1st TEONGRAV international workshop on the

# Theory of Gravitational Waves

# *"A novel Lagrange-multiplier approach to the effective-one-body dynamics of binary systems in post-Minkowskian gravity"*

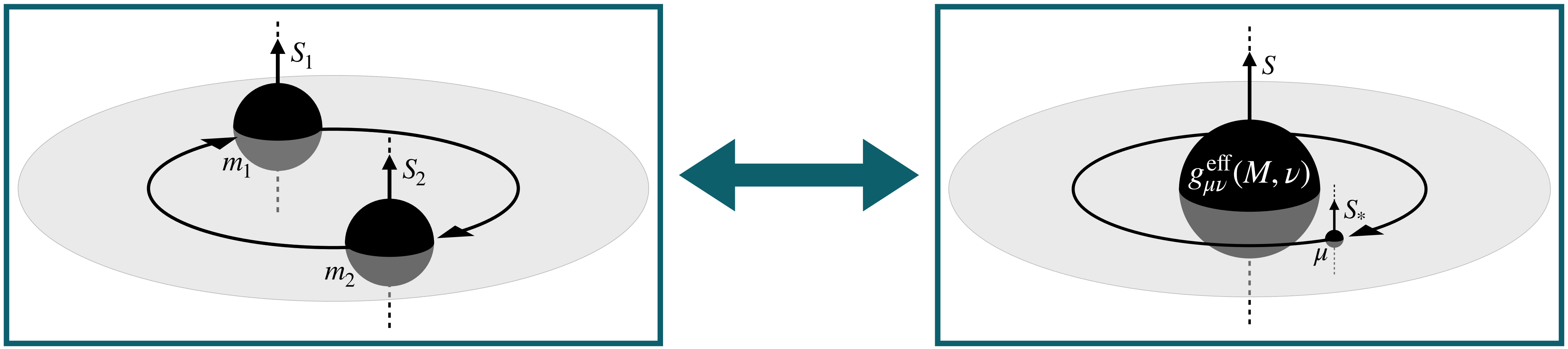
Based on **[T. Damour, A. Nagar, AP, P. Rettegno; 2024]** (in preparation)

16/09/2024

**Andrea Placidi**

# EOB approach in a nutshell


[A. Buonanno, T. Damour; 1999]




$$M = m_1 + m_2, \quad \mu = m_1 m_2/M, \quad \nu \equiv \mu/M$$

$$S = S_1 + S_2, \quad S_* = \frac{m_2}{m_1}S_1 + \frac{m_1}{m_2}S_2$$

**Two basic building blocks in the conservative dynamics:**

- EOB energy map

$$E_{\rm real} = M\sqrt{1 + 2\nu\left(\frac{E_{\rm eff}}{\mu} - 1\right)}$$

- Relativistic mass-shell condition for the effective particle

$$g^{\mu\nu}_{\rm eff}(X^\rho)P_\mu P_\nu + \mu^2 + Q(X^\mu, P_\mu) = 0$$

# EOB and post-Minkowskian information

In standard EOB models based on **post-Newtonian** information:

$$g_{\text{eff}}^{\mu\nu}(X^\rho)P_\mu P_\nu + \mu^2 + Q(X^\mu, P_\mu) = 0$$

+

$$g_{\mu\nu}^{\text{eff}}(X^\rho)dX^\mu dX^\nu = -A(R)c^2dT^2 + B(R)dR^2 + R^2C(R)(d\theta^2 + \sin^2\theta d\phi^2)$$

which is solved for $P_0 = -E_{\text{eff}}$ assuming $Q(X^\mu, P_\mu) = Q(X^i, P_i)$

$H_{\text{eff}} = E_{\text{eff}}$ → $H_{\text{eff}} = H_{\text{eff}}(A, B, C, Q)$ Energy map → $H_{\text{EOB}} = M\sqrt{1 + 2\nu\left(\frac{H_{\text{eff}}}{\mu} - 1\right)}$

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With **post-Minkowskian** (PM) information, which is inherently energy dependent, at least one function among $\{A, B, C, Q\}$ must be allowed to depend on $E_{\text{eff}}$

$H_{\text{eff}} = E_{\text{eff}}$ → $H_{\text{eff}} = H_{\text{eff}}(A, B, C, Q, H_{\text{eff}})$ $\rightarrow$ **recursive definition** (the self dependence appears at 2PM)

[T. Damour; 2018] [A. Antonelli et al.; 2019]
[M. Khalil et al.; 2022] [A. Buonanno et al.; 02/2024] [A. Buonanno et al.; 05/2024]

$\{A, B, C, Q\}$ are then determined from the knowledge of the PM-expanded scattering angle

$$\chi_{\text{PM}}(j, \gamma) = \sum_n 2\frac{\chi_n(\gamma, \nu)}{j^n}$$

**intricate dependence** on $\gamma = E_{\text{eff}}/\mu$
- 3PM $\rightarrow$ hyperbolic functions
- 4PM $\rightarrow$ elliptic integrals, polylogs

# LEOB: a novel Lagrange-multiplier approach

$$S[X^\mu, P_\mu] = \int [P_\mu dX^\mu]^{\text{on-shell}} = \int P_i dX^i - H_{\text{eff}}(X^i, P_i) dT_{\text{eff}}$$

replaced by

$$S[X^\mu, P_\mu, e_L] = \int P_\mu dX^\mu - e_L \mathscr{C}\left(X^\mu, P_\mu\right) d\tau$$

Lagrange multiplier

Evolution parameter associated to $e_L$

EOB mass-shell constraint $\mathscr{C} = g^{\mu\nu}_{\text{eff}}(X^\rho) P_\mu P_\nu + \mu^2 + Q(X^\mu, P_\mu)$

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Lagrange multiplier: $e_{\text{L}}$

Evolution parameter associated to $e_L$: $\tau$

EOB mass-shell constraint $\mathscr{C} = g_{\text{eff}}^{\mu\nu}(X^\rho) P_\mu P_\nu + \mu^2 + Q(X^\mu, P_\mu)$

From the variational principle:

$$\delta S[X^\mu, P_\mu, e_{\text{L}}] = 0 \quad \rightarrow \quad \frac{dX^\mu}{d\tau} = e_{\text{L}} \frac{\partial \mathscr{C}}{\partial P_\mu}, \qquad \frac{dP_\mu}{d\tau} = -\, e_{\text{L}} \frac{\partial \mathscr{C}}{\partial X^\mu}, \qquad \mathscr{C} = 0$$

