Modular Invariance and the Strong CP Problem

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Modular invariance and the QCD angle

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Outline

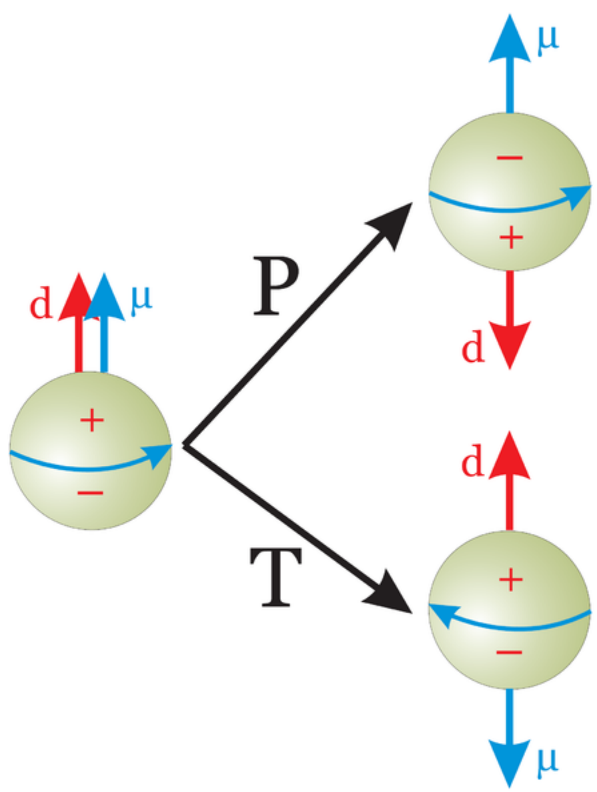
1. Strong CP problem
2. Existing solutions
3. Modular invariance and global supersymmetry
4. Modular anomalies and their cancellation
5. Corrections to $\bar{\theta} = 0$
6. Conclusions

The strong CP problem

$$\mathscr{L}_{\text{QCD}} = \bar{q}\left(i\not{D} - M_q\right) q - \frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} + \theta_{\text{QCD}} \frac{g_3^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu}$$

$\boxed{\bar{\theta} = \theta_{\text{QCD}} + \arg\det M_q}$ CPV parameter

Neutron EDM d



$d = 2.4 \times 10^{-16}\ \bar{\theta}\ e \cdot$ cm Pospelov, Ritz, hep-ph/9908508v4

$|d| \leq 1.8 \times 10^{-26}\ e \cdot$ cm (90% C.L.) Abel et al., 2001.11966

$\boxed{|\bar{\theta}| \lesssim 10^{-10}}$

Why so small???

… and the CPV phase in the CKM matrix $\delta_{\text{CKM}} \approx 1.2$

Solution 1: the Axion

Promote $\bar{\theta}$ to a dynamical scalar field a, the axion, which washes out CP violation in QCD

$$\mathscr{L}_a = \frac{1}{2}(\partial_\mu a)^2 + \frac{g_3^2}{32\pi^2}\frac{a}{f_a}G\tilde{G} + \dots$$

$$\bar{\theta} = \frac{\langle a \rangle}{f_a} \qquad \text{with} \qquad \langle a \rangle = 0$$

New global $\mathrm{U(1)_{PQ}}$

Peccei, Quinn, PRL **38** (1977) 1440; PRD **16** (1977) 1791

- spontaneously broken => the axion is a NGB
- anomalous under QCD $(\partial_\mu J^\mu_{\mathrm{PQ}} \propto G\tilde{G})$ => the axion is a pNGB

Quality problem

- Corrections of order $(f_a/M_{\mathrm{Pl}})^{\#}$ from higher-dimensional operators
- $\mathrm{U(1)_{PQ}}$ should be an accidental symmetry in a complete model

Solution 2: CP (P) is symmetry of UV

- CP (P) is a symmetry of the UV
- It is broken spontaneously in such a way that $\bar{\theta} = 0$ and $\delta_{\rm CKM} = \mathcal{O}(1)$

Nelson—Barr models

Nelson, PLB **136** (1984) 387; Barr, PRL **53** (1984) 329

New heavy vector-like quarks Q and scalars η with CPV complex VEVs $\langle\eta\rangle$

$$(q_R\ Q_R)\, M_q \begin{pmatrix} q_L \\ Q_L \end{pmatrix} = (q_R\ Q_R) \begin{pmatrix} y\mathrm{v}_H & y'\langle\eta\rangle \\ 0 & \mu \end{pmatrix} \begin{pmatrix} q_L \\ Q_L \end{pmatrix}$$

- CP is a symmetry => $\theta_{\rm QCD} = 0$ and the couplings (y, y', μ) are real
- $\det M_q = y\mathrm{v}_H\mu$ is real (and positive) => $\arg\det M_q = 0$
- Effective light quark mass matrix depends on $\langle\eta\rangle$ => $\delta_{\rm CKM} \neq 0$

Additional matter, tuning, loop corrections…

Dine, Draper, 1506.05433

Our solution: CP + modular invariance

1. CP is a symmetry => $\theta_{\mathrm{QCD}} = 0$ (and real Lagrangian couplings)

2. Modular invariance/anomaly cancellation => $\arg\det M_q = 0$

3. CP is broken spontaneously by the VEV of a single complex field, the modulus τ => $\delta_{\mathrm{CKM}} = \mathcal{O}(1)$

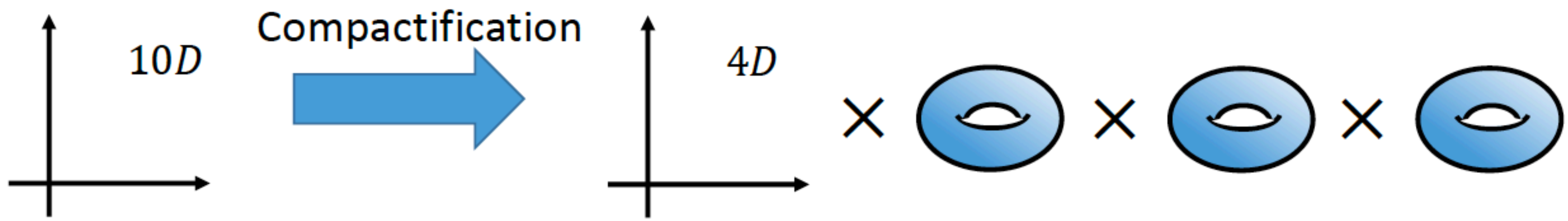
4. Quark mass hierarchies and mixing angles are reproduced by $\mathcal{O}(1)$ parameters

