The conundrum of tidal Love numbers

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Outline

- The static and dynamic tidal Love numbers.
- Dynamical tidal Love numbers for black holes
- Dynamical tidal Love numbers for reflective compact object.

References

- Nair, **SC** and Sarkar, **PRD 107**, 124041 (2023) + Work in Progress.
- **SC**, Maggio, Pani and Silvestrini, arXiv:2310.06023 + Work in Progress.
- Bhatt, **SC** and Bose, **PRD 108**, 084013 (2023) + arXiv 2406.09543.

Tides are Everywhere

- Tides are ubiquitous and captures the true nature of gravity, as it depends on the Riemann.
- Deformation produced by the tides depends on the detail of the constituent of the object being deformed.
- What about black holes? Can they be deformed?
- In the context of GW, the effects due to tidal deformation appears at 5 pN, thus unless the deformation is large, detection can be challenging.
- What happens if we consider objects as compact as BHs, but with reflective surface? Can they be deformed?
- Does deformation depends on asymptotic properties of spacetime? Can a Schwarzschild-de Sitter BH be deformed?

Newtonian Deformation

- Newtonian gravity is solely governed by the potential.
- The tidal field arises from double derivative of the potential.

$$\mathcal{E}_{ij} = \frac{\partial^2 \Phi_{\rm ext}}{\partial x^i \partial x^j}.$$

- The total potential will consist of potential due to the external tidal field and the potential of the deformed object. **[Thorne, Phys. Rev. D 58, 124031 (1998)]**

$$\frac{(1-g_{tt})}{2} = -\frac{M}{r} - \frac{3Q_{ij}}{2r^3}\left(n^i n^j - \frac{1}{3}\delta^{ij}\right) + O\left(\frac{1}{r^3}\right) + \frac{1}{2}\mathcal{E}_{ij}x^i x^j + O\left(r^3\right),$$

- Quadrupole moment of the deformed object arises due to tidal field.

$$Q_{ij} = -\lambda \mathcal{E}_{ij}.$$

$$k_2 = \frac{3}{2} G \lambda R^{-5}.$$

Static Love number of BHs

- BH perturbation theory becomes directly applicable.

$$g_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h_{\alpha\beta},$$

$$h_{\alpha\beta} = \mathrm{diag}\left[e^{-\nu(r)}H_0(r),\ e^{\lambda(r)}H_2(r),\ r^2K(r),\ r^2\sin^2\theta K(r)\right] Y_{2m}(\theta,\varphi).$$

- Among the three unknown functions, perturbed Einstein's equations demand $H_2 = H_0 \equiv H,$ and K can be expressed in terms of H.

$$\left(x^2-1\right)H'' + 2xH' - \left(6 + \frac{4}{x^2-1}\right)H = 0.$$

[Hinderer, arXiv: 0711.2420]

- Exact solution exist, in terms of associated Legendre polynomials.

$$H = c_1\left(\frac{r}{M}\right)^2\left(1-\frac{2M}{r}\right)\left[-\frac{M(M-r)(2M^2+6Mr-3r^2)}{r^2(2M-r)^2} + \frac{3}{2}\log\left(\frac{r}{r-2M}\right)\right] + 3c_2\left(\frac{r}{M}\right)^2\left(1-\frac{2M}{r}\right).$$

[near horizon]

[infinity]

$$H = \frac{8}{5}\left(\frac{M}{r}\right)^3 c_1 + O\left(\left(\frac{M}{r}\right)^4\right) + 3\left(\frac{r}{M}\right)^2 c_2 + O\left(\left(\frac{r}{M}\right)\right),$$

Love number of compact objects

- Generically, compact objects are described by two parameters

$$r_0 = r_+(1+\epsilon)\,,$$

$$\mathcal{R}(\omega) = \left[\frac{1 - \frac{i}{\bar{\omega}}\left(\frac{1}{X}\frac{dX}{dr_*}\right)}{1 + \frac{i}{\bar{\omega}}\left(\frac{1}{X}\frac{dX}{dr_*}\right)}\right]_{r_*^0}$$

- In the static case, the reflectivity makes sense for Dirichlet and Neumann conditions on the surface.