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Fixing $\tau = T_{\text{real}}$ while considering $\dfrac{dT_{\text{eff}}}{dT_{\text{real}}} = \dfrac{dE_{\text{real}}}{dE_{\text{eff}}} = \dfrac{M}{E_{\text{real}}}$ and $X^0 = T_{\text{eff}}$

$$\frac{M}{E_{\text{real}}} = e_{\text{L}} \frac{\partial \mathscr{C}}{\partial P_0} = -\, e_{\text{L}} \frac{\partial \mathscr{C}}{\partial E_{\text{eff}}} \quad \longrightarrow \quad e_L = -\frac{M}{E_{\text{real}}} \left( \frac{d\mathscr{C}}{dE_{\text{eff}}} \right)^{-1}$$

# LEOB equations of motion

In terms of mass-rescaled quantities, the resulting Euler-Lagrange equations are:

$$\frac{dx^i}{dt_{\rm real}} = -\frac{1}{h}\left(\frac{\partial\hat{\mathscr{C}}}{\partial\gamma}\right)^{-1}\frac{\partial\hat{\mathscr{C}}}{\partial p_i}, \qquad \frac{dp_i}{dt_{\rm real}} = \frac{1}{h}\left(\frac{\partial\hat{\mathscr{C}}}{\partial\gamma}\right)^{-1}\frac{\partial\hat{\mathscr{C}}}{\partial x^i}, \qquad \frac{d\gamma}{dt_{\rm real}} = 0$$

$$t_{\rm real} = \frac{T_{\rm real}}{GM}, \qquad x^\mu = \frac{X^\mu}{GM}, \qquad p_\mu = \frac{P_\mu}{\mu}, \qquad \gamma = \frac{E_{\rm eff}}{\mu}, \qquad h = \frac{E_{\rm real}}{M}, \qquad \hat{\mathscr{C}} = \frac{\mathscr{C}}{\mu^2}$$

plus the equation $\hat{\mathscr{C}} = 0$ that only affects the determination of the initial conditions.

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plus the equation $\hat{\mathscr{C}} = 0$ that only affects the determination of the initial conditions.

To go **beyond the conservative dynamics**, we generalize the evolution equation of $p_\mu$ by adding a **radiation-reaction force** $\mathscr{F}_\mu$, i.e.

$$\frac{dx^i}{dt_{\rm real}} = -\frac{1}{h}\left(\frac{\partial \hat{\mathscr{C}}}{\partial \gamma}\right)^{-1}\frac{\partial \hat{\mathscr{C}}}{\partial p_i}, \qquad \frac{dp_i}{dt_{\rm real}} = \frac{1}{h}\left(\frac{\partial \hat{\mathscr{C}}}{\partial \gamma}\right)^{-1}\frac{\partial \hat{\mathscr{C}}}{\partial x^i} + \mathscr{F}_i, \qquad \frac{d\gamma}{dt_{\rm real}} = -\mathscr{F}_0$$

with the condition $\frac{dx^\mu}{dt_{\rm real}}\mathscr{F}_\mu = 0$ ensuring that $\hat{\mathscr{C}} = 0$ holds along the whole radiation-reacted evolution

# LEOB gauge flexibility

The LEOB approach makes it easy to admit any functional dependence on $\gamma$.

Without any loss of generality one can either make the **effective metric energy independent**, with the constraint

$$\hat{\mathscr{C}} = g_{\text{eff}}^{\mu\nu}(x^\rho) p_\mu p_\nu + 1 + Q(x^\mu, \gamma)$$

or choose a **geodesic-like constraint** of the type

$$\hat{\mathscr{C}} = g_{\text{eff}}^{\mu\nu}(x^\rho, \gamma) p_\mu p_\nu + 1$$

In both cases there is still a large leftover gauge freedom.

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Focusing on the second, in fact, we have $g_{\text{eff}}^{\mu\nu}(x^\rho, \gamma) = g_{\text{eff}}^{\mu\nu}[A(r,\gamma), B(r,\gamma), C(r,\gamma)]$ whereas the scattering angle $\chi_{\text{PM}}(j, \gamma)$ can just fix one of the three metric functions. This is due to:

- The usual **coordinate freedom** of general relativity
- The fact that, just like an Hamiltonian, $\hat{\mathscr{C}}$ admits the set of **canonical transformations as symmetries**

We have thus to impose **two** conditions on $\{A, B, C\}$

# Gauge fixed constraint and fluxes

We select the constraint $\hat{\mathscr{C}} = g^{\mu\nu}_{\text{eff}}(x^\rho, \gamma) p_\mu p_\nu + 1$ and choose:

- $C(r,\gamma) = 1 \;\rightarrow\;$ Schwarzschild-like coordinates
- $A(r,\gamma)B(r,\gamma) = 1 \;\rightarrow\;$ Resulting constraint in simple form

$$\rightarrow \quad \boxed{\hat{\mathscr{C}} = -\frac{\gamma^2}{A(r,\gamma)} + p_r^2 A(r,\gamma) + \frac{p_\varphi^2}{r^2} + 1}$$

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Remaining metric function $A(r,\gamma)$:

$\chi_{\le 3\rm PM}(j,\gamma) \;\rightarrow\; A_{\le 3\rm PM}(r,\gamma)$

$\chi_{\rm 4PM,local}(j,\gamma) \;\rightarrow\; A_{\rm 4PM,local}(r,\gamma)$

+

PN completion $\Delta A_{\ge 4\rm PN}(r,\gamma)$, comprensive of nonlocal tail-transported effects
→ [Piero Rettegno's talk on Wednesday]

$$A(r,\gamma) = A_{\le 3\rm PM}(r,\gamma) + A_{\rm 4PM,local}(r,\gamma) + \Delta A_{\ge 4\rm PN}(r,\gamma)$$

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Remaining metric function $A(r,\gamma)$:

$\chi_{\leq 3\text{PM}}(j,\gamma) \;\rightarrow\; A_{\leq 3\text{PM}}(r,\gamma)$

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Fluxes and waveform:

$\mathscr{F}_r = 0$ [quasi-circular (qc) orbits]

$\mathscr{F}_\gamma = -\dot{\varphi}\mathscr{F}_\varphi$

$\mathscr{F}_\varphi \rightarrow$, $h_{\ell m} \rightarrow$ PN-based factorized and resummed prescription of the qc iteration of TEOBResumS

[T. Damour, B. R. Iyer. A. Nagar; 2009]