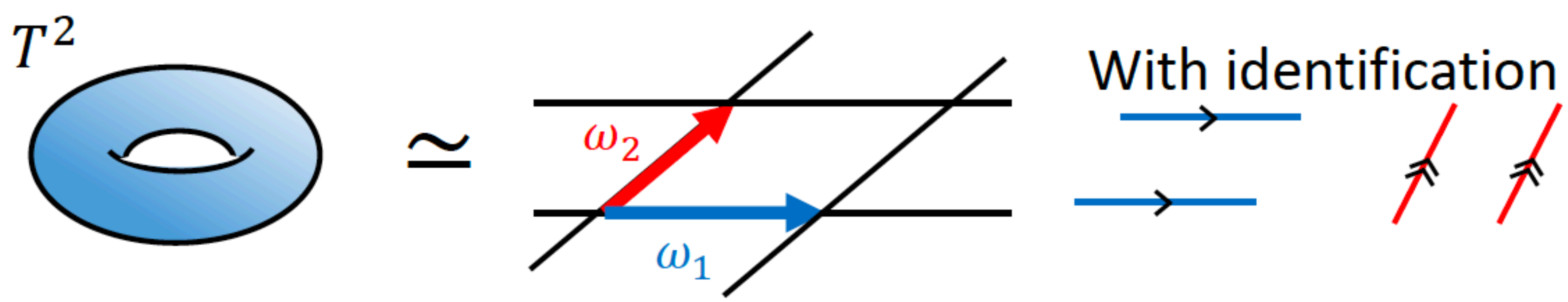
5. Corrections to $\bar{\theta} = 0$ are small under certain assumptions on SUSY breaking

Modular invariance

String theory requires extra dimensions

Images: Takuya H. Tatsuishi





$$\tau = \frac{\omega_2}{\omega_1}, \quad \mathrm{Im}\,(\tau) > 0$$

modulus

Lattice left invariant by modular transformations

$$\tau \to \gamma\tau \equiv \frac{a\tau + b}{c\tau + d} \qquad a, b, c, d \in \mathbb{Z} \qquad ad - bc = 1$$

These transformations form an infinite discrete group

Modular group

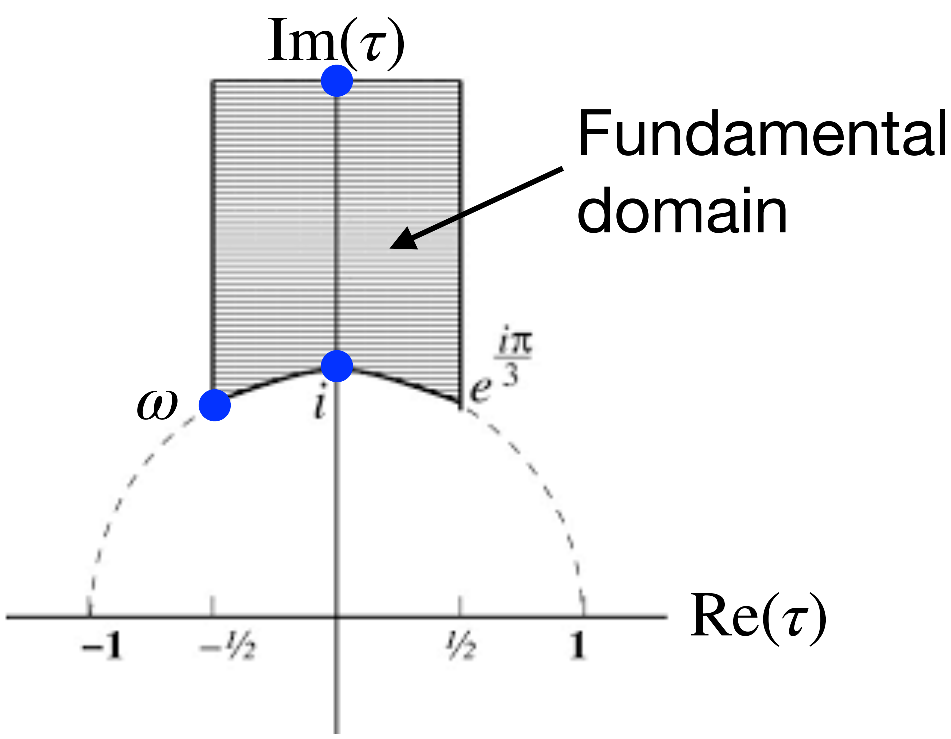
Homogeneous modular group

$$\Gamma = \langle S,\ T \mid S^4 = (ST)^3 = I \rangle \cong \mathrm{SL}(2,\mathbb{Z})$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad \tau \xrightarrow{T} \tau + 1$$

duality — discrete shift symmetry



Special points

- $\tau = i: \quad i \xrightarrow{S} -\frac{1}{i} = i \quad \Rightarrow \quad Z_4^S$
- $\tau = \omega \equiv e^{\frac{2\pi i}{3}}: \quad \omega \xrightarrow{ST} -\frac{1}{\omega + 1} = \omega \quad \Rightarrow \quad Z_3^{ST} \times Z_2^{S^2}$
- $\tau = i\infty: \quad i\infty \xrightarrow{T} i\infty + 1 = i\infty \quad \Rightarrow \quad Z^T \times Z_2^{S^2}$

Modular forms

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under Γ as

$$\boxed{f\left(\gamma\tau\right) = \left(c\tau + d\right)^k f(\tau)\,, \qquad \gamma \in \Gamma}$$

k is weight, a non-negative even integer

Normalised Eisenstein series

$$E_k(\tau) = \frac{1}{2\zeta(k)} \sum_{(m,n)\neq(0,0)} \frac{1}{(m+n\tau)^k}$$

Each modular form can be written as a polynomial in E_4 and E_6

$$f(\tau) = \sum_{a,b\geq 0} c_{ab}\, E_4^a(\tau)\, E_6^b(\tau) \qquad \text{with} \qquad 4a + 6b = k$$

