Neutrino Oscillations

Lecture IV

Eligio Lisi

(INFN, Bari, Italy)

Outline of lectures I-IV:

Lecture I

Pedagogical introduction + warm-up exercise

Lecture II

3ν osc. in vacuum and matter: notation and basic math

Lecture III

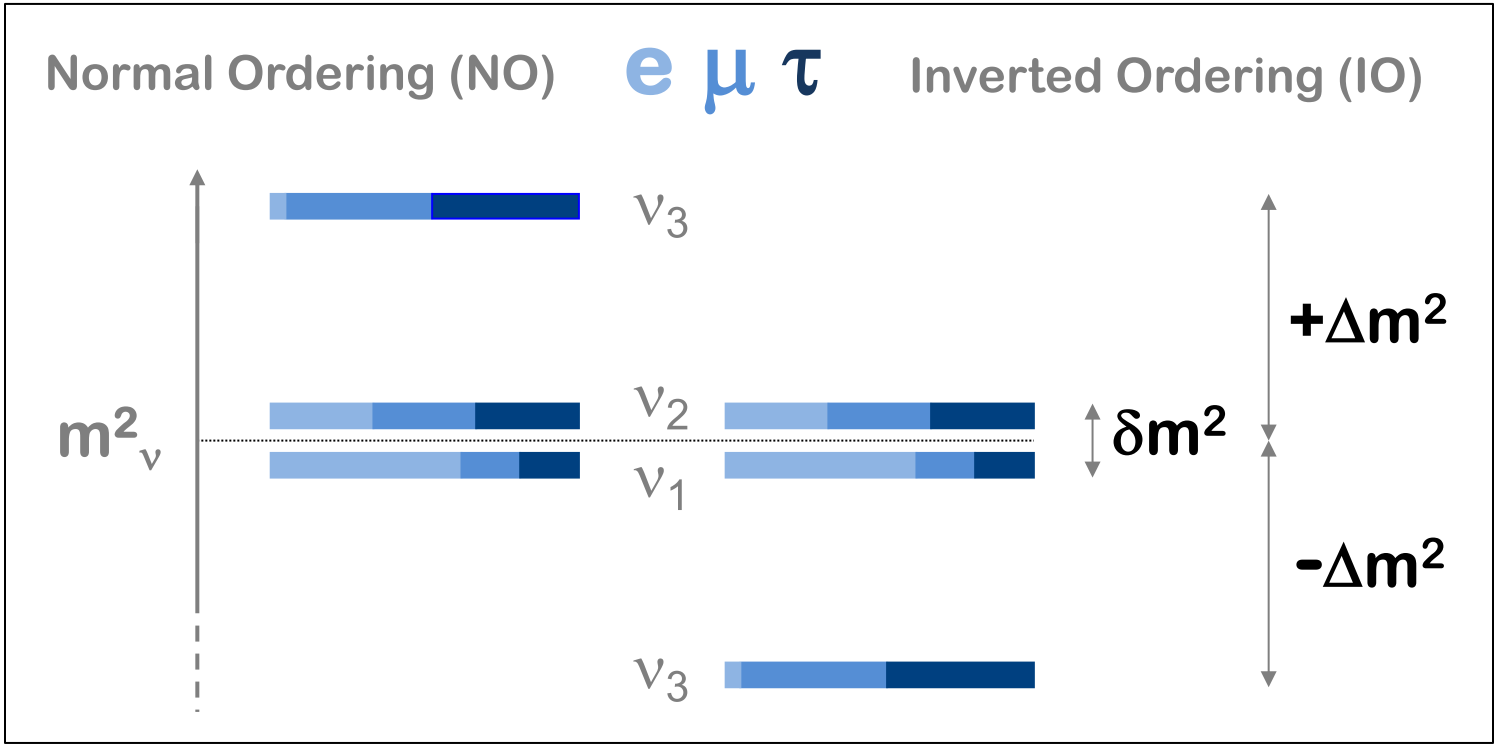
2ν approximations of phenomenological interest

Lecture IV

Back to 3ν oscillations: Status and Perspectives

Recap: 3ν status

5 knowns:			5 unknowns:
δm^2	$\sim 7 \times 10^{-5}$ eV^2	*Oscillations*	**δ CPV** Dirac phase
$\|\Delta m^2\|$	$\sim 2 \times 10^{-3}$ eV^2	*Oscillations*	**sign(Δm^2)** → NO/IO
$\sin^2\theta_{12}$	~ 0.3	*Oscillations*	θ_{23} **octant** degeneracy
$\sin^2\theta_{23}$	~ 0.5	*Non-oscillat.*	**absolute mass** scale
$\sin^2\theta_{13}$	~ 0.02	*Non-oscillat.*	**Dirac/Majorana** nature



Each known parameter probed by at least two different kinds of experiments!

How do $\nu_\mu \rightarrow \nu_e$ oscillation searches probe CPV?

Volume 72B, number 3 PHYSICS LETTERS 2 January 1978

TIME REVERSAL VIOLATION IN NEUTRINO OSCILLATION

Nicola CABIBBO*
*Laboratoire de Physique Théorique et Hautes Energies, Paris, France***

Received 11 October 1977

We discuss the possibility of CP or T violation in neutrino oscillation CP requires $\nu_\mu \leftrightarrow \nu_e$ and $\bar{\nu}_\mu \leftrightarrow \bar{\nu}_e$ oscillations to be equal Time reversal invariance requires the oscillation probability to be an even function of time Both conditions can be violated, even drastically, if more than two neutrinos exist

For two neutrinos, no CPV:

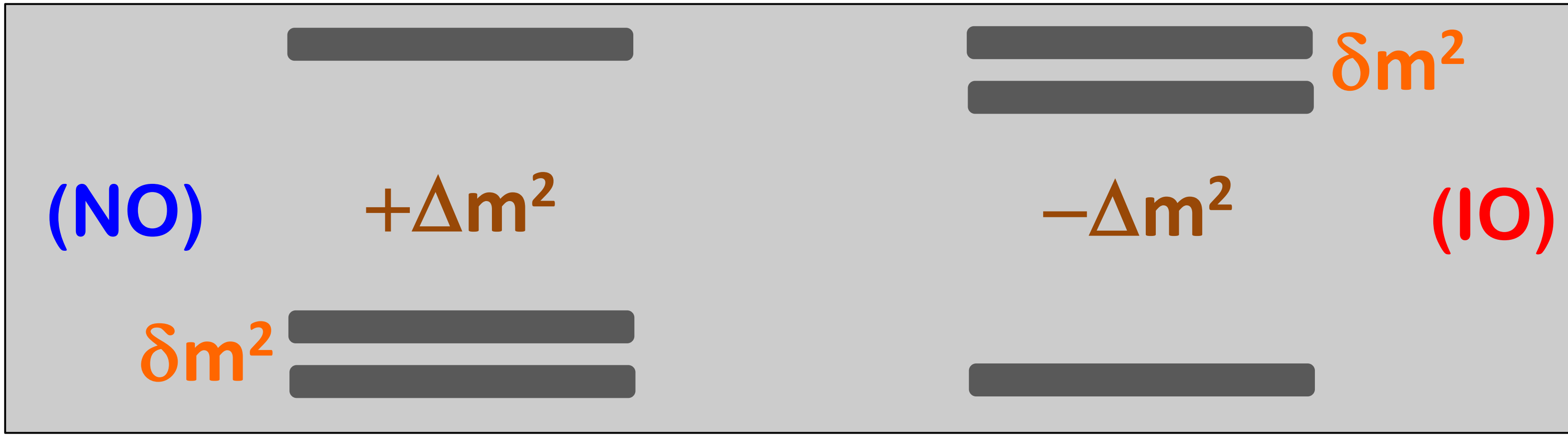
$$\overset{(-)}{\nu}_e = \cos\theta_{12}\,\nu_1 + \sin\theta_{12}\,\nu_2$$

For three neutrinos: new possible CPV phase δ, tested via ν versus $\bar{\nu}$

$$\overset{(-)}{\nu}_e = \cos\theta_{13}\,(\cos\theta_{12}\,\nu_1 + \sin\theta_{12}\,\nu_2) + e^{\pm i\delta}\sin\theta_{13}\,\nu_3$$

CPV is a genuine 3ν effect →
all oscillation parameters (known & unknown) are involved/entangled

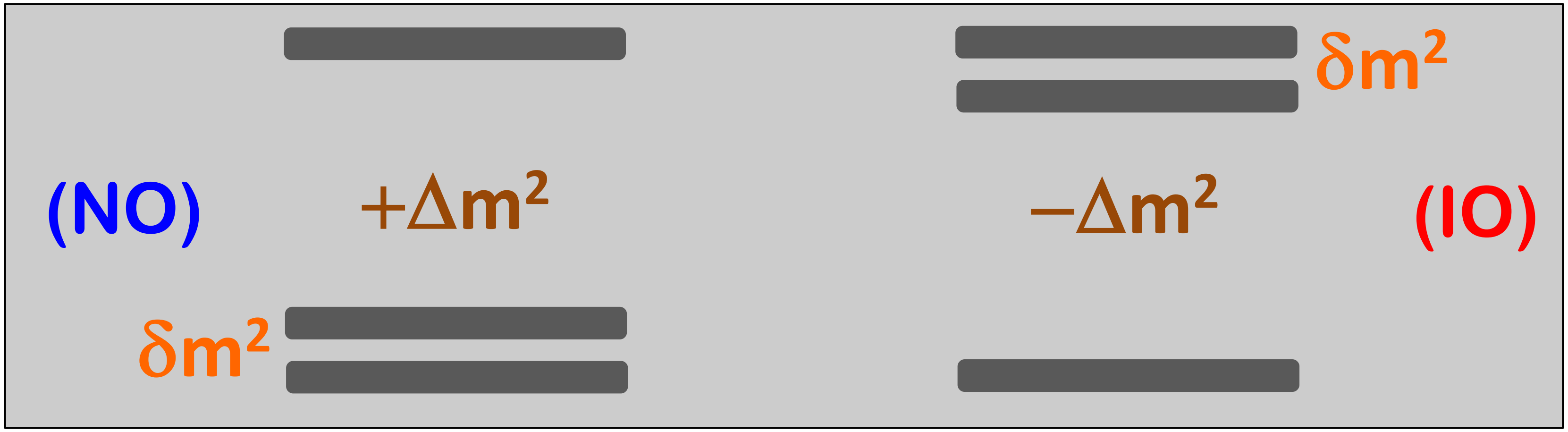
How do oscillation searches probe mass ordering?



Observe interference effects of oscill. driven by $\pm\Delta m^2$ with oscill. driven by another quantity Q with <u>known sign</u>. Options:

$Q \sim \delta m^2$	medium-baseline reactors → JUNO
$Q \sim G_F N_e E$	ν−matter effects → atm & LBL accel. expt.
[$Q \sim G_F N_\nu E$	ν−ν collective effects → core-collapse SN]

How do oscillation searches probe mass ordering?