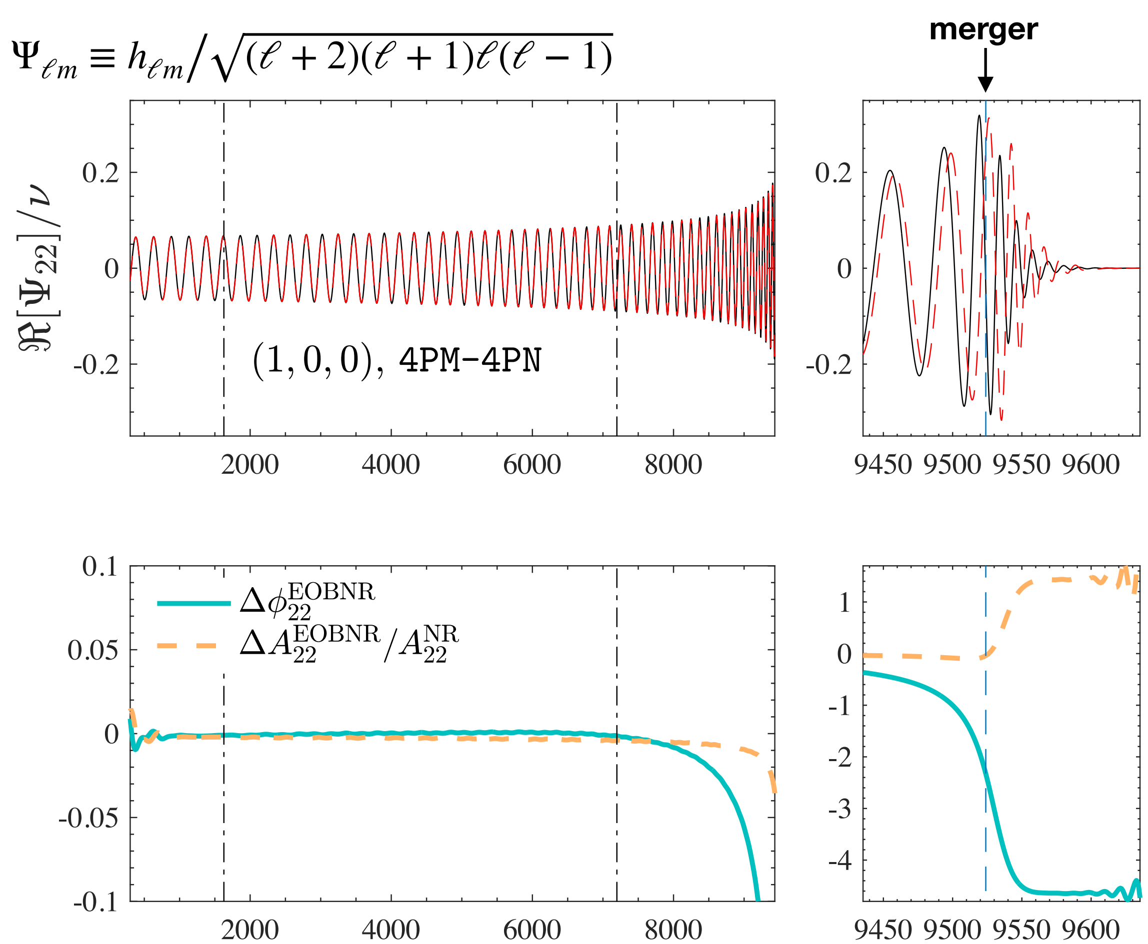
# Testing LEOB: phasings

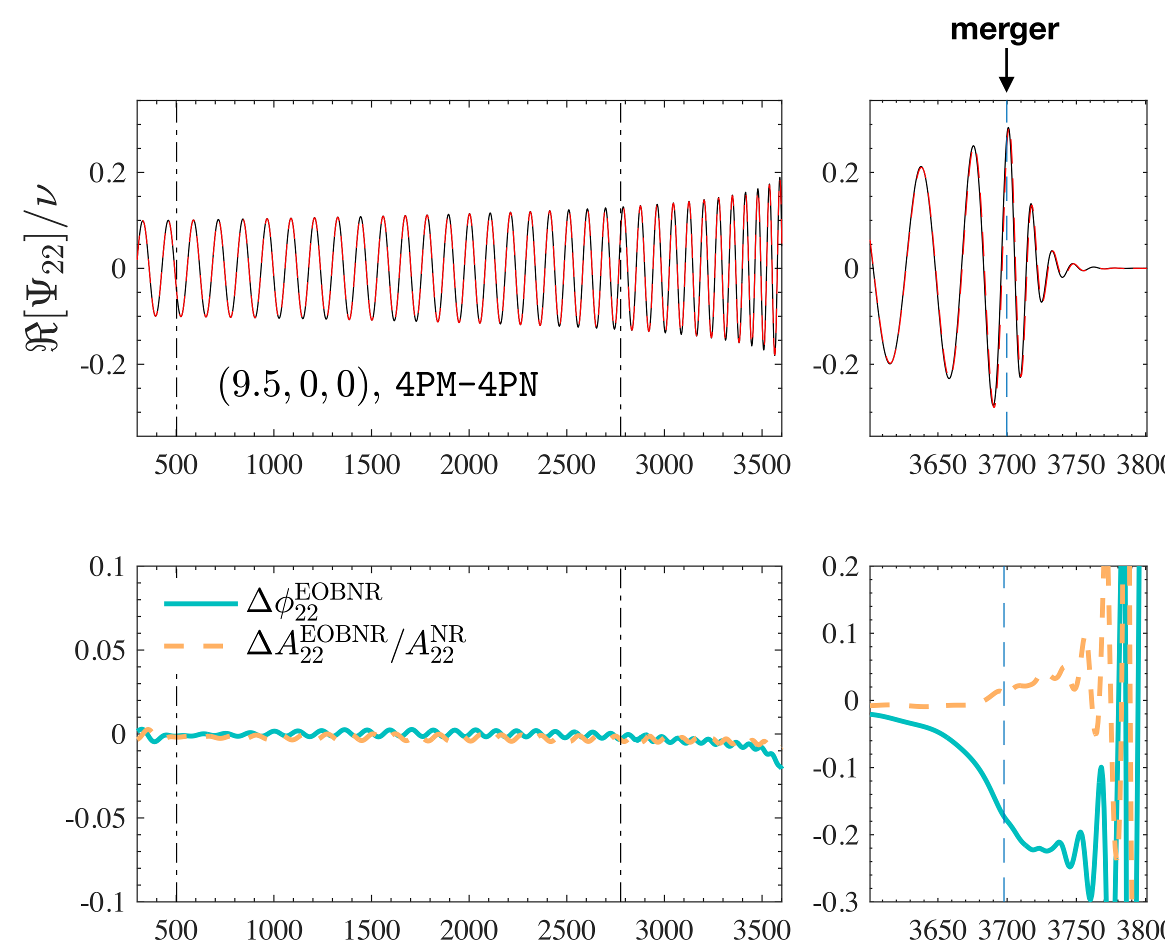
**No numerical relativity (NR) tuning in the dynamics!** NR just in NQCs and ringdown