Modular weight k	0	2	4	6	8	10	12	14
Modular forms	1	–	E_4	E_6	$E_8 = E_4^2$	$E_{10} = E_4 E_6$	E_4^3, E_6^2	$E_{14} = E_4^2 E_6$

Modular-invariant SUSY theories

$\mathcal{N} = 1$ global SUSY action

$$\mathcal{L} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\; K\left(\tau, e^{2V}\Phi, \tau^\dagger, \Phi^\dagger\right) + \left[\int \mathrm{d}^2\theta\; W(\tau, \Phi) + \frac{1}{16}\int \mathrm{d}^2\theta\; f(\tau)\, \mathcal{G}\,\mathcal{G} + \text{h.c.}\right]$$

Kähler potential K
(kinetic terms,
gauge interactions)

Superpotential W
(Yukawa interactions)

Gauge kinetic function f

$$f_3 = \frac{1}{g_3^2} - i\frac{\theta_{\mathrm{QCD}}}{8\pi^2}$$

Under modular transformations $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ \Phi \to \left(c\tau + d\right)^{-k_\Phi}\Phi \\ V \to V \end{cases}$$

τ is promoted to a (dimensionless) superfield

matter supermultiplets

vector supermultiplets

Modular symmetry acts non-linearly

Ferrara et al., PLB **225** (1989) 363; PLB **233** (1989) 147; Feruglio, 1706.08749

Modular-invariant SUSY theories

$\mathcal{N} = 1$ global SUSY action

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Kähler potential K
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Superpotential W
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Gauge kinetic function f

$$f_3 = \frac{1}{g_3^2} - i\frac{\theta_{\text{QCD}}}{8\pi^2}$$

Modular invariance of the action requires

$$\begin{cases} K(\tau, \Phi, \tau^\dagger, \Phi^\dagger) \to K(\tau, \Phi, \tau^\dagger, \Phi^\dagger) + f_K(\tau, \Phi) + \overline{f_K}\left(\tau^\dagger, \Phi^\dagger\right) \\ W(\tau, \Phi) \to W(\tau, \Phi) \\ f(\tau) \to f(\tau) \end{cases}$$

Ferrara et al., PLB **225** (1989) 363; PLB **233** (1989) 147; Feruglio, 1706.08749

Modular-invariant SUSY theories

Minimal Kähler potential

$$K = -h^2 \log(-i\tau + i\tau^\dagger) + \sum_\Phi \frac{\Phi^\dagger e^{2V} \Phi}{(-i\tau + i\tau^\dagger)^{k_\Phi}}$$

Superpotential

$$W = Y^u_{ij}(\tau)\, u_{Ri}\, Q_j\, H_u + Y^d_{ij}(\tau)\, d_{Ri}\, Q_j\, H_d$$

τ-dependent Yukawa couplings

$$Y^q_{ij}(\tau) \to (c\tau + d)^{k^q_{ij}}\, Y^q_{ij}(\tau) \qquad \text{with} \qquad k^q_{ij} = k_{q_{Ri}} + k_{Q_j} + k_{H_q}$$

are modular forms!

$$\boxed{Y^q_{ij}(\tau) = c^q_{ij}\, F_{k^q_{ij}}(\tau) \qquad \text{with} \qquad c^q_{ij} \in \mathbb{R} \qquad \text{because of CP}}$$

Gauge kinetic function

$$f = \frac{1}{g_3^2} \qquad\qquad \theta_{\mathrm{QCD}} = 0 \quad \text{because of CP}$$

Modular invariance and CP

Fields

$$\tau \xrightarrow{\mathrm{CP}} -\tau^\dagger \qquad \text{and} \qquad \Phi \xrightarrow{\mathrm{CP}} \Phi^\dagger$$

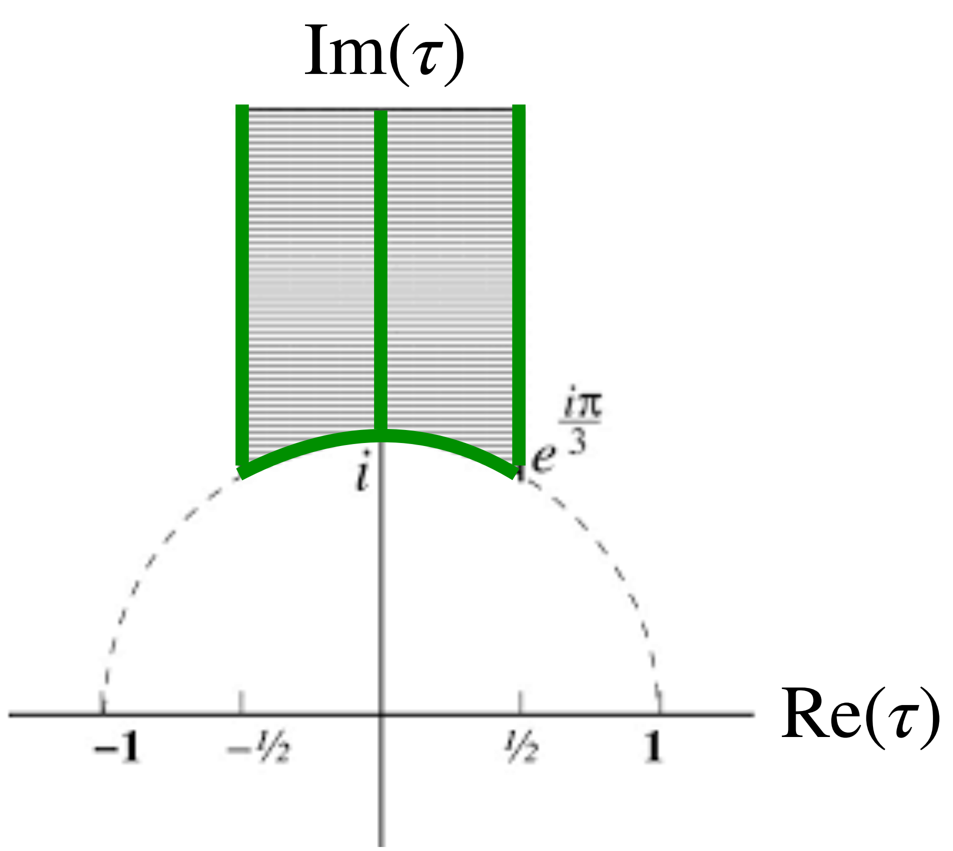
Modular forms

$$F(\tau) \xrightarrow{\mathrm{CP}} F(-\tau^*) = F(\tau)^*$$

CP-conserving values of τ

$\tau \xrightarrow{\mathrm{CP}} \boxed{-\tau^* = \gamma\tau}$ (goes to itself up to γ)