Observe **interference effects** of oscill. driven by $\pm\Delta m^2$ with oscill. driven by another quantity **Q** with known sign. Options:

$Q \sim \delta m^2$	medium-baseline reactors → JUNO
$Q \sim G_F N_e E$	ν–matter effects → atm & LBL accel. expt.
[$Q \sim G_F N_\nu E$	ν–ν collective effects → core-collapse SN]

Additional tool: synergy of $|\Delta m^2|$ data from different experiments, e.g. two or more data from reactor + accelerator + atmospheric (should converge better in the true ordering than in the wrong one)

→ **It makes sense to perform global analyses of all neutrino oscillation data, to squeeze information on subleading 3ν effects and to exploit correlations**

Useful analysis sequence:

LBL Accel + Solar + KL (KamLAND)

minimal set sensitive to all osc. param. δm^2, Δm^2, θ_{13}, θ_{23}, θ_{12}, δ, NO/IO

LBL Accel + Solar + KL + SBL Reactor

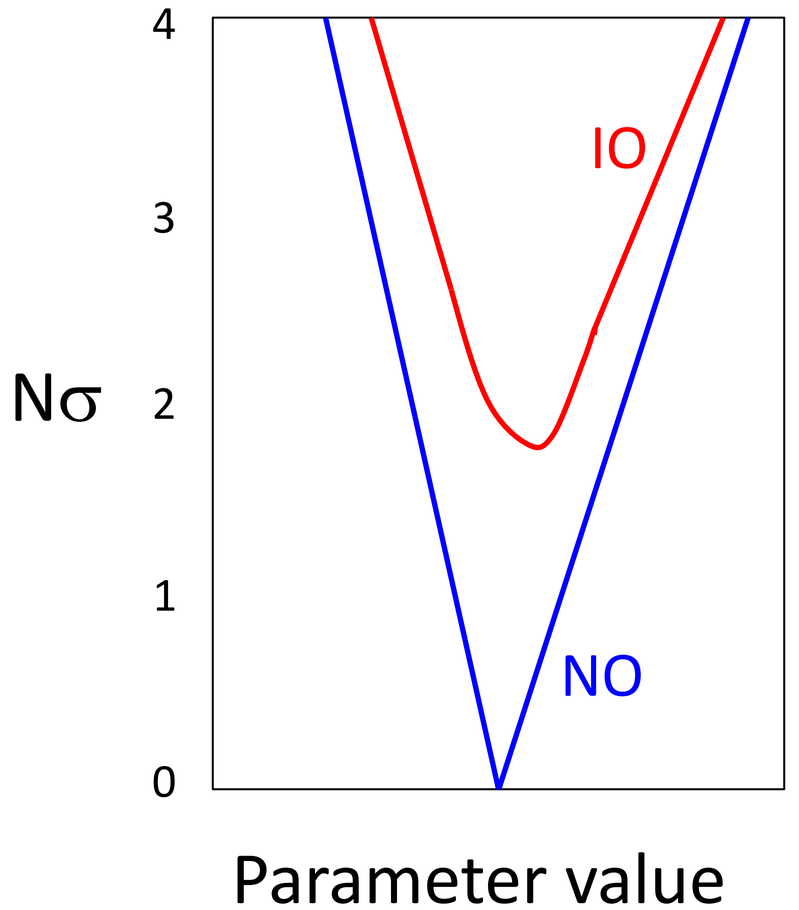
add sensitivity to Δm^2, θ_{13} and affect other parameters via correlations

LBL Accel + Solar + KL + SBL Reactor + Atmosph.

add sensitivity to Δm^2, θ_{23}, δ, NO/IO (but: entangled information in atmos.)

$\Delta\chi^2$ statistics adopted for all datasets: $N\sigma = \sqrt{\Delta\chi^2}$ →

E.g.,



In the following: results from the 2021 global data analysis:
“Unfinished fabric of the three neutrino paradigm”, Capozzi et al., hep-ph 2107.00532
(similar results from NuFit and Valencia groups in 2021)

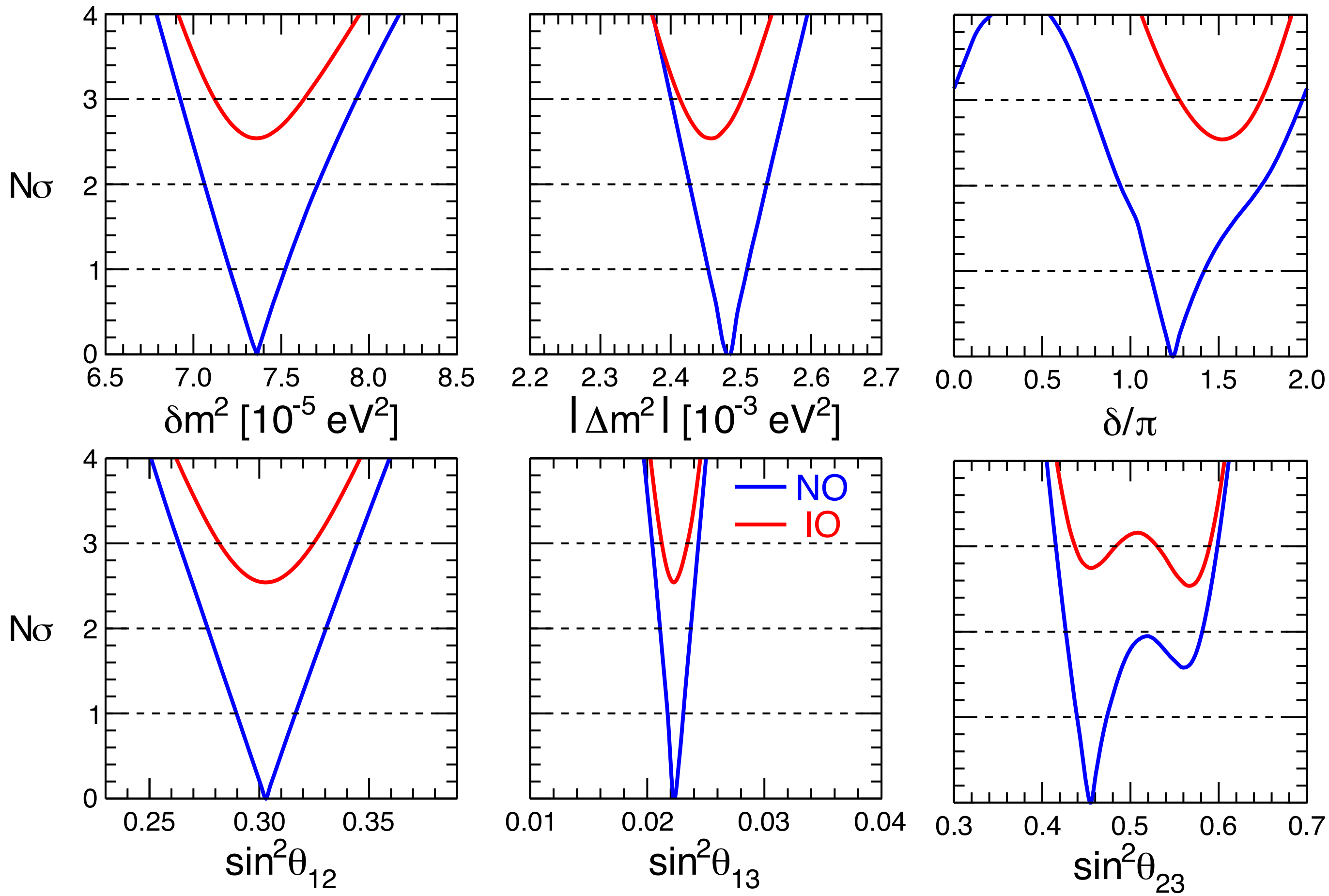
+ educated guesses about the impact of sparse data presented in 2022-2023
→ need to be checked by future global analyses (work in progress)

Status of known and unknown 3ν oscillation parameters, circa 2021(*)

1σ error of **known** parameters

$\|\Delta m^2\|$	1.1%
δm^2	2.3%
θ_{13}	3.0%
θ_{12}	4.5%
θ_{23}	~ 6%

All ν oscillation data



Hints on oscillation **unknowns** **(2021)**

NO	~99% CL
$\sin\delta < 0$	~90% CL
$\theta_{23} < \pi/4$	~90% CL

(*) Next relevant update after Neutrino 2024…

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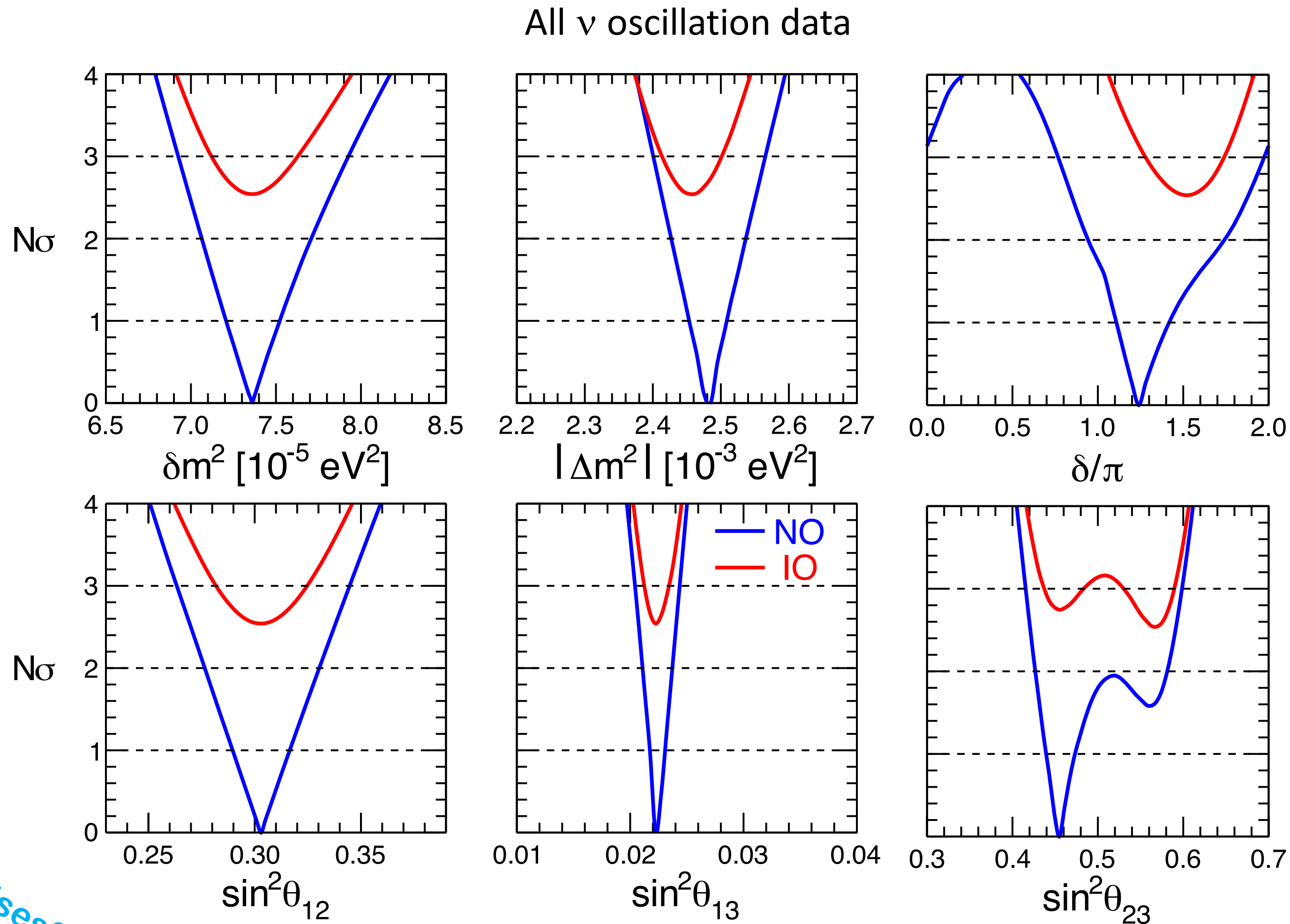