4PM-4PN $\rightarrow$ the PN completion $\Delta A_{\geq 4\text{PN}}(r, \gamma)$ is stopped at the 4PN order

$q = 1$ $q = 9.5$

$\Psi_{\ell m} \equiv h_{\ell m} / \sqrt{(\ell+2)(\ell+1)\ell(\ell-1)}$




$h_{22} = A_{22} e^{i\phi_{22}}, \quad \Delta\phi_{22}^{\text{EOBNR}} = \phi_{22}^{\text{LEOB}} - \phi_{22}^{\text{NR}}, \quad \Delta A_{22}^{\text{EOBNR}} = A_{22}^{\text{LEOB}} - A_{22}^{\text{NR}}$

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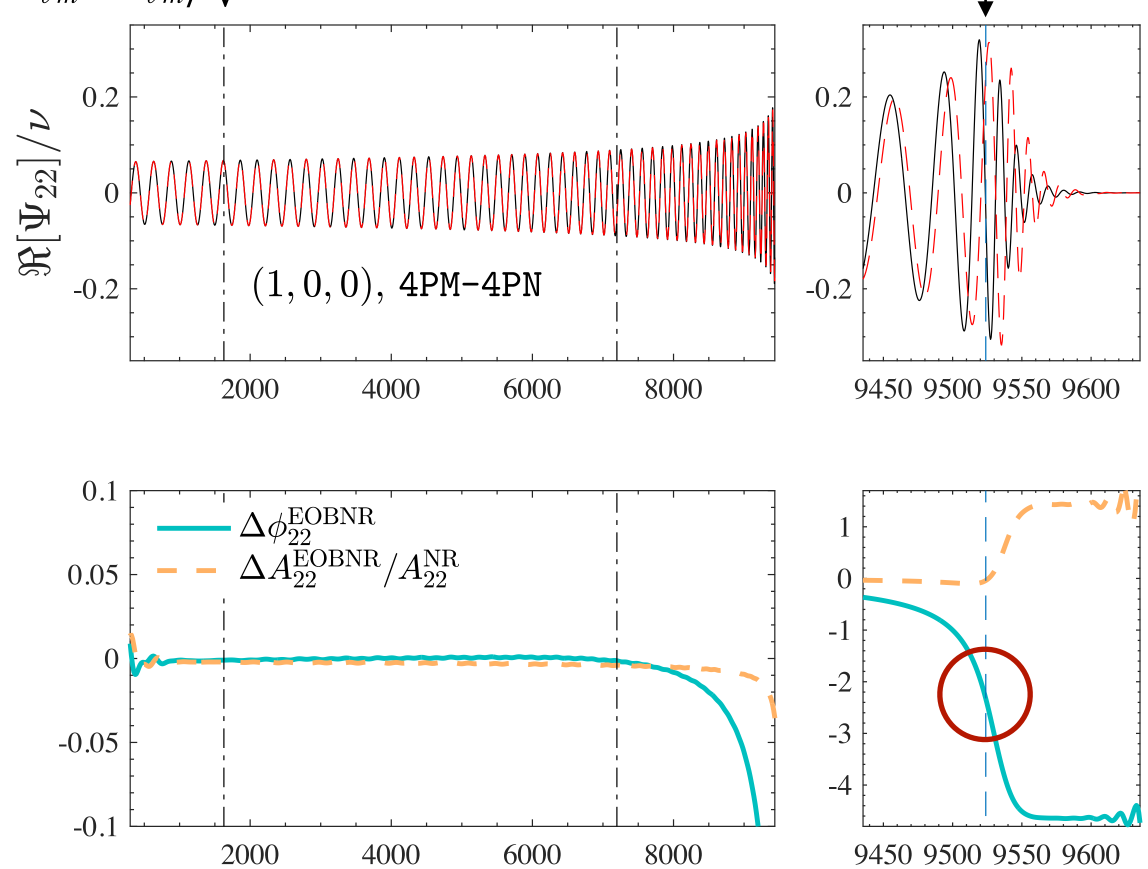
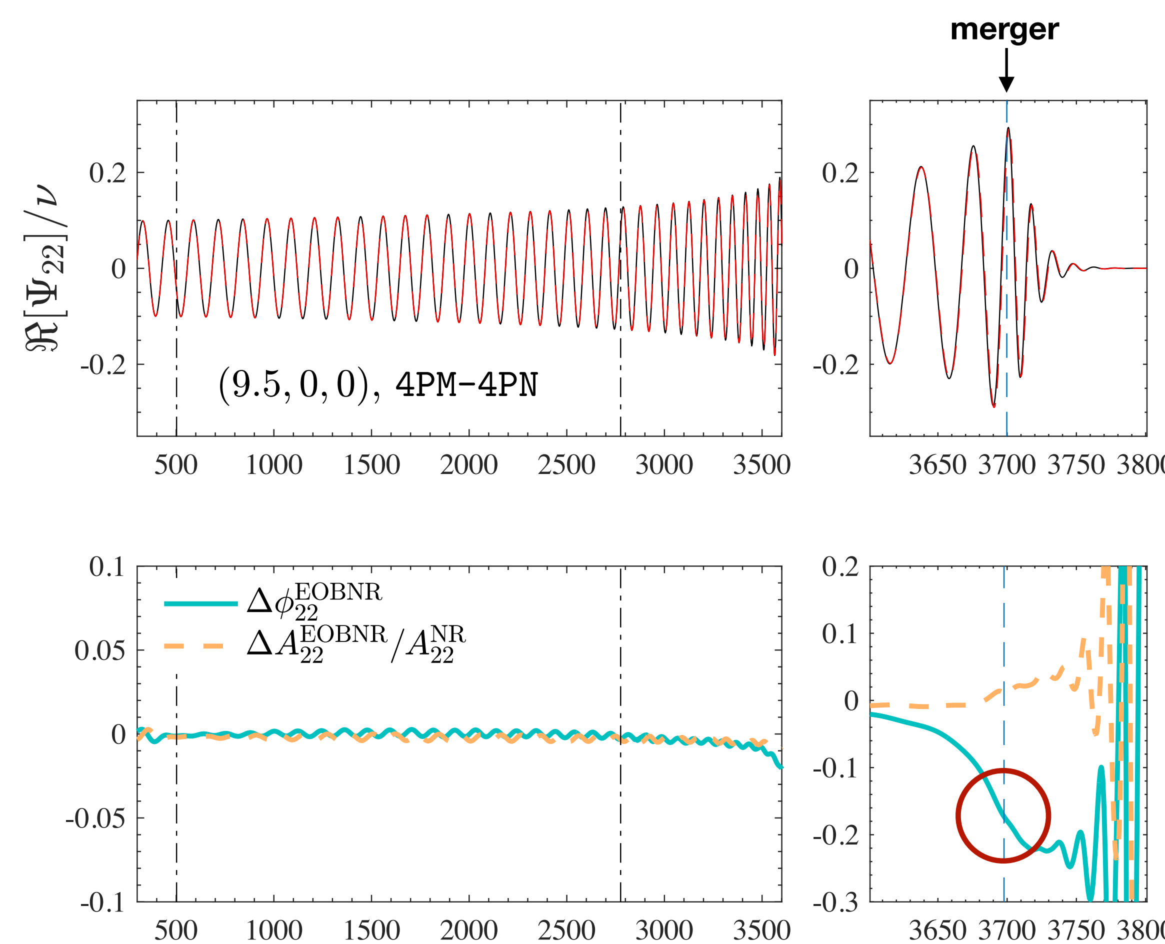



