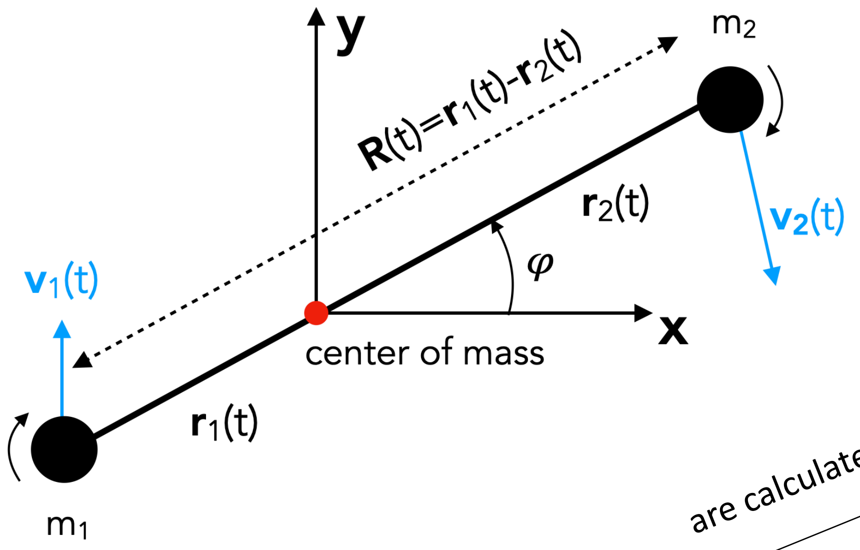
Reduce the computational costs for producing gravitational waveforms in compact binary systems







Preliminaries



are calculated through

Model parameters

$$m_1 \geq m_2, \quad M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{M}, \quad \nu = \frac{\mu}{M},$$

$$\boldsymbol{v}_1 = \frac{\mathrm{d}\boldsymbol{r}_1}{\mathrm{d}t}, \quad \boldsymbol{v}_2 = \frac{\mathrm{d}\boldsymbol{r}_2}{\mathrm{d}t}, \quad \boldsymbol{R} = \boldsymbol{r}_1(t) - \boldsymbol{r}_2(t), \quad \boldsymbol{V} = \boldsymbol{v}_1(t) - \boldsymbol{v}_2(t)$$

Conserved energy and angular momentum

$$E = E_0 + \frac{1}{c^2} E_1 + \mathrm{O}\left(c^{-4}\right),$$

$$\boldsymbol{J} = \boldsymbol{J}_0 + \frac{1}{c^2} \boldsymbol{J}_1 + \mathrm{O}\left(c^{-4}\right)$$

$$J = J_0 + \frac{1}{c^2} J_1 + \mathrm{O}\left(c^{-4}\right)$$

Analytical expression of the radius by DD85

$$R(\varphi) = R_0(\varphi) + \frac{1}{c^2} R_1(\varphi) + \mathrm{O}\left(c^{-4}\right)$$

$$R_0 = R_0(\varphi; E, J, m_1, m_2), \quad R_1 = R_1(\varphi; E, J, m_1, m_2)$$

Initial data

$$R(t_{\mathrm{in}}) = R_{\mathrm{in}}, \quad \frac{\mathrm{d}R}{\mathrm{d}t}(t_{\mathrm{in}}) = \dot{R}_{\mathrm{in}},$$

$$\varphi(t_{\mathrm{in}}) = \varphi_{\mathrm{in}}, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t_{\mathrm{in}}) = \beta\sqrt{\frac{GM}{R_{\mathrm{in}}^3}},$$

