

PRIMORDIAL BLACK HOLE FORMATION FROM THE COLLAPSE OF A MASSLESS SCALAR FIELD

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A BRIEF INTRODUCTION

- **Primordial Black Holes** are formed in the very early Universe spanning a wide range of masses in order of magnitudes.
- Cosmological perturbations with an amplitude larger than threshold δ_c collapse into PBHs after re-entering the cosmological horizon.
- The aim of this work is to study the **threshold for PBH formation** from the collapse of adiabatic perturbations of a massless scalar field (massive field in the future).
- This scenario was investigated by *T. Harada & B. Carr*: [Growth of primordial black holes in a universe containing a massless scalar field](#), PRD (2005)

MATHEMATICAL FORMALISM

- We formulate our project working with the 3+1 conformal decomposition with the metric line element:

$$ds^2 = -\alpha^2 dt^2 + a^2(t)e^{2\zeta}\tilde{\gamma}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

- Stress-energy tensor for a massless scalar field:

$$T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + \partial_\mu\phi\partial_\nu\phi$$

- The **conjugate momentum**: $\Pi = \frac{1}{\alpha}(\partial_t - \beta^i\partial_i)\phi$

- In this formalism the Klein-Gordon equation could be decomposed:

$$\begin{cases} (\partial_t - \beta^i\partial_i)\Pi = \alpha\Delta\phi + \alpha K\Pi + D_i\phi D^i\phi \\ (\partial_t - \beta^i\partial_i)\phi = \alpha\Pi \end{cases}$$

SCALAR FIELD-PERFECT FLUID EQUIVALENCE

- In spherical symmetry the metric in **cosmic time slicing** is:

$$ds^2 = -A^2(t, r)dt^2 + B^2(t, r)dr^2 + R^2(t, r)d\Omega^2$$

- Misner-Sharp differential operators:

$$\begin{cases} D_t = \frac{1}{A}\partial_t \\ D_r = \frac{1}{B}\partial_r \end{cases} \Rightarrow \begin{cases} \eta = D_t\phi \\ \psi = D_r\phi \end{cases}$$

- The stress-energy tensor becomes: $T_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}(\psi^2 - \eta^2) + \partial_\mu\phi\partial_\nu\phi$

$$\text{In the comoving gauge } \psi = 0 \text{ and } E := T^{00} = \frac{\eta^2}{2} = P := T^{11}$$

The scalar field behaves concisely as a perfect fluid with an equation of state $p = \rho$ ONLY during the nearly linear regime.

GRADIENT EXPANSION

- We applied the **gradient expansion approach**, expanding each variable in spatial gradient of the curvature perturbation on super horizon scales.
- According to *Shibata-Sasaki* (1999)

$$\varepsilon(t) \propto \frac{k}{H(t)a(t)} \ll 1$$

is measuring the order of the gradient expansion.

- Adiabatic cosmological perturbations of the scalar field ϕ and its conjugate momentum Π

$$\phi(t, r) = \frac{1}{3} \sqrt{\frac{3}{4\pi}} \ln(t) + \lambda(t, r) + O(\varepsilon^3)$$

$$\Pi(t, r) = \frac{1}{3t} \sqrt{\frac{3}{4\pi}} + \Omega(t, r) + O(\varepsilon^3)$$

CURVATURE PERTURBATION PROFILE

- At **super-horizon scales** the perturbed Universe is described by the asymptotic form of the spatial metric:

$$dl^2 = a^2(t)e^{2\zeta(r)} [dr^2 + r^2d\Omega^2]$$

- The **perturbation amplitude** is defined as the **peak of the compaction function** in the comoving gauge:

