**Theory of Gravitational Waves** 



# "A novel Lagrange-multiplier approach to the effective-onebody dynamics of binary systems in post-Minkowskian gravity"

Based on [T. Damour, A. Nagar, AP, P. Rettegno; 2024] (in preparation)



16/09/2024



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# EOB approach in a nutshell





[A. Buonanno, T. Damour; 1999]



$$M = m_1 + m_2, \quad \mu = m_1 m_2 / M, \quad \nu \equiv \mu / M$$
$$S = S_1 + S_2, \quad S_* = \frac{m_2}{m_1} S_1 + \frac{m_1}{m_2} S_2$$

Two basic building blocks in the conservative dynamics:

• EOB energy map

$$E_{\text{real}} = M \sqrt{1 + 2\nu \left(\frac{E_{\text{eff}}}{\mu} - 1\right)}$$

• Relativistic mass-shell condition for the effective particle

 $g_{\rm eff}^{\mu\nu}(X^{\rho})P_{\mu}P_{\nu} + \mu^2 + Q(X^{\mu}, P_{\mu}) = 0$ 

 $g_{\rm eff}^{\mu\nu}(X^{\rho})P_{\mu}P_{\nu} + \mu^{2} + Q(X^{\mu}, P_{\mu}) = 0$ 

 $H_{\rm eff}$ 

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## **EOB** and post-Minkowskian information

In standard EOB models based on **post-Newtonian** information:

$$g_{\mu\nu}^{\text{eff}}(X^{\rho})dX^{\mu}dX^{\nu} = -A(R)c^{2}dT^{2} + B(R)dR^{2}$$
$$+R^{2}C(R)(d\theta^{2} + \sin^{2}\theta d\phi^{2})$$

which is solved for  $P_0 = -E_{\text{eff}}$  assuming  $Q(X^{\mu}, P_{\mu}) = Q(X^i, P_i)$ 

$$H_{\rm eff} = E_{\rm eff}$$
  $H_{\rm eff} = H_{\rm eff}(A, B, C, Q)$  Energy map  $H_{\rm EOB} = M_{\rm V} \sqrt{1 + 2\nu \left(\frac{H_{\rm eff}}{\mu} - 1\right)}$ 

With **post-Minkowskian** (PM) information, which is inherently energy dependent, at least one function among  $\{A, B, C, Q\}$  must be allowed to depend on  $E_{eff}$ 

$$H_{\text{eff}} = H_{\text{eff}}(A, B, C, Q, H_{\text{eff}}) \rightarrow \text{recursive definition}$$

[T. Damour; 2018] [A. Antonelli et al.; 2019] (M. Khalil et al.; 2022] [A. Buonanno et al.; 02/2024] [A. Buonanno et al.; 05/2024]

 $\{A, B, C, Q\} \text{ are then determined from the knowledge of the PM-expanded} \\ \text{scattering angle} \\ \chi_{\text{PM}}(j, \gamma) = \sum_{n} 2 \frac{\chi_n(\gamma, \nu)}{j^n} \\ \text{intricate dependence on } \gamma = E_{\text{eff}}/\mu \\ \text{o 3PM } \rightarrow \text{hyperbolic functions} \\ \text{o 4PM } \rightarrow \text{elliptic integrals, polylogs} \end{cases}$ 

### LEOB: a novel Lagrange-multiplier approach

$$S[X^{\mu}, P_{\mu}] = \int [P_{\mu}dX^{\mu}]^{\text{on-shell}} = \int P_{i}dX^{i} - H_{\text{eff}}(X^{i}, P_{i})dT_{\text{eff}}$$

From the variational principle:

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### **LEOB** equations of motion

In terms of mass-rescaled quantities, the resulting Euler-Lagrange equations are:

$$\frac{dx^{i}}{dt_{\text{real}}} = -\frac{1}{h} \left(\frac{\partial \hat{\mathscr{C}}}{\partial \gamma}\right)^{-1} \frac{\partial \hat{\mathscr{C}}}{\partial p_{i}}, \qquad \frac{dp_{i}}{dt_{\text{real}}} = \frac{1}{h} \left(\frac{\partial \hat{\mathscr{C}}}{\partial \gamma}\right)^{-1} \frac{\partial \hat{\mathscr{C}}}{\partial x^{i}}, \qquad \frac{d\gamma}{dt_{\text{real}}} = 0$$
$$t_{\text{real}} = \frac{T_{\text{real}}}{GM}, \qquad x^{\mu} = \frac{X^{\mu}}{GM}, \qquad p_{\mu} = \frac{P_{\mu}}{\mu}, \qquad \gamma = \frac{E_{\text{eff}}}{\mu}, \qquad h = \frac{E_{\text{real}}}{M}, \qquad \hat{\mathscr{C}} = \frac{\mathscr{C}}{\mu^{2}}$$

plus the equation  $\hat{\mathscr{C}} = 0$  that only affects the determination of the initial conditions.