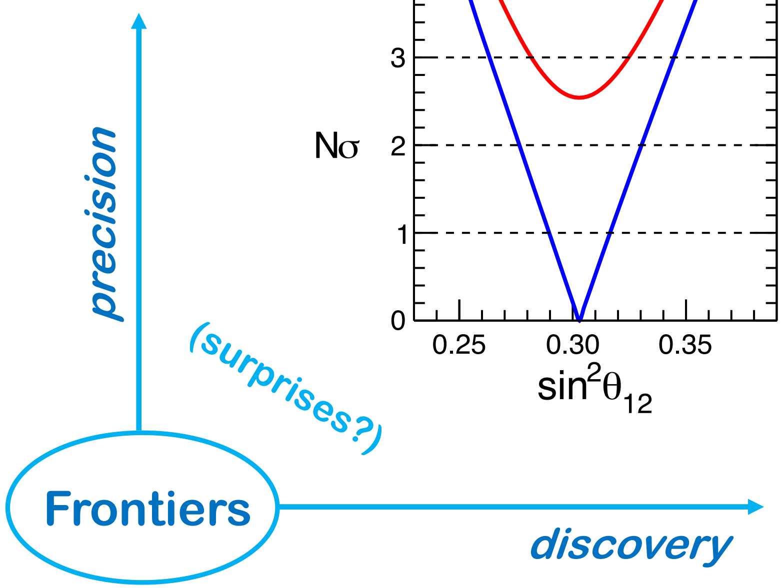

Hints on oscillation **unknowns** **(2021)**

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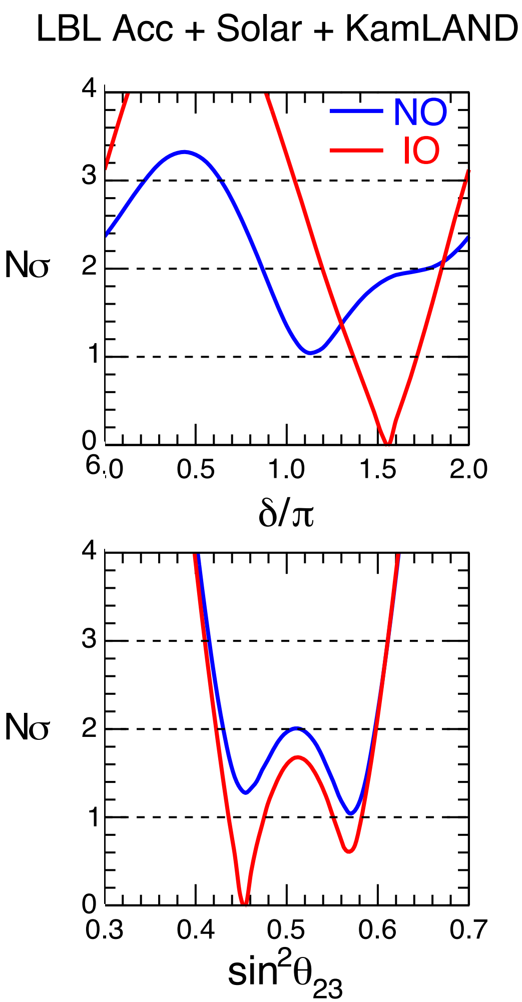
Focus on the three oscillation unknowns: NO/IO, δ, θ_{23} octant degen.



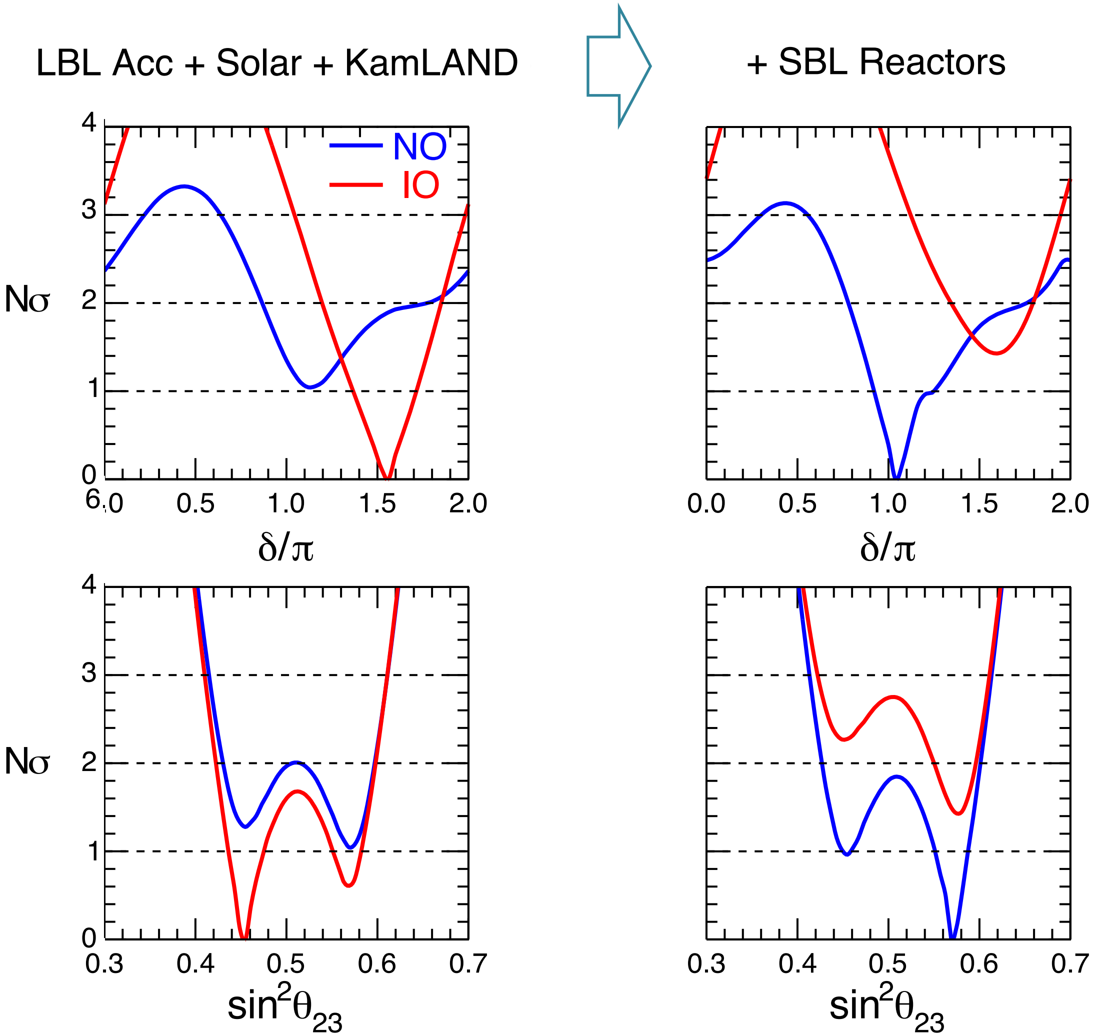
IO favored (~1σ)

δ~1.5π (IO), ~π (NO)

θ_{23} octants ~degenerate

[confusing T2K-NOvA tension]

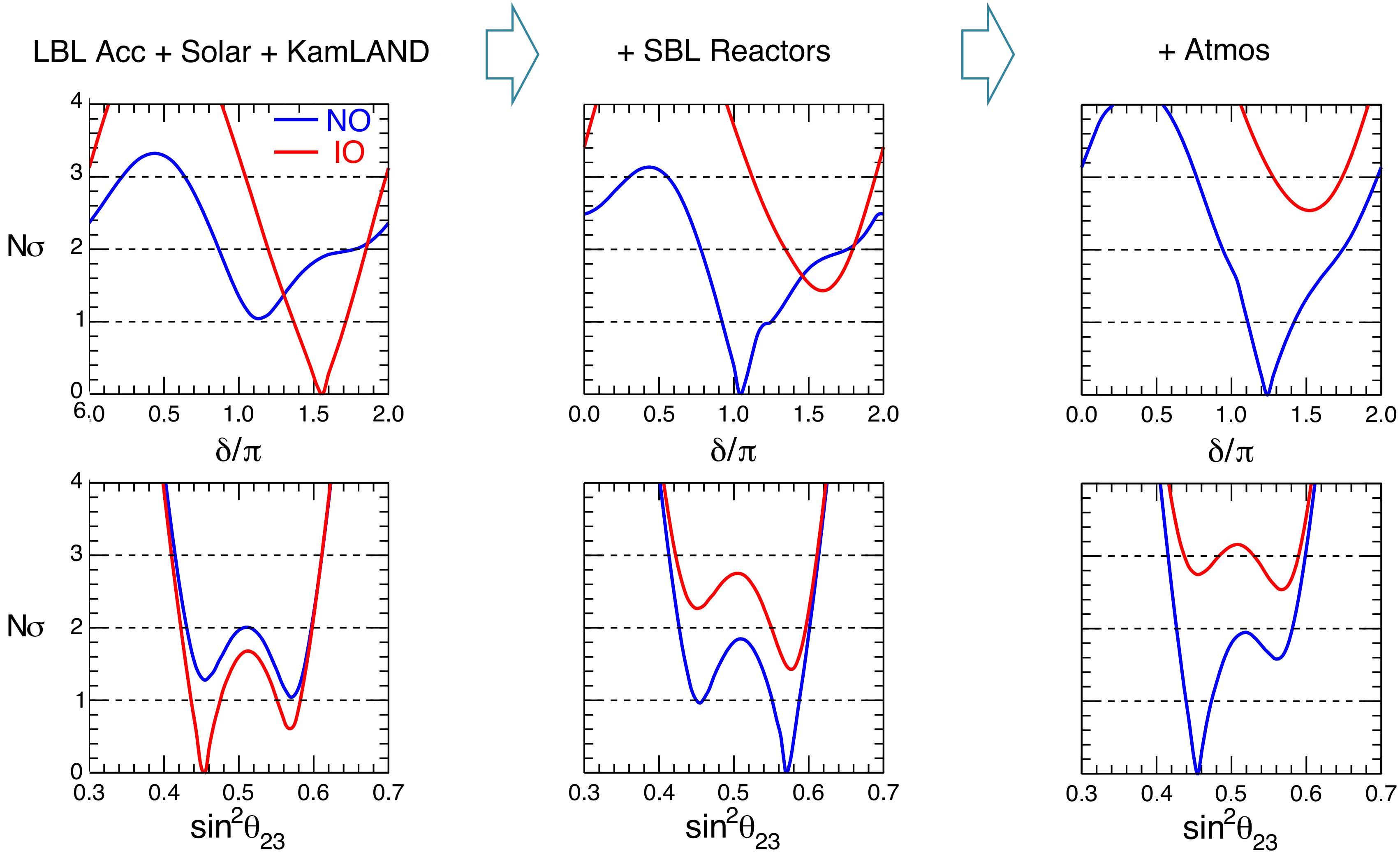
Focus on the three oscillation unknowns: NO/IO, δ, θ_{23} octant degen.