$h_{22} = A_{22} e^{i\phi_{22}}\,, \qquad \Delta\phi_{22}^{\mathrm{EOBNR}} = \phi_{22}^{\mathrm{LEOB}} - \phi_{22}^{\mathrm{NR}}\,, \qquad \Delta A_{22}^{\mathrm{EOBNR}} = A_{22}^{\mathrm{LEOB}} - A_{22}^{\mathrm{NR}}$

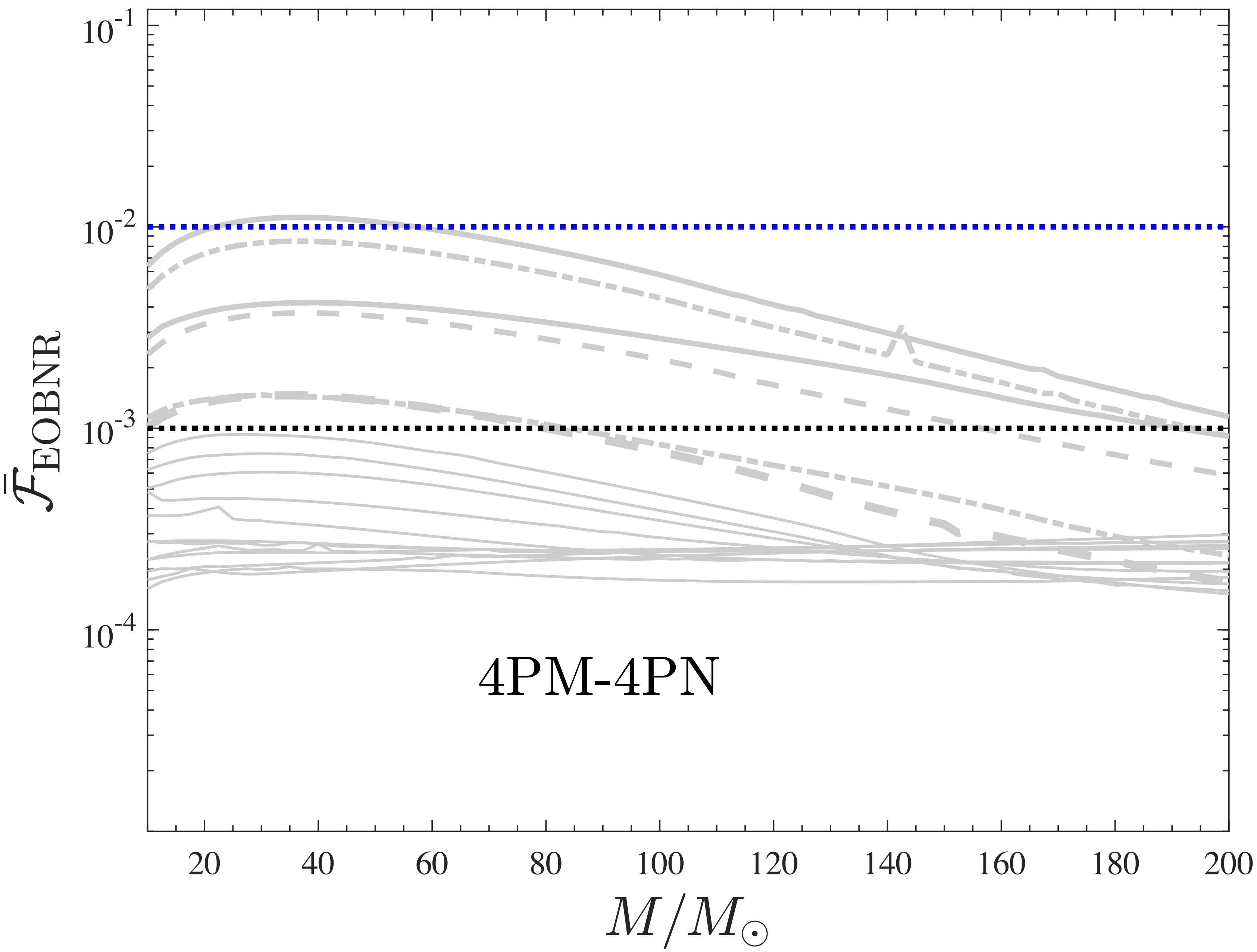
$\Delta\phi_{22}^{\mathrm{EOBNR}}$ at merger ~-2.2 rad for $q = 1$ and ~-0.18 rad for $q = 9.5$

# Testing LEOB: unfaithfulness

**No numerical relativity (NR) tuning in the dynamics!** NR just in NQCs and ringdown

4PM-4PN → the PN completion $\Delta A_{\geq 4\text{PN}}(r, \gamma)$ is stopped at the 4PN order.