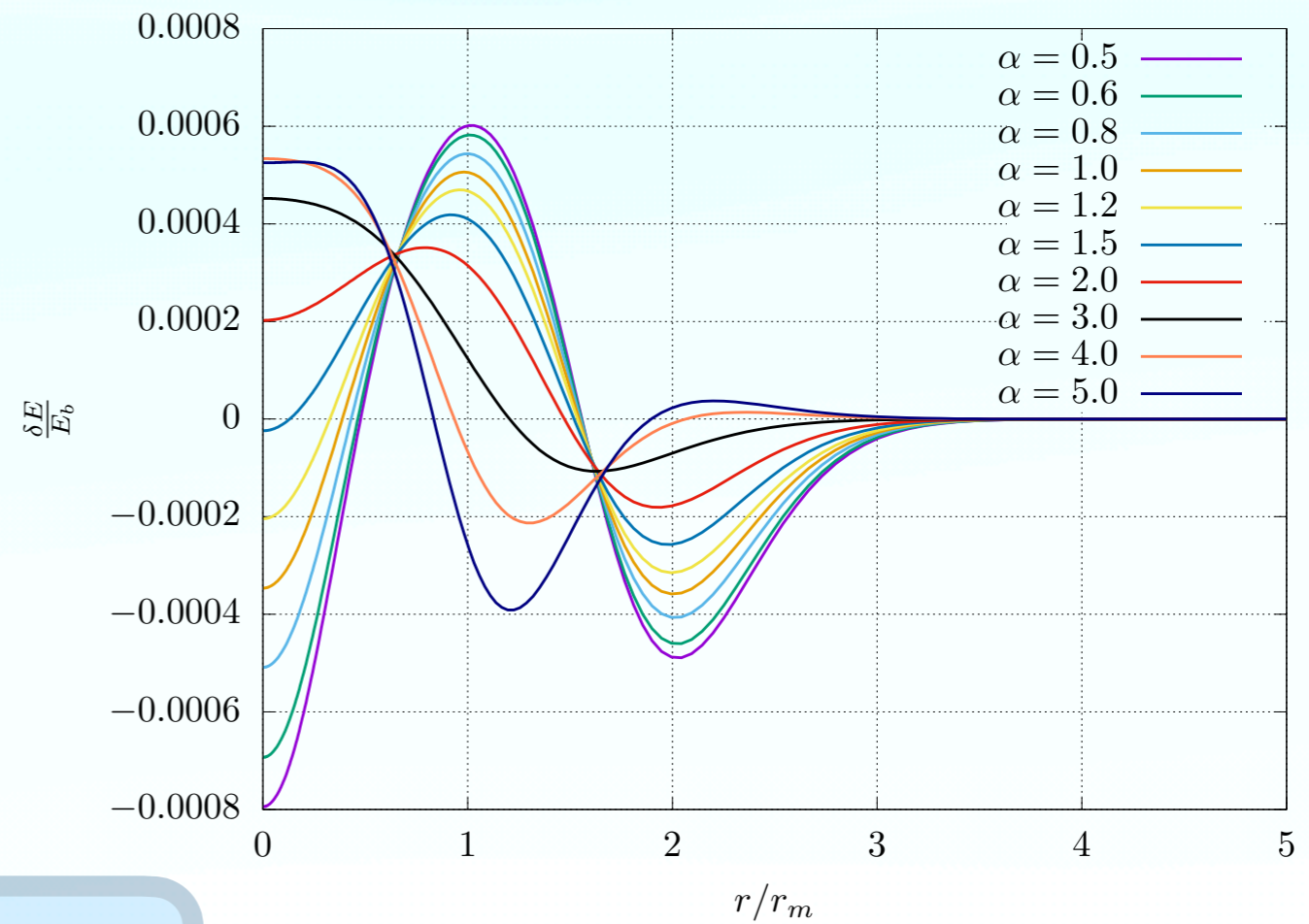
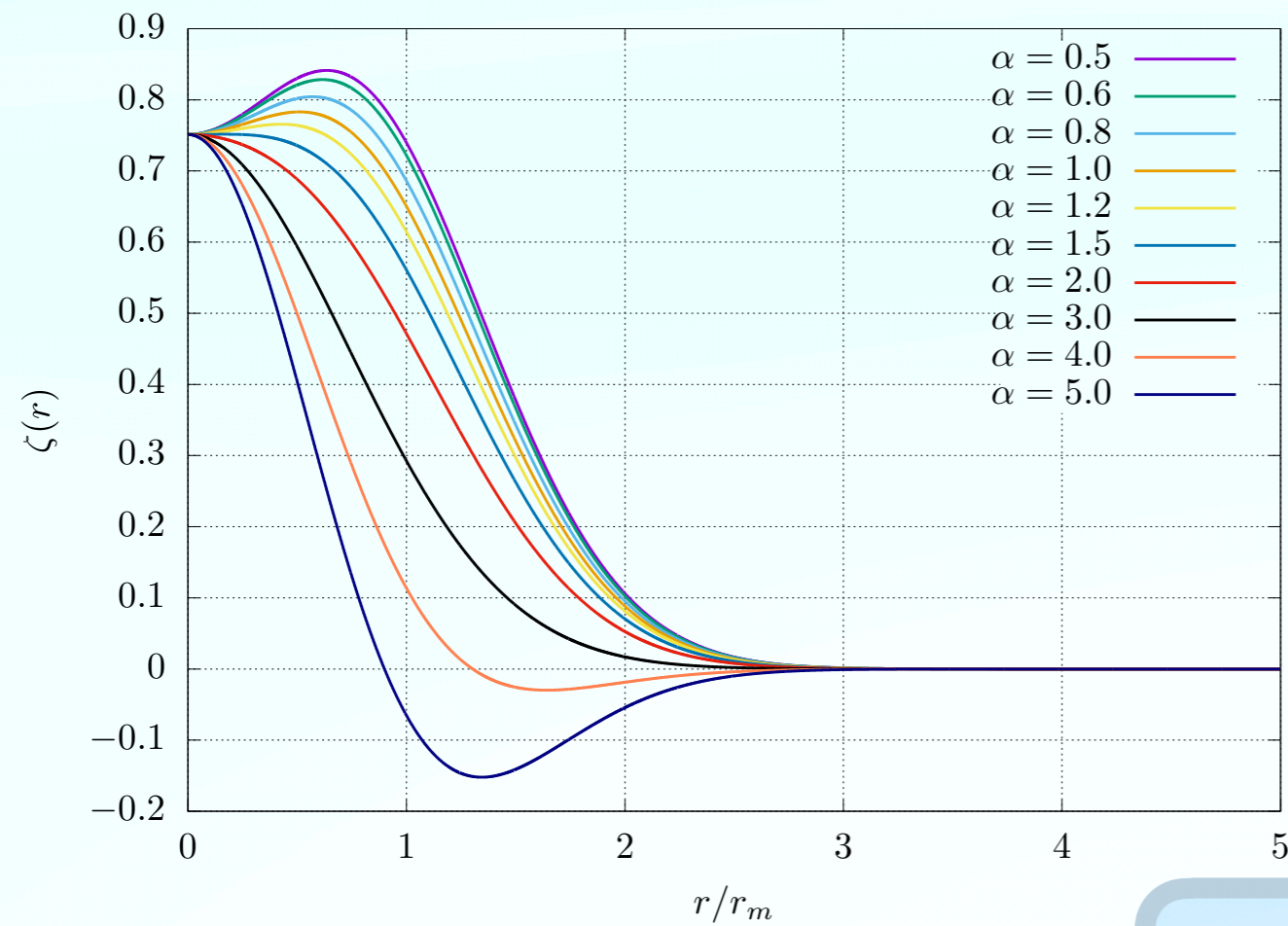
$$C(r) = -\frac{3}{4}r\zeta'(r) [2 + r\zeta'(r)] \quad \Rightarrow \quad \delta := C(r_m) \quad \text{where} \quad C'(r_m) = 0$$