IO favored ($\sim 1\sigma$)
$\delta \sim 1.5\pi$ (IO), $\sim\pi$ (NO)
θ_{23} octants ~degenerate

NO favored ($\sim 1.5\sigma$)
$\delta \sim \pi$ (NO), $\sim 1.5\pi$ (IO)
$\theta_{23} \sim 0.57$ favored ($\sim 1\sigma$)

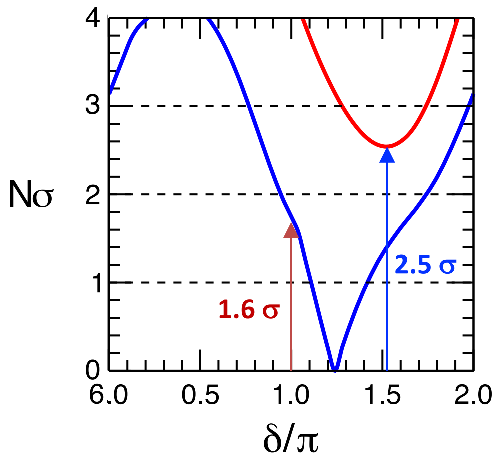
Focus on the three oscillation unknowns: NO/IO, δ, θ_{23} octant degen.

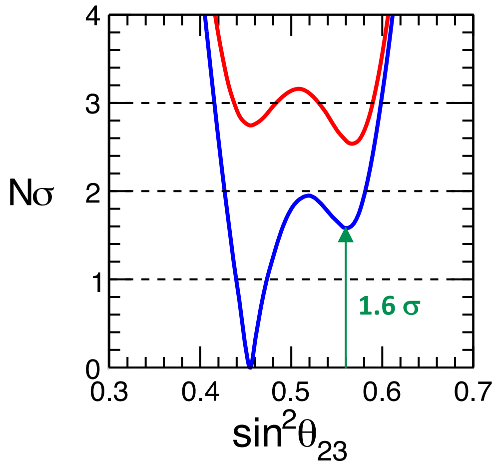


IO favored (~1σ)	**NO** favored (~1.5σ)	**NO** favored (~2.5σ)
δ~1.5π (IO), ~π (NO)	δ~π (NO), ~1.5π (IO)	δ~1.2π (NO) fav. (~1.6σ)
θ_{23} octants ~degenerate	θ_{23}~0.57 favored (~1σ)	θ_{23}~0.46 favored (~1.6σ)

Hints on oscillation **unknowns, 2021…**

- **NO** ~99% CL
- **$\sin\delta < 0$** ~90% CL
- $\theta_{23} < \pi/4$ ~90% CL





Hints on oscillation
unknowns,
2021…

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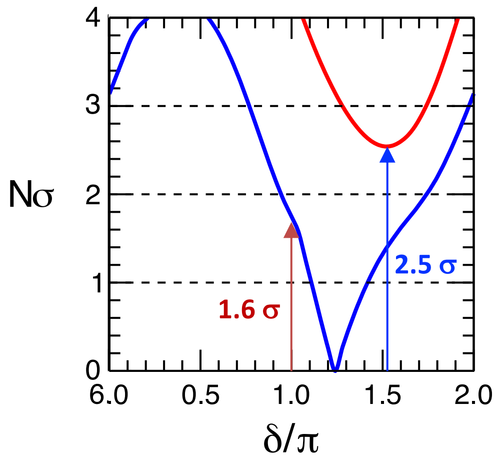
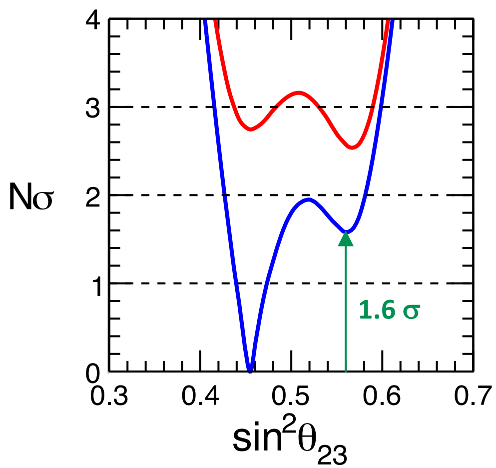




…Educated guess on
unknowns,
after 2022-2023 data

→ presumably >99% CL
→ presumably >90% CL
→ presumably flipped to $> \pi/4$

Main impact expected from ***new SK atm. data in combination with T2K****, which may win over the "T2K-NOvA tension" and other small changes*

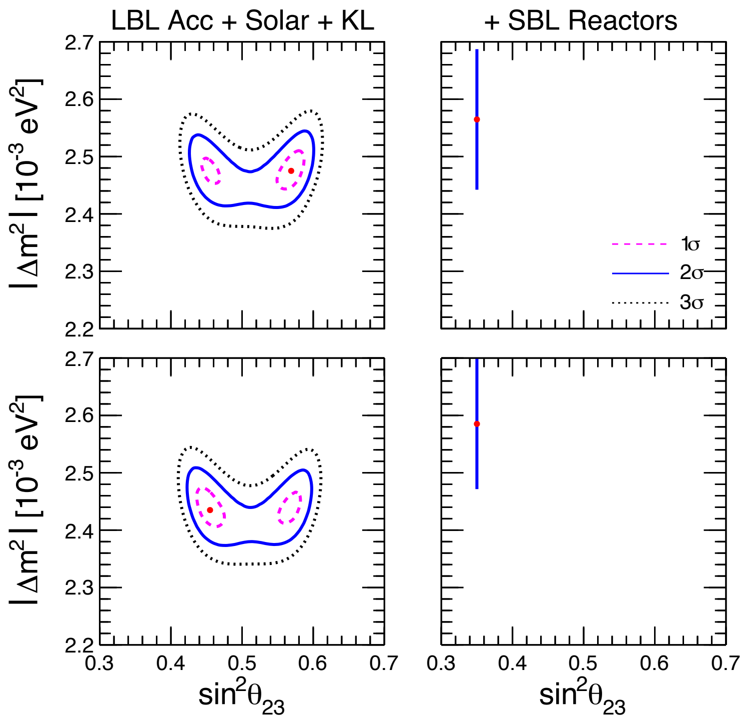
Also: new IC-DC atm. analysis for NO, wait for analysis w/ IO

Also: T2K+NOvA joint fit!

Watch for synergy of various ***|Δm^2| measurements:*** *convergence / divergence in true / wrong mass ordering*

($\pm\Delta m^2$, θ_{23}) pair: data synergy

Standalone SBL reactor data ($\pm 2\sigma$)



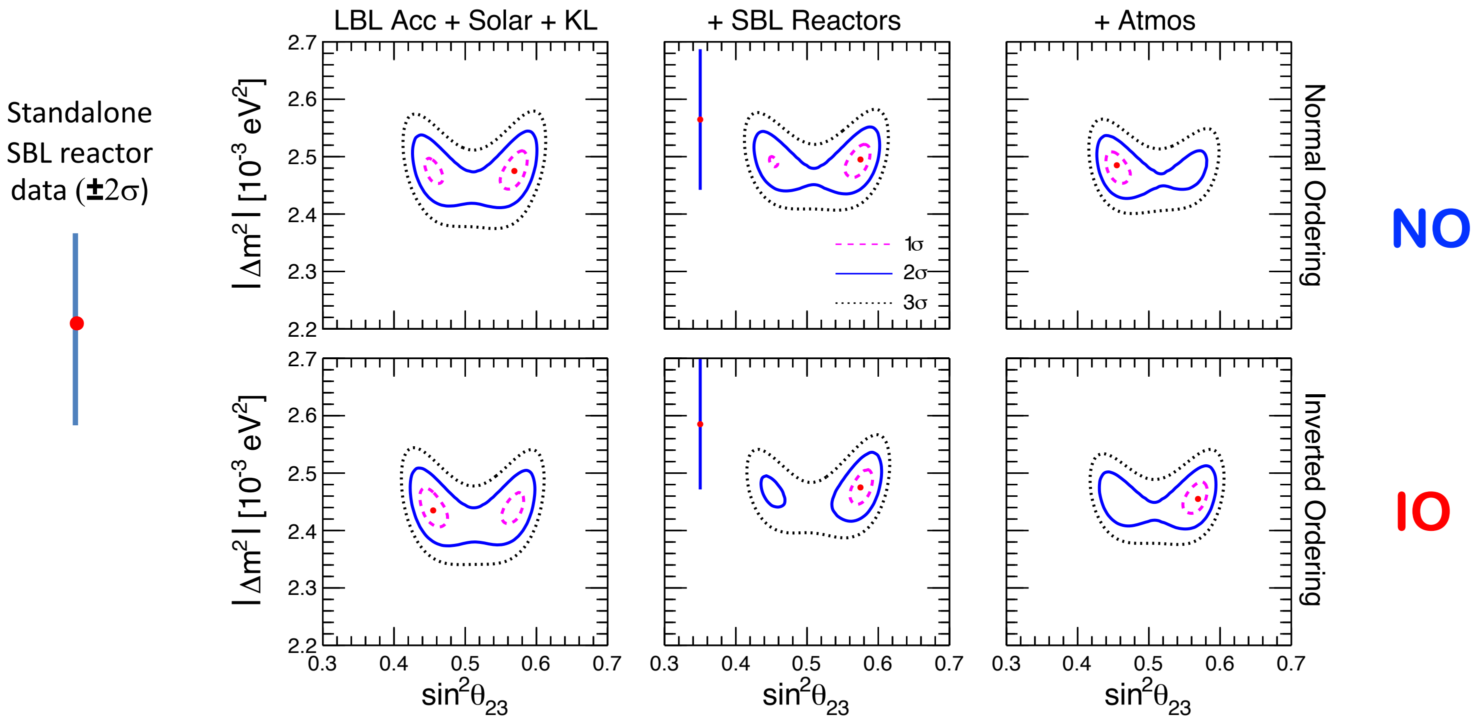
NO

IO

SBL reactors prefer **higher** Δm^2 than LBL accel. (and atmos.) expts.