We consider 17 nonspinning NR waveforms from the SXS public catalogue and the $q = 15$ nonspinning waveform of [J. You et al.;2022]



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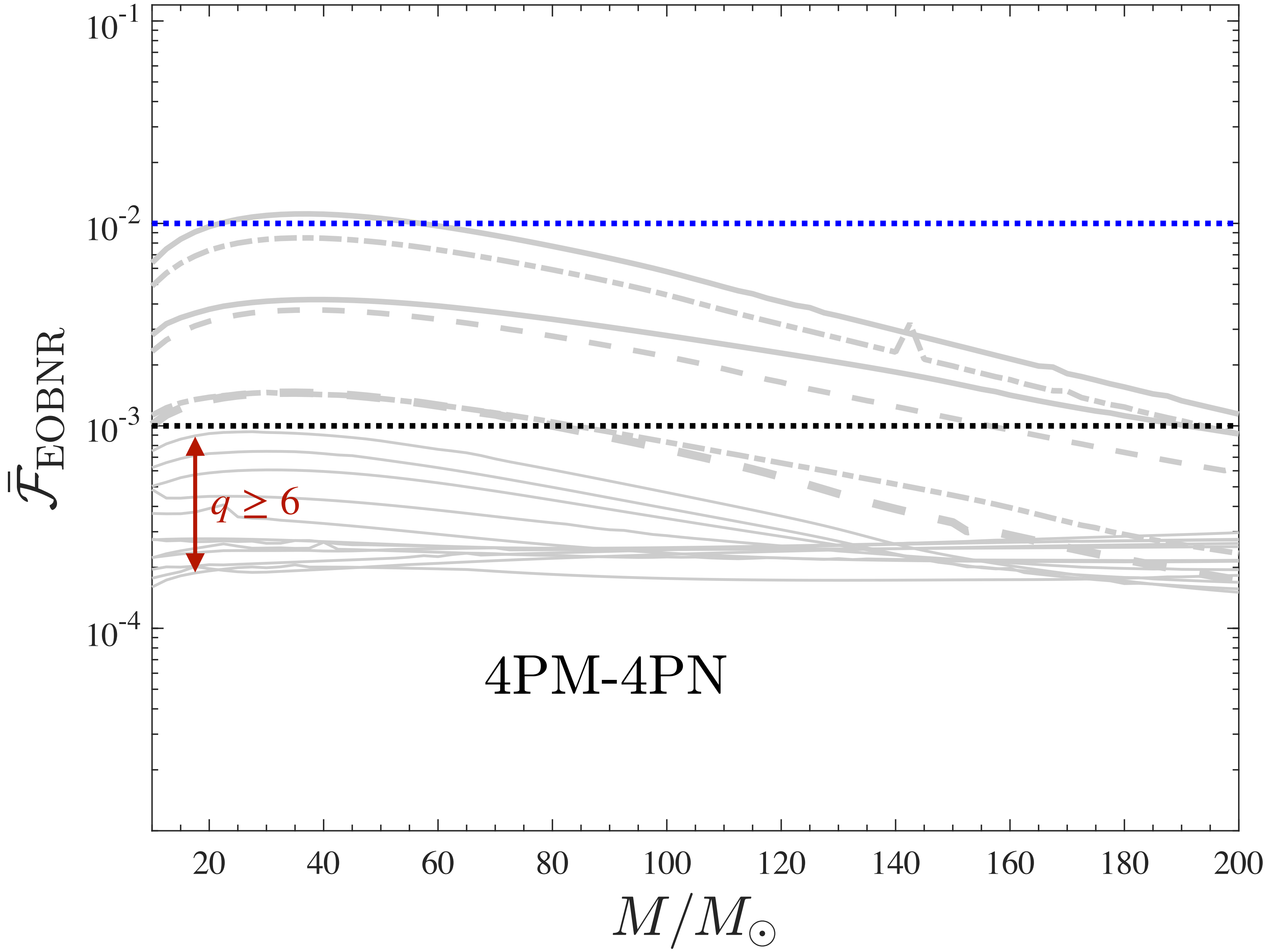

# Conclusions

With LEOB, our novel Lagrange-multiplier approach, we are able to provide a complete description, within the EOB framework and in the form of Euler-Lagrange equations, of the dynamical evolution of a binary system of black holes.

Crucially, it avoids the need to solve the EOB mass-shell constraint for an effective Hamiltonian, at the rather small cost of having one additional evolution equation for the energy $\gamma$.

The simplification and flexibility brought about by the LEOB approach have a notable impact in the development of PM-based EOB models built upon the 4PM analytical information that is currently available, and we expect even more benefits when higher-order PM perturbative results will be achieved.

When applied to a complete waveform model, the conservative dynamics of the LEOB approach yields good results up to merger (especially for $q \geq 6$), where the phase difference with NR remains quite low even without NR tuning the dynamics. This is also confirmed by looking at the unfaithfulness.

LEOB can be also applied to spinning binary black holes (work in progress)
→ [Piero Rettegno’s talk on Wednesday]

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***Thanks for your attention!***

# Backup Slides

# Link between LEOB and the $H$-based EOB

The traditional Hamiltonian EOB dynamics is regained when using a mass-shell constraint in the explicitly solved form

$$\hat{\mathscr{C}}_H \equiv \hat{H}_{\rm eff}(x^i, p_i) - \gamma$$

which implies

$$\frac{\partial \hat{\mathscr{C}}_H}{\partial \gamma} = -1\,, \qquad \frac{\partial \hat{\mathscr{C}}_H}{\partial p_i} = \frac{\partial \hat{H}_{\rm eff}}{\partial p_i}\,, \qquad \frac{\partial \hat{\mathscr{C}}_H}{\partial x^i} = \frac{\partial \hat{H}_{\rm eff}}{\partial x^i}$$

We have in fact:

$$\frac{dx^i}{dt_{\rm real}} = -\frac{1}{h}\left(\frac{\partial \hat{\mathscr{C}}_H}{\partial \gamma}\right)^{-1}\frac{\partial \hat{\mathscr{C}}_H}{\partial p_i}, \qquad \frac{dp_i}{dt_{\rm real}} = \frac{1}{h}\left(\frac{\partial \hat{\mathscr{C}}_H}{\partial \gamma}\right)^{-1}\frac{\partial \hat{\mathscr{C}}_H}{\partial x^i} + \mathscr{F}_i, \qquad \frac{d\gamma}{dt_{\rm real}} = -\mathscr{F}_0$$

