Eccentricity: a recipe ~~for~~ from catastrophe

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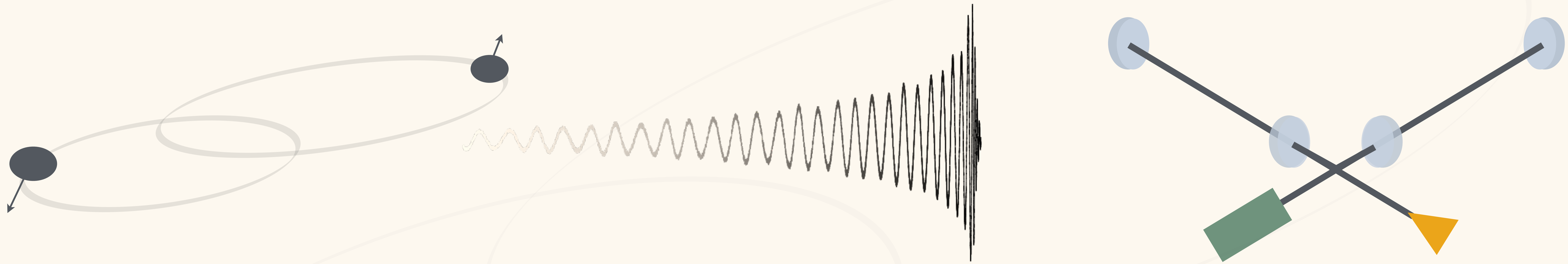
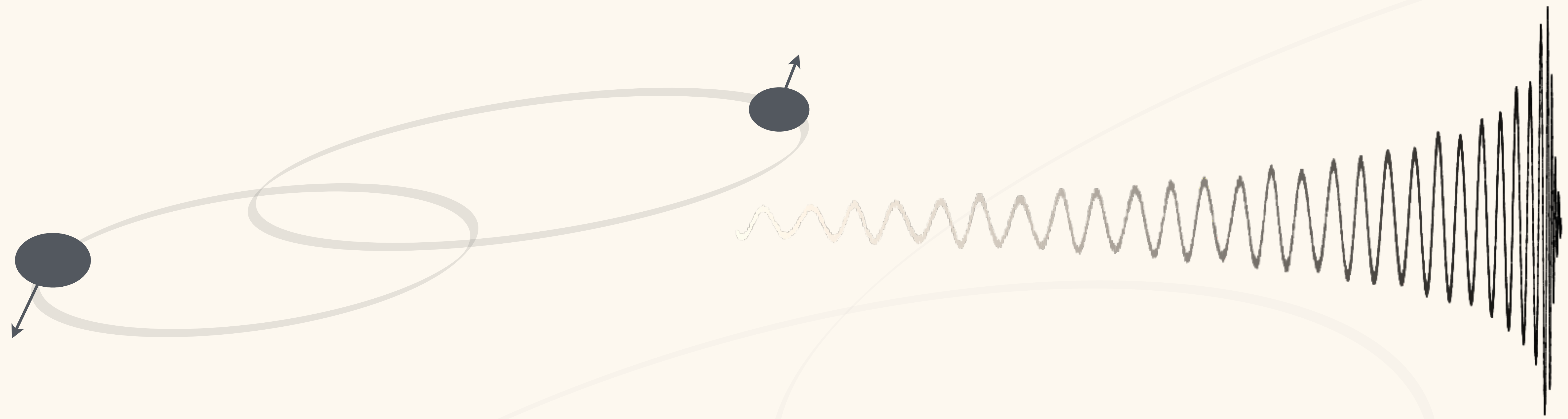
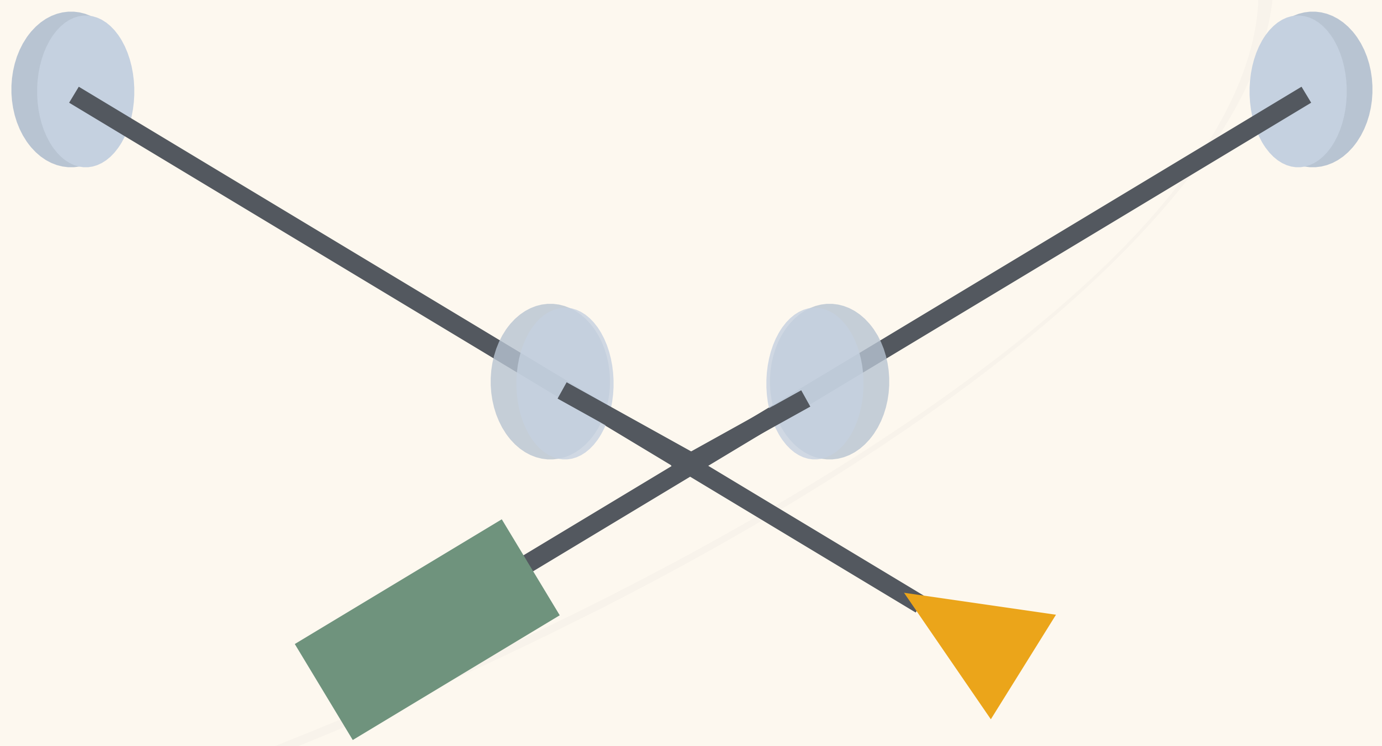
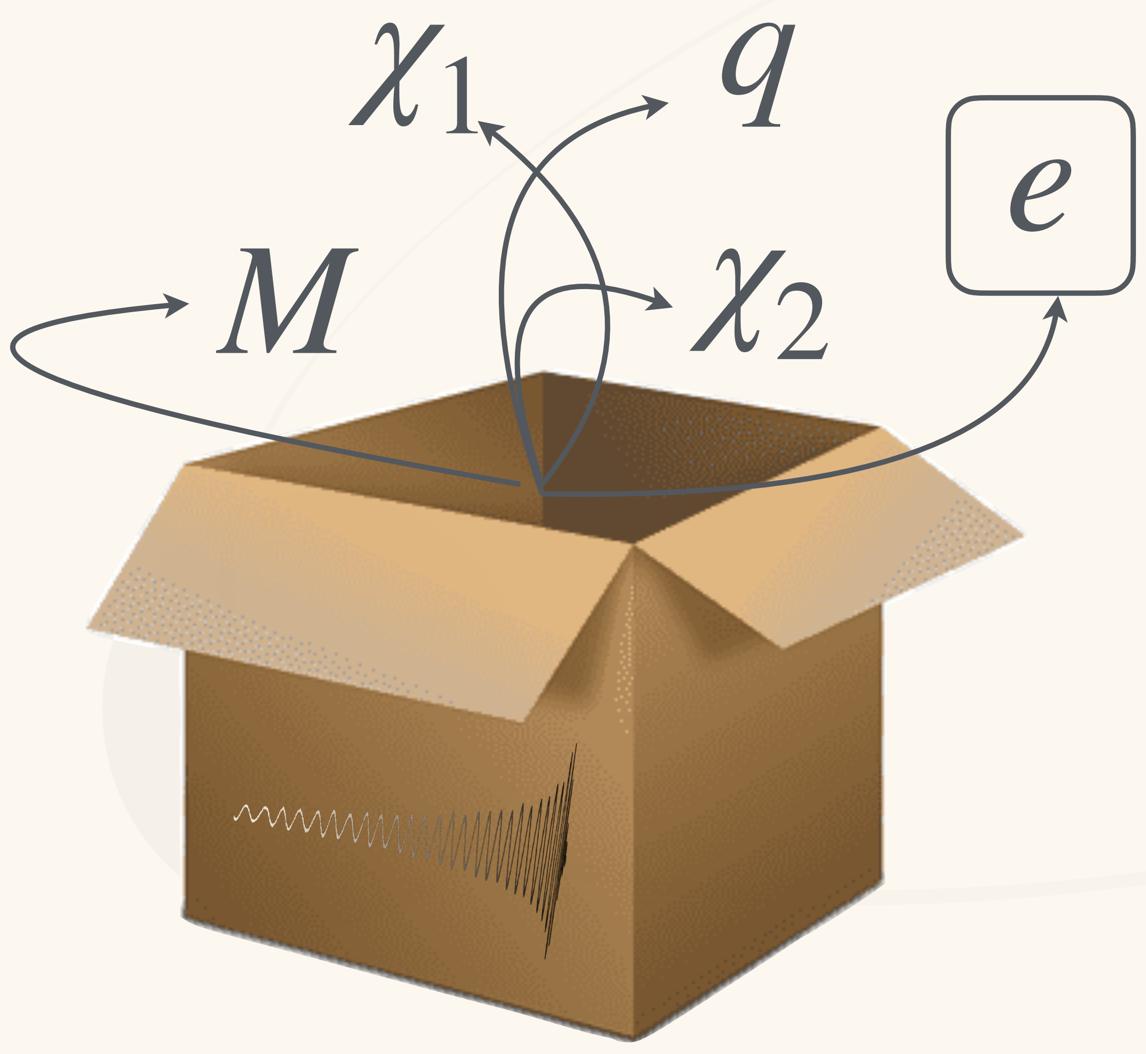
Eccentricity in GW astronomy