[Cardoso +, arXiv: 1701.01116]

$$k_2^{\text{polar}} = \frac{8}{5\,(7 + 3\ln\epsilon)}$$

- Thus, departure from BH leads to non-zero static Love number with a Logarithmic behaviour.

Questions to Address

- **Q1**: How to remove gauge ambiguities? Can we provide a formalism based on scalars?
- **Q2**: What happens in a dynamical situation?
- The GW observations of tidal Love number happens in the in-spiral phase, which is inherently dynamical.
- **Q3**: Do BHs have vanishing dynamical tidal Love numbers?
- **Q4**: Does Logarithmic behaviour for compact objects retained in dynamical context as well?

Tidal Love number from scalar

- For the Love number to be gauge invariant, one solves the Teukolsky equation for the Weyl scalars and impose boundary conditions near the horizon.

$$\lim_{c\to\infty} c^2\psi_0 = \sum_{\ell m} \alpha_{\ell m}(t)\, r^{\ell-2}\left[1 + 2k_{\ell m}\left(\frac{R}{r}\right)^{2\ell+1}\right] {}_2Y_{\ell m}(\theta,\varphi)\,,$$

$$F_{\ell m}(\omega) = 2\,k_{\ell m} + i\omega\tau_0\,\nu_{\ell m} + \mathcal{O}(\omega^2)\,.$$

- The real part determines the tidal Love numbers, while imaginary part determines the dissipation.

[Chia, arXiv:2010.07300]

[Le Tiec +, arXiv:2010.15795]

Dynamical Love number for BHs

- Starting point is the radial Teukolsky equation in the ingoing null coordinate.
- In small frequency and near horizon limit, it can be solved exactly.

$$R(z) = (z+1)^{2-N_3}\left[c_1 z^2\, {}_2F_1\left(3+l-N_2, 2-l-N_1; 3+2iP_+; -z\right)\right. \\ \left. + c_2 z^{-2iP_+}\, {}_2F_1\left(l-2iP_+-N_2+1, -l-2iP_+-N_1; -1-2iP_+; -z\right)\right]$$

[Bhatt, SC and Bose, arXiv: 2406.09543]

- For BH, imposing purely ingoing boundary condition near the horizon, the second solution identically vanishes. Also, N_1 and N_2 are both $\mathcal{O}(M\omega)$.

$$P_\pm = \frac{am - 2r_\pm M\omega}{r_+ - r_-}$$

$$N_3 = \frac{12\omega\,(r_+ - M)^2}{am + 2i(r_+ - M)}$$

Dynamical Love for Kerr BH

- The hypergeometric function can be further expanded in the intermediate region.

$$\Psi_4^{\text{intermediate}} \propto z^{l-2+N_1-N_3} \left\{ \frac{\Gamma(3+2iP_+)\,\Gamma(2l+N_1-N_2+1)}{\Gamma(l+2iP_++N_1+1)\,\Gamma(3+l-N_2)} \right\} \times \left[1 + z^{-2l-1+N_2-N_1} \left\{ \frac{\Gamma(-2l-N_1+N_2-1)\,\Gamma(l+2iP_++N_1+1)\,\Gamma(3+l-N_2)}{\Gamma(2-l-N_1)\,\Gamma(-l+2iP_++N_2)\,\Gamma(2l+N_1-N_2+1)} \right\} \right]$$

- Assuming l to be a complex number, the Gamma functions can be simplified. For zero frequency, the response function for Kerr BH becomes

$$F_{lm}^{\text{static}} = -\left(\frac{iam}{r_+-r_-}\right) \frac{(l-2)!\,(l+2)!}{(2l+1)!\,(2l)!} \prod_{j=1}^{l} \left[j^2 + \left(\frac{2am}{r_+-r_-}\right)^2 \right]$$

[Bhatt, SC and Bose, arXiv: 2406.09543]