1. $\tau = iy \xrightarrow{\mathrm{CP}} iy$
2. $\tau = -\dfrac{1}{2} + i\,y \xrightarrow{\mathrm{CP}} \dfrac{1}{2} + i\,y = T\tau$
3. $\tau = e^{i\varphi} \xrightarrow{\mathrm{CP}} -e^{-i\varphi} = S\tau$



Novichkov, Penedo, Petcov, **AT**, 1905.11970; Baur, Nilles, Trautner, Vaudrevange, 1901.03251

Determinant of quark mass matrix

$$M_u = \mathrm{v}_u Y^u \qquad M_d = \mathrm{v}_d Y^d$$

$$\det M_q = \det M_u \det M_d \propto \det Y^u \det Y^d$$

$$Y^q(\tau) = \begin{pmatrix} F_{k_{11}^q} & F_{k_{12}^q} & F_{k_{13}^q} \\ F_{k_{21}^q} & F_{k_{22}^q} & F_{k_{23}^q} \\ F_{k_{31}^q} & F_{k_{32}^q} & F_{k_{33}^q} \end{pmatrix} \quad \Rightarrow \quad \det Y^q(\tau) \text{ is a modular form of weight } k_{\det}^q$$

$$k_{\det}^q = k_{11}^q + k_{22}^q + k_{33}^q = \ldots = \sum_{i=1}^{3} \left(k_{q_{Ri}} + k_{Q_i}\right) + 3k_{H_q}$$

And $\det Y^u(\tau) \det Y^d(\tau)$ is a modular form of weight $k_{\det}$

$$k_{\det} = k_{\det}^u + k_{\det}^d = \sum_{i=1}^{3} \left(2k_{Q_i} + k_{u_{Ri}} + k_{d_{Ri}}\right) + 3\left(k_{H_u} + k_{H_d}\right)$$

$$k_{\det} = 0 \quad \Rightarrow \quad \det Y^u(\tau) \det Y^d(\tau) = \text{(real) constant}$$

Matter fields and canonical normalisation

Gauge quantum numbers

	Q	u_R	d_R	L	e_R	H_u	H_d
$\mathrm{SU}(3)_C$	$\mathbf{3}$	$\overline{\mathbf{3}}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\mathrm{SU}(2)_L$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$
$\mathrm{U}(1)_Y$	$\frac{1}{6}$	$-\frac{2}{3}$	$\frac{1}{3}$	$-\frac{1}{2}$	1	$\frac{1}{2}$	$-\frac{1}{2}$

Canonical normalisation

$$K \supset \frac{\Phi^\dagger \Phi}{(-i\tau + i\tau^\dagger)^{k_\Phi}} = \Phi^\dagger_{\mathrm{can}} \Phi_{\mathrm{can}} \qquad \Phi_{\mathrm{can}} = \{\phi_{\mathrm{can}}, \psi_{\mathrm{can}}\}$$

$$\psi_{\mathrm{can}} \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{-\frac{k_\Phi}{2}} \psi_{\mathrm{can}} = e^{-ik_\Phi \alpha(\tau)} \psi_{\mathrm{can}} \qquad \alpha(\tau) = \arg(c\tau + d)$$

Modular symmetry acts on canonically normalised fields
as a τ-dependent phase rotation

Cancellation of modular anomalies

Conditions for modular-gauge anomaly cancellation

$$\mathrm{SU}(3)_C: \quad A \equiv \sum_{i=1}^{3} \left(2k_{Qi} + k_{u_{Ri}} + k_{d_{Ri}}\right) = 0$$

$$\mathrm{SU}(2)_L: \quad \sum_{i=1}^{3} \left(3k_{Qi} + k_{Li}\right) + k_{H_u} + k_{H_d} = 0$$

$$\mathrm{U}(1)_Y: \quad \sum_{i=1}^{3} \left(k_{Q_i} + 8k_{u_{Ri}} + 2k_{d_{Ri}} + 3k_{L_i} + 6k_{e_{Ri}}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

Simplest solution

$$\boxed{k_Q = k_{u_R} = k_{d_R} = k_L = k_{e_R} = (-k, 0, k) \qquad \text{and} \qquad k_{H_u} + k_{H_d} = 0}$$

Cancellation of modular-QCD anomaly along with $k_{H_u} + k_{H_d} = 0$ implies

$$k_{\det} = \sum_{i=1}^{3} \left(2k_{Q_i} + k_{u_{Ri}} + k_{d_{Ri}}\right) + 3\left(k_{H_u} + k_{H_d}\right) = 0$$

Simplest example: quarks

Simplest non-trivial example giving $k_{\text{det}} = 0$ and $A = 0$

$$k_Q = k_{u_R} = k_{d_R} = (-6, 0, 6) \quad \text{and} \quad k_{H_u} = k_{H_d} = 0$$

Yukawa matrices

$$Y^q = \begin{pmatrix} 0 & 0 & c^q_{13} \\ 0 & c^q_{22} & c^q_{23} E_6 \\ c^q_{31} & c^q_{32} E_6 & c^q_{33} E_4^3 + c'^q_{33} E_6^2 \end{pmatrix} \quad \Rightarrow \quad Y^q|_{\text{can}} = \begin{pmatrix} 0 & 0 & c^q_{13} \\ 0 & c^q_{22} & c^q_{23}\,(2\text{Im}\tau)^3\, E_6 \\ c^q_{31} & c^q_{32}\,(2\text{Im}\tau)^3\, E_6 & (2\text{Im}\tau)^6 \left[c^q_{33} E_4^3 + c'^q_{33} E_6^2\right] \end{pmatrix}$$

$$\det Y^q|_{\text{can}} = -\, c^q_{13} c^q_{22} c^q_{31} \in \mathbb{R}$$

Fixing $\tau = 1/8 + i$ and $\tan\beta = 10$

$$c^u_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.56 \\ 0 & -1.86 & 0.87 \\ 1.29 & 4.14 & 3.51,\ 1.40 \end{pmatrix} \qquad c^d_{ij} \approx 10^{-3} \begin{pmatrix} 0 & 0 & 1.55 \\ 0 & -2.59 & 4.59 \\ 0.378 & 0.710 & 0.734,\ 1.76 \end{pmatrix}$$