- The threshold δ_c depends on the **shape parameter** α :

$$\tilde{r} = re^{\zeta(r)}, \quad \alpha = -\frac{C''(\tilde{r}_m) \tilde{r}_m^2}{4C(\tilde{r}_m)} \quad \Rightarrow \quad \alpha = -\frac{C''(r_m) r_m^2}{4C(r_m) \left[1 - \frac{3}{2}C(r_m)\right]}$$

CURVATURE PERTURBATION PROFILE

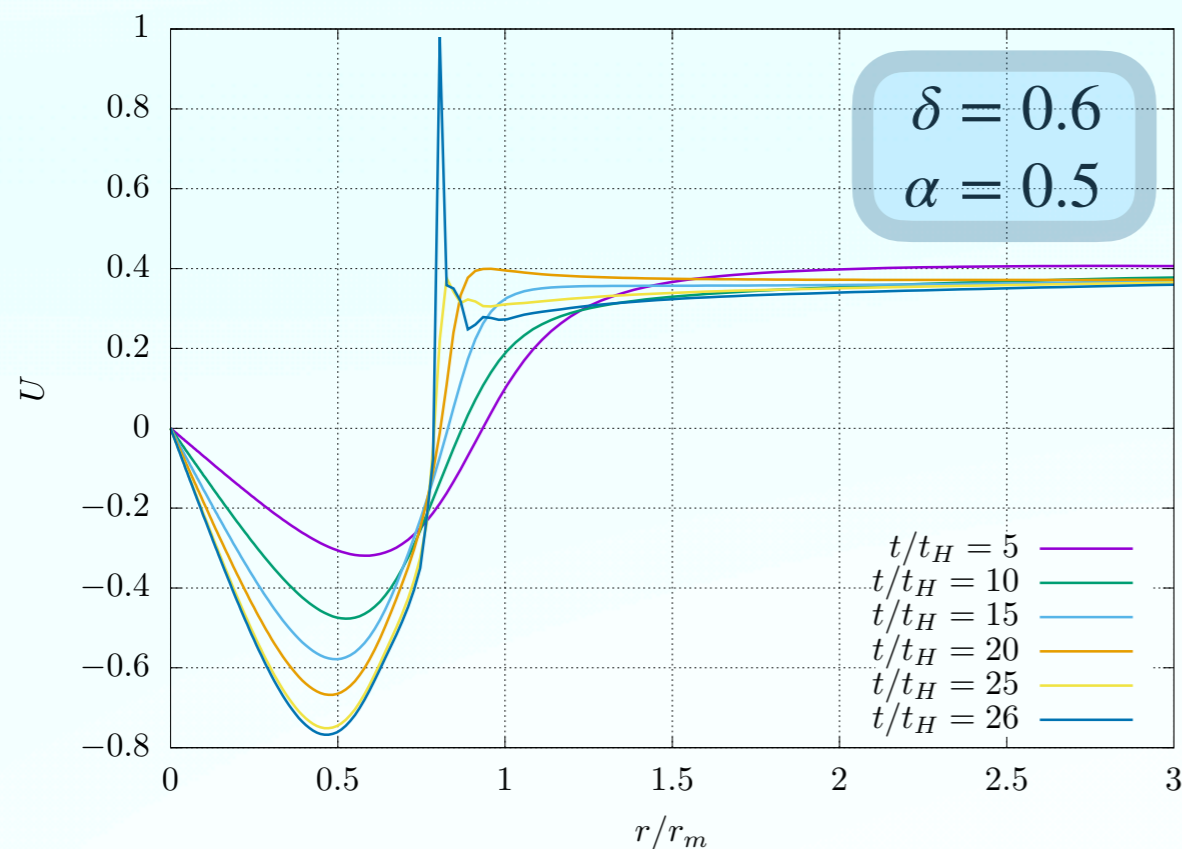
$$\zeta(r) = A \left[1 + 2(1 - \beta) \frac{r^2}{r_m^2} \right] e^{-\frac{r^2}{r_m^2}}; \quad A = \frac{e}{2} \left(1 - \sqrt{1 - \frac{4}{3}\delta} \right), \quad \beta = \alpha \left(1 - \frac{A}{e} \right) \left(1 - \frac{2A}{e} \right)$$