Relative difference is **smaller** for **NO** and for non-maximal θ_{23} mixing

($\pm\Delta m^2$, θ_{23}) pair: data synergy



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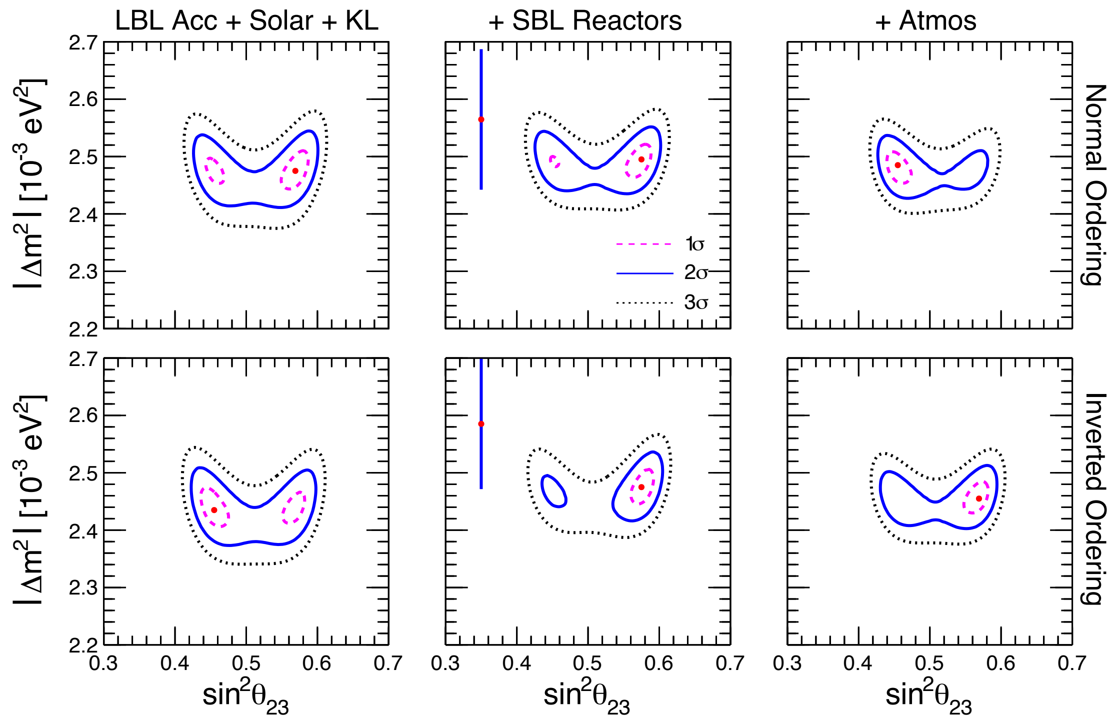
Relative difference is **smaller** for **NO** and for non-maximal θ_{23} mixing

→ Better agreement reached for **NO** & **nonmax θ_{23} at intermediate Δm^2**

→ SBL reactor data not sensitive to sign(Δm^2) and θ_{23}, but affect their likelihood!

$(\pm\Delta m^2, \theta_{23})$ pair: data synergy

Standalone SBL reactor data ($\pm 2\sigma$)



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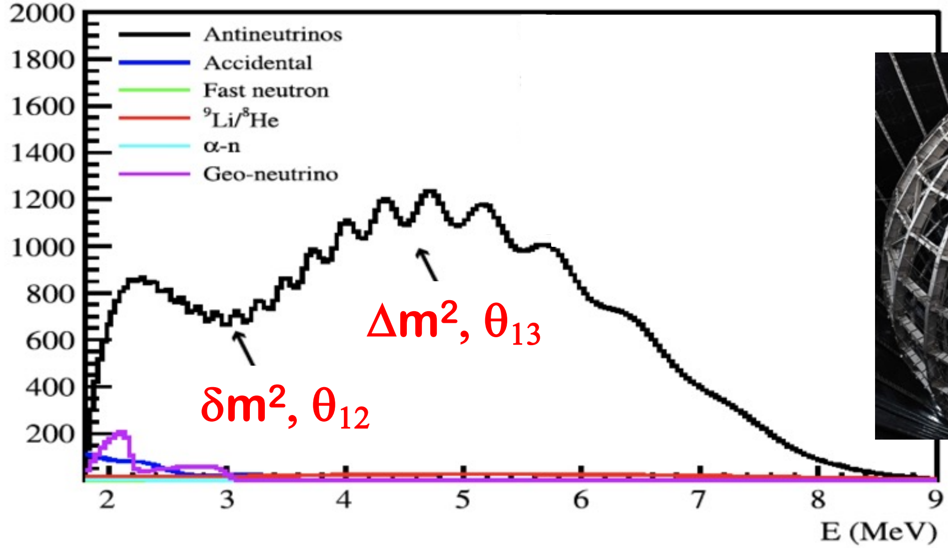
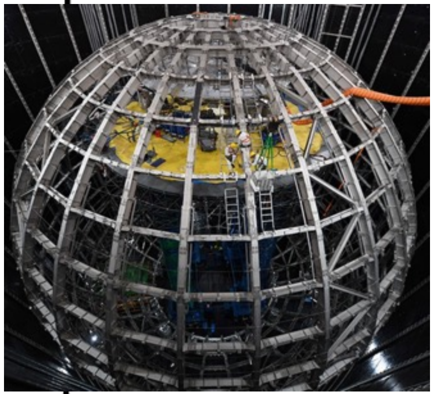
Near Future: incremental progress from Daya Bay + T2K + NOvA + SK + IC-DC…
Farther Future: decisive progress with JUNO + DUNE + HK + IC + KM3…

E.g.: Frontiers for the JUNO reactor experiment [1507.05613]

At "medium" baseline ~50 km, will probe two oscillations in ~vacuum

Main **discovery** goal: distinguish **NO vs IO at 3-4σ** in 6y.





Significant better **precision** expected on 3 out of 4 oscillation parameters:

Parameter	1σ, now	JUNO in ~6y
δm^2	2.3 %	0.6 %
$\sin^2\theta_{12}$	4.4 %	0.7 %
Δm^2	1.1 %	0.4 %
$\sin^2\theta_{13}$	3.0 %	comparable

arXiv:1507.05613v2

Neutrino Physics with JUNO

JUNO is designed to resolve the neutrino MH using precision spectral measurements of reactor antineutrino oscillations. Before giving the quantitative calculation of the MH sensitivity, we shall briefly review the principle of this method. The electron antineutrino survival probability in vacuum can be written as [69, 79, 94]:

$$
\begin{aligned}
P_{\bar{\nu}_e \to \bar{\nu}_e} &= 1 - \sin^2 2\theta_{13}(\cos^2\theta_{12}\sin^2\Delta_{31} + \sin^2\theta_{12}\sin^2\Delta_{32}) - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21} \qquad (2.1)\\
&= 1 - \frac{1}{2}\sin^2 2\theta_{13}\left[1 - \sqrt{1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}}\cos(2|\Delta_{ee}| \pm \phi)\right] - \cos^4\theta_{13}\sin^2 2\theta_{12}\sin^2\Delta_{21},
\end{aligned}
$$

where $\Delta_{ij} \equiv \Delta m^2_{ij} L/4E$, in which L is the baseline, E is the antineutrino energy,

$$
\sin\phi = \frac{c^2_{12}\sin(2s^2_{12}\Delta_{21}) - s^2_{12}\sin(2c^2_{12}\Delta_{21})}{\sqrt{1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}}}, \quad \cos\phi = \frac{c^2_{12}\cos(2s^2_{12}\Delta_{21}) + s^2_{12}\cos(2c^2_{12}\Delta_{21})}{\sqrt{1 - \sin^2 2\theta_{12}\sin^2\Delta_{21}}},
$$

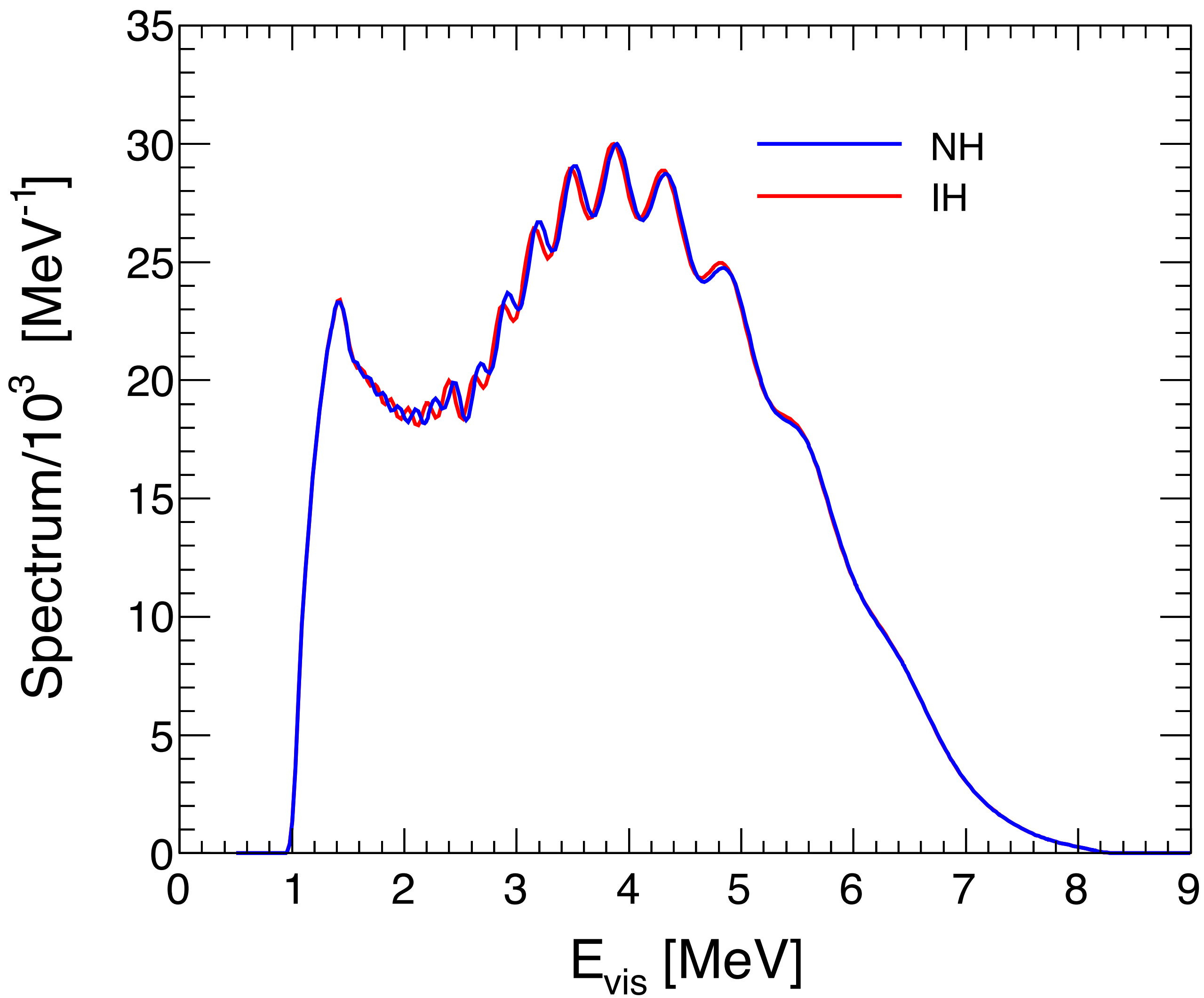
and [95, 96]

$$
\Delta m^2_{ee} = \cos^2\theta_{12}\Delta m^2_{31} + \sin^2\theta_{12}\Delta m^2_{32}\,. \qquad (2.2)
$$

The $\pm$ sign in the last term of Eq. (2.1) is decided by the MH with plus sign for the normal MH and minus sign for the inverted MH.