To go **beyond the conservative dynamics**, we generalize the evolution equation of  $p_{\mu}$  by adding a **radiation-reaction force**  $\mathscr{F}_{\mu}$ , i.e.

$$\frac{dx^{i}}{dt_{\text{real}}} = -\frac{1}{h} \left(\frac{\partial \hat{\mathscr{C}}}{\partial \gamma}\right)^{-1} \frac{\partial \hat{\mathscr{C}}}{\partial p_{i}}, \qquad \frac{dp_{i}}{dt_{\text{real}}} = \frac{1}{h} \left(\frac{\partial \hat{\mathscr{C}}}{\partial \gamma}\right)^{-1} \frac{\partial \hat{\mathscr{C}}}{\partial x^{i}} + \mathscr{F}_{i}, \qquad \frac{d\gamma}{dt_{\text{real}}} = -\mathscr{F}_{0}$$

with the condition  $\frac{dx^{\mu}}{dt_{real}}\mathscr{F}_{\mu} = 0$  ensuring that  $\hat{\mathscr{C}} = 0$  holds along the whole radiation-reacted evolution

# LEOB gauge flexibility

The LEOB approach makes it easy to admit any functional dependence on  $\gamma$ .

Without any loss of generality one can either make the **effective metric energy independent**, with the constraint

$$\hat{\mathscr{C}} = g_{\text{eff}}^{\mu\nu}(x^{\rho})p_{\mu}p_{\nu} + 1 + Q(x^{\mu},\gamma)$$

or choose a geodesic-like constraint of the type

$$\hat{\mathscr{C}} = g_{\text{eff}}^{\mu\nu}(x^{\rho}, \gamma)p_{\mu}p_{\nu} + 1$$

In both cases there is still a large leftover gauge freedom.

Focusing on the second, in fact, we have  $g_{\text{eff}}^{\mu\nu}(x^{\rho}, \gamma) = g_{\text{eff}}^{\mu\nu}[A(r, \gamma), B(r, \gamma), C(r, \gamma)]$ whereas the scattering angle  $\chi_{\text{PM}}(j, \gamma)$  can just fix one of the three metric functions. This is due to:

- The usual **coordinate freedom** of general relativity
- The fact that, just like an Hamiltonian, 
   <sup>2</sup>
   *â* admits the set of canonical transformations as symmetries

We have thus to impose **two** conditions on  $\{A, B, C\}$ 

## Gauge fixed constraint and fluxes

We select the constraint  $\hat{\mathscr{C}} = g_{\text{eff}}^{\mu\nu}(x^{\rho}, \gamma)p_{\mu}p_{\nu} + 1$  and choose:

- $C(r, \gamma) = 1 \rightarrow$  Schwarzschild-like coordinates
- $A(r, \gamma)B(r, \gamma) = 1 \rightarrow$  Resulting constraint in simple form

$$\hat{\mathscr{C}} = -\frac{\gamma^2}{A(r,\gamma)} + p_r^2 A(r,\gamma) + \frac{p_{\varphi}^2}{r^2} + 1$$

Remaining metric function  $A(r, \gamma)$ :

$$\chi_{\leq 3PM}(j,\gamma) \to A_{\leq 3PM}(r,\gamma)$$
$$\chi_{4PM,local}(j,\gamma) \to A_{4PM,local}(r,\gamma)$$

PN completion  $\Delta A_{\geq 4PN}(r, \gamma)$ , comprensive of nonlocal tail-transported effects [Piero Rettegno's talk on Wednesday]

$$A(r, \gamma) = A_{\leq 3\text{PM}}(r, \gamma) + A_{4\text{PM,local}}(r, \gamma) + \Delta A_{\geq 4\text{PN}}(r, \gamma)$$

Fluxes and waveform:

 $\mathcal{F}_{\gamma} = -\dot{\varphi}\mathcal{F}_{\omega}$ 

$$\mathcal{F}_r = 0$$
 [quasi-circular (qc) orbits]

$$\mathscr{F}_{\varphi} \rightarrow \ \ \, \mathbb{P}$$
N-based factorized and  
resummed prescription of the  
qc iteration of TEOBResumS