$$\delta = 0.6$$

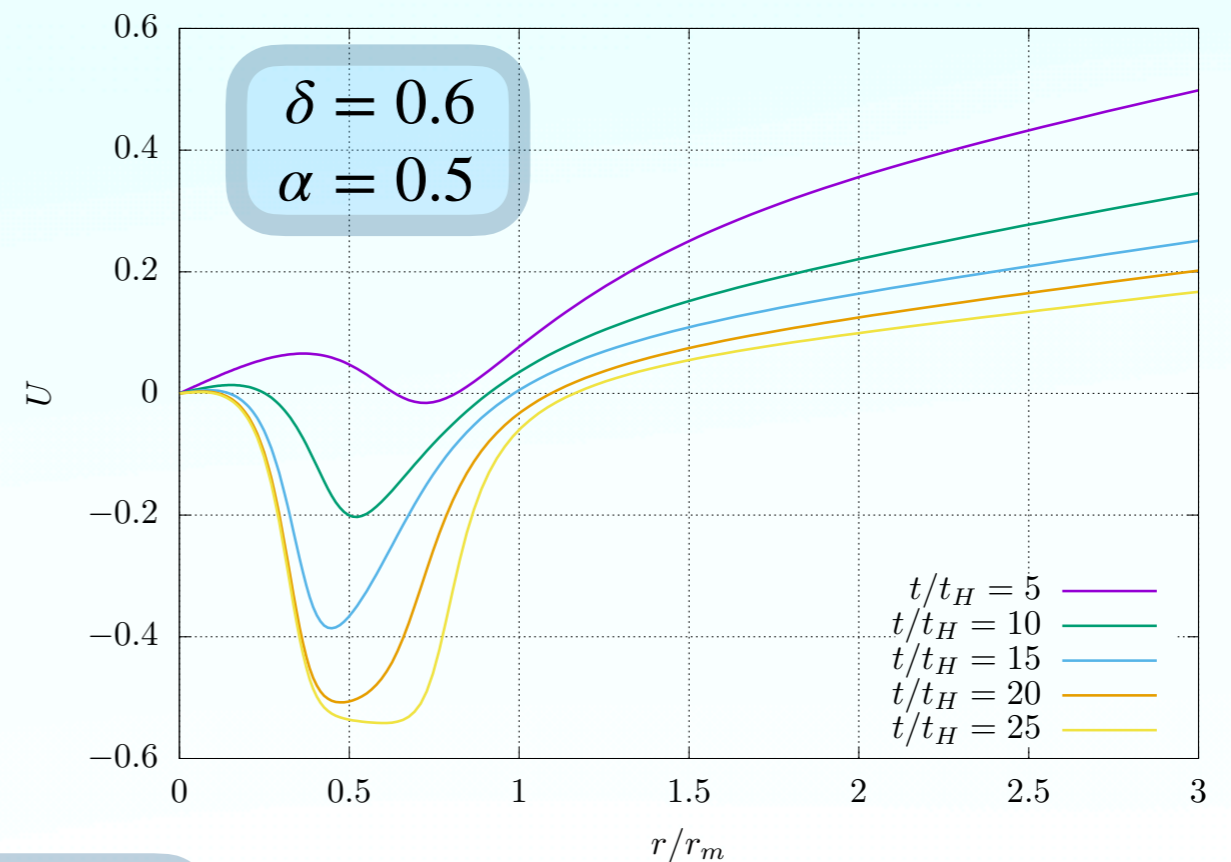
NUMERICAL SIMULATIONS

- **Spherically symmetric relativistic hydrodynamical code** based on the BSSN formalism using a non-uniform (exponential growth) grid, extended up to 10 perturbation length-scales.
- **Comoving against CMC gauge:** the comoving gauge is failing because of a coordinate singularity forming during the collapse and before the formation of the apparent horizon.