Eccentricity in GW astronomy







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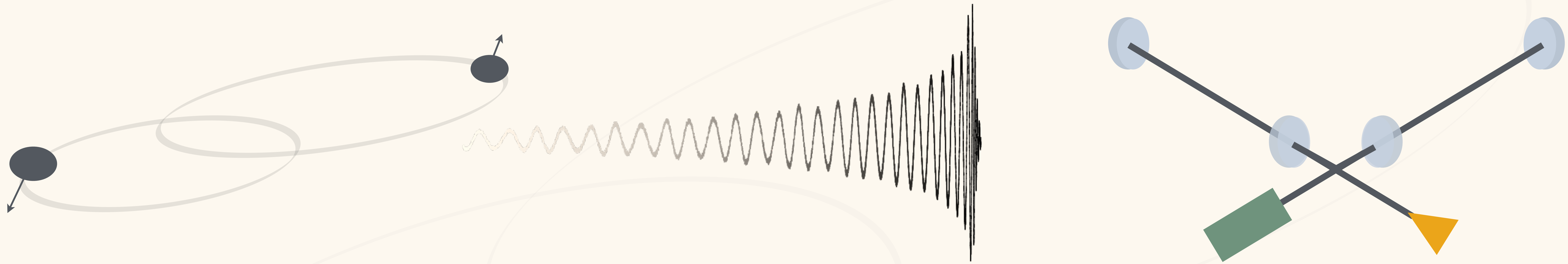
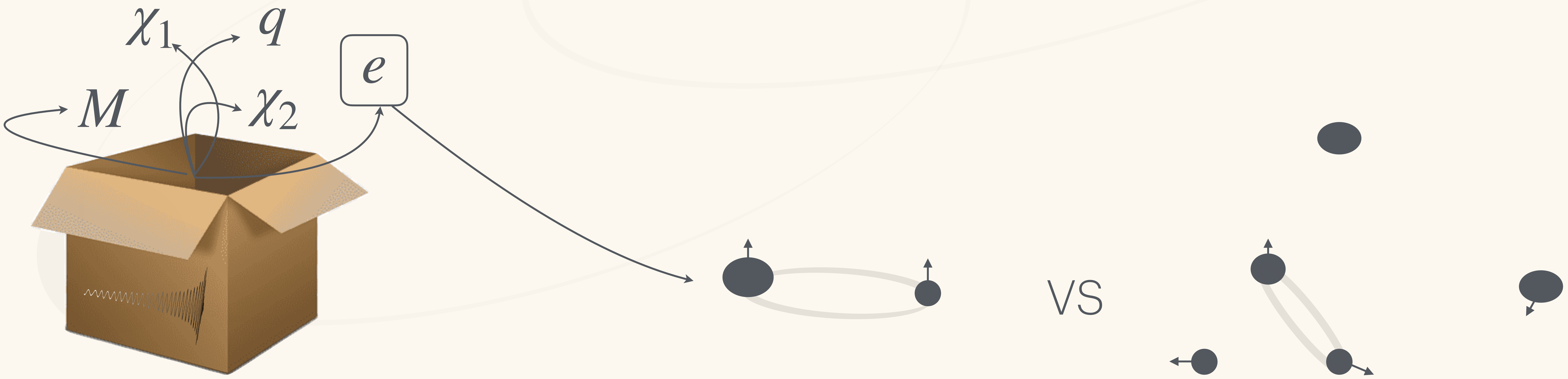


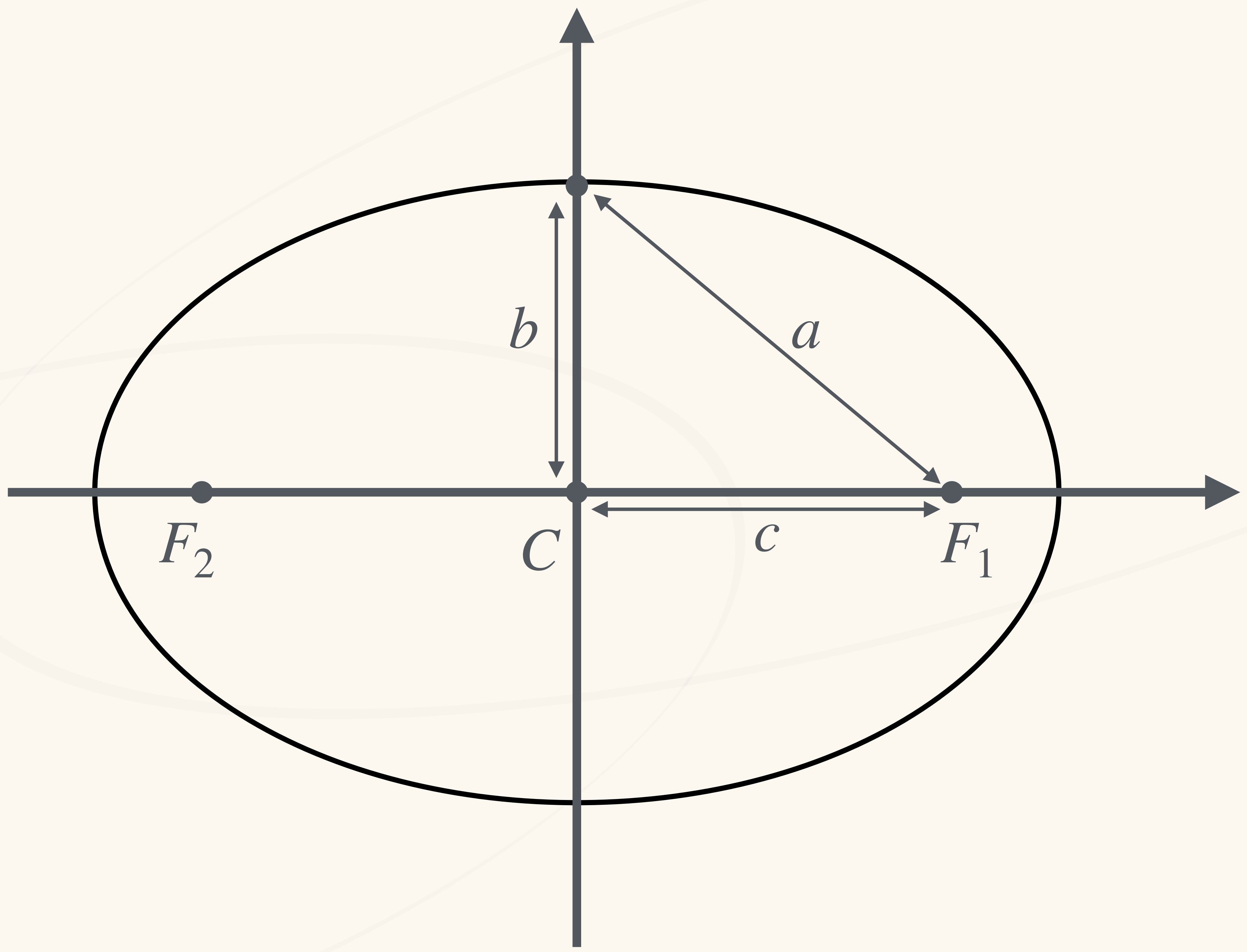


Eccentricity in GW astronomy

What is e?