Love of Arbitrary Rotating BH

- For an arbitrary rotating BH, the response function with all linear-in-frequency term become:

[Bhatt, SC and Bose, arXiv: 2406.09543]

$$F_{lm}=\frac{\Gamma\left(-2l-N_1+N_2-1\right)\Gamma\left(l+2iP_{+}+N_1+1\right)\Gamma\left(3+l-N_2\right)}{\Gamma\left(2-l-N_1\right)\Gamma\left(-l+2iP_{+}+N_2\right)\Gamma\left(2l+N_1-N_2+1\right)}$$

- The Gamma functions can be simplified using reflection formula, as well as doing a power series expansion, leading to di-gamma functions.

- The above response function turns out to have a real part. The associated tidal Love numbers become

$$k_{lm}=(am)^2\,k_{lm}^{(0)}+am\omega\,k_{lm}^{(1)}+\mathcal{O}(M^2\omega^2)$$

Compact Objects can be deformed

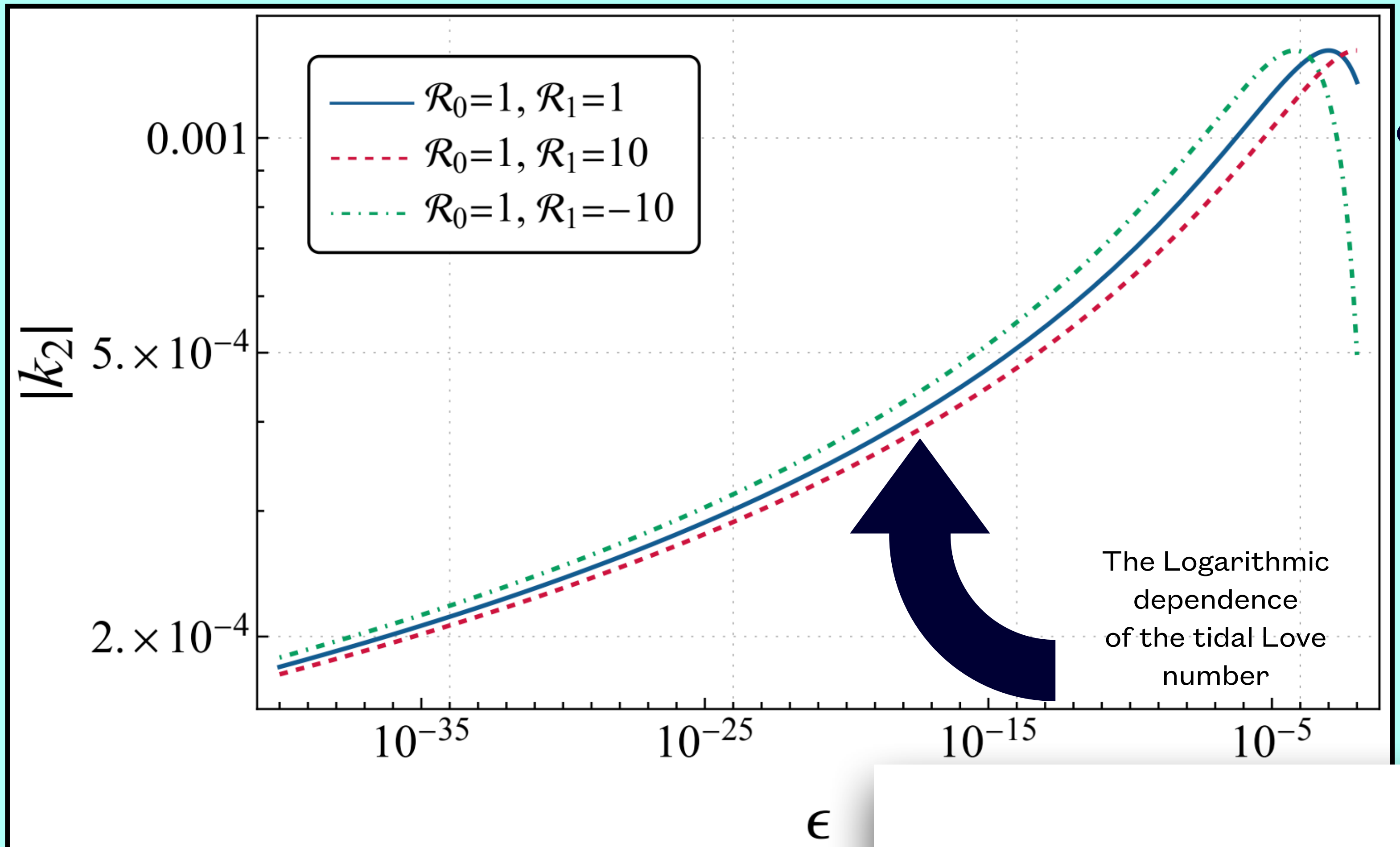
- Compact Objects having non-zero reflectivity, in general, have non-zero Love number. [SC, Maggio, Silvestrini and Pani, arXiv: 2310.06023]