Comoving gauge

$$U = D_t R$$

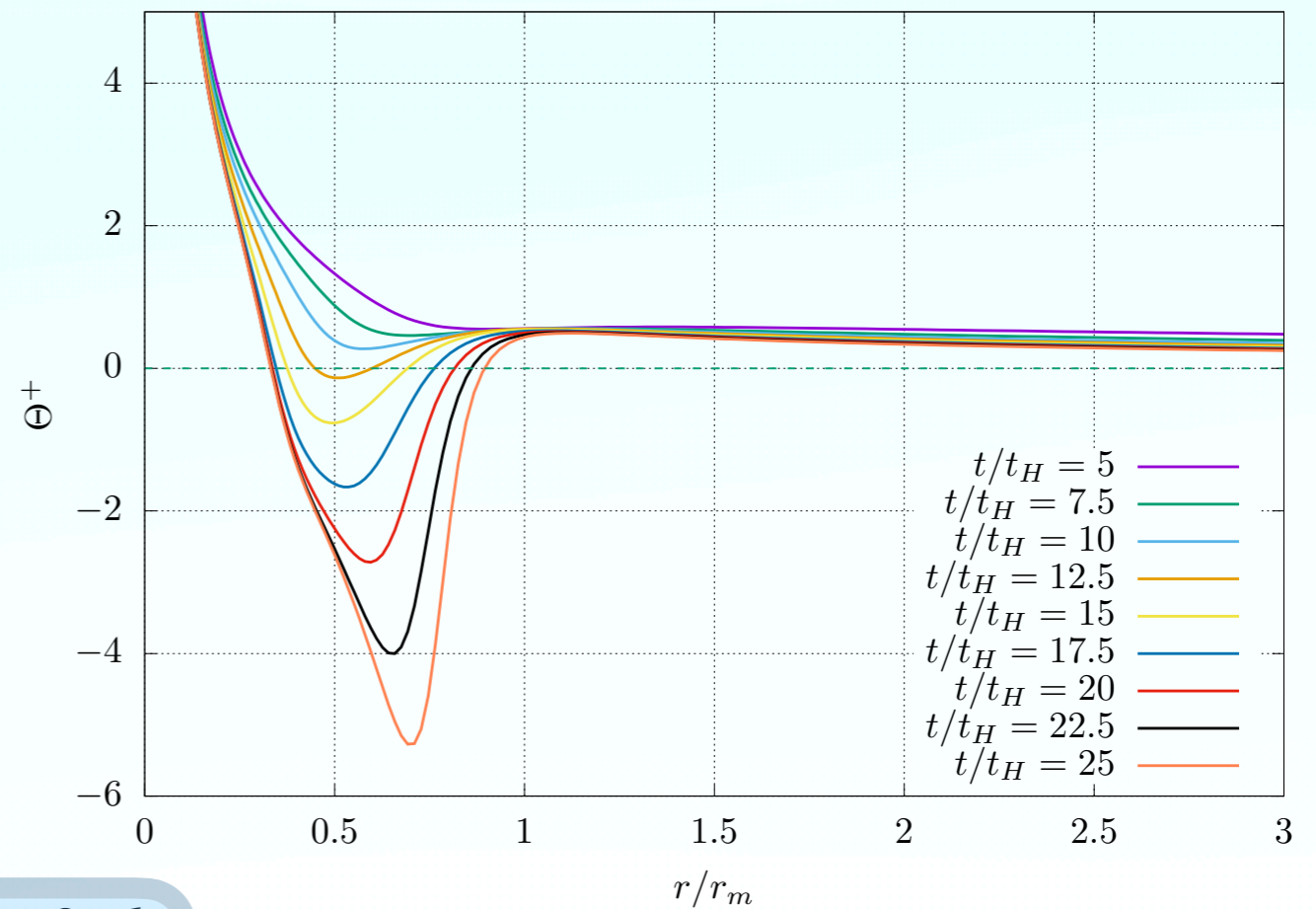
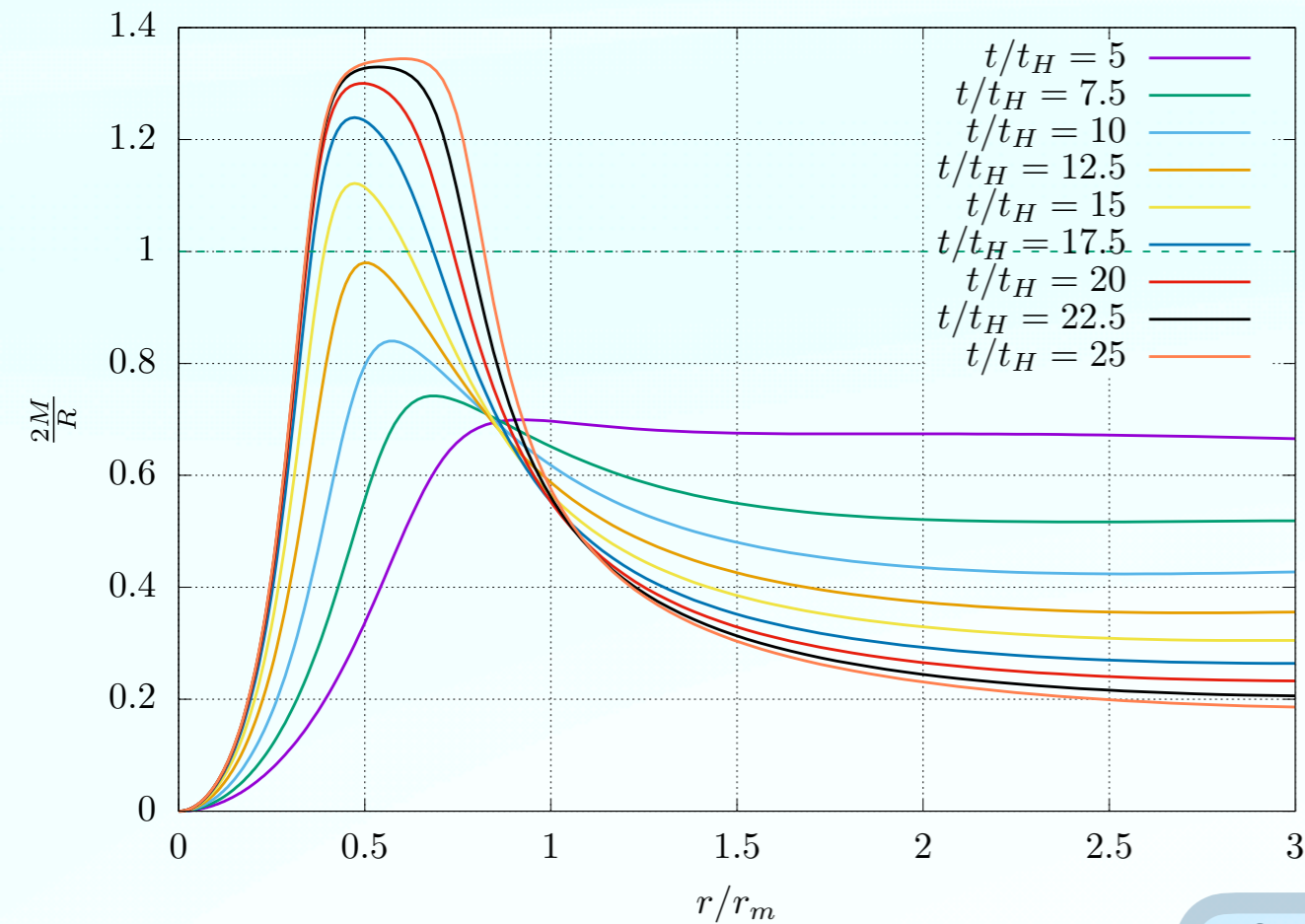