Math eccentricity

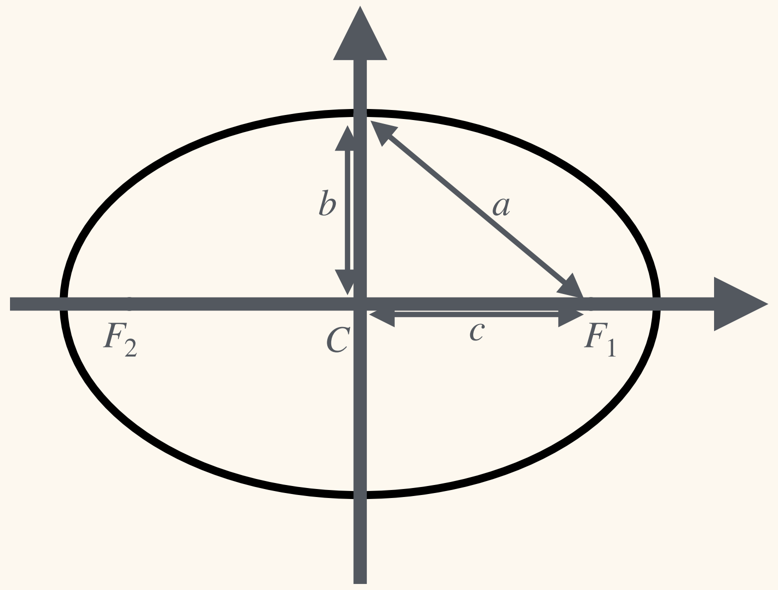
The easy thing



$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

Newtonian eccentricity

Still manageable



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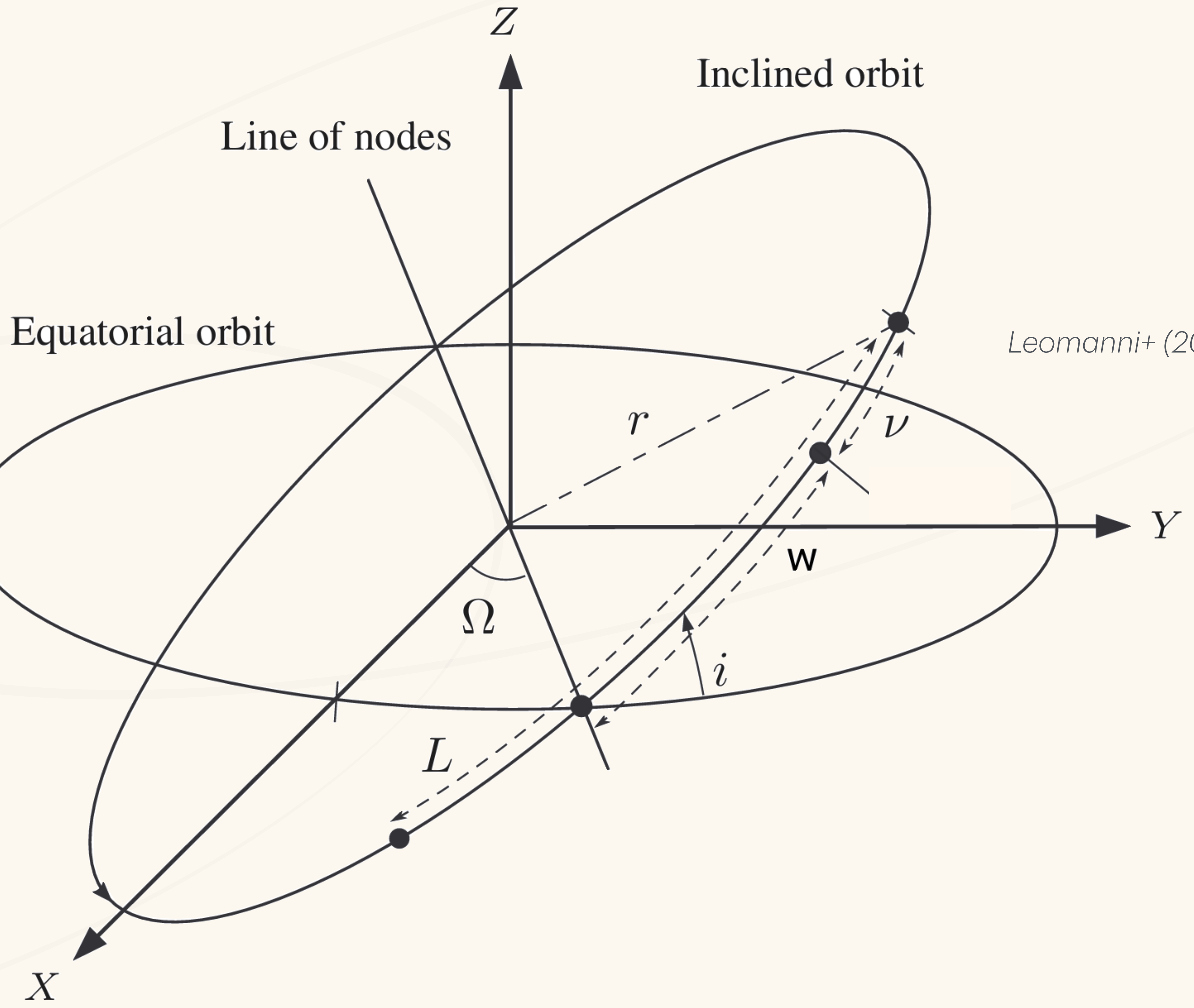
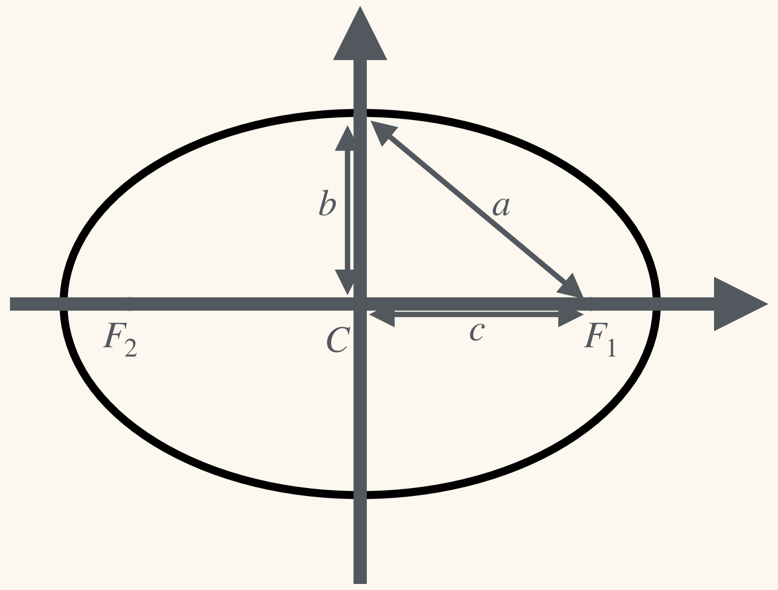


Leomanni+ (2019)

$$e = \sqrt{1 + \frac{2\epsilon h^2}{\mu^2}} \qquad e = \frac{r_a - r_p}{r_a + r_p}$$

Eccentricity in GR

A tricky thing



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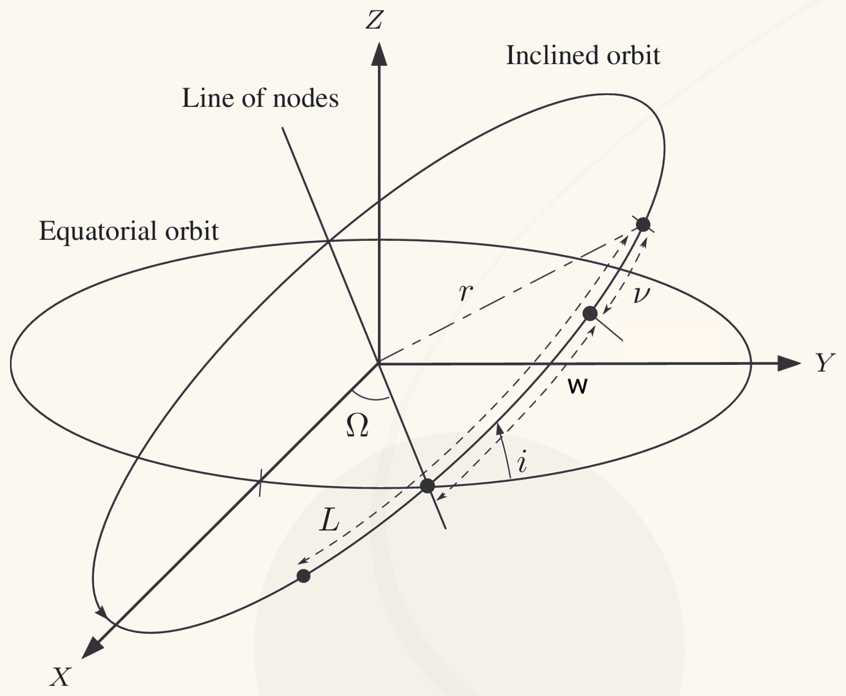
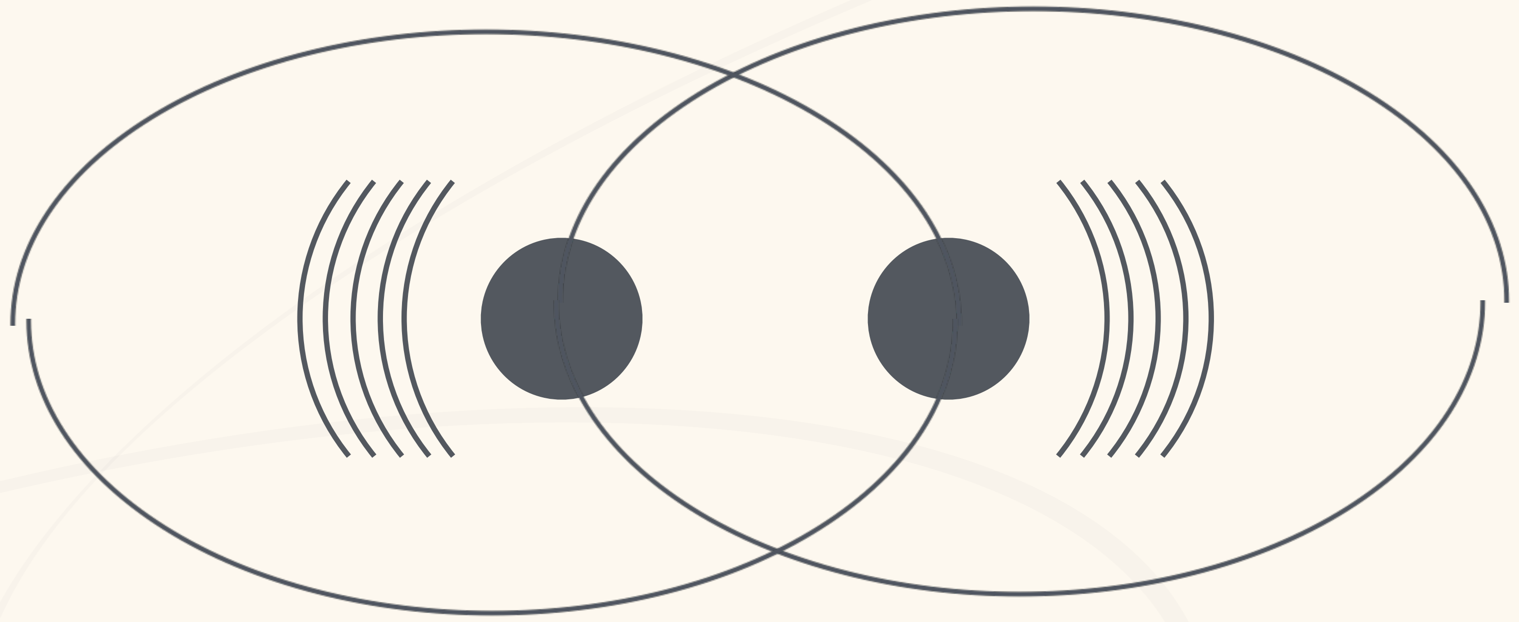
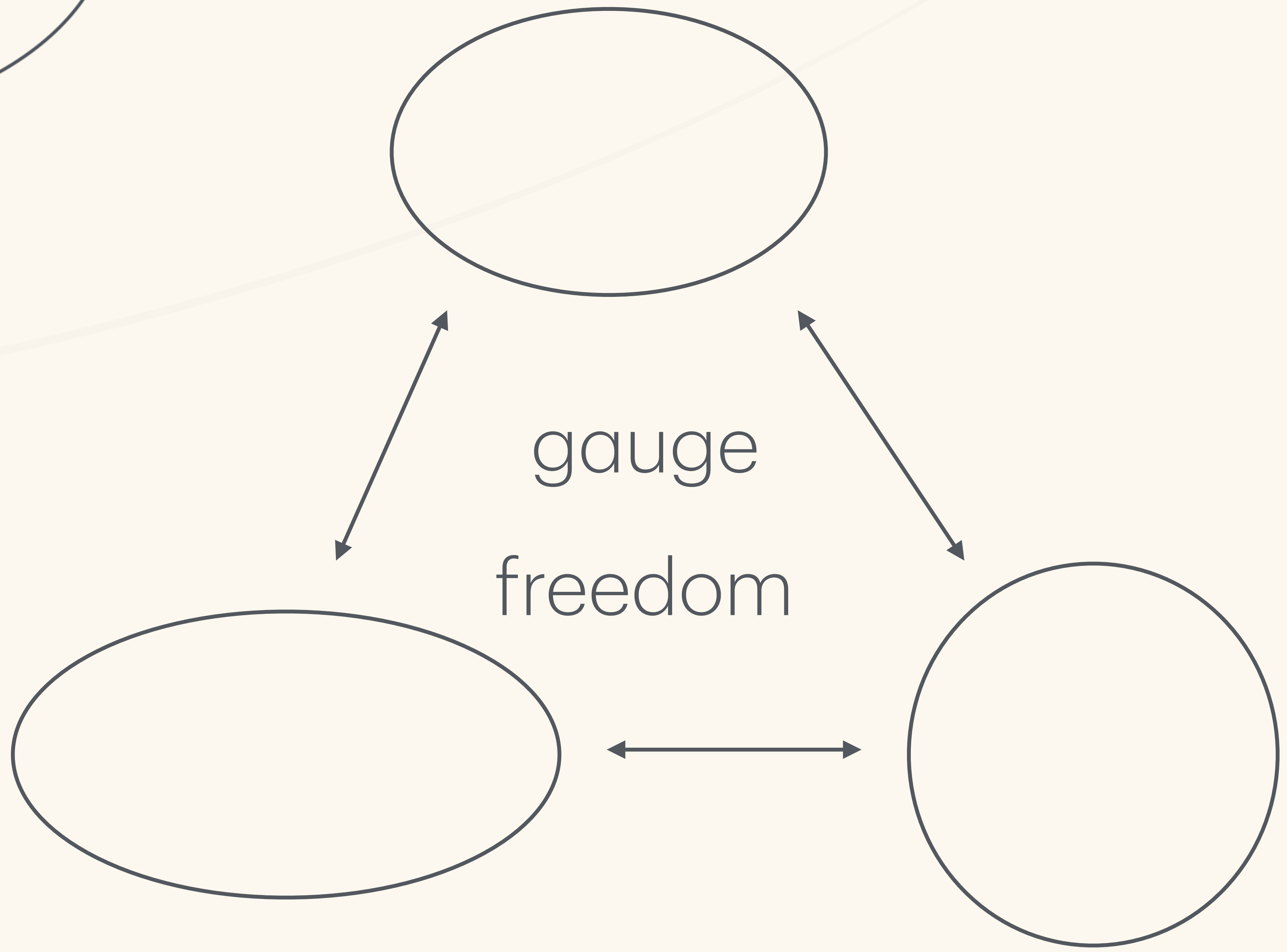