$$k_2 = \frac{2}{15}\mathrm{Re}\left[\frac{1}{-2\mathcal{R}_1 + \{7 + 16i\pi + 8(\epsilon + \ln\epsilon)\}}\right]$$

- Reflectivity is defined in terms of the Detweiler function, which can be related to the Teukolsky function.

- It turns out that the Love number is non-zero, if and only if

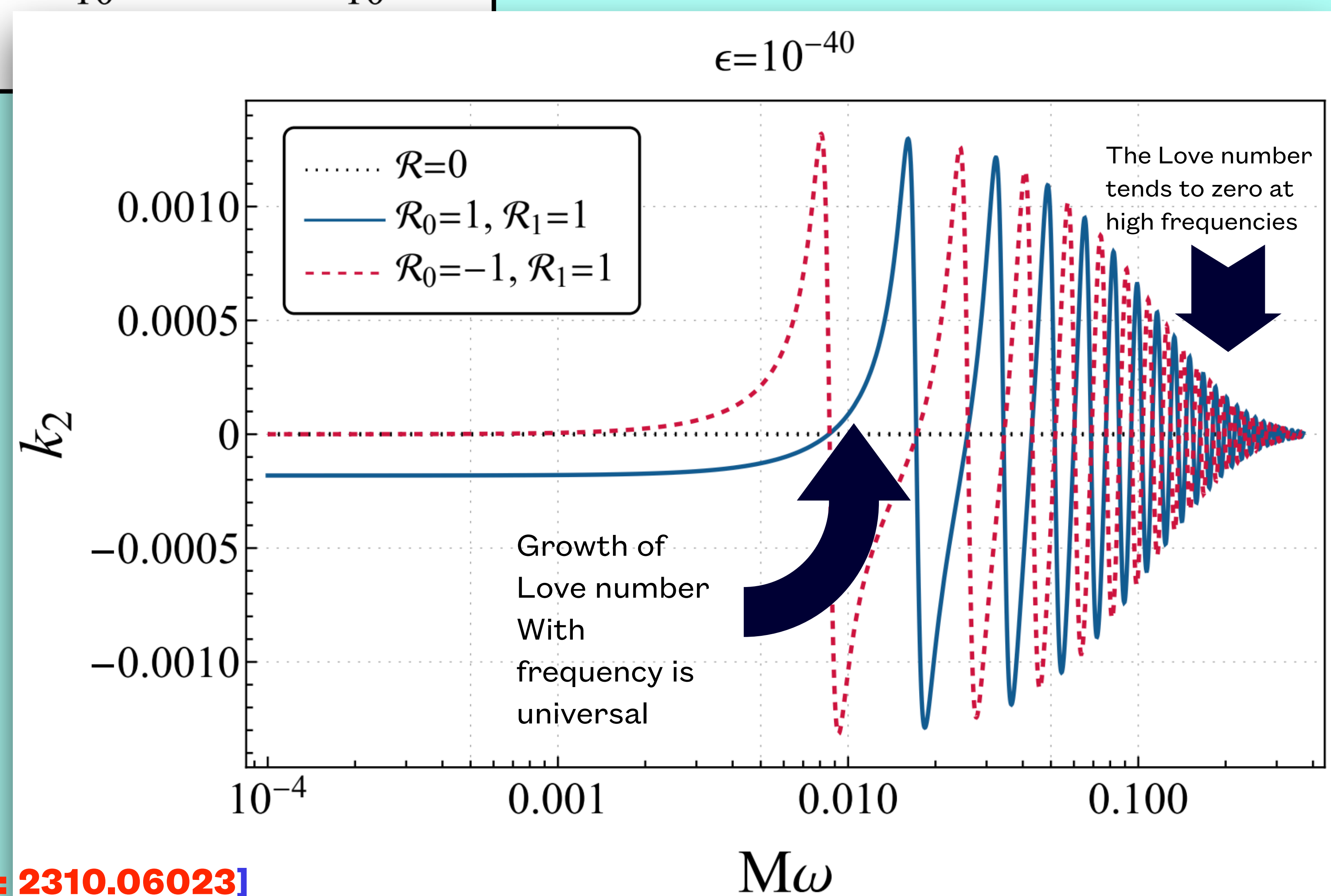
$$\mathcal{R}(\omega) = 1 + iM\omega\mathcal{R}_1$$