CMC gauge

APPARENT HORIZON FORMATION

$$R = 2M$$

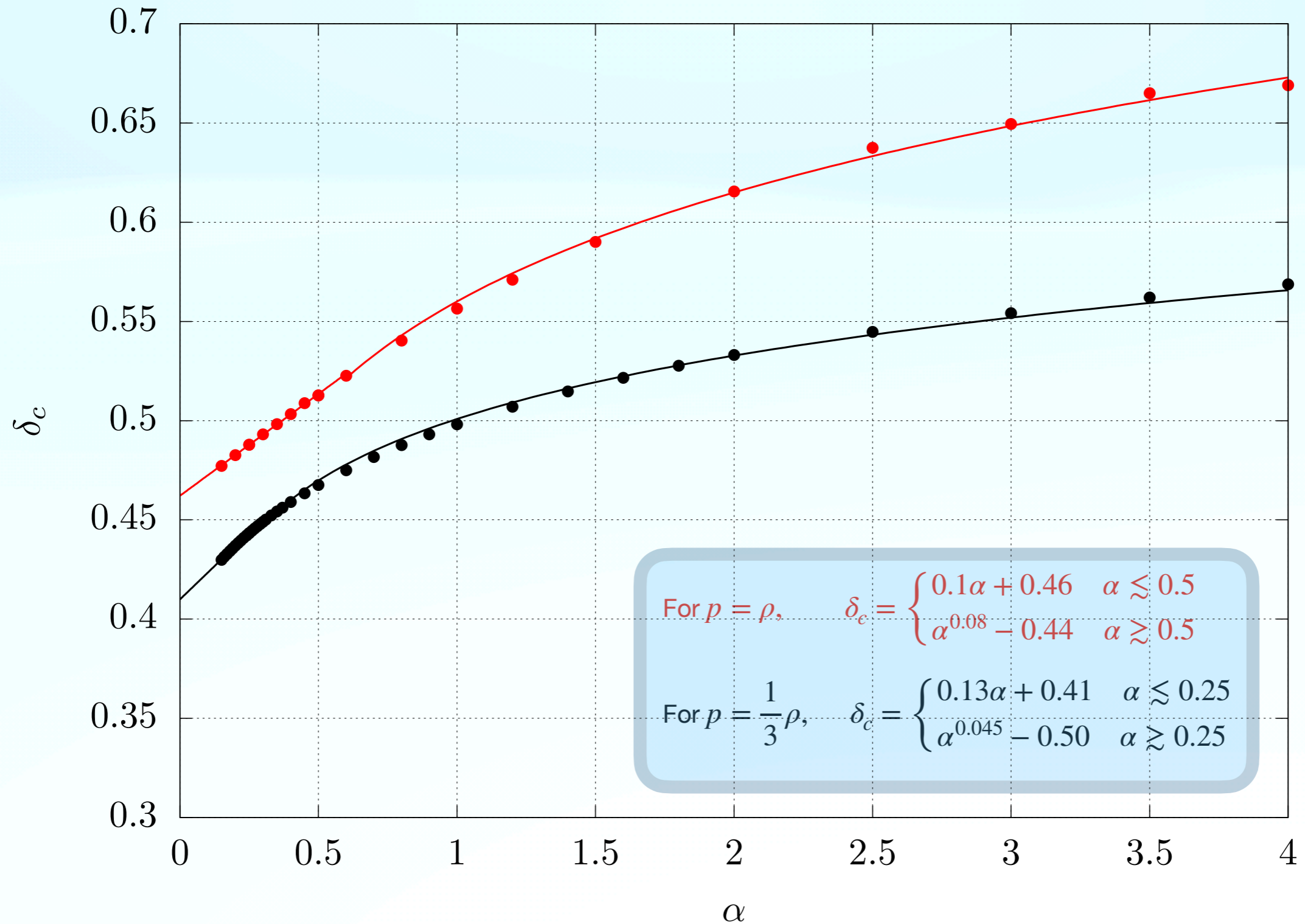
$$\theta^+ = 0$$



$$\delta = 0.6$$

$$\alpha = 0.5$$

THRESHOLD FOR PBH FORMATION



SUMMARY AND FUTURE

This is a work in progress...

1. We have investigated to which extent the collapse of adiabatic perturbations of a massless scalar field can be treated as a perfect fluid ($w=1$), computing the threshold: the equivalence holds only in the linear regime when the perturbation is expanding, before the onset of the gravitational collapse.
2. These results will be used in a follow up work to study the critical collapse, computing the **mass function** and the **abundance** of the population of PBHs formed from the collapse of a massless scalar field, comparing with the LIGO/Virgo catalog
3. In the future we would like to extend this analysis for **massive scalar fields** (dark matter models).

Thanks for your attention!