$$e = \sqrt{1 + \frac{2\epsilon h^2}{\mu^2}}$$

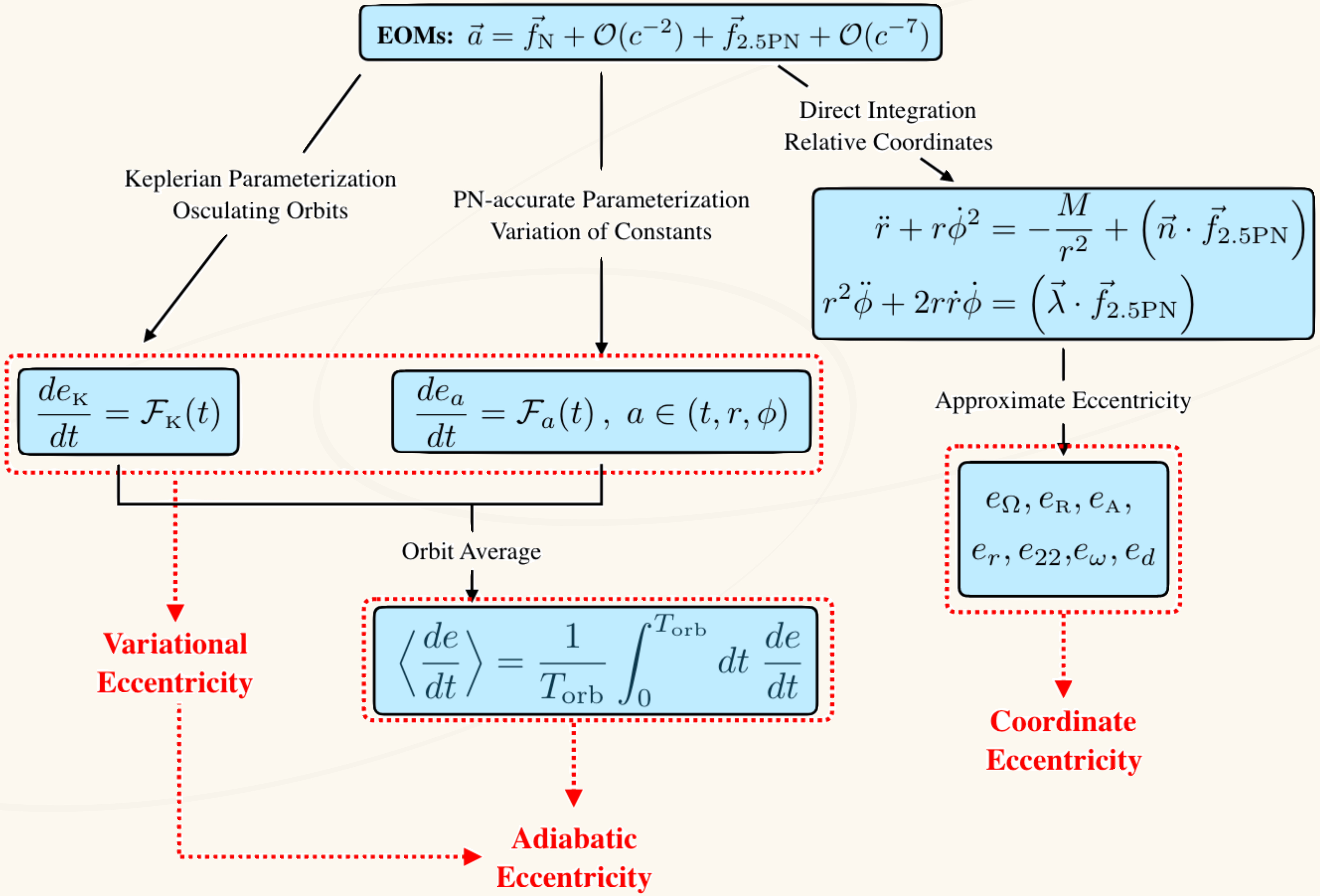
$$e = \frac{r_a - r_p}{r_a + r_p}$$



$$e = ??$$

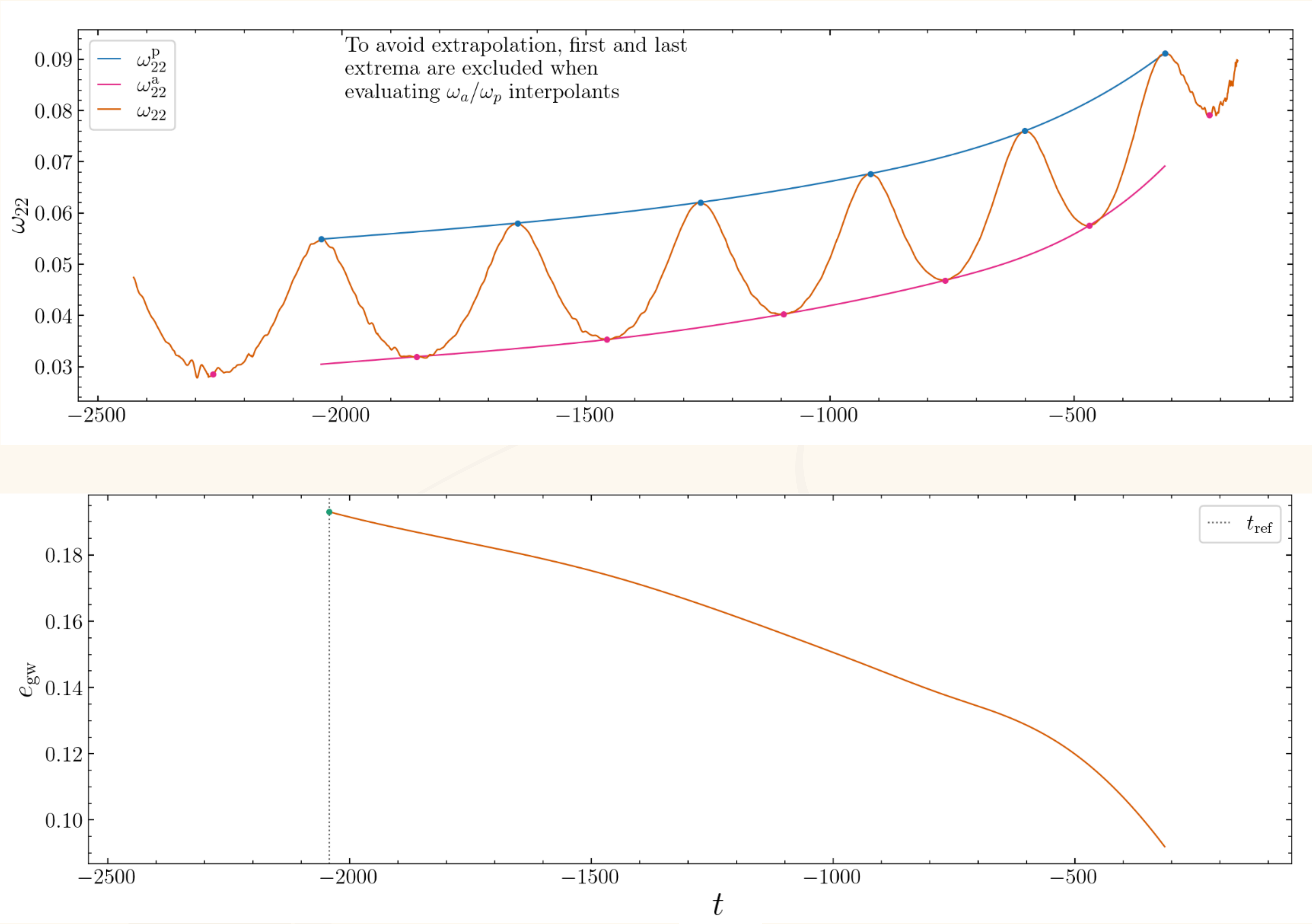


Many eccentricities in GR...



Loutrel+ (2019)

One of the latest



$$e_{\Omega_{orb}} = \frac{\sqrt{\Omega^p_{orb}} - \sqrt{\Omega^a_{orb}}}{\sqrt{\Omega^p_{orb}} + \sqrt{\Omega^a_{orb}}}$$

$$e_{\omega_{22}} = \frac{\sqrt{\omega^p_{22}} - \sqrt{\omega^a_{22}}}{\sqrt{\omega^p_{22}} + \sqrt{\omega^a_{22}}}$$

$$e_{gw} = \cos(\Psi/3) - \sqrt{3}\sin(\Psi/3)$$

$$\Psi = \arctan\left(\frac{1 - e^2_{\omega_{22}}}{2e_{\omega_{22}}}\right)$$

Gw-eccentricity, Shaikh+ (2023)

Ramos-Buades+ (2022)

Catastrophe theory

In a nutshell

- Function in n state variable $f(x_1, \ldots, x_n)$
    - stationary points $\nabla f = 0 \to \boldsymbol{x}_\star$ and their nature H

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- Function in n state variable and m control variable $f(x_1, ..., x_n \,|\, \lambda_1, ..., \lambda_m)$
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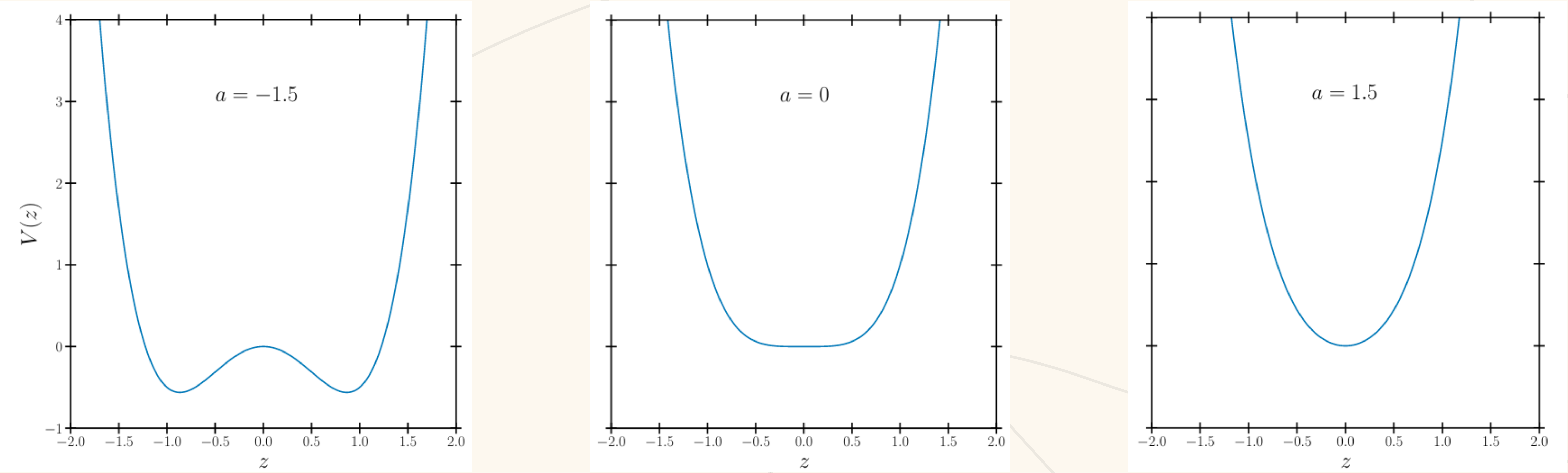
- A catastrophe happens when also $H \Big|_{\boldsymbol{\lambda}} (x_\star) = 0$