reproduce the quark masses, mixing angles and δ_{CKM} at the GUT scale

Simplest example: leptons

$k_L = k_{e_R} = (-6, 0, 6)$

Weinberg operator $\mathscr{C}^{\nu}_{ij}(L_i H_u)(L_j H_u)$ for neutrino masses

Charged lepton Yukawa matrix and coefficient of the Weinberg operator

$$Y^e = \begin{pmatrix} 0 & 0 & c^e_{13} \\ 0 & c^e_{22} & c^e_{23}E_6 \\ c^e_{31} & c^e_{32}E_6 & c^e_{33}E_4^3 + c'^e_{33}E_6^2 \end{pmatrix} \qquad \mathscr{C}^{\nu} = \begin{pmatrix} 0 & 0 & c^{\nu}_{13} \\ 0 & c^{\nu}_{22} & c^{\nu}_{23}E_6 \\ c^{\nu}_{31} & c^{\nu}_{32}E_6 & c^{\nu}_{33}E_4^3 + c'^{\nu}_{33}E_6^2 \end{pmatrix}$$

Fixing $\tau = 1/8 + i$ and $\tan\beta = 10$

$$c^e_{ij} = 10^{-3}\begin{pmatrix} 0 & 0 & 1.29 \\ 0 & 5.95 & 0.35 \\ -2.56 & 1.47 & 1.01,\ 1.32 \end{pmatrix} \qquad c^{\nu}_{ij} = \frac{1}{10^{16}\ \text{GeV}}\begin{pmatrix} 0 & 0 & 3.4 \\ 0 & 7.1 & 1.2 \\ 3.4 & 1.2 & 0.19,\ 0.95 \end{pmatrix}$$

reproduce the lepton masses and mixings, including δ_{PMNS}

Models with larger modular charges

Yukawa matrices	Modular weights			Alternative bigger weights		
$Y_{u,d}$	$(u_L, d_L)_{1,2,3}$	$u_{R1,2,3}$	$d_{R1,2,3}$	$(u_L, d_L)_{1,2,3}$	$u_{R1,2,3}$	$d_{R1,2,3}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_6 \\ 1 & E_6 & E_4^3 + E_6^2 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 0 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -4 \\ 2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ -2 \\ 4 \end{pmatrix}$	$\begin{pmatrix} -8 \\ -2 \\ 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^2 \\ 1 & E_4 & E_4^3 + E_6^2 \end{pmatrix}$	$\begin{pmatrix} -6 \\ -2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 2 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -4 \\ -4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^2 \\ 1 & E_4^2 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 0 \\ 8 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4 E_6 \\ 1 & E_6 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ -2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 2 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 2 \\ 8 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & E_4^3 + E_6^2 \\ 1 & E_4 & E_4(E_4^3 + E_6^2) \end{pmatrix}$				$\begin{pmatrix} -8 \\ -4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix}$	$\begin{pmatrix} -8 \\ 4 \\ 8 \end{pmatrix}$

Corrections to $\bar{\theta} = 0$

$M_{\rm Pl}$

SUSY unbroken

$\Lambda_{\rm flavour/CP}$

- Modular invariance determines completely (up to real couplings) the functional dependence $W(\tau)$
- It is not the case for K, but $\bar{\theta}$ is insensitive to K
- No-renormalisation theorems Ellis, Ferrara, Nanopoulos, PLB **114** (1982) 231

$\Lambda_{\rm SUSY}$

SUSY breaking corrections

- In general, can be large

$m_{\rm SUSY}$

- Small if $\Lambda_{\rm flavour/CP} \gg \Lambda_{\rm SUSY}$ (as e.g. in gauge mediation) and soft breaking terms respect the flavour structure of the SM

$\rm v$

$$\bar{\theta} \lesssim \frac{M_t^4 M_b^4 M_c^2 M_s^2}{\rm v^{12}} J_{\rm CP} \tan^6\beta \sim 10^{-28} \tan^6\beta$$

SM corrections are negligible

- $\bar{\theta} \sim 10^{-18}$ at four loops Khriplovich, PLB **173** (1986) 193
Ellis, Gaillard, NPB **150** (1979) 141

Conclusions

- Modular invariance is inherent to toroidal compactifications in string theory
- It can be consistently implemented in a supersymmetric QFT
- The VEV of the modulus τ is the only source of spontaneous CP violation

$$\bar{\theta} = \theta_{\text{QCD}} + \arg\det M_q$$

- $\theta_{\text{QCD}} = 0$ because the UV theory is CP-conserving
- $\arg\det M_q = 0$ because of anomaly-free modular symmetry
- Corrections to $\bar{\theta} = 0$ are small under certain assumptions on SUSY breaking

Back-up slides

Phenomenology and cosmology

- Couplings to matter are suppressed by $1/h$ ($1/M_{\mathrm{Pl}}$ in SUGRA)
- No couplings to gauge bosons in the exact SUSY limit
- $m_\tau \gtrsim 10$ TeV not to spoil BBN
- Fermionic component of τ could be LSP and maybe DM
- Scalar potential $V(\tau) = V(-\tau^*)$ => CP-conjugated minima
  (domain walls are inflated away if CP breaking occurs before inflation)

Modular group

Homogeneous modular group

$$\Gamma = \langle S,\, T \mid S^4 = (ST)^3 = I \rangle \cong \mathrm{SL}(2,\mathbb{Z})$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad\qquad \tau \xrightarrow{T} \tau + 1$$

duality   discrete shift symmetry

Inhomogeneous modular group

$$\overline{\Gamma} = \langle S,\, T \mid S^2 = (ST)^3 = I \rangle \cong \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{I, -I\}$$

In other words, $\mathrm{SL}(2,\mathbb{Z})$ matrices γ and $-\gamma$ are identified

$$\tau \xrightarrow{\gamma} \gamma\tau = \frac{a\tau + b}{c\tau + d} \qquad\qquad \tau \xrightarrow{-\gamma} (-\gamma)\,\tau = \frac{-a\tau - b}{-c\tau - d} = \gamma\tau$$