The frequency dependent Part of the reflectivity takes over

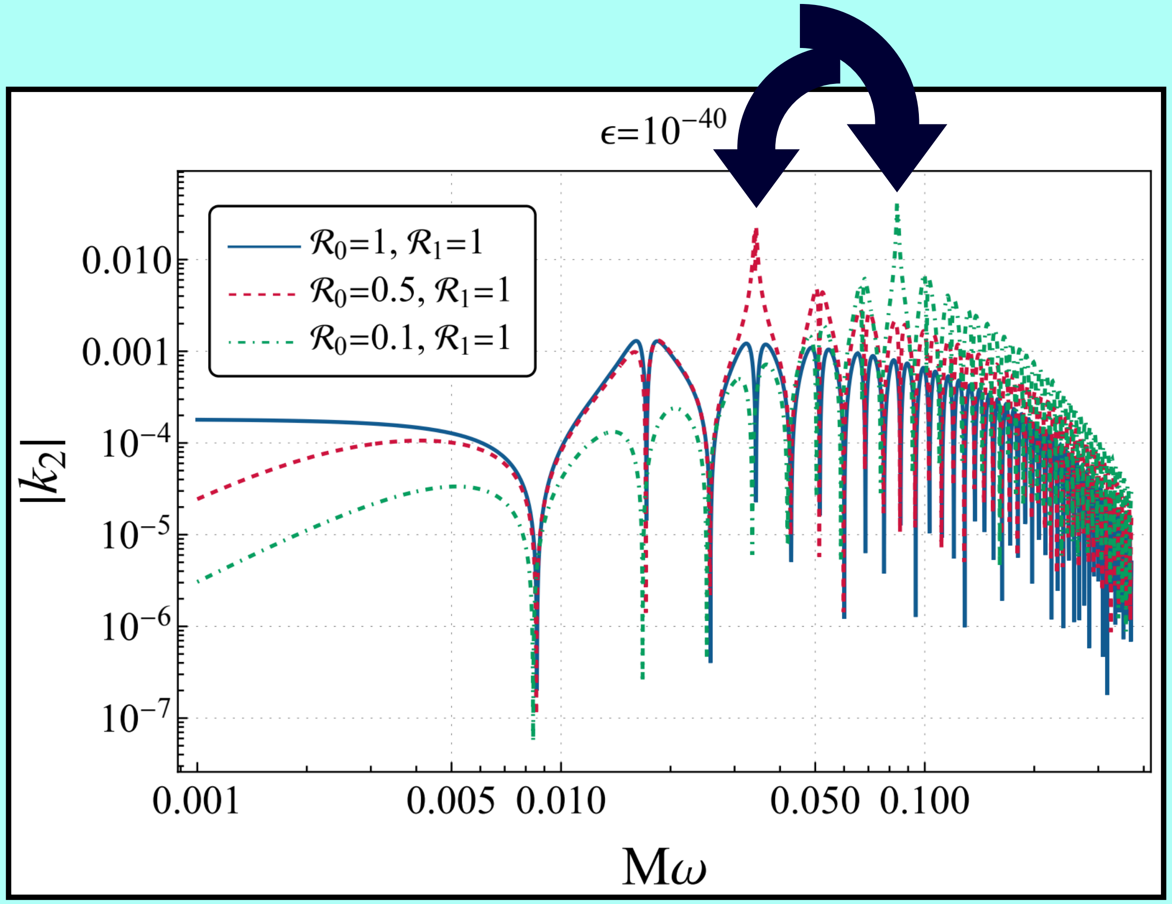
- The Logarithmic behaviour is specific to unit reflectivity.
- Love number heavily depends on the phase of the reflectivity.