Catastrophe theory

In a nutshell

$$V(x \mid a) = z^4 - az^2$$



Catastrophe theory

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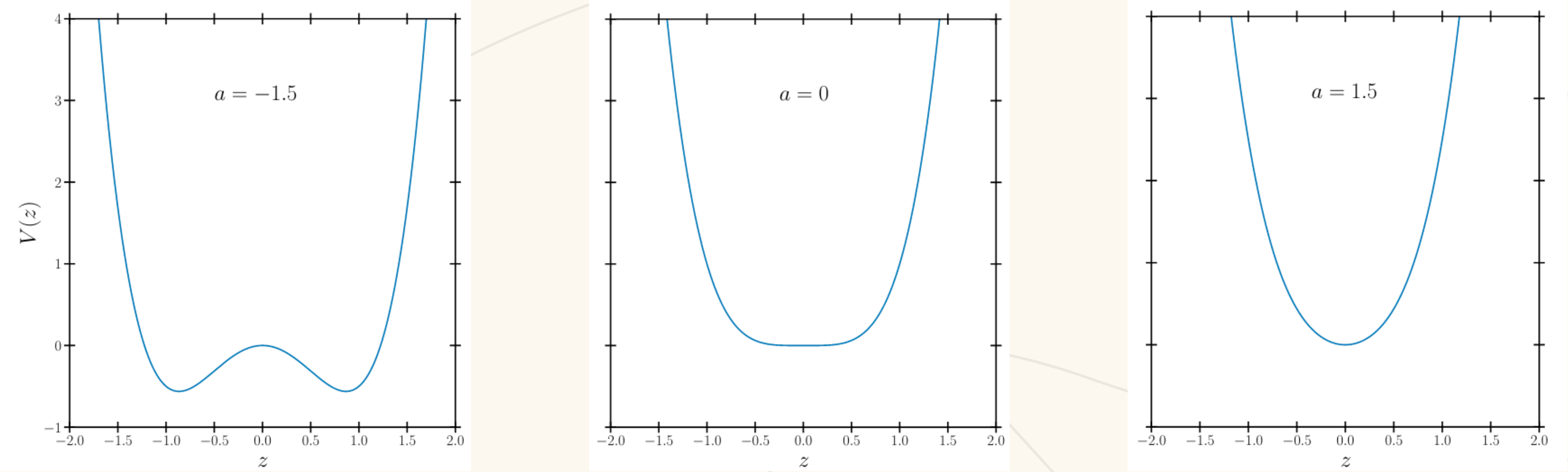

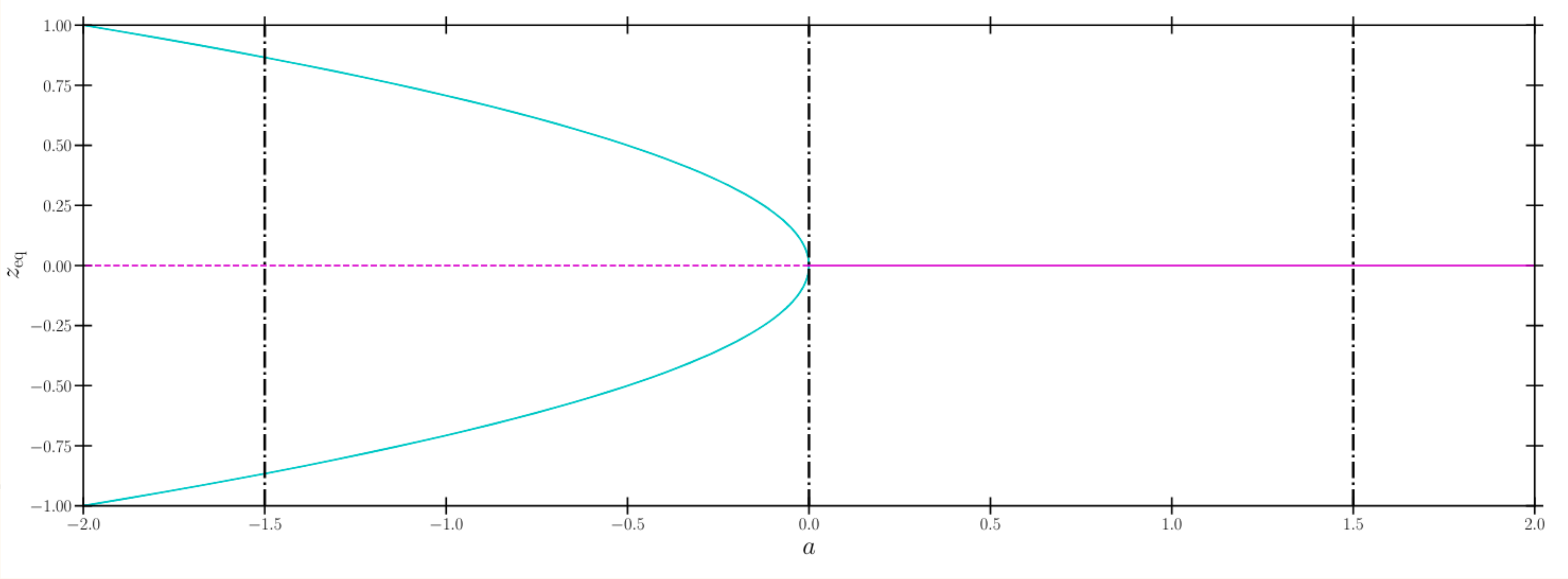

Loutrel (2023)

Eccentric catastrophes

In Keplerian integrals

$$E_m(f) \sim \int_{-\pi}^{+\pi} dt\, e^{2\pi i f t \pm i m V(t)}$$

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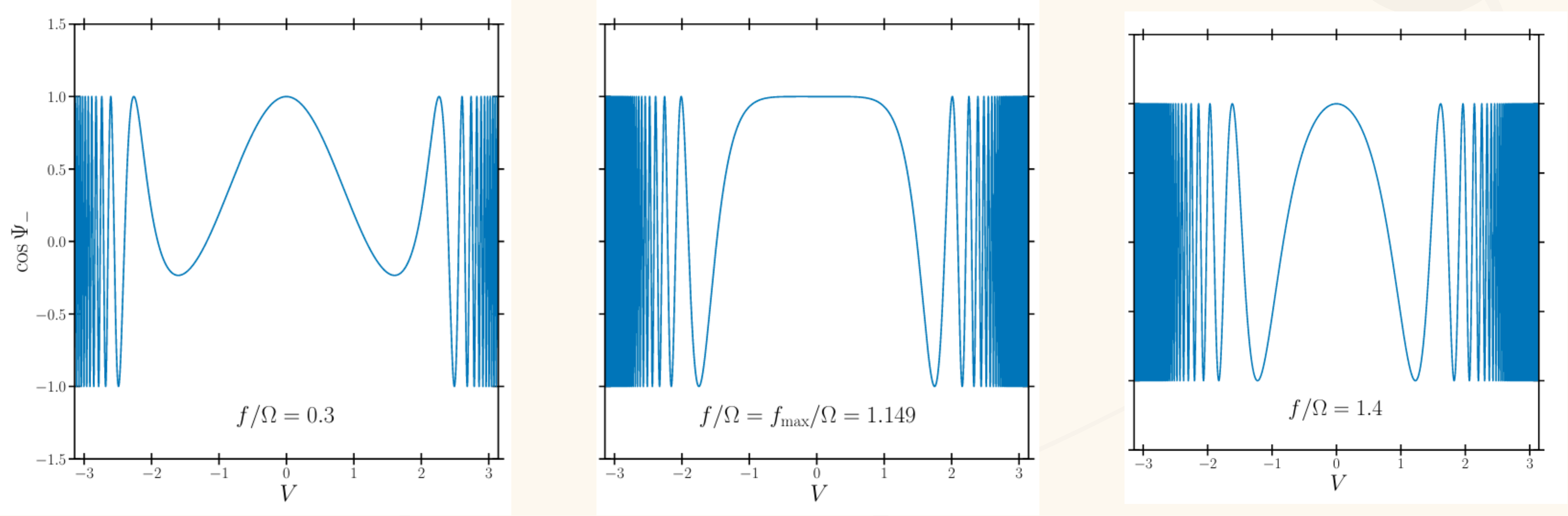
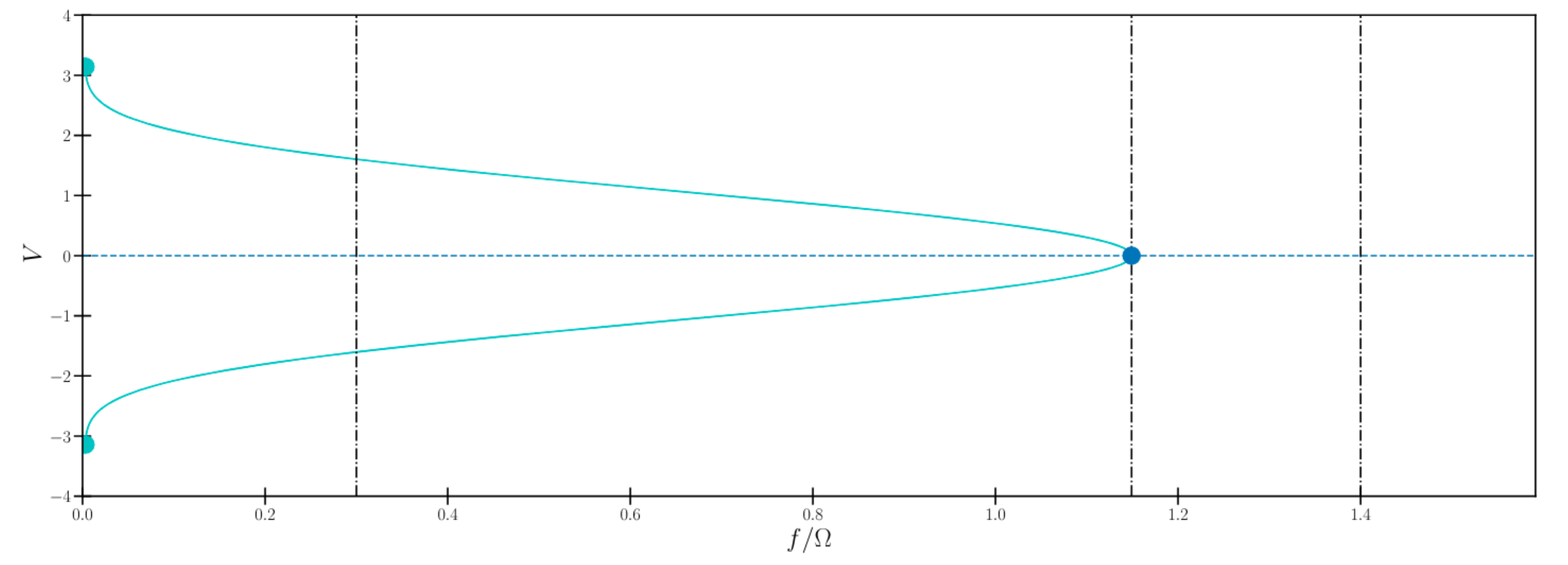
$$V_\star^{1,2} = \pm \cos^{-1}\left[\frac{1}{e}\left(\sqrt{\frac{\mp 2\pi f}{m\Omega}} - 1\right)\right]$$