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# Relations between $A(r,\gamma)$ and $\chi_{\rm PM}(r,\gamma)$

$$A - A_s = \sum_n \frac{\bar{a}_n}{r^n}\,, \qquad \Delta\chi_{\rm PM} = \sum_n 2\frac{\chi_n(\gamma,\nu) - \chi_n^S(\gamma,\nu)}{j^n}$$

$$\bar{a}_1(\gamma,\nu) = 0\,,$$

$$\bar{a}_2(\gamma,\nu) = -\frac{8\Delta\chi_2}{\pi\left(3\gamma^2-1\right)}\,,$$

$$\bar{a}_3(\gamma,\nu) = \frac{24\left(8\gamma^4-8\gamma^2+1\right)\Delta\chi_2}{\pi\left(3\gamma^2-1\right)\left(4\gamma^2-1\right)p_\infty^2} - \frac{3\Delta\chi_3}{\left(4\gamma^2-1\right)p_\infty}\,,$$

$$\bar{a}_4(\gamma,\nu) = -\frac{12\left(140\gamma^8-235\gamma^6+123\gamma^4-13\gamma^2+1\right)\Delta\chi_2}{\pi\left(3\gamma^2-1\right)\left(4\gamma^2-1\right)\left(5\gamma^2-1\right)p_\infty^4} + \frac{16\left(35\gamma^4-30\gamma^2+3\right)\Delta\chi_2^2}{\pi^2\left(3\gamma^2-1\right)^2\left(5\gamma^2-1\right)p_\infty^2}$$
$$+ \frac{3\left(35\gamma^4-30\gamma^2+3\right)\Delta\chi_3}{\left(4\gamma^2-1\right)\left(5\gamma^2-1\right)p_\infty^3} - \frac{32\Delta\chi_4}{3\pi\left(5\gamma^2-1\right)p_\infty^2}\,.$$

$$p_\infty = \sqrt{\gamma^2-1}$$

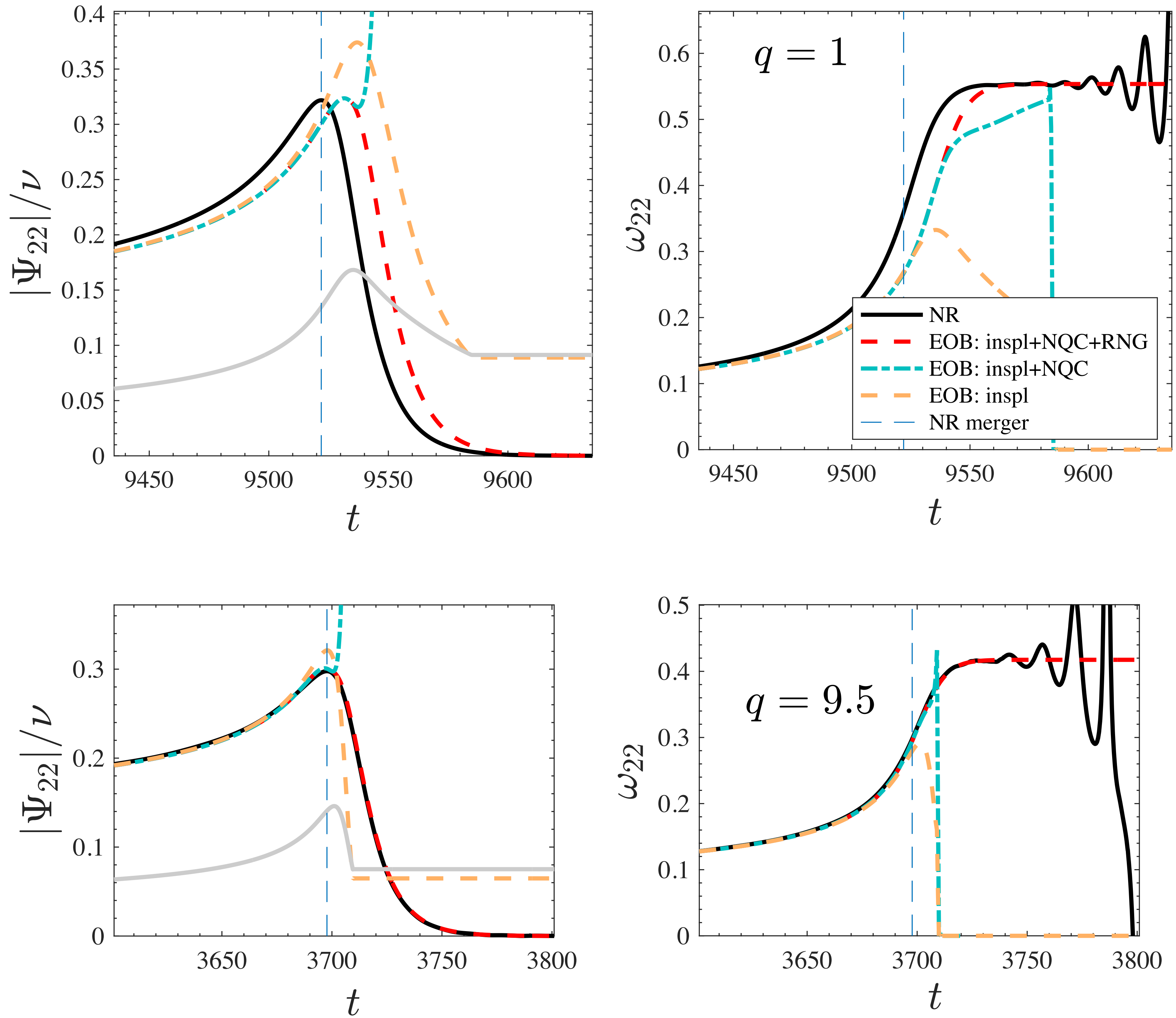
# Numerical waveforms used in the tests

TABLE IV: Properties of the 17 nonspinning public NR SXS waveforms used to test the new LEOB waveform model for nonspinning binaries. From left to right, the columns report: the identification number; the SXS classification name; the mass ratio $q \equiv m_1/m_2 \geq 1$; the symmetric mass ratio $\nu \equiv m_1 m_2/(m_1+m_2)^3$; the number of orbits between $t=0$ and the time when a common event horizon is formed; the orbital eccentricity $\epsilon$ determined at relaxed measurement time; the $\ell=m=2$ phase-difference $\delta\phi^{\rm NR}_{\rm mrg}$ between the highest and second highest resolution accumulated between $t=600M$ and the peak amplitude of the highest resolution data.