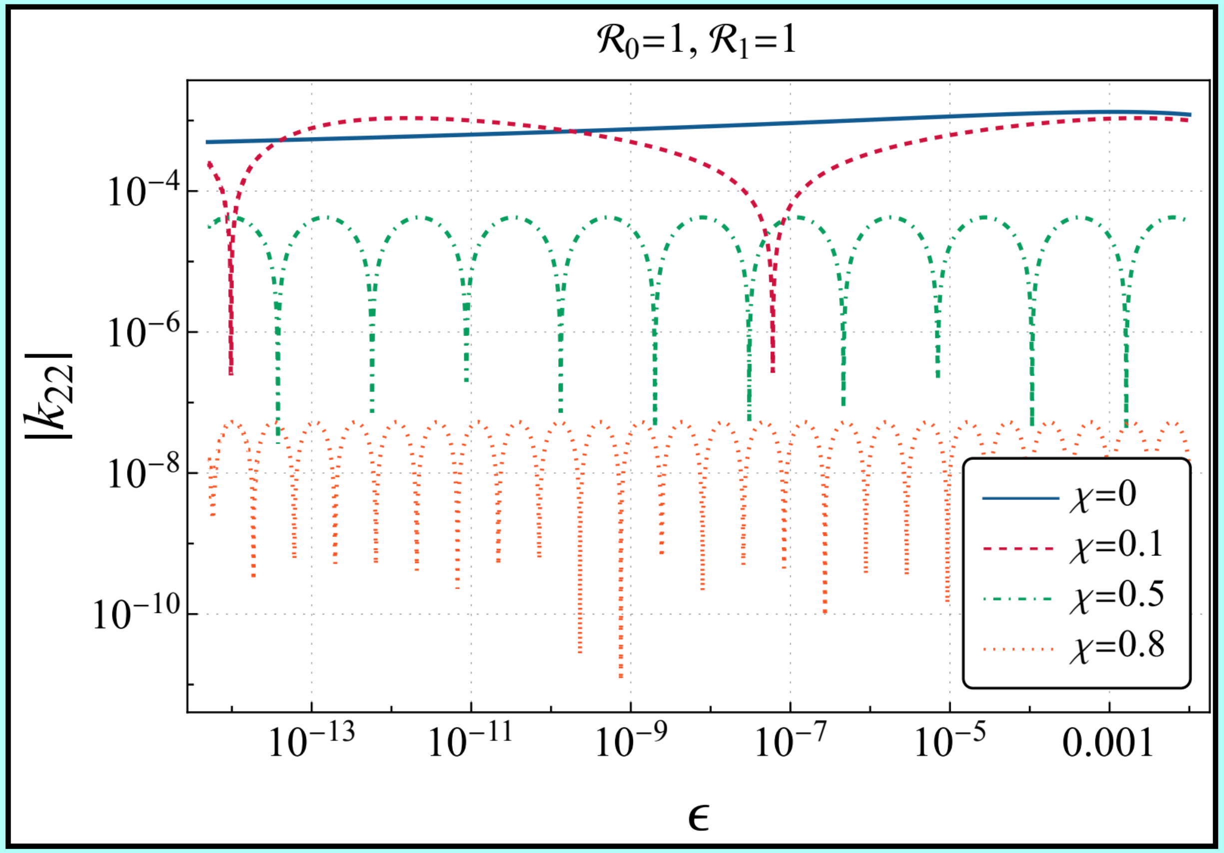
- The tidal Love number grows with frequency.
- At higher frequencies the Love number tends to zero → the compact object effectively behaves as a BH.