Worked out as an exercise, see extra slides

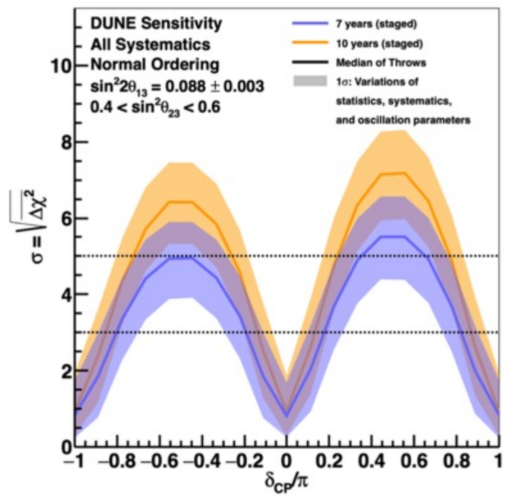
Need high statistics & high resolution

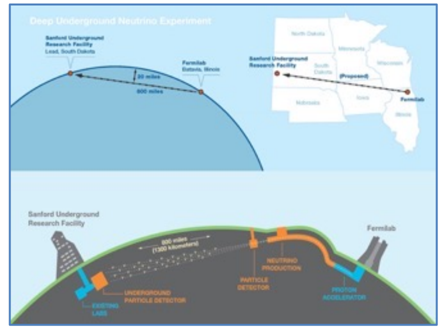


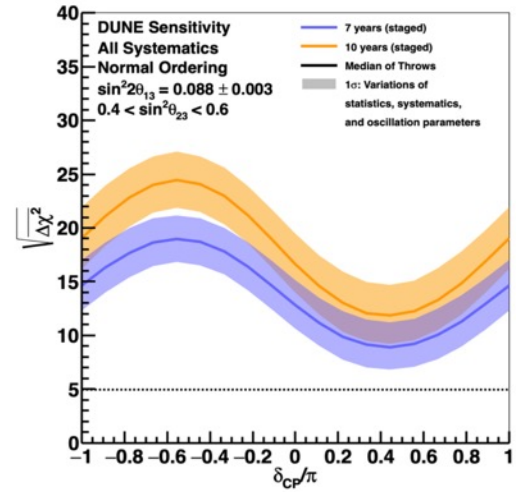
E.g., frontiers for DUNE, LBL acceler. expt [2002.03005]

Disapp. + app., nu/antinu mode, CPV & NO/IO & matter effects at L~1300 km

CPV ← **Discovery goals** → **NO vs IO**







Precision frontier

↓

Parameter	1σ, now	DUNE in ~10 y
Δm^2	1.1 %	factor ~1/4 reduction
$\sin^2\theta_{23}$	~ 6 %	factor ~1/4 reduction
$\sin^2\theta_{13}$	3.0 %	comparable

T2HK: same ballpark. DUNE & T2HK will need precise cross sections!

Worldwide activity to better understand **nuclear response to ν probes**

In the LBL accel. Context, one often refers to the following 3ν appear. probability...

$$\nu_\mu \leftrightarrow \nu_e$$

$$\begin{aligned}
P_{app} \simeq\ & \sin^2 2\theta_{13} \sin^2\theta_{23} \frac{\sin^2[(1-\hat{A})\Delta]}{(1-\hat{A})^2} \\
\pm\ & \alpha \sin 2\theta_{13}\, \xi \sin\delta_{CP} \sin(\Delta) \frac{\sin(\hat{A}\Delta)}{\hat{A}} \frac{\sin[(1-\hat{A})\Delta]}{(1-\hat{A})} \\
+\ & \alpha \sin 2\theta_{13}\, \xi \cos\delta_{CP} \cos(\Delta) \frac{\sin(\hat{A}\Delta)}{\hat{A}} \frac{\sin[(1-\hat{A})\Delta]}{(1-\hat{A})} \\
+\ & \alpha^2 \cos^2\theta_{23} \sin^2 2\theta_{12} \frac{\sin^2(\hat{A}\Delta)}{\hat{A}^2},
\end{aligned}$$

Slide from: Walter Winter

$$\alpha \equiv \frac{\Delta m^2_{21}}{\Delta m^2_{31}} \simeq \pm 0.03,\ \Delta \equiv \frac{\Delta m^2_{31} L}{4E},\ \xi \equiv \sin 2\theta_{12} \sin 2\theta_{23},\ \hat{A} \equiv \pm \frac{2\sqrt{2} G_F n_e E}{\Delta m^2_{31}}$$

(Cervera et al. 2000; Freund, Huber, Lindner, 2000; Freund, 2001)

Complicated, but all the interesting information is there:

θ_{13}, θ_{23} octant, δ_{CP}, mass ordering, matter effects

Note that CP phase and matter term change sign for $\nu \rightarrow \bar{\nu}$

May be written in several equivalent forms, e.g.:

arXiv:1512.06148v2

Long-Baseline Neutrino Facility (LBNF) and Deep Underground Neutrino Experiment (DUNE)

Conceptual Design Report

The oscillation probability of $\nu_\mu \to \nu_e$ through matter in a constant density approximation is, to first order [13]:

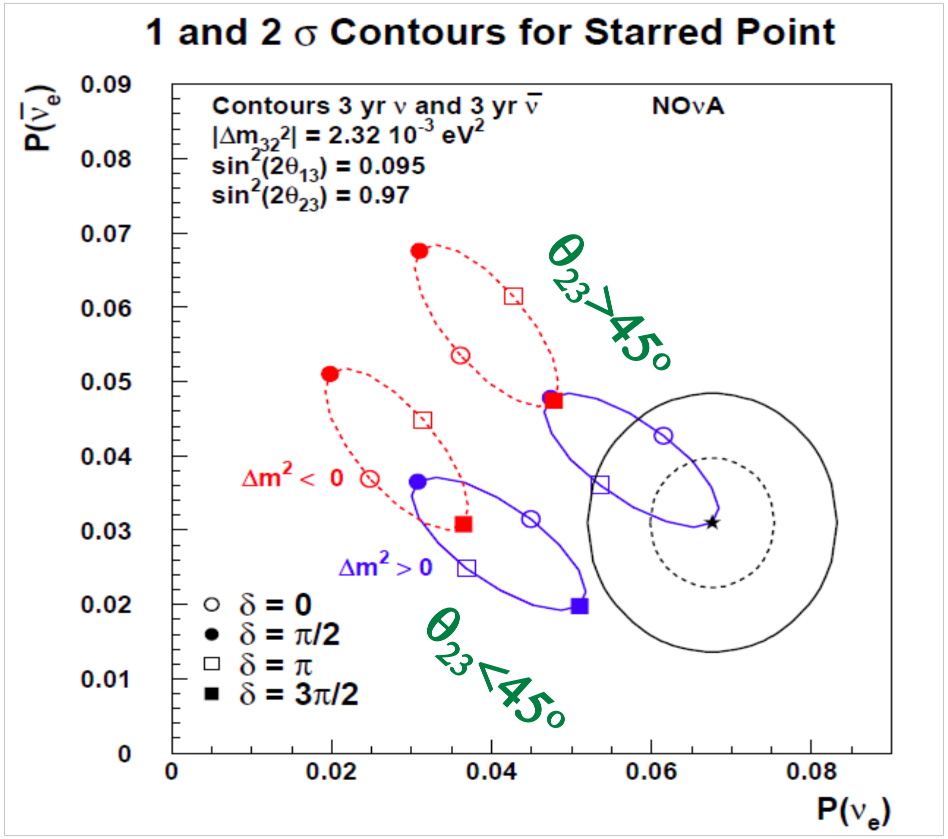
$$P(\nu_\mu \to \nu_e) \simeq \sin^2\theta_{23}\sin^2 2\theta_{13}\frac{\sin^2(\Delta_{31}-aL)}{(\Delta_{31}-aL)^2}\Delta_{31}^2 \qquad (3.5)$$
$$+\sin 2\theta_{23}\sin 2\theta_{13}\sin 2\theta_{12}\frac{\sin(\Delta_{31}-aL)}{(\Delta_{31}-aL)}\Delta_{31}\frac{\sin(aL)}{(aL)}\Delta_{21}\cos(\Delta_{31}+\delta_{\rm CP})$$
$$+\cos^2\theta_{23}\sin^2 2\theta_{12}\frac{\sin^2(aL)}{(aL)^2}\Delta_{21}^2,$$

where $\Delta_{ij} = \Delta m_{ij}^2 L/4E_\nu$, $a = G_F N_e/\sqrt{2}$, G_F is the Fermi constant, N_e is the number density of electrons in the Earth, L is the baseline in km, and E_ν is the neutrino energy in GeV. In the

Worked out as an exercise, see extra slides

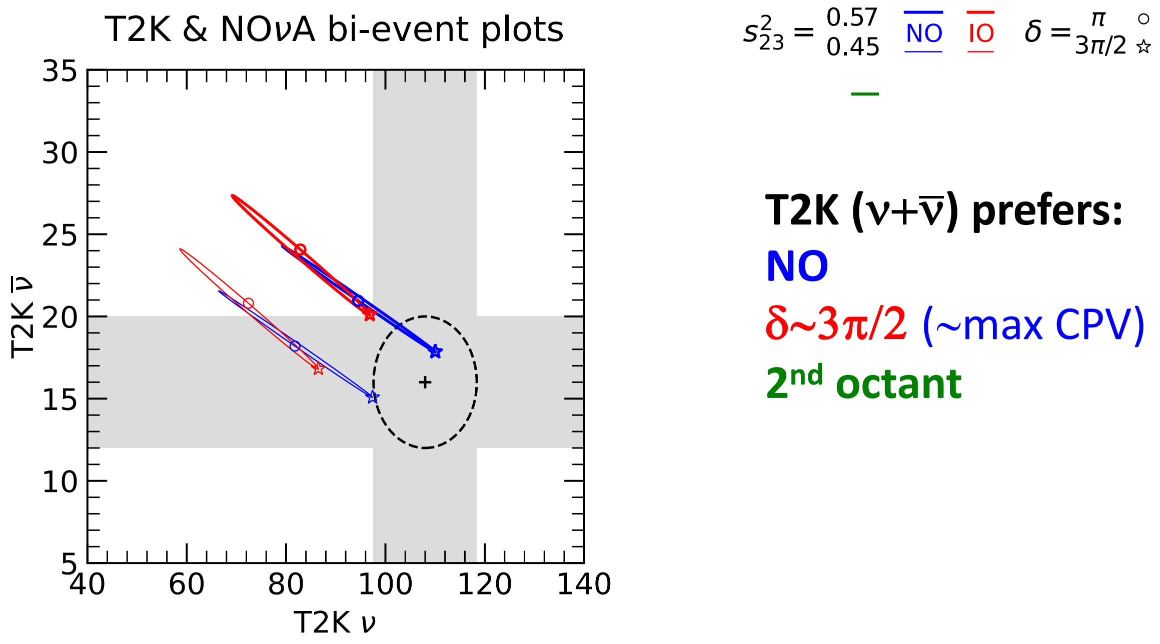
At fixed L, E and N_e, $P(\nu)$ and $P(\bar{\nu})$ depend parametrically on δ (cyclic)