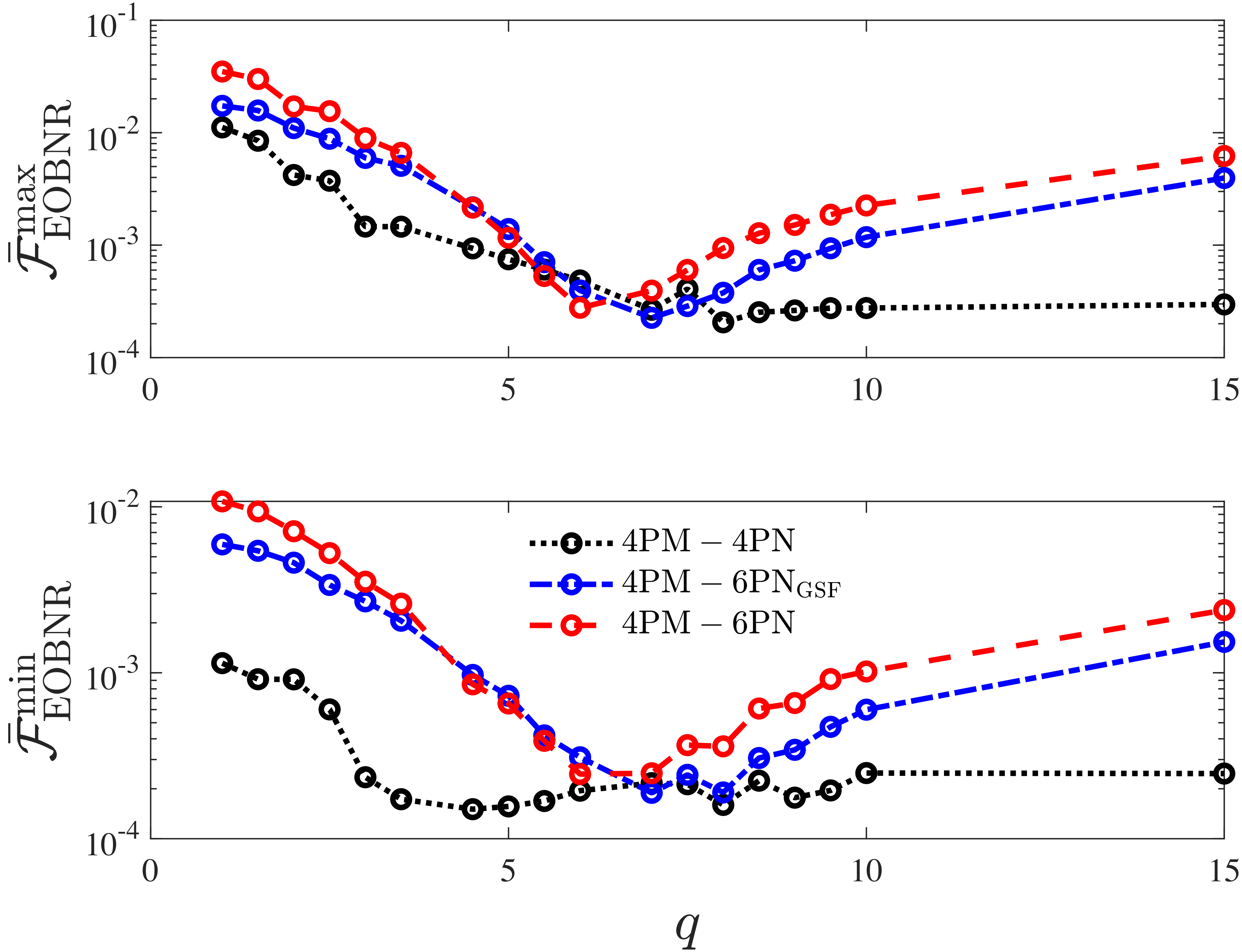
| # | name | $q$ | $\nu$ | # orbits | $\epsilon\,[10^{-3}]$ | $\delta\phi^{\rm NR}_{\rm mrg}$ [rad] |
|---|---|---|---|---|---|---|
| 1 | SXS:BBH:0180 | 1 | 0.2500 | 28.1825 | 0.0514 | −0.4247 |
| 2 | SXS:BBH:0007 | 1.5 | 0.2400 | 29.0922 | 0.4200 | −0.0186 |
| 3 | SXS:BBH:0169 | 2 | 0.2222 | 15.6805 | 0.1200 | −0.0271 |
| 4 | SXS:BBH:0259 | 2.5 | 0.2041 | 28.5625 | 0.0590 | −0.0080 |
| 5 | SXS:BBH:0030 | 3 | 0.1875 | 18.2228 | 2.0100 | −0.0870 |
| 6 | SXS:BBH:0167 | 4 | 0.1600 | 15.5908 | 0.0990 | −0.5165 |
| 7 | SXS:BBH:0295 | 4.5 | 0.1488 | 27.8067 | 0.0520 | +0.2397 |
| 8 | SXS:BBH:0056 | 5 | 0.1389 | 28.8102 | 0.4900 | +0.4391 |
| 9 | SXS:BBH:0296 | 5.5 | 0.1302 | 27.9335 | 0.0520 | +0.4427 |
| 10 | SXS:BBH:0166 | 6 | 0.1224 | 21.5589 | 0.0440 | … |
| 11 | SXS:BBH:0298 | 7 | 0.1094 | 19.6757 | 0.0610 | −0.0775 4 |
| 12 | SXS:BBH:0299 | 7.5 | 0.1038 | 20.0941 | 0.0590 | −0.0498 |
| 13 | SXS:BBH:0063 | 8 | 0.0988 | 25.8255 | 0.2800 | +1.0094 |
| 14 | SXS:BBH:0300 | 8.5 | 0.0942 | 18.6953 | 0.0570 | −0.0804 |
| 15 | SXS:BBH:0301 | 9 | 0.0900 | 18.9274 | 0.0550 | −0.1641 |
| 16 | SXS:BBH:0302 | 9.5 | 0.0862 | 19.1169 | 0.0600 | +0.0206 |
| 17 | SXS:BBH:0303 | 10 | 0.0826 | 19.2666 | 0.0510 | +0.2955 |

+ the $q = 15$ nonspinning waveform of [J. You et al.;2022]

# Effects of NQC corrections and ringdown



# Max unfaithfulness versus $q$



# Unfaithfulness with NR tuning at 5PM-5PN