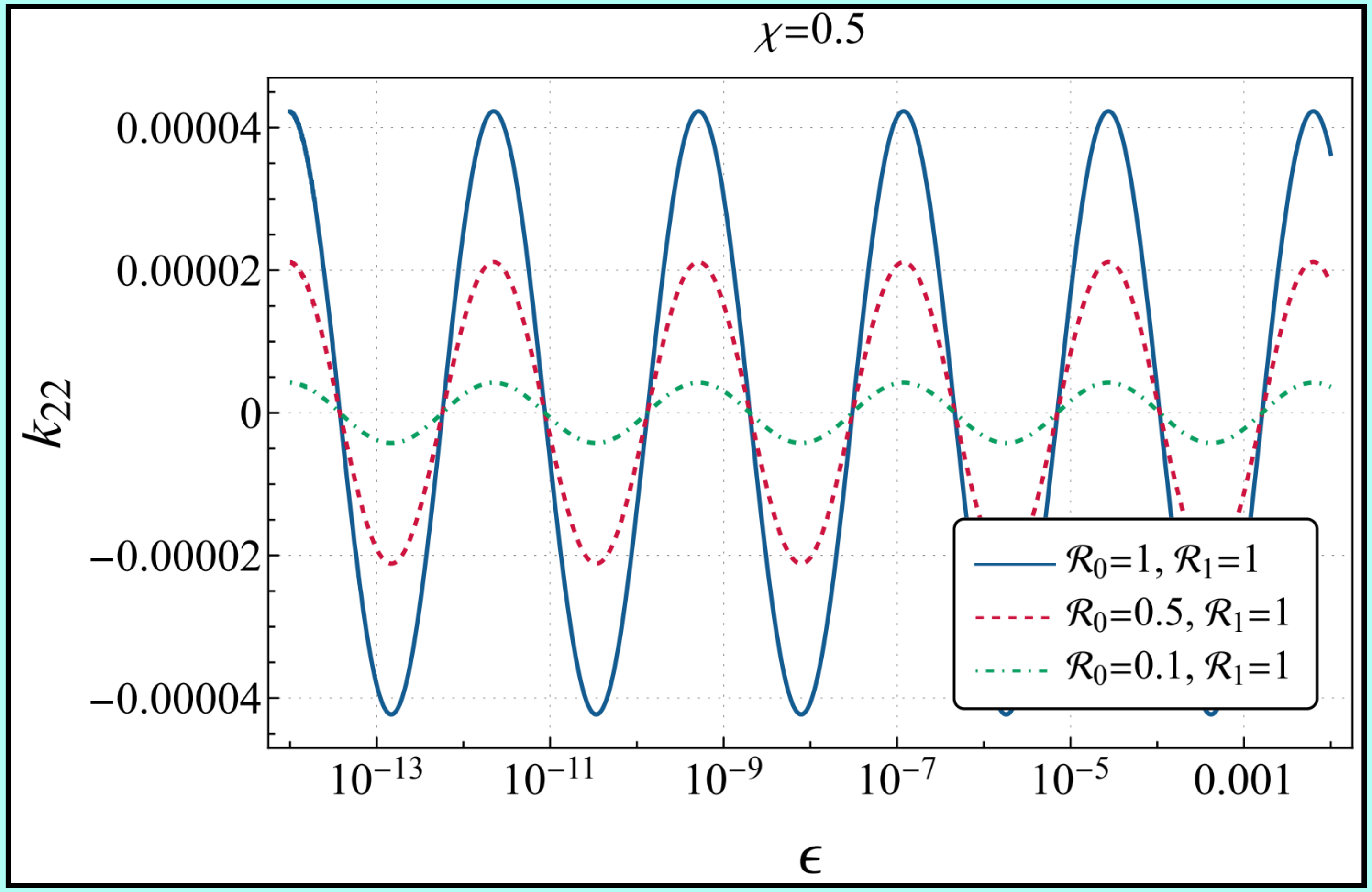


[SC, Maggio, Silvestrini and Pani, arXiv: 2310.06023]