$$f_{min} = \frac{m\Omega}{2\pi}(1-e)^2$$

$$f_{max} = \frac{m\Omega}{2\pi}(1+e)^2$$



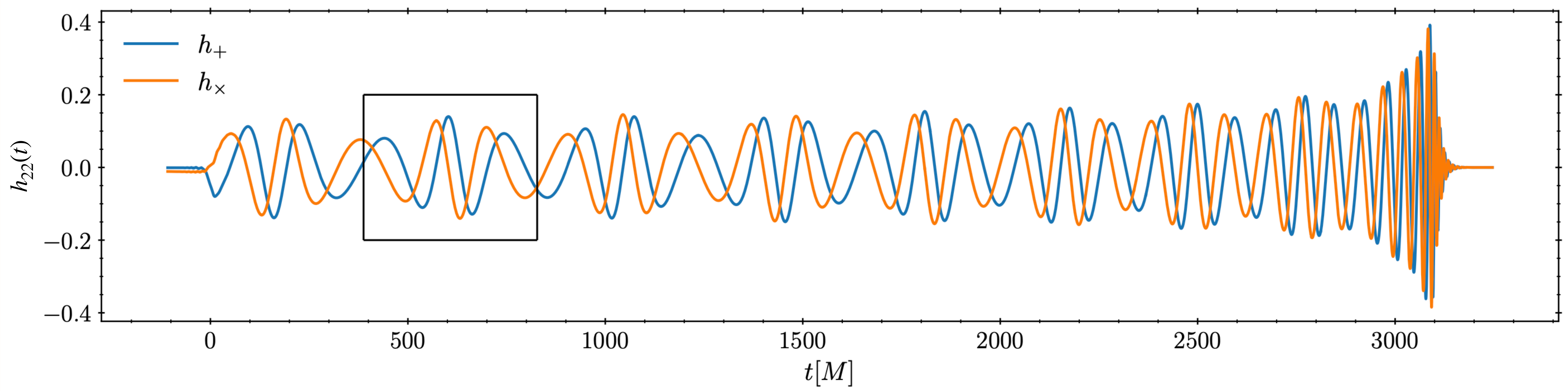


Loutrel (2023)

Eccentric catastrophes

In NR waveform

SXS:BBH:1360, 10.5281/zenodo.3326460, Hinder+(2019)

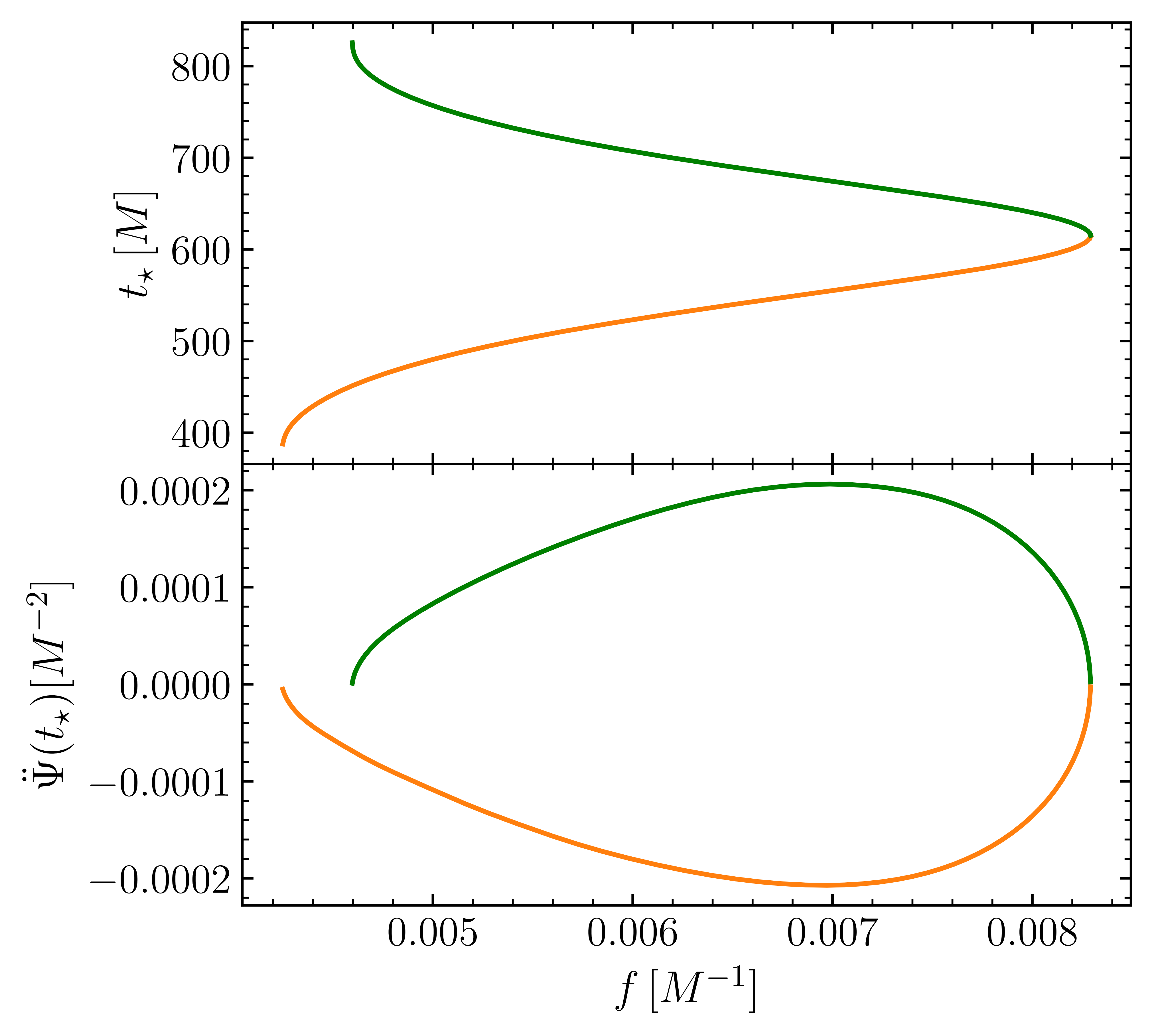


$$h_{22}^{+,\times}(t) \longrightarrow h(t) = A(t)\, e^{i\phi(t)}$$

$$\Psi(t) = 2\pi f t - \phi(t) \longrightarrow \dot{\Psi} = 0, \ddot{\Psi}$$

Eccentric catastrophes

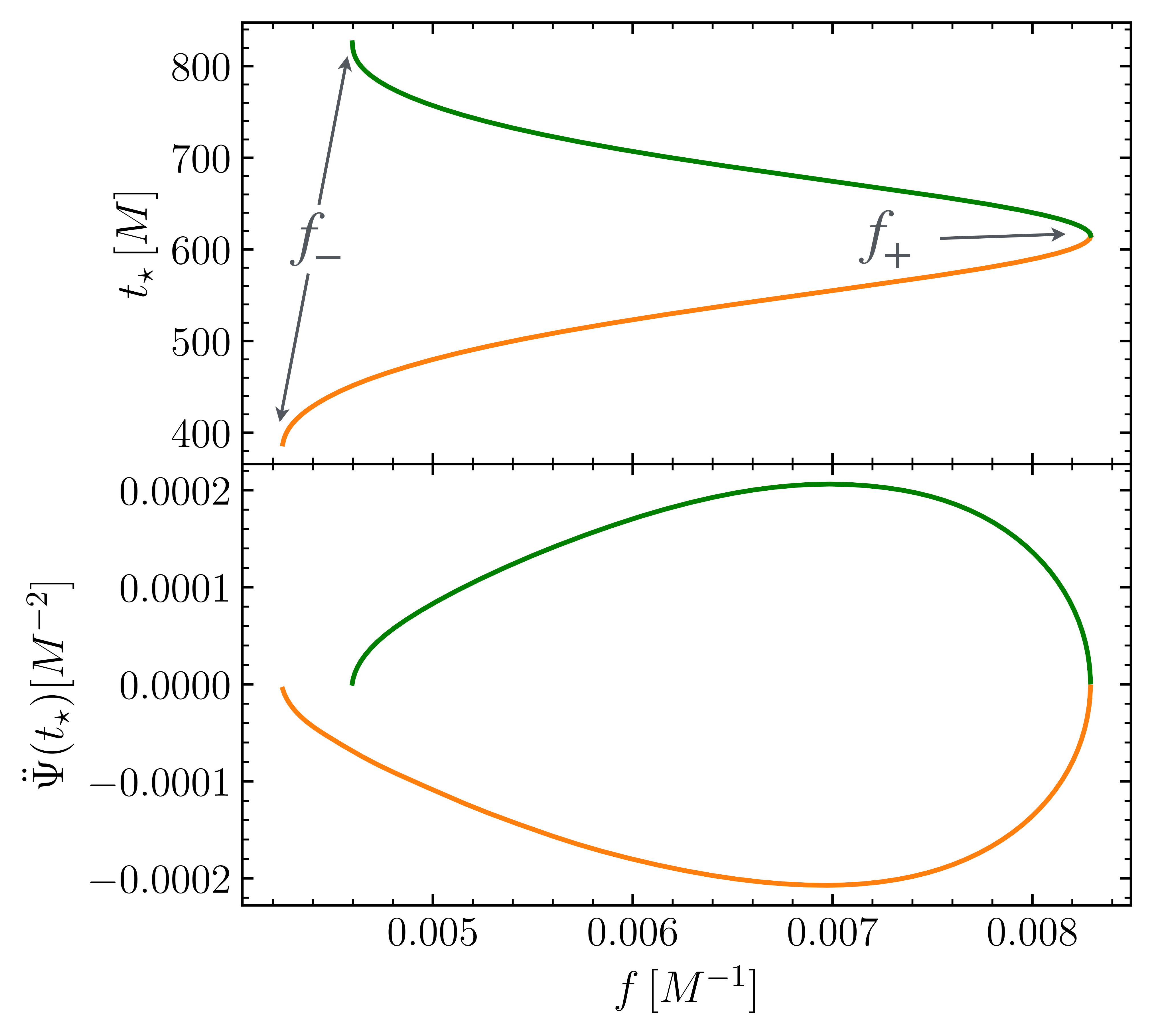
In NR waveform



MB+ (in prep.)