Numerical integration

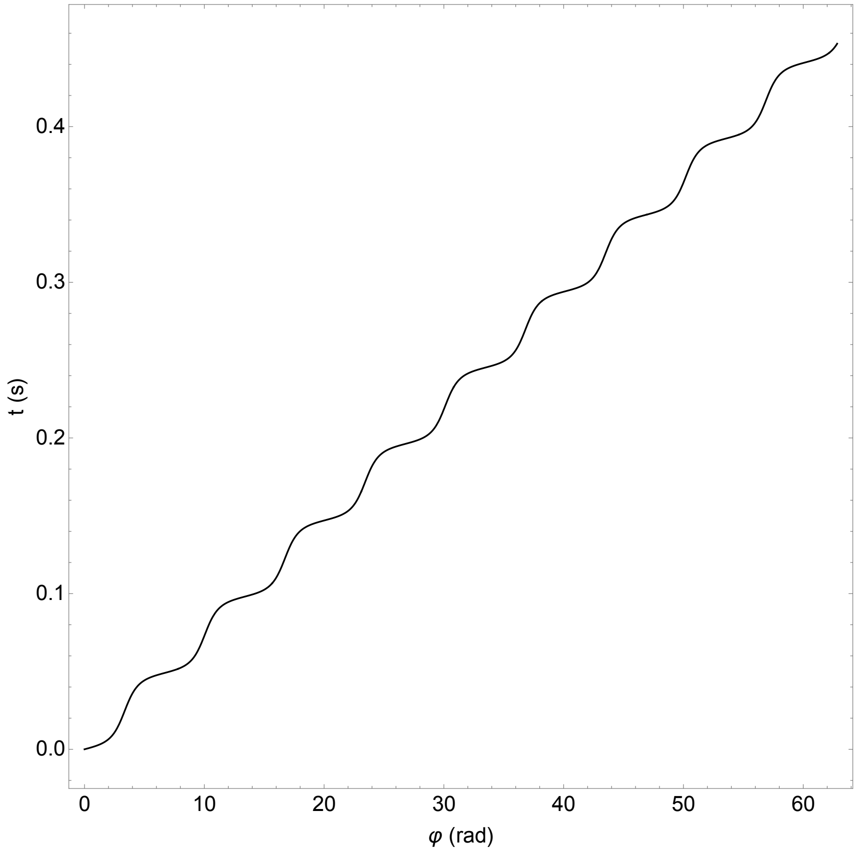
From DD85, we have

$$\mathrm{d}t = \frac{\mathrm{d}\varphi}{\frac{H}{R^2} + \frac{I}{R^3}} \qquad H = J\left[1 + (3\nu - 1)\frac{E}{c^2}\right], \quad I = (2\nu - 4)\frac{GMJ}{c^2}.$$

INTEGRATING FUNCTION (*)

$$\mathrm{d}t = \frac{R_0^2}{J_0}\left\{1 + \frac{1}{c^2}\left[E_0(1 - 3\nu) + \frac{2R_1}{R_0} - 2Q_0(\nu - 2) - \frac{J_1}{J_0}\right]\right\}\mathrm{d}\varphi$$

Integrating it numerically



We obtain a monotonically increasing function, as we would expect from the coordinate time

Analytical formula

Starting from (*), we must integrate

$$t = \int g(\tilde{K}\bar{\varphi})\mathrm{d}\bar{\varphi} \qquad \bar{\varphi} = \varphi - \varphi_{\mathrm{in}}$$

1PN-expansion of the perihelion shift term

Dividing the integrating function in three parts

$$g(\tilde{K}\bar{\varphi}) = g_1(\tilde{K}\bar{\varphi}) + g_2(\tilde{K}\bar{\varphi}) + g_3(\tilde{K}\bar{\varphi})$$

$$g_1(\tilde{K}\bar{\varphi}) = A_1 R_0,$$
$$g_2(\tilde{K}\bar{\varphi}) = A_2 R_0^2,$$
$$g_3(\tilde{K}\bar{\varphi}) := A_3 \cos(\tilde{K}\bar{\varphi}) R_0^3.$$