→ "bi-probability ellipses", testable via nu-antinu comparison



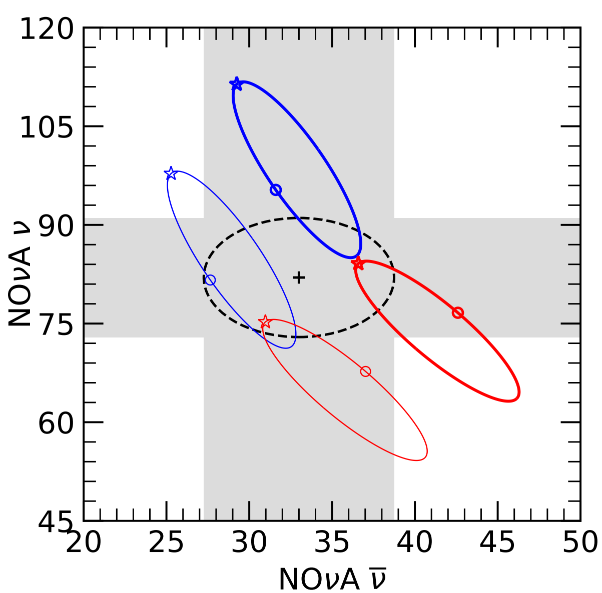
→ Discussion of LBL accelerator experiments in terms of **energy-integrated "bi-rate" or "bi-event" plots** for electron flavor appearance in nu and antinu channels

Bi-event plots for current T2K & NOvA results and "T2K-NOvA tension"



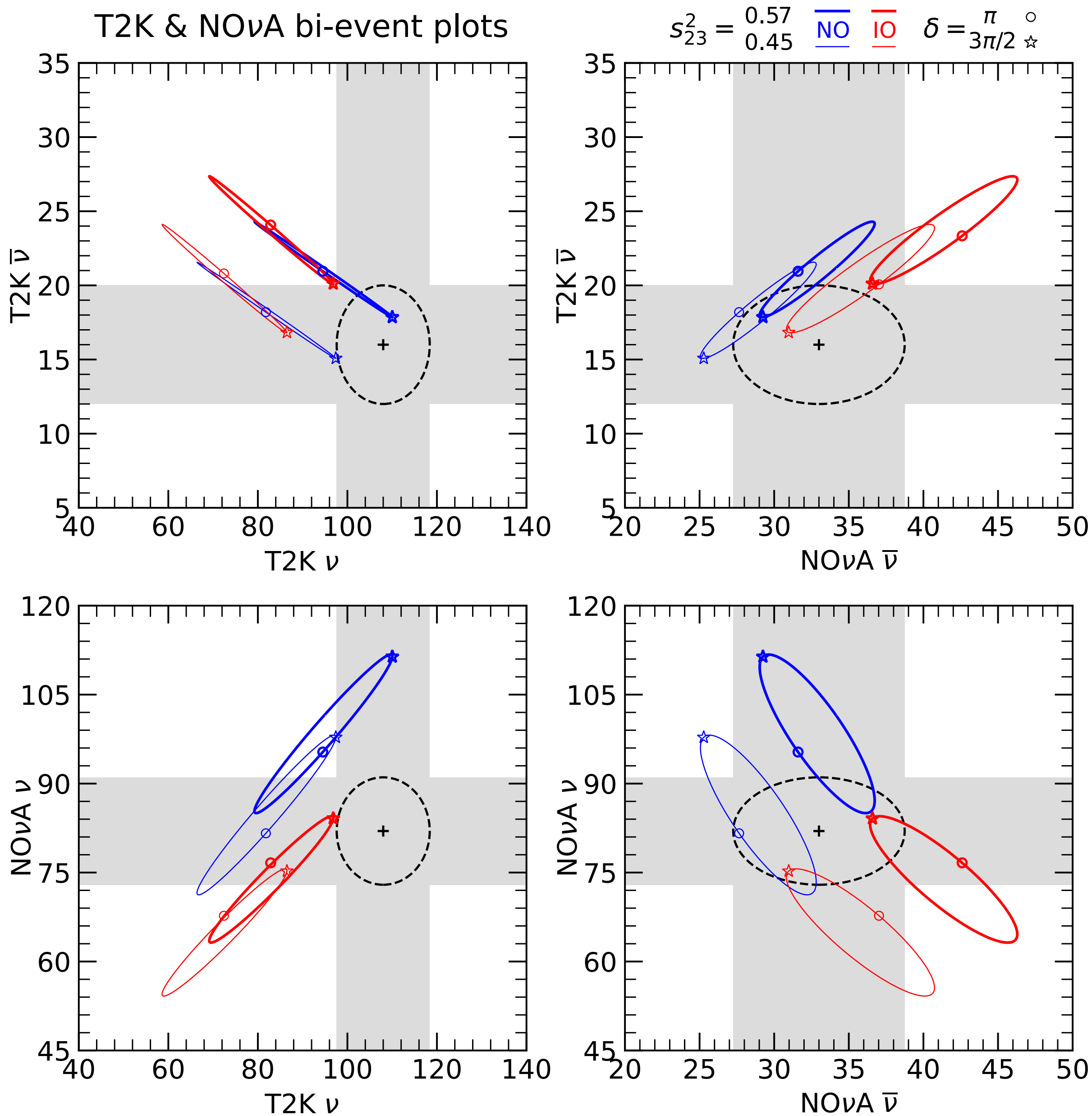
T2K ($\nu+\bar{\nu}$) prefers:
NO
δ~3π/2 (~max CPV)
2nd octant

NOVA ($\nu+\bar{\nu}$) prefers:
NO
CP conservation
octants ~degenerate

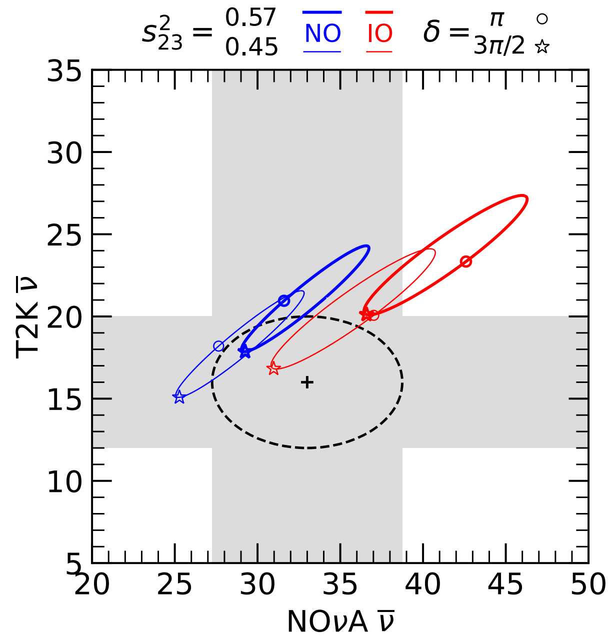


→ **T2K and NOVA, separately: NO preferred; CP and octant ambiguous**

The same info can be reorganized in terms of T2K vs NOvA:



T2K & NOνA bi-event plots

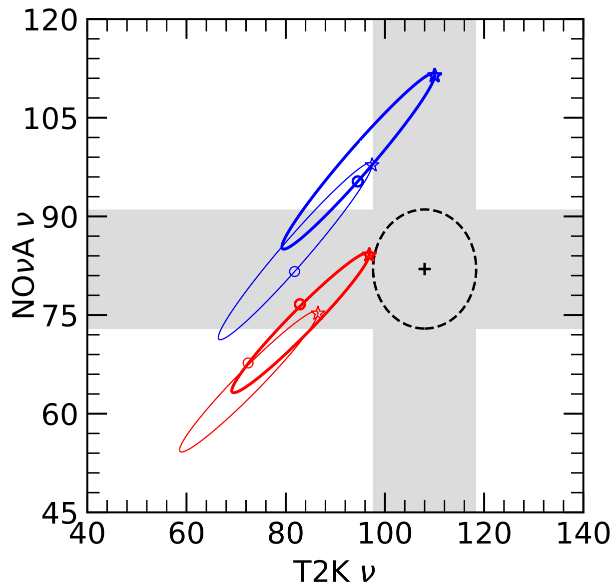


T2K+NOvA (ν) prefer:

IO

δ~3π/2

1st octant



T2K+NOvA ($\bar{\nu}$) prefer:

IO

δ~3π/2

2nd octant

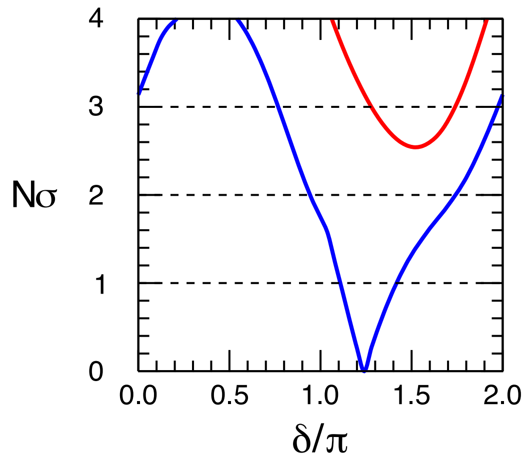
→ T2K and NOVA, jointly: IO and CPV preferred; octant ambiguous

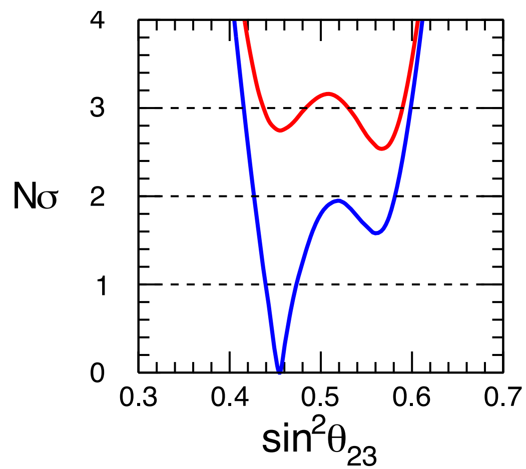
...In the T2K+NOvA combination, still unstable results on three unknowns:

mass ordering (NO vs IO), θ_{23} octant and CP phase δ

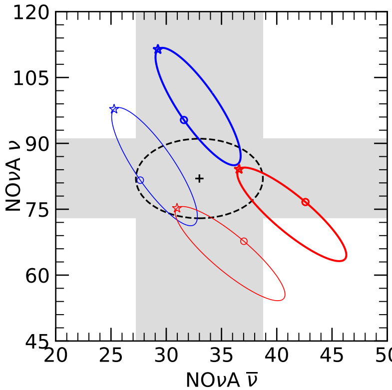
Further data may tilt the current balance, or even point to new physics (NSI?)