Eccentric catastrophes

In NR waveform



$$f_-, f_+ \longrightarrow \frac{f_+ - f_-}{f_+ + f_-} \overset{?}{\longrightarrow} e$$

MB+ (in prep.)

Eccentric catastrophes

PN analysis

PN analytic waveforms

$(2,2)_{0PN}$ ⟶ $$f_{ratio} = \frac{1}{2} e \left(3 - e^2\right)$$

$(3,3)_{0.5PN}$ ⟶ $$f_{ratio} = \frac{396e - 46e^3}{243 + 107e^2}$$

$(4,4)_{1PN}$ ⟶ $$f_{ratio} = \frac{2e(6944 - 5700e^2 + 261e^4)}{8192 - 1513e^2 - 3669e^4}$$

MB+ (in prep.)

Eccentric catastrophes

PN analysis

PN analytic waveforms

$(2,2)_{0PN}$ → $$\Theta = \arccos\left(\frac{\sqrt{1-f_{ratio}^2}}{f_{ratio}}\right) \qquad e_c^{(2,2)} = \cos(\Theta/3) - \sqrt{3}\sin(\Theta/3)$$

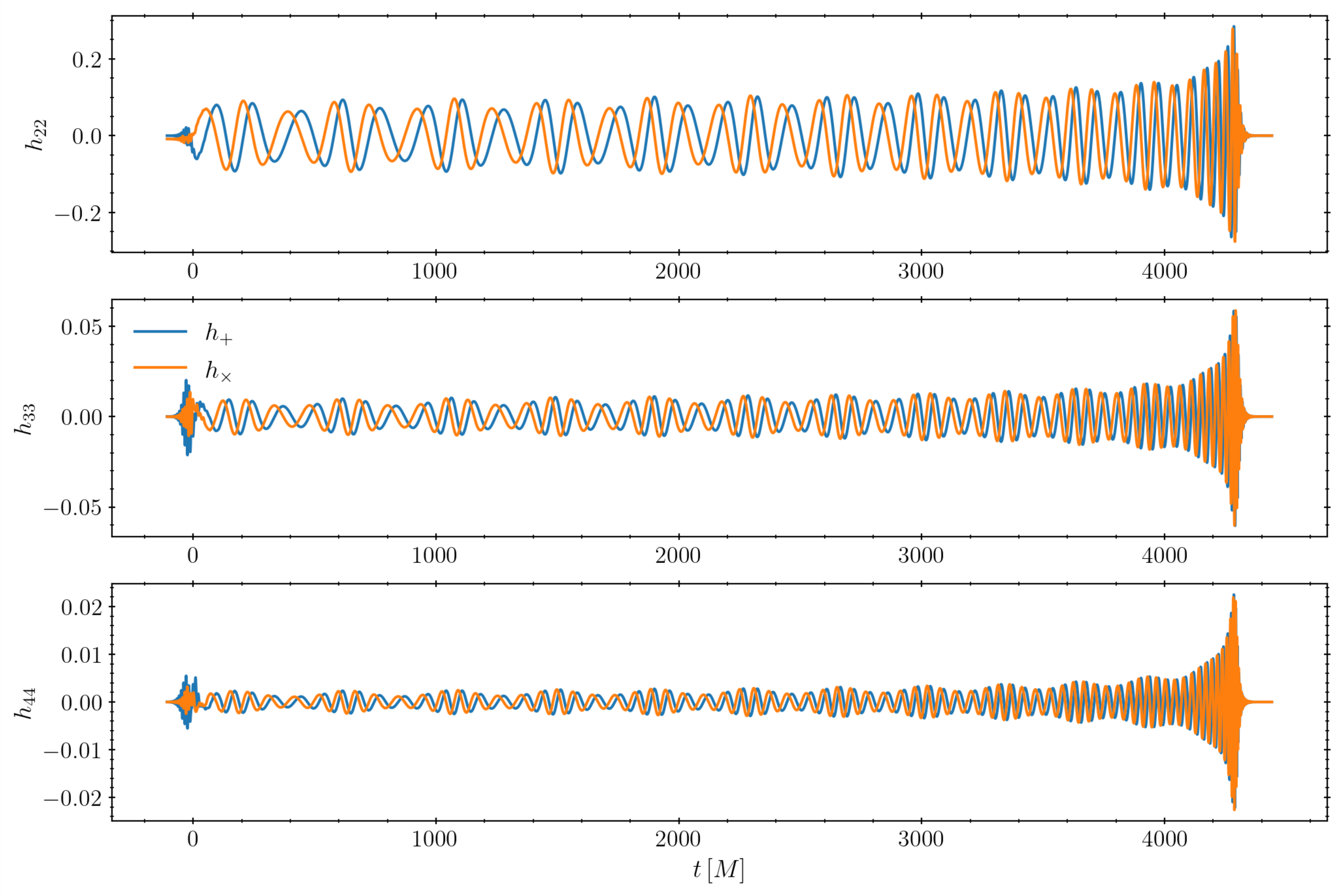
$(3,3)_{0.5PN}$ → $$\Theta = \arccos\left(\frac{1242\sqrt{3}\sqrt{35266176 - 31184003 f_{ratio}^2 - 3675129 f_{ratio}^4}}{15712542 f_{ratio} + 1225043 f_{ratio}^3}\right)$$

$$e_c^{(3,3)} = \frac{1}{138}\Big(-107 f_{ratio} + \sqrt{54648 + 11449 f_{ratio}^2}\cos(\Theta/3) + -\sqrt{163944 + 34347 f_{ratio}^2}\sin(\Theta/3)\Big)$$

$(4,4)_{1PN}$ → Numerical inversion $e_c^{(4,4)} = \ldots$

MB+ (in prep.)

Eccentric catastrophes



SXS:BBH:1372

10.5281/zenodo.3326488

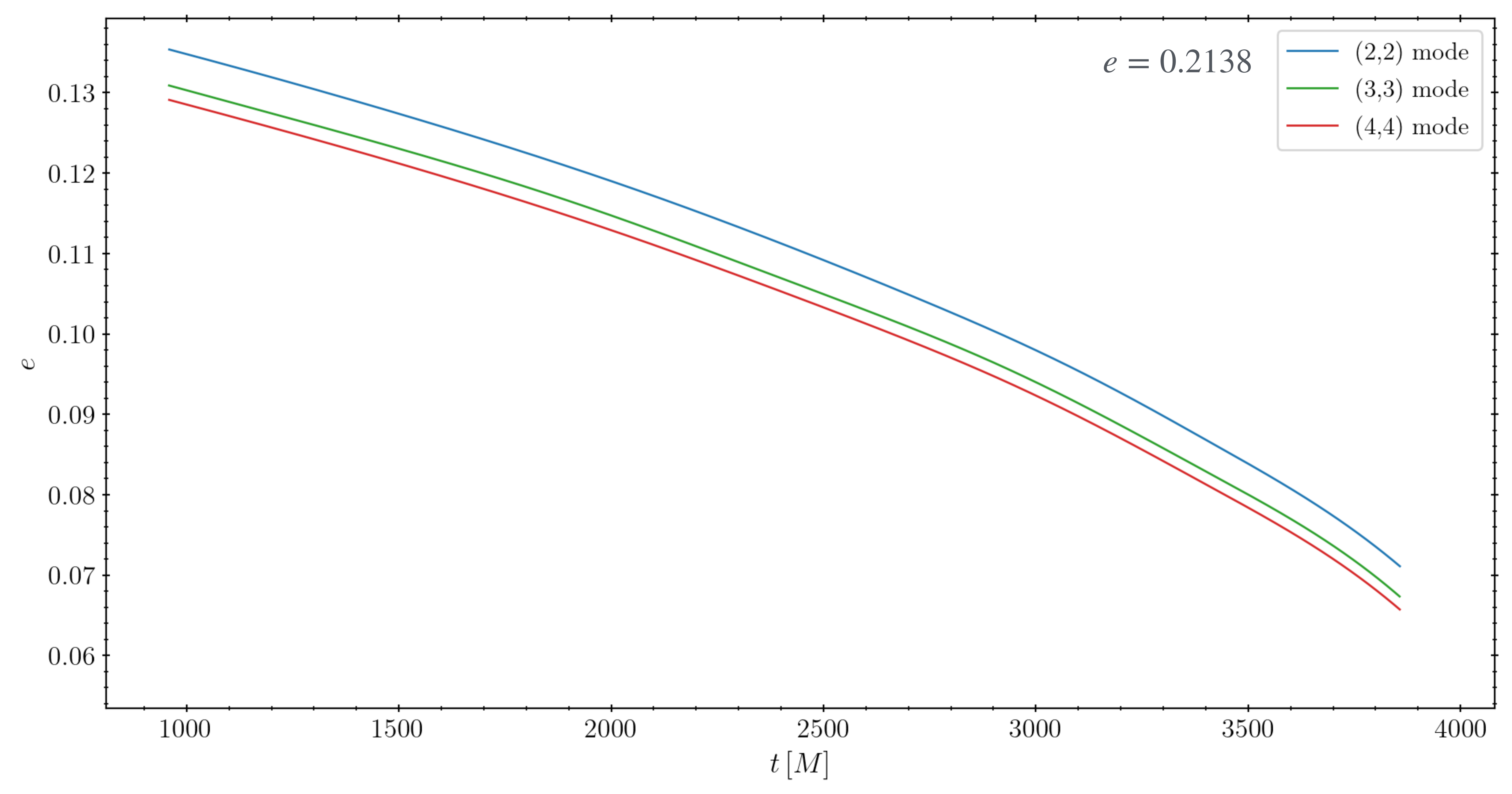
$q = 0.33$

$e = 0.2138$

$\chi_1 = \left[2.20 \times 10^{-8},\ 2.35 \times 10^{-8},\ 3.95 \times 10^{-5}\right]$

$\chi_1 = \left[3.36 \times 10^{-8},\ -4.11 \times 10^{-8},\ 4.56 \times 10^{-6}\right]$