3+1 conformal decomposition

$$\tilde{\Delta} \psi = \frac{\tilde{R}_k^k}{8} \psi - 2\pi\psi^5 a^2 E - \frac{\psi^5 a^2}{8} \left(\tilde{A}_{ij} \tilde{A}^{ij} - \frac{2}{3} K^2 \right)$$

$$\tilde{D}^j \left(\psi^6 \tilde{A}_{ij} \right) - \frac{2}{3} \psi^6 \tilde{D}_i K = 8\pi\psi^6 J_i$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta \right) \psi = -\frac{\dot{a}}{2a} \psi + \frac{\psi}{6} \left(\overline{\mathcal{D}}_i \beta^i - \alpha K \right)$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta \right) K = \alpha \left(\tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} K^2 \right) - \mathcal{D}^i \mathcal{D}_i \alpha + 4\pi\alpha (E + S)$$

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \mathcal{L}_\beta \right) \tilde{A}_{ij} = & \frac{1}{a^2 \psi^4} \left[\alpha \left(R_{ij} - \frac{1}{3} \gamma_{ij} R_{(sc)} \right) - \left(\mathcal{D}_i \mathcal{D}_j \alpha - \frac{\gamma_{ij}}{3} \mathcal{D}_k \mathcal{D}^k \alpha \right) \right] + \alpha \left(K \tilde{A}_{ij} - 2A_{ik} A_j^k \right) + \\ & -\frac{2}{3} \tilde{A}_{ij} \overline{\mathcal{D}}_k \beta^k - \frac{8\pi\alpha}{a^2 \psi^4} \left(S_{ij} - \frac{\gamma_{ij}}{3} S_k^k \right) \end{aligned}$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta \right) \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} - \frac{2}{3} \tilde{\gamma}_{ij} \overline{\mathcal{D}}_k \beta^k$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta \right) \Pi = \alpha \Delta \phi + \alpha K \Pi + D_i \phi D^i \phi$$

$$\left(\frac{\partial}{\partial t} - \mathcal{L}_\beta \right) \phi = \alpha \Pi$$

Perfect fluid - scalar field equivalence

$$\frac{D_t B}{B} = \frac{D_r U}{\Gamma}$$

$$D_t \eta + \frac{2U}{R} \eta + \frac{\eta}{\Gamma} D_r U = 0$$

$$D_r M = 4\pi \Gamma E R^2$$

$$D_t \Gamma = \frac{U D_r A}{A}$$

$$D_t M = -4\pi R^2 P U$$



$$D_t U = - \left[\frac{\Gamma}{e+p} D_r p + \frac{M}{R^2} + 4\pi R p \right]$$

$$\frac{D_t \rho}{\rho} = - \frac{1}{\Gamma R^2} D_r (R^2 U)$$

$$D_t e = \frac{e+p}{\rho} D_t \rho$$

$$D_r A = - \frac{A}{e+p} D_r p$$

$$D_r M = 4\pi \Gamma e R^2$$

$$D_t \Gamma = \frac{U D_r A}{A}$$

$$D_t M = -4\pi R^2 p U$$

Lapse equation

- Starting from the definition of $D_t\phi = \frac{1}{A}\partial_t\phi$, and differentiating with respect D_r , we obtain:

$$\frac{D_r A}{A} = -\frac{D_r \eta}{\eta}$$

- From the definition of $E = P = \frac{\eta^2}{2}$, we finally obtain:

$$\frac{D_r A}{A} = -\frac{D_r P}{E + P}$$

10-Einstein equation

- We can mix the 10-Einstein equation with the Klein-Gordon equation:

$$\frac{D_t \eta}{\eta} = - \frac{1}{\Gamma R^2} D_r (R^2 U)$$

Continuity equation

- Starting from the definition of E and differentiating with respect to D_t , we get:

$$\frac{D_t \eta}{\eta} = \frac{D_t E}{2E}$$

- In particular we can rewrite for the fluid:

$$D_t e = \frac{e + p}{\rho} D_t \rho \quad \Rightarrow \quad \frac{D_t \rho}{\rho} = \frac{D_t e}{2e}$$

11-Einstein equation

- We can rewrite the 11-Einstein equation for the scalar field in the following way:

$$D_t U = - \left[\frac{\Gamma}{E + P} D_r P + \frac{M}{R^2} + 4\pi R P \right]$$

- While for the fluid, assuming the definition of the Misner-Sharp mass we obtain:

$$D_t U = - \left[\frac{\Gamma}{e + p} D_r p + \frac{M}{R^2} + 4\pi R p \right]$$

Gradient expansion equations

$$\left\{ \begin{array}{l} \zeta(t, r) = \zeta_b(r) + \xi(t, r) + \mathcal{O}(\epsilon^3) \\ \tilde{\gamma}_{ij}(t, r) = \eta_{ij} + h_{ij}(t, r) + \mathcal{O}(\epsilon^3) \\ K(t, r) = K_b(t)(1 + \kappa(t, r)) + \mathcal{O}(\epsilon^3) \\ \alpha(t, r) = 1 + \chi(t, r) + \mathcal{O}(\epsilon^3) \\ \beta^i(t, r) = \mathcal{O}(\epsilon) \\ \phi(t, r) = \Phi(t) + \lambda(t, r) + \mathcal{O}(\epsilon^3) \\ \Pi(t, r) = \Pi_b(t) + \Omega(t, r) + \mathcal{O}(\epsilon^3) \end{array} \right.$$