A_1,A_2,A_3 coefficients

We rewrite R_0 as

$$R_0 = \frac{1}{B_1 + B_2\cos(\tilde{K}\bar{\varphi})},$$

coefficients

$$B_1 \geq B_2 \geq 0$$

analytically integrating

$$f_i(\varphi) = \int g_i(\tilde{K}\bar{\varphi})\mathrm{d}\bar{\varphi}, \quad i = 1, 2, 3.$$

$$f_1(\varphi) = \frac{2A_1}{\tilde{K}} \frac{\arctan\left(\sqrt{\frac{C_2}{C_1}}\tau\right)}{\sqrt{C_1 C_2}},$$

$$f_2(\varphi) = \frac{2A_2}{\tilde{K}}\left[\frac{C_1 + C_2}{2(C_1C_2)^{3/2}}\arctan\left(\sqrt{\frac{C_2}{C_1}}\tau\right) - \frac{\tau(C_1 - C_2)}{2C_1C_2\left(C_1 + C_2\tau^2\right)}\right],$$

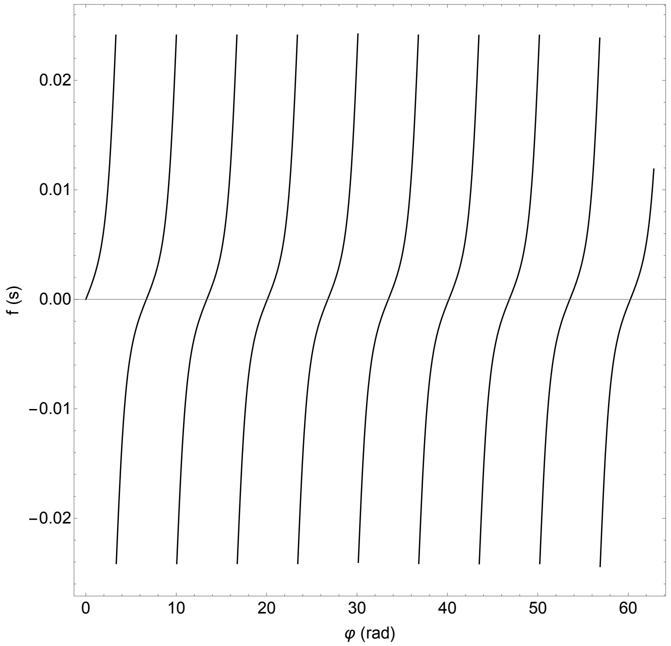
$$f_3(\varphi) = \frac{2A_3}{\tilde{K}}\left[-\frac{3\left(C_1^2 - C_2^2\right)\arctan\left(\sqrt{\frac{C_2}{C_1}}\tau\right)}{8(C_1C_2)^{5/2}} + \frac{\tau C_1(3C_1^2 + 5C_2^2) + (5C_1^2 + 3C_2^2)C_2\tau^3}{8C_1^2C_2^2\left(C_1 + C_2\tau^2\right)^2}\right].$$

$$C_1 = B_1 + B_2 \geq 0,$$
$$C_2 = B_1 - B_2 \geq 0$$

$$\tau = \tan(x/2) \qquad x = \tilde{K}\bar{\varphi}$$

Plotting the function (**)

$$f(\varphi) = f_1(\varphi) + f_2(\varphi) + f_3(\varphi)$$



Accumulation function

The function (**) must be made continuous, connecting smoothly the different periodic branches

This can be achieved through, what we called, an ***accumulation function***

We introduce a ***characteristic period***

$$P_\varphi = \frac{\pi}{\tilde{K}}$$

accumulation function

$$F_n(\varphi) := \begin{cases} 0, & \text{if } \bar{\varphi} \in [0, P_\varphi], \\ 2n f(P_\varphi), & \text{if } \bar{\varphi} \in [P_\varphi(2n+1), P_\varphi(2n+2)], \end{cases}$$