Modular forms

Holomorphic functions on $\mathscr{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}(\tau) > 0\}$ transforming under Γ as

$$\boxed{f\left(\gamma\tau\right) = \left(c\tau + d\right)^k f(\tau)\,, \qquad \gamma \in \Gamma}$$

k is weight, a non-negative even integer

$$\gamma = -I \quad \Rightarrow \quad f(\tau) = (-1)^k f(\tau) \quad \Rightarrow \quad k \text{ is even}$$

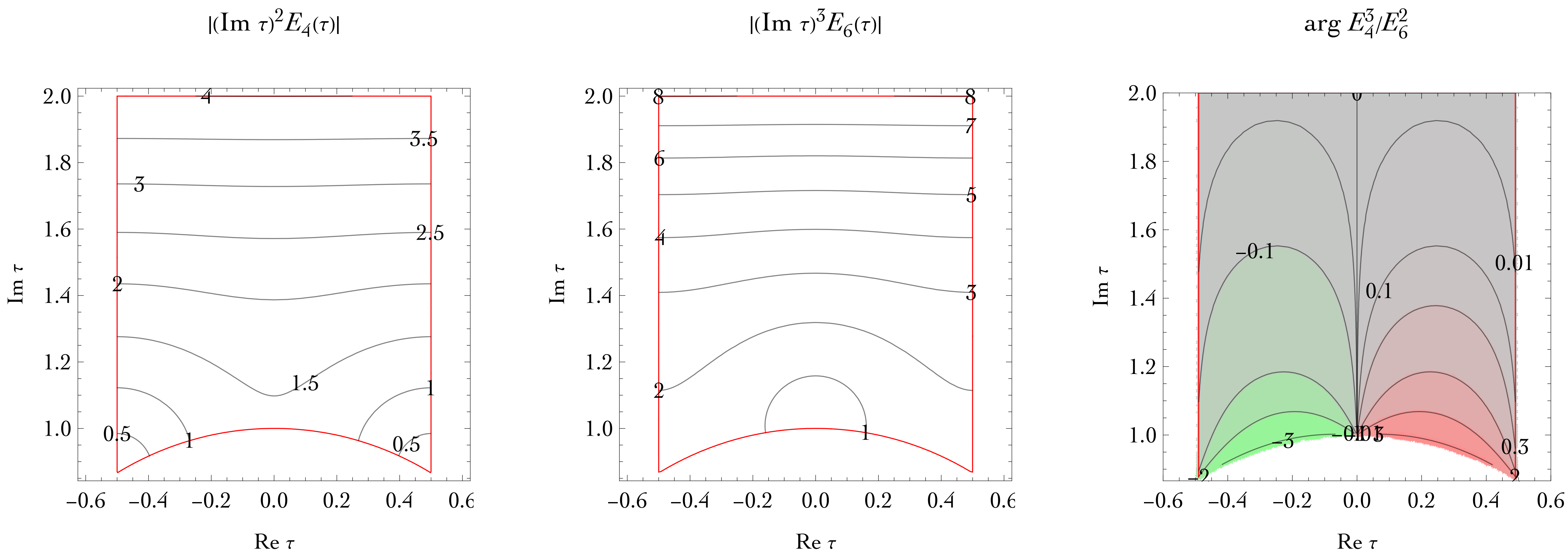
Modular forms are periodic and admit q-expansions

$$\gamma = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad f(\tau + 1) = f(\tau) \quad \Rightarrow \quad f(\tau) = \sum_{n=0}^{\infty} a_n q^n\,, \quad q = e^{2\pi i\tau}$$

Modular forms of weight k form a linear space $\mathscr{M}_k$ of finite dimension

$$\dim \mathscr{M}_k = \begin{cases} 0 & \text{if } k \text{ is negative or odd} \\ \lfloor k/12 \rfloor & \text{if } k = 2 \pmod{12} \\ \lfloor k/12 \rfloor + 1 & \text{if } k \neq 2 \pmod{12} \end{cases}$$

E4 and E6



$|(\mathrm{Im}\ \tau)^2 E_4(\tau)|$
$|(\mathrm{Im}\ \tau)^3 E_6(\tau)|$
$\arg E_4^3/E_6^2$
Im τ
Re τ
2.0
1.8
1.6
1.4
1.2
1.0
−0.6
−0.4
−0.2
0.0
0.2
0.4
0.6
4
3.5
3
2.5
2
1.5
1
0.5
8
7
6
5
−0.1
0.01
0.1
−3
0.3

Heavy quarks and singularities

- Heavy quarks are not needed for the mechanism to work, but assume they exist

$$k_q = (-6,\ -2, 0,\ +2,\ +6) \quad \text{and} \quad k_{H_u} = k_{H_d} = 0$$

Light chiral quarks Heavy vector-like quarks

- In the full theory $f_{\mathrm{UV}} \in \mathbb{R}$ and $\det M_{\mathrm{all}} \in \mathbb{R}$ => $\bar{\theta} = 0$

$$M_{\mathrm{all}} = \begin{pmatrix} M_{LL} & M_{LH} \\ M_{HL} & M_{HH} \end{pmatrix}$$

$$M_{\mathrm{light}} \approx M_{LL} - M_{LH} M_{HH}^{-1} M_{HL} \qquad M_{\mathrm{heavy}} \approx M_{HH}$$

Singularities where $\det M_{\mathrm{heavy}}(\tau) = 0$
(breakdown of EFT)

$$\det M_{\mathrm{all}} = \det M_{\mathrm{light}} \det M_{\mathrm{heavy}}$$

$$\det M_{\mathrm{light}} \to (c\tau + d)^{k_{\mathrm{light}}} \det M_{\mathrm{light}} \qquad \det M_{\mathrm{heavy}} \to (c\tau + d)^{k_{\mathrm{heavy}}} \det M_{\mathrm{heavy}}$$

$$k_{\mathrm{all}} = k_{\mathrm{light}} + k_{\mathrm{heavy}} = 0$$

EFT of light quarks

In the EFT of light quarks

$$\bar{\theta} = \theta_{\text{QCD}} + \arg\det M_{\text{light}} = -\,8\pi^2\,\text{Im}\,f_{\text{IR}} + \arg\det M_{\text{light}}$$