Eccentric catastrophes



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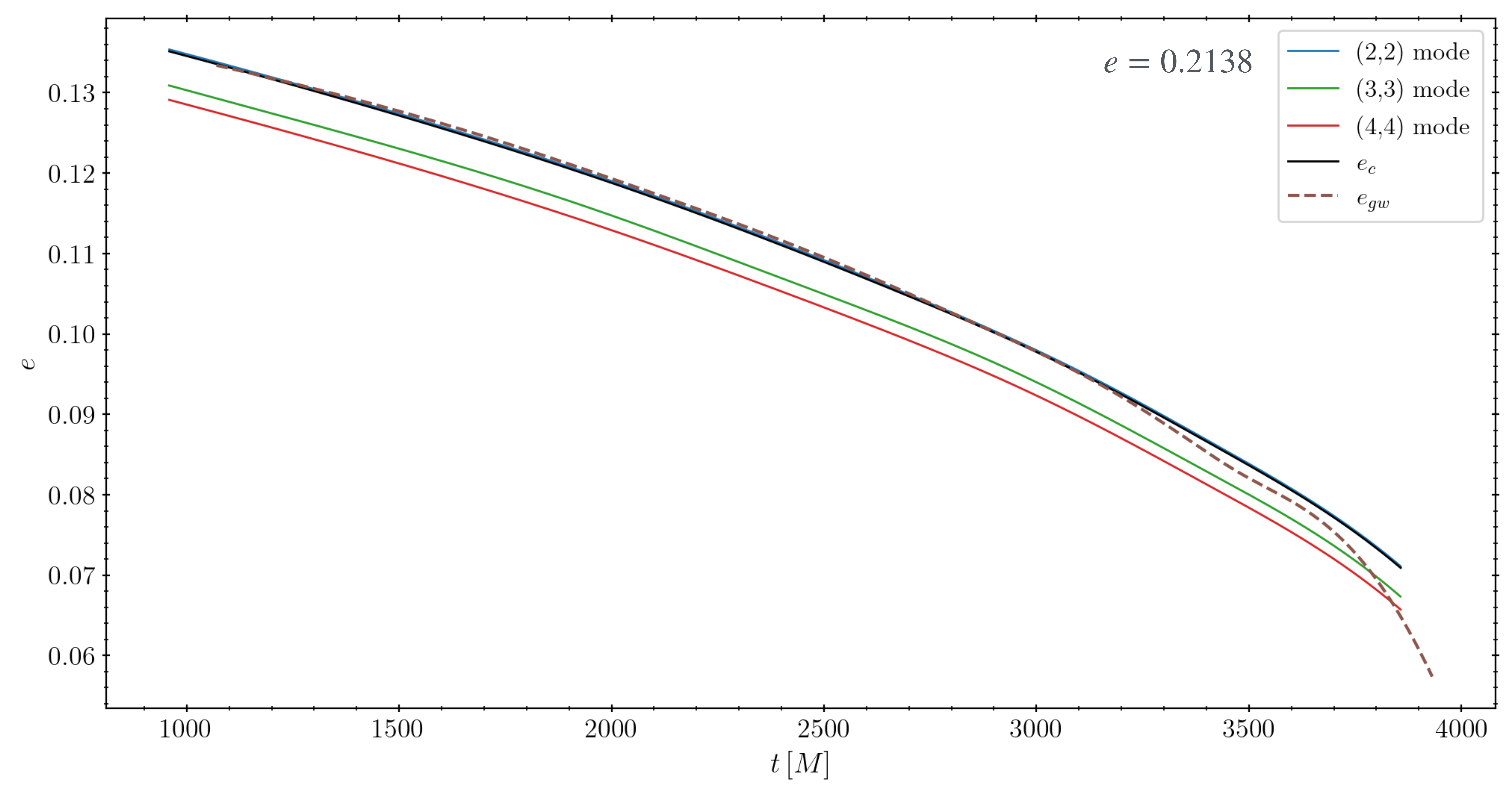
Eccentric catastrophes



MB+ (in prep.)

Summary

Eccentricity in GR is challenging

NR waveforms can be studied using catastrophe theory

Gauge-free definition of eccentricity

Future work

Paper&code

More PN terms and modes

Eccentricity estimate for GW events

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Thank you!

Backup slides

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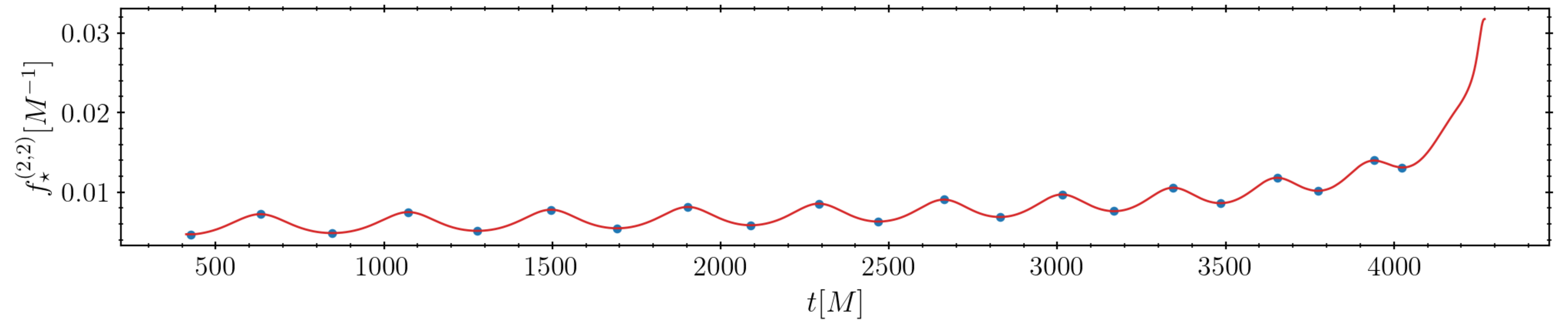
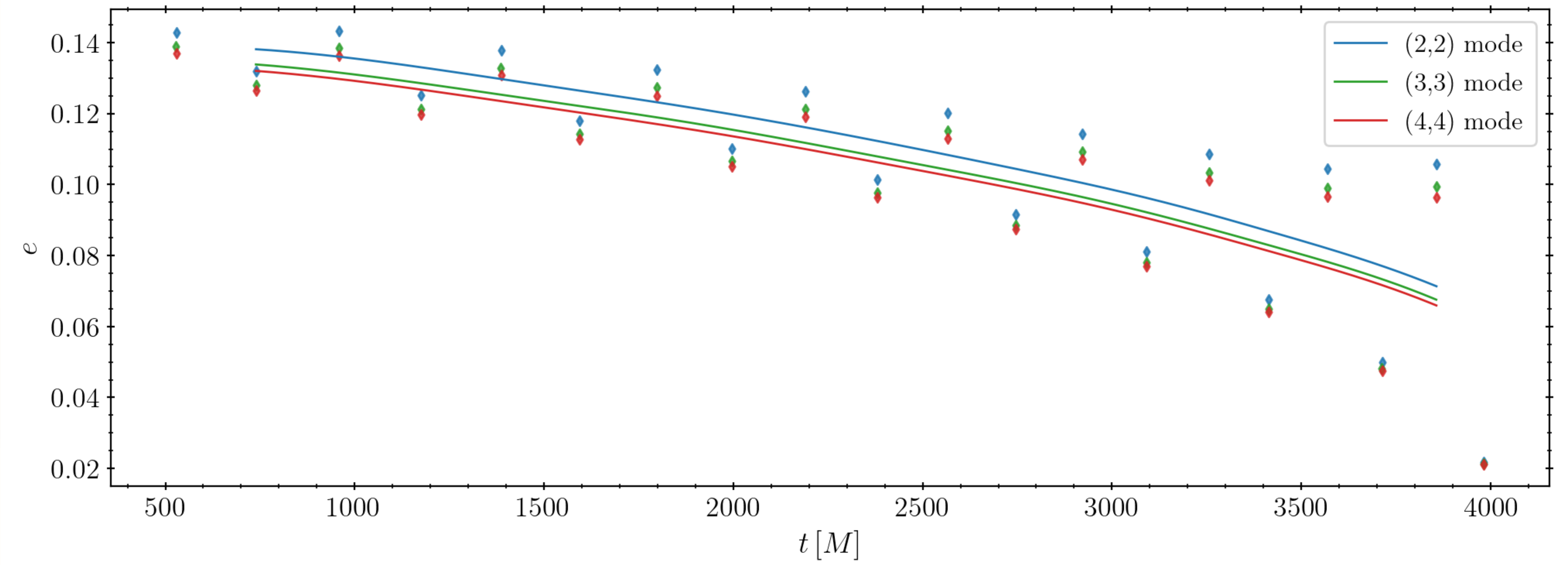
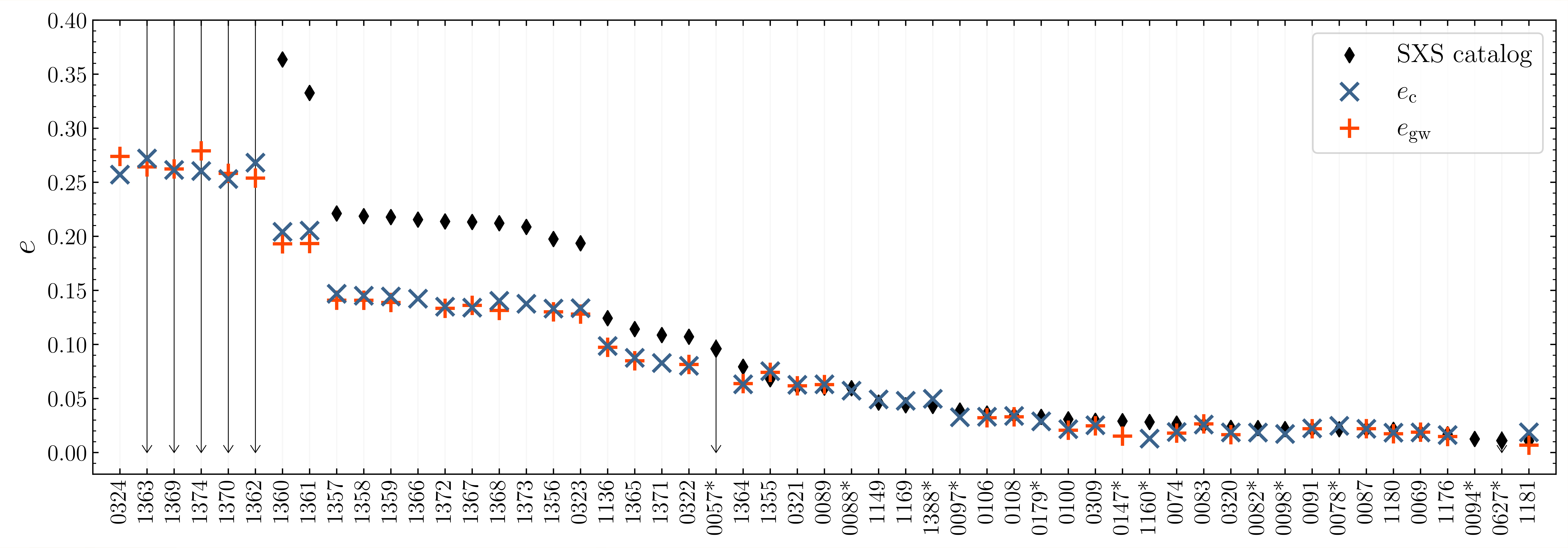




SXS eccentric catalog



RIT eccentric catalog