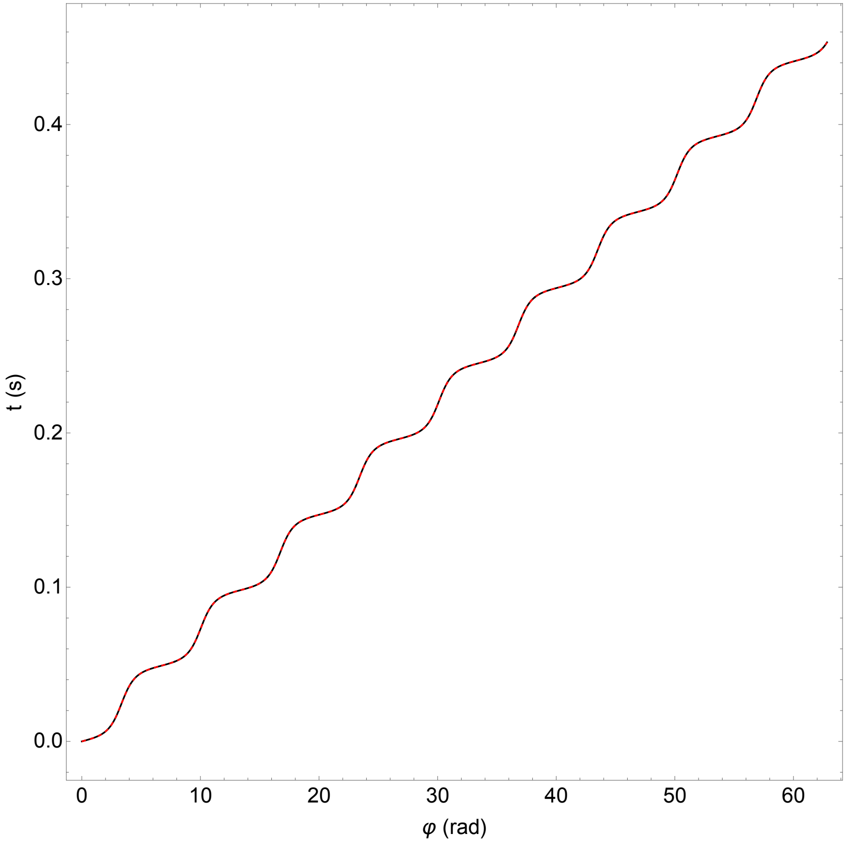
INTEGER PART

$n \in \mathbb{N}$ is calculated as follows $q = [(\bar{\varphi} - P_\varphi)/P_\varphi]$

$$\begin{cases} q \text{ is an even number} & n = (q+2)/2, \\ q \text{ is an odd number} & n = (q+1)/2. \end{cases}$$

final analytical formula

$$t(\varphi) = f(\varphi) + F_n(\varphi) + \mathrm{O}\left(c^{-4}\right)$$



Perfect agreement between ***numerical integration*** and ***analytical formula***

Resumé of the results

- The formula t(φ) at 1PN represents a new analytical result in the GR PN literature.
- The strength of our method relies upon two fundamental steps:

(1) computing integrals (symbolic program);

(2) making the solution (1) continuous via an accumulation function.

- The accumulation function, never proposed in the literature, is the original part of our work.
- We have proposed this analytical result at 1PN level, as the starting point.
- Our finding can be applied for timing the orbital period of compact binary systems during the inspiral phase. Furthermore, it is also useful for extracting information from binary systems, as well as for providing significant tests of gravity.

Conclusions

ADAVANTAGES OF OUR RESULT

Our formula can be used to replace numerical schemes in TEMPO and LIGO due to its simpler and faster implementation with no approximation costs

Our method can be generalized and extended to higher PN orders, where an analytical formula (in whatever form is presented) may improve speed and accuracy

LIMITS OF OUR RESULT

We have presented the lowest PN order

Pulsar timing recipes often make use of $\varphi(t)$, whose calculation still requires a numerical inversion. However our solution can speed up such computations

FUTURE PERSPECTIVE

Extending our method to the 2PN order

THANK YOU FOR YOUR ATTENTION