The EFT has anomalous field content with $k_q = (-6,\,-2,0)$

Anomaly is cancelled by a new contribution to the gauge kinetic function arising from the integration over the heavy quarks

$$f_{\text{IR}} = f_{\text{UV}} - \frac{1}{8\pi^2}\log\det M_{\text{heavy}}$$

Thus

$$\bar{\theta} = \arg\det M_{\text{heavy}} + \arg\det M_{\text{light}} = \arg\det M_{\text{all}} = 0$$

Modular invariance and SUGRA

$\mathcal{N} = 1$ SUGRA action depends on

$$G = \frac{K}{M_{\rm Pl}^2} + \log\left|\frac{W}{M_{\rm Pl}^3}\right|^2$$

For G to be invariant, both K and W have to transform

$$K \to K + M_{\rm Pl}^2\left(F + F^\dagger\right) \qquad \text{and} \qquad W \to e^{-F}W$$

In the case of modular transformations

$$F = \frac{h^2}{M_{\rm Pl}^2}\log(c\tau + d)$$

$$\boxed{W \to \left(c\tau + d\right)^{-k_W} W \qquad \text{with} \qquad k_W = \frac{h^2}{M_{\rm Pl}^2} > 0}$$

The superpotential is a modular function, having singularities at some values of τ

$k_W \to 0$ rigid SUSY limit

Modular invariance and SUGRA

$$W = Y^u_{ij}(\tau)\, u_{Ri}\, Q_j\, H_u + Y^d_{ij}(\tau)\, d_{Ri}\, Q_j\, H_d$$

$$Y^q_{ij}(\tau) \to (c\tau + d)^{k^q_{ij}}\, Y^q_{ij}(\tau) \qquad \text{with} \qquad k^q_{ij} = k_{q_{Ri}} + k_{Q_j} + k_{H_q} - k_W$$

Furthermore, the Kähler transformation must be accompanied by a U(1) rotation

$$\psi \to e^{\frac{F - F^\dagger}{4}}\, \psi \qquad\qquad \lambda \to e^{-\frac{F - F^\dagger}{4}}\, \lambda$$

how gaugino enters the game

$$\psi_{\text{can}} \to \left(\frac{c\tau + d}{c\tau^\dagger + d} \right)^{\frac{k_W}{4} - \frac{k_\Phi}{2}} \psi_{\text{can}} \qquad\qquad \lambda \to \left(\frac{c\tau + d}{c\tau^\dagger + d} \right)^{-\frac{k_W}{4}} \lambda$$

Modular-QCD anomaly modifies as

$$A = \sum_{i=1}^{3} \left(2k_{Qi} + k_{u_{Ri}} + k_{d_{Ri}} - 2k_W \right) + Ck_W$$

$C = 3$ is quadratic Casimir of $\mathbf{8}$ of $\mathrm{SU}(3)_C$

Gluino mass

$$\bar{\theta} = \theta_{\text{QCD}} + \arg\det M_q + C \arg M_3$$

Assume $k_{\text{det}} = 0$ and the quark contribution to A vanishes. Then

$$\bar{\theta} = \theta_{\text{QCD}} + C \arg M_3$$

Gluino mass requires SUSY breaking

$$M_3 = \frac{g_3^2}{2} e^{K/2M_{\text{Pl}}^2} K^{i\bar{j}} D_{\bar{j}} W^\dagger f_i$$

Assuming $D_\tau W = 0$ and no additional phases from SUSY breaking

$$\arg M_3 = -\arg W$$

$$W = \ldots + \frac{c_0 M_{\text{Pl}}^3}{\eta(\tau)^{2k_W}} \qquad \text{and} \qquad f = \ldots + \frac{Ck_W}{4\pi^2} \log \eta(\tau)$$

$$\bar{\theta} = -8\pi^2 \text{Im} f - C \arg W = 0$$

More on modular invariance in SUGRA

$$\det M_q \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{\frac{k_{\det}}{2}} \det M_q \qquad k_{\det} = \sum_{i=1}^{3}\left(2k_{Q_i} + k_{u_{Ri}} + k_{d_{Ri}} - 2k_W\right) + 3\left(k_{H_u} + k_{H_d}\right)$$

$$M_3 \to \left(\frac{c\tau + d}{c\tau^\dagger + d}\right)^{\frac{k_W}{2}} M_3$$

(gluino mass arises only if SUSY is broken)

Quark masses and mixings

At the GUT scale of 2×10^{16} GeV,
assuming MSSM with $\tan\beta = 10$ and SUSY breaking scale of 10 TeV

m_u/m_c	$(1.93 \pm 0.60) \times 10^{-3}$
m_c/m_t	$(2.82 \pm 0.12) \times 10^{-3}$
m_d/m_s	$(5.05 \pm 0.62) \times 10^{-2}$
m_s/m_b	$(1.82 \pm 0.10) \times 10^{-2}$
$\sin^2\theta_{12}$	$(5.08 \pm 0.03) \times 10^{-2}$
$\sin^2\theta_{13}$	$(1.22 \pm 0.09) \times 10^{-5}$
$\sin^2\theta_{23}$	$(1.61 \pm 0.05) \times 10^{-3}$
δ/π	0.385 ± 0.017

$$m_t = 87.46 \text{ GeV}$$
$$m_b = 0.9682 \text{ GeV}$$

Antusch, Maurer, 1306.6879
Yao, Lu, Ding, 2012.13390

Lepton masses and mixings

NuFIT 5.2 (2022)

		Normal Ordering (best fit)		Inverted Ordering ($\Delta\chi^2 = 6.4$)	
		bfp $\pm 1\sigma$	3σ range	bfp $\pm 1\sigma$	3σ range
with SK atmospheric data	$\sin^2\theta_{12}$	$0.303^{+0.012}_{-0.012}$	$0.270 \to 0.341$	$0.303^{+0.012}_{-0.011}$	$0.270 \to 0.341$
	$\theta_{12}/^\circ$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$	$33.41^{+0.75}_{-0.72}$	$31.31 \to 35.74$
	$\sin^2\theta_{23}$	$0.451^{+0.019}_{-0.016}$	$0.408 \to 0.603$	$0.569^{+0.016}_{-0.021}$	$0.412 \to 0.613$
	$\theta_{23}/^\circ$	$42.2^{+1.1}_{-0.9}$	$39.7 \to 51.0$	$49.0^{+1.0}_{-1.2}$	$39.9 \to 51.5$
	$\sin^2\theta_{13}$	$0.02225^{+0.00056}_{-0.00059}$	$0.02052 \to 0.02398$	$0.02223^{+0.00058}_{-0.00058}$	$0.02048 \to 0.02416$
	$\theta_{13}/^\circ$	$8.58^{+0.11}_{-0.11}$	$8.23 \to 8.91$	$8.57^{+0.11}_{-0.11}$	$8.23 \to 8.94$
	$\delta_{\mathrm{CP}}/^\circ$	232^{+36}_{-26}	$144 \to 350$	276^{+22}_{-29}	$194 \to 344$
	$\dfrac{\Delta m^2_{21}}{10^{-5}\ \mathrm{eV}^2}$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$	$7.41^{+0.21}_{-0.20}$	$6.82 \to 8.03$
	$\dfrac{\Delta m^2_{3\ell}}{10^{-3}\ \mathrm{eV}^2}$	$+2.507^{+0.026}_{-0.027}$	$+2.427 \to +2.590$	$-2.486^{+0.025}_{-0.028}$	$-2.570 \to -2.406$