LBL Acc + Solar + KamLAND
(current)

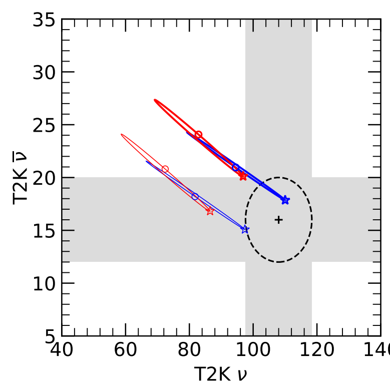




NOvA close to different
options within 1σ...



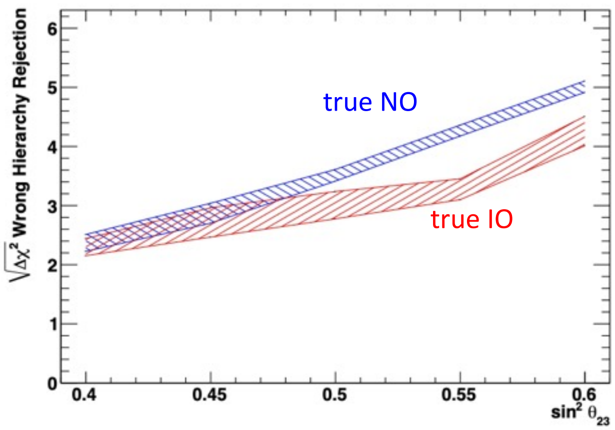
T2K close to the edge of
its expected sensitivity...

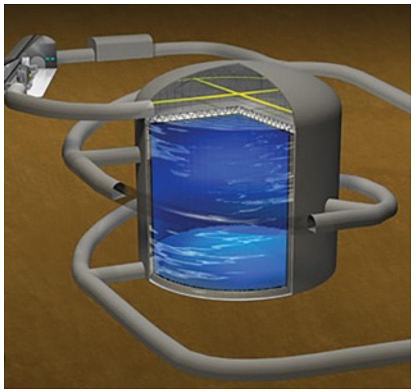


T2K + NOvA Joint analysis?

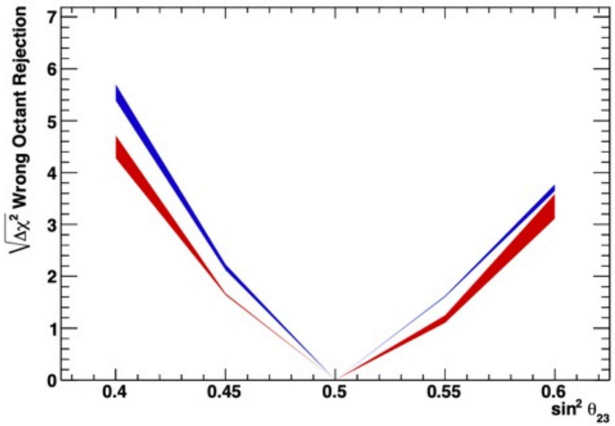
Further frontiers, e.g., Hyper-Kamiokande atmosph. [2002.03005]

Mass ordering





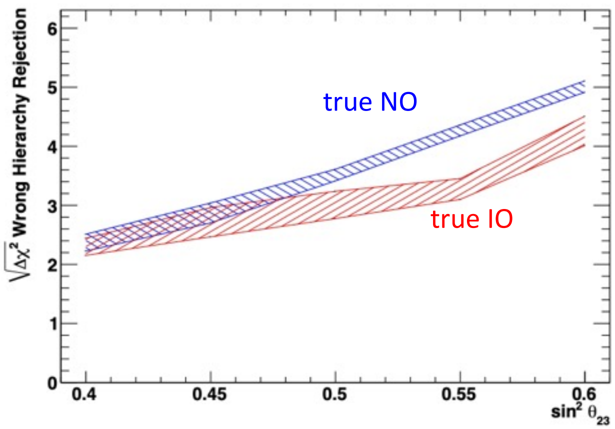
Octant resolution

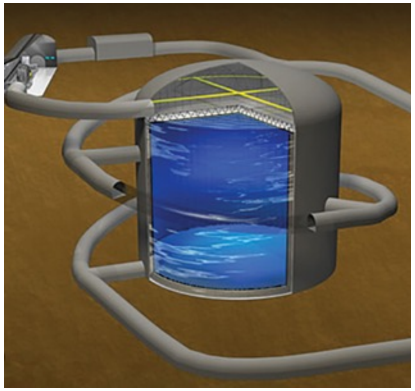


+ IceCube upgrade, +Km3NeT, +…

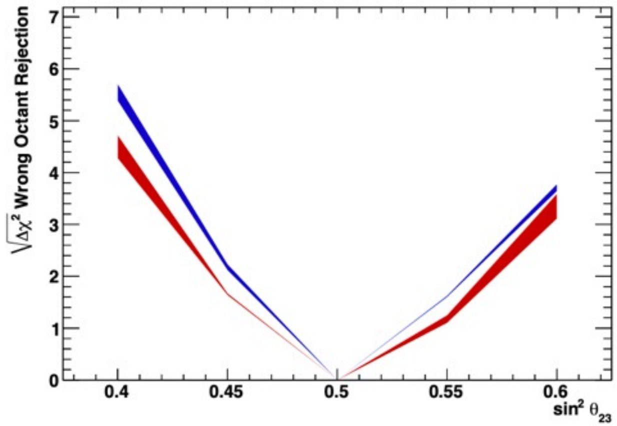
Further frontiers, e.g., Hyper-Kamiokande atmosph. [2002.03005]

Mass ordering





Octant resolution



+ IceCube upgrade, +Km3NeT, +…

… + surprises?

While advancing the precision and discovery frontiers, JUNO, DUNE, (T2)HK, … might either converge on consistent discoveries and precision parameters, or find anomalous results → new neutrino states, nonstandard interactions?
+ heavy neutrino signals in other searches, e.g., DM or HEP experiments?

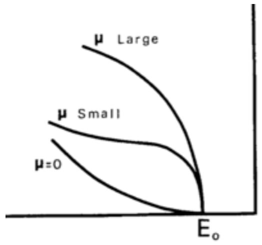
E.g., already in current data:

- **Saga of possible indications of sterile (~RH) neutrino state(s) at O(eV) scale**
- **4-fermion-like interactions ~$\varepsilon_{\alpha\beta}G_F$ weakly preferred by recent SK solar data**

Connect with non-oscillation ν mass observables: (m_β , $m_{\beta\beta}$, Σ)

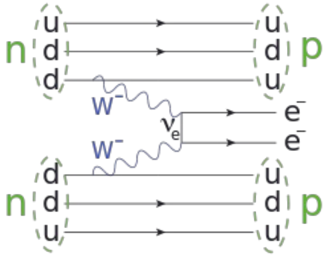
β decay, sensitive to the "effective electron neutrino mass":

$$m_\beta = \left[c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2\right]^{\frac{1}{2}}$$



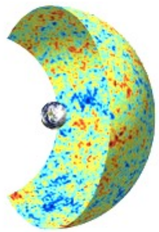
0νββ decay: only if Majorana. "Effective Majorana mass":

$$m_{\beta\beta} = \left|c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 m_2 e^{i\phi_2} + s_{13}^2 m_3 e^{i\phi_3}\right|$$



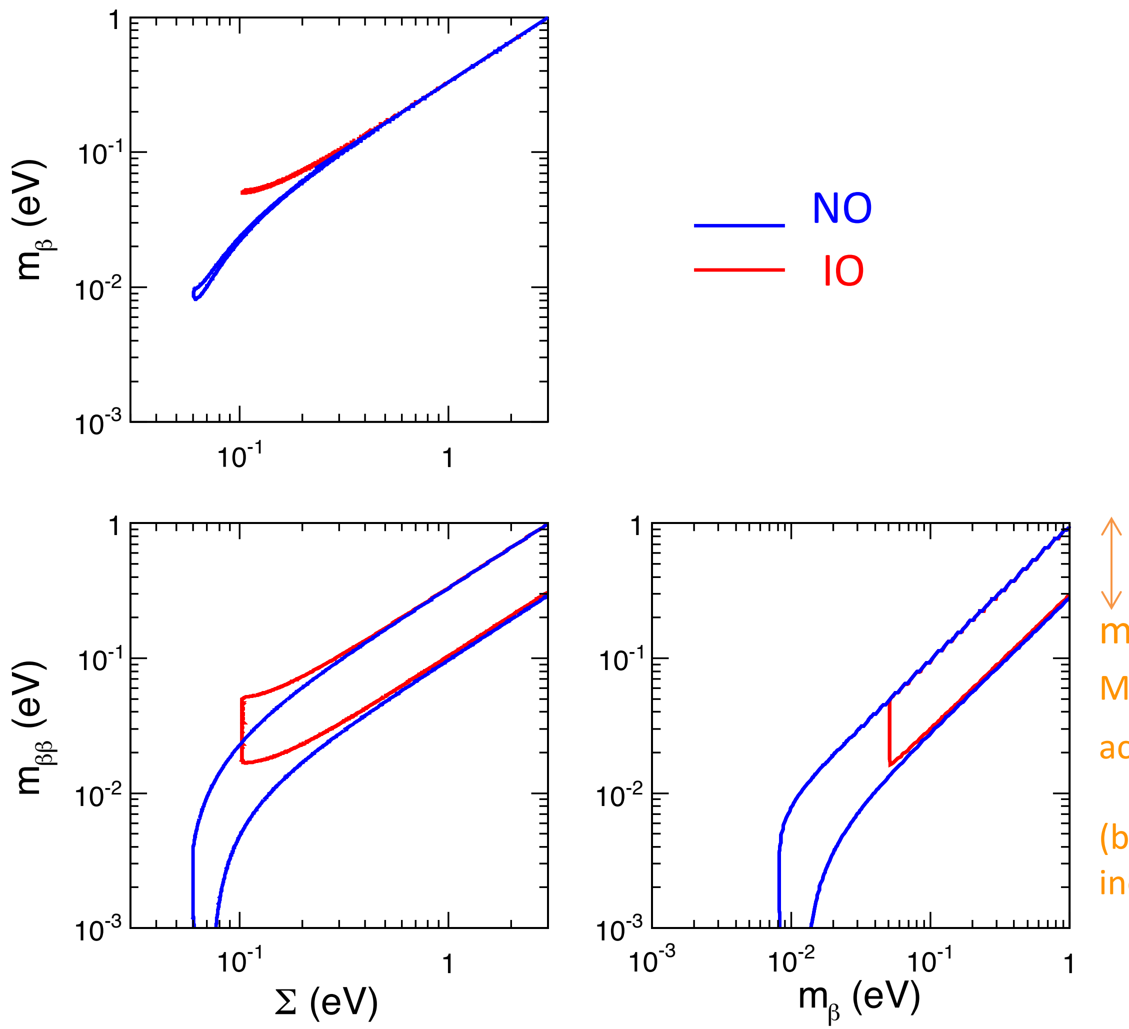
Cosmology: Dominantly sensitive to sum of neutrino masses:

$$\Sigma = m_1 + m_2 + m_3$$



May provide additional handles to distinguish NO/IO!