[T. Damour, B. R. Iyer. A. Nagar; 2009]

# **Testing LEOB: phasings**

No numerical relativity (NR) tuning in the dynamics! NR just in NQCs and ringdown

4PM-4PN  $\rightarrow$  the PN completion  $\Delta A_{\geq 4PN}(r, \gamma)$  is stopped at the 4PN order



 $\Delta \phi_{22}^{\text{EOBNR}}$  at merger ~-2.2 rad for q = 1 and ~-0.18 rad for q = 9.5

# Testing LEOB: unfaithfulness

No numerical relativity (NR) tuning in the dynamics! NR just in NQCs and ringdown

4PM-4PN  $\rightarrow$  the PN completion  $\Delta A_{>4PN}(r, \gamma)$  is stopped at the 4PN order.

We consider 17 nonspinning NR waveforms from the SXS public catalogue and the q = 15 nonspinning waveform of [J. You et al.;2022]



## Conclusions

With LEOB, our novel Lagrange-multiplier approach, we are able to provide a complete description, within the EOB framework and in the form of Euler-Lagrange equations, of the dynamical evolution of a binary system of black holes.

Crucially, it avoids the need to solve the EOB mass-shell constraint for an effective Hamiltonian, at the rather small cost of having one additional evolution equation for the energy  $\gamma$ .

The simplification and flexibility brought about by the LEOB approach have a notable impact in the development of PM-based EOB models built upon the 4PM analytical information that is currently available, and we expect even more benefits when higher-order PM perturbative results will be achieved.

When applied to a complete waveform model, the conservative dynamics of the LEOB approach yields good results up to merger (especially for  $q \ge 6$ ), where the phase difference with NR remains quite low even without NR tuning the dynamics. This is also confirmed by looking at the unfaithfulness.

LEOB can be also applied to spinning binary black holes (work in progress)

[Piero Rettegno's talk on Wednesday]





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# Link between LEOB and the $H\text{-}\mathsf{based}$ EOB

The traditional Hamiltonian EOB dynamics is regained when using a mass-shell constraint in the explicitly solved form

$$\hat{\mathscr{C}}_{H} \equiv \hat{H}_{\text{eff}}(x^{i}, p_{i}) - \gamma$$

which implies

$$\frac{\partial \hat{\mathscr{C}}_H}{\partial \gamma} = -1, \qquad \frac{\partial \hat{\mathscr{C}}_H}{\partial p_i} = \frac{\partial \hat{H}_{\text{eff}}}{\partial p_i}, \qquad \frac{\partial \hat{\mathscr{C}}_H}{\partial x^i} = \frac{\partial \hat{H}_{\text{eff}}}{\partial x^i}$$

We have in fact:

$$\frac{dx^{i}}{dt_{\text{real}}} = -\frac{1}{h} \left( \frac{\partial \hat{\mathscr{C}}_{H}}{\partial \gamma} \right)^{-1} \frac{\partial \hat{\mathscr{C}}_{H}}{\partial p_{i}}, \qquad \frac{dp_{i}}{dt_{\text{real}}} = \frac{1}{h} \left( \frac{\partial \hat{\mathscr{C}}_{H}}{\partial \gamma} \right)^{-1} \frac{\partial \hat{\mathscr{C}}_{H}}{\partial x^{i}} + \mathscr{F}_{i}, \qquad \frac{d\gamma}{dt_{\text{real}}} = -\mathscr{F}_{0}$$
$$\frac{dx^{i}}{dt_{\text{real}}} = \frac{1}{h} \frac{\partial \hat{H}_{\text{eff}}}{\partial p_{i}}, \qquad \frac{dp_{i}}{dt_{\text{real}}} = -\frac{1}{h} \frac{\partial \hat{H}_{\text{eff}}}{\partial x^{i}} + \mathscr{F}_{i}$$