Esteban et al., 2007.14792 and www.nu-fit.org

$$m_e/m_\mu = 0.0048 \pm 0.0002$$

$$m_\mu/m_\tau = 0.0565 \pm 0.0045$$

Finite modular groups

Infinite normal subgroups of $SL(2,\mathbb{Z})$, $N = 2, 3, 4, \dots$

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

Principal congruence subgroups of the modular group

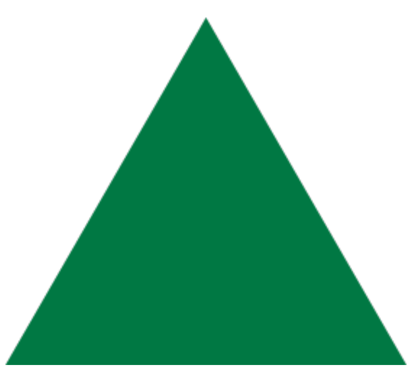
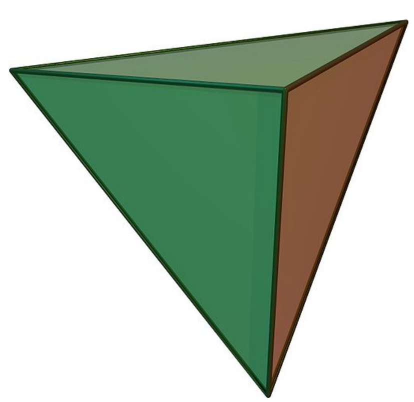
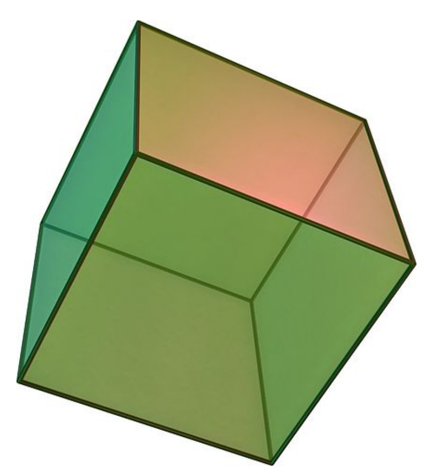
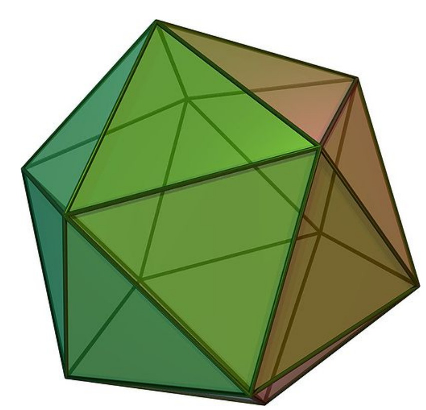
$$\overline{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\} \qquad \overline{\Gamma}(N) \equiv \Gamma(N), \quad N > 2$$

Finite modular groups

$$\Gamma_N \equiv \overline{\Gamma}/\overline{\Gamma}(N)$$

$$\Gamma_N = \left\langle S, T \mid S^2 = (ST)^3 = T^N = I \right\rangle, \quad N = 2, 3, 4, 5$$

$\Gamma_2 \cong S_3$ $\Gamma_3 \cong A_4$ $\Gamma_4 \cong S_4$ $\Gamma_5 \cong A_5$

Theories based on finite modular groups

$\mathcal{N} = 1$ rigid SUSY matter action

$$\mathcal{S} = \int d^4x\, d^2\theta\, d^2\bar{\theta}\; K(\tau,\bar{\tau},\psi,\bar{\psi}) + \int d^4x\, d^2\theta\; W(\tau,\psi) + \int d^4x\, d^2\bar{\theta}\; \overline{W}(\bar{\tau},\bar{\psi})$$

Ferrara, Lust, Shapere, Theisen, PLB **225** (1989) 363

Ferrara, Lust, Theisen, PLB **233** (1989) 147

$$\begin{cases} \tau \to \dfrac{a\tau + b}{c\tau + d} \\ \psi_i \to \left(c\tau + d\right)^{-k_i} \underbrace{\rho_i(\tilde{\gamma})}_{\text{unitary representation of } \Gamma_N} \psi_i \end{cases} \Rightarrow \begin{cases} W(\tau,\psi) \to W(\tau,\psi) \\ K(\tau,\bar{\tau},\psi,\bar{\psi}) \to K(\tau,\bar{\tau},\psi,\bar{\psi}) + f_K(\tau,\psi) + \overline{f_K}(\bar{\tau},\bar{\psi}) \end{cases}$$

Feruglio, 1706.08749

$$W(\tau,\psi) = \sum_n \sum_{\{i_1,\ldots,i_n\}} \sum_s g_{i_1\ldots i_n,s} \left(Y_{i_1\ldots i_n,s}(\tau)\, \psi_{i_1}\ldots\psi_{i_n} \right)_{\mathbf{1},s}$$

$$\boxed{Y(\tau) \xrightarrow{\gamma} (c\tau + d)^{k_Y} \rho_Y(\tilde{\gamma})\, Y(\tau)}$$

$$k_Y = k_{i_1} + \ldots + k_{i_n}$$

$$\rho_Y \otimes \rho_{i_1} \otimes \ldots \otimes \rho_{i_n} \supset \mathbf{1}$$

Yukawa couplings are modular forms!