$$\left\{ \begin{array}{l} \partial_t \lambda = \Omega + \chi \Pi_b + \mathcal{O}(\epsilon^3) \\ \partial_t \Omega = -3H_b \Pi_b \left(\chi + \kappa + \frac{\Omega}{\Pi_b} \right) + \mathcal{O}(\epsilon^3) \\ \partial_t h_{ij} = -2\tilde{A}_{ij} - \frac{2}{3}\eta_{ij} \bar{D}_k \beta^k + \bar{D}_j \beta_i + \bar{D}_i \beta_j + \beta^k \eta_{ij,k} + \mathcal{O}(\epsilon^3) \\ 3\dot{\xi} - 3\beta^k \partial_k \zeta_b - 3H_b \chi - 3H_b \kappa - \bar{D}_i \beta^i = \mathcal{O}(\epsilon^3) \\ \kappa \left(2 + \frac{\partial_t H_b}{H_b^2} \right) + \frac{1}{H_b} \dot{\kappa} + 3\chi + \frac{8\Omega}{H_b} \sqrt{\frac{\pi}{3}} = \mathcal{O}(\epsilon^3) \\ \partial_t \tilde{A}_{ij} + 3\frac{\dot{a}}{a} \tilde{A}_{ij} = a^{-2} e^{-2\zeta_b} \mathfrak{R}_{ij}(\zeta_b) + \mathcal{O}(\epsilon^4) \\ \bar{\Delta} \zeta_b + \frac{1}{2} \bar{D}^k \zeta_b \bar{D}_k \zeta_b = 4\pi a^2 e^{2\zeta_b} \Pi_b^2 \left(\kappa - \frac{\Omega}{\Pi_b} \right) + \mathcal{O}(\epsilon^4) \\ \bar{D}_j \tilde{A}_i^j + 3\tilde{A}_i^j \bar{D}_j \zeta_b + \Pi_b \left(2\sqrt{\frac{4\pi}{3}} \bar{D}_i \kappa + 8\pi \partial_i \lambda \right) = \mathcal{O}(\epsilon^5) \end{array} \right.$$

$$\mathfrak{R}_{ij}(\zeta_b) = \bar{D}_i \zeta_b \bar{D}_j \zeta_b + \frac{1}{3} \eta_{ij} \bar{\Delta} \zeta_b - \bar{D}_i \bar{D}_j \zeta_b - \frac{1}{3} \eta_{ij} \bar{D}^k \zeta_b \bar{D}_k \zeta_b$$

$$\bar{\Delta} \zeta_b = \frac{e^{\frac{\zeta_b}{2}}}{4} \left(\bar{D}^k \zeta_b \bar{D}_k \zeta_b + 2\bar{\Delta} \zeta_b \right)$$

Gradient expansion solution

$$\Omega = -\frac{3}{16a_0^2} \sqrt{\frac{3}{\pi}} e^{-2\zeta_b} \left(\overline{D}^k \zeta_b \overline{D}_k \zeta_b + 2\overline{\Delta} \zeta_b \right) t^{\frac{1}{3}}$$

$$\kappa = \frac{3}{8a_0^2} e^{-2\zeta_b} \left(\overline{D}^k \zeta_b \overline{D}_k \zeta_b + 2\overline{\Delta} \zeta_b \right) t^{\frac{4}{3}}$$

$$\chi = \frac{9}{8a_0^2} e^{-2\zeta_b} \left(\overline{D}^k \zeta_b \overline{D}_k \zeta_b + 2\overline{\Delta} \zeta_b \right) t^{\frac{4}{3}}$$

$$\lambda = 0$$

$$\hat{\xi} = \frac{3e^{-2\zeta_b}}{8a_0^2} \left(\overline{D}^k \zeta_b \overline{D}_k \zeta_b + 2\overline{\Delta} \zeta_b \right) t^{\frac{4}{3}}$$

$$\tilde{A}_{ij} = \frac{3}{4a_0^2} \zeta_{ij}(\zeta_b) e^{-2\zeta_b} t^{\frac{1}{3}}$$

$$h_{ij} = -\frac{9}{8a_0^2} \mathfrak{R}_{ij}(\zeta_b) e^{-2\zeta_b} t^{\frac{4}{3}}$$

$$\Omega = -\frac{1}{4a_0^2} \sqrt{\frac{3}{\pi}} e^{-2\zeta_b} \left(\overline{D}^k \zeta_b \overline{D}_k \zeta_b + 2\overline{\Delta} \zeta_b \right) t^{\frac{1}{3}}$$

$$\kappa = 0$$

$$\chi = \frac{2e^{-2\zeta_b}}{a_0^2} \left(\overline{D}^k \zeta_b \overline{D}_k \zeta_b + 2\overline{\Delta} \zeta_b \right) t^{\frac{4}{3}}$$

$$\lambda = \frac{1}{16a_0^2} \sqrt{\frac{3}{\pi}} e^{-2\zeta_b} \left(\overline{D}^k \zeta_b \overline{D}_k \zeta_b + 2\overline{\Delta} \zeta_b \right) t^{\frac{4}{3}}$$

$$\hat{\xi} = \frac{e^{-2\zeta_b}}{2a_0^2} \left(\overline{D}^k \zeta_b \overline{D}_k \zeta_b + 2\overline{\Delta} \zeta_b \right) t^{\frac{4}{3}}$$

$$\tilde{A}_{ij} = \frac{3}{4a_0^2} \mathfrak{R}_{ij}(\zeta_b) e^{-2\zeta_b} t^{\frac{1}{3}}$$

$$h_{ij} = -\frac{9}{8a_0^2} \mathfrak{R}_{ij}(\zeta_b) e^{-2\zeta_b} t^{\frac{4}{3}}$$