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# **Relations between** $A(r, \gamma)$ and $\chi_{PM}(r, \gamma)$

$$A - A_s = \sum_n \frac{\bar{a}_n}{r^n}, \qquad \Delta \chi_{\rm PM} = \sum_n 2 \frac{\chi_n(\gamma, \nu) - \chi_n^S(\gamma, \nu)}{j^n}$$

$$\begin{split} \bar{a}_{1}(\gamma,\nu) &= 0\,,\\ \bar{a}_{2}(\gamma,\nu) &= -\frac{8\Delta\chi_{2}}{\pi\left(3\gamma^{2}-1\right)}\,,\\ \bar{a}_{3}(\gamma,\nu) &= \frac{24\left(8\gamma^{4}-8\gamma^{2}+1\right)\Delta\chi_{2}}{\pi\left(3\gamma^{2}-1\right)\left(4\gamma^{2}-1\right)p_{\infty}^{2}} - \frac{3\Delta\chi_{3}}{\left(4\gamma^{2}-1\right)p_{\infty}}\,,\\ \bar{a}_{4}(\gamma,\nu) &= -\frac{12\left(140\gamma^{8}-235\gamma^{6}+123\gamma^{4}-13\gamma^{2}+1\right)\Delta\chi_{2}}{\pi\left(3\gamma^{2}-1\right)\left(4\gamma^{2}-1\right)\left(5\gamma^{2}-1\right)p_{\infty}^{4}} + \frac{16\left(35\gamma^{4}-30\gamma^{2}+3\right)\Delta\chi_{2}^{2}}{\pi^{2}\left(3\gamma^{2}-1\right)^{2}\left(5\gamma^{2}-1\right)p_{\infty}^{2}} \\ &+ \frac{3\left(35\gamma^{4}-30\gamma^{2}+3\right)\Delta\chi_{3}}{\left(4\gamma^{2}-1\right)\left(5\gamma^{2}-1\right)p_{\infty}^{3}} - \frac{32\Delta\chi_{4}}{3\pi\left(5\gamma^{2}-1\right)p_{\infty}^{2}}\,. \end{split}$$

$$p_{\infty} = \sqrt{\gamma^2 - 1}$$

#### Numerical waveforms used in the tests

TABLE IV: Properties of the 17 nonspinning public NR SXS waveforms used to test the new LEOB waveform model for nonspinning binaries. From left to right, the columns report: the identification number; the SXS classification name; the mass ratio  $q \equiv m_1/m_2 \geq 1$ ; the symmetric mass ratio  $\nu \equiv m_1 m_2/(m_1 + m_2)^3$ ; the number of orbits between t = 0 and the time when a common event horizon is formed; the orbital eccentricity  $\epsilon$  determined at relaxed measurement time; the  $\ell = m = 2$  phasedifference  $\delta \phi_{mrg}^{NR}$  between the highest and second highest resolution accumulated between t = 600M and the peak amplitude of the highest resolution data.

#	name	q	$\nu$	# orbits	$\epsilon \left[ 10^{-3} \right]$	$\delta \phi_{ m mrg}^{ m NR}  [ m rad]$
1	SXS:BBH:0180	1	0.2500	28.1825	0.0514	-0.4247
2	SXS:BBH:0007	1.5	0.2400	29.0922	0.4200	-0.0186
3	SXS:BBH:0169	2	0.2222	15.6805	0.1200	-0.0271
4	SXS:BBH:0259	2.5	0.2041	28.5625	0.0590	-0.0080
5	SXS:BBH:0030	3	0.1875	18.2228	2.0100	-0.0870
6	SXS:BBH:0167	4	0.1600	15.5908	0.0990	-0.5165
7	SXS:BBH:0295	4.5	0.1488	27.8067	0.0520	+0.2397
8	SXS:BBH:0056	5	0.1389	28.8102	0.4900	+0.4391
9	SXS:BBH:0296	5.5	0.1302	27.9335	0.0520	+0.4427
10	SXS:BBH:0166	6	0.1224	21.5589	0.0440	
11	SXS:BBH:0298	7	0.1094	19.6757	0.0610	-0.0775 4
12	SXS:BBH:0299	7.5	0.1038	20.0941	0.0590	-0.0498
13	SXS:BBH:0063	8	0.0988	25.8255	0.2800	+1.0094
14	SXS:BBH:0300	8.5	0.0942	18.6953	0.0570	-0.0804
15	SXS:BBH:0301	9	0.0900	18.9274	0.0550	-0.1641
16	SXS:BBH:0302	9.5	0.0862	19.1169	0.0600	+0.0206
17	SXS:BBH:0303	10	0.0826	19.2666	0.0510	+0.2955

+ the q = 15 nonspinning waveform of [J. You et al.;2022]

### Effects of NQC corrections and ringdown



#### Max unfaithfulness versus q



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### **Unfaithfulness with NR tuning at 5PM-5PN**