Resonances at QNM frequencies







- There are resonances in the Love number at QNM frequencies.
- Increasing rotation and decreasing reflectivity decreases the Love number drastically.

[SC, Maggio, Silvestrini and Pani, arXiv: 2310.06023]

Conclusion

- Non-rotating BHs have zero tidal Love number.
- Rotating BHs have non-zero tidal Love number.
- Tidal Love number is gauge dependent.
- BHs can have non-zero tidal Love number at quadratic order in the frequency.
- Static limit of tidal Love number must be taken carefully.
- Tidal Love number depends on asymptotics.
- Asymptotically de Sitter BH has non-zero tidal Love number.

Thank You

Love for extremal black hole

- The above approach is not directly applicable for extremal black hole. In which case, the response function becomes: [Bhatt, SC and Bose, arXiv: 2406.09543]

$$F_{lm} = -\{2i(m-2M\omega)\}^{2l+1-2N} \frac{\Gamma(-2l+2N)\,\Gamma(3+l-2iM\omega-N)}{\Gamma(2-l-2iM\omega+N)\,\Gamma(2l+2-2N)}$$

- In the static case, the above response function reduces to,

$$F_{lm}^{\text{static}} = (-1)^{-1-l} im(2im)^{2l} \frac{\Gamma(3+l)\Gamma(-1+l)}{\Gamma(2l+1)\Gamma(2l+2)}$$

- While the dynamical response function has a real part:

$$\begin{aligned} k_{lm} = (2m)^{2l-2N-1}\{m-(2l+1)2M\omega\}\left(\frac{\Gamma(-1+l)\,\Gamma(3+l)}{\Gamma(1+2l)\,\Gamma(2l+2)}\right) \\ \times\Bigg[-\frac{2l(l+1)(2l+1)}{(l^2+l+4)}\{1+N(2\psi(1+2l)+2\psi(2+2l)-\psi(-1+l)-\psi(3+l))\} \\ +4Mm\omega\left\{\psi(-1+l)-\psi(3+l)-\frac{\pi m\,(l^2+l+4)}{l(l+1)(2l+1)}\right\}\Bigg]. \end{aligned}$$

Near-Horizon and Love Number

- Schwarzschild dS has two horizons.

$$f(R) = 1 - \frac{2M}{R} - H^2R^2$$

$$HR_{\rm c} = \left(1 - HM - \frac{3}{2}H^2M^2 + O(H^3M^3)\right)$$

$$R_{\rm h} = M\left(2 + 8H^2M^2 + \mathcal{O}(H^3M^3)\right)$$

- Keeping only leading order terms of $z = (R - R_{\rm h})/R_{\rm h}$, the perturbation equation can be solved.

- Subsequent expansion revealed a non-zero Love number, dependent on the dimensionless combination H^2M^2.

$$k_\ell^{(2)} = -\frac{24\ell(\ell+1)}{2\ell+1}H^2M^2\,\mathrm{Im}\left(\mathcal{F}_\ell^{\rm Sch}\right)$$

$$^{\rm near}R^{(2)}(y) = (1+y)^{2iM\omega}y^{-2iM\omega}\,{}_2F_1[-\tilde{\ell}, \tilde{\ell}+1, 1-4iM\omega, -y]$$

$$\tilde{\ell} = \ell + \frac{24\ell(\ell+1)}{2\ell+1}H^2M^2 \, .$$

[Nair, SC and Sarkar, arXiv: 2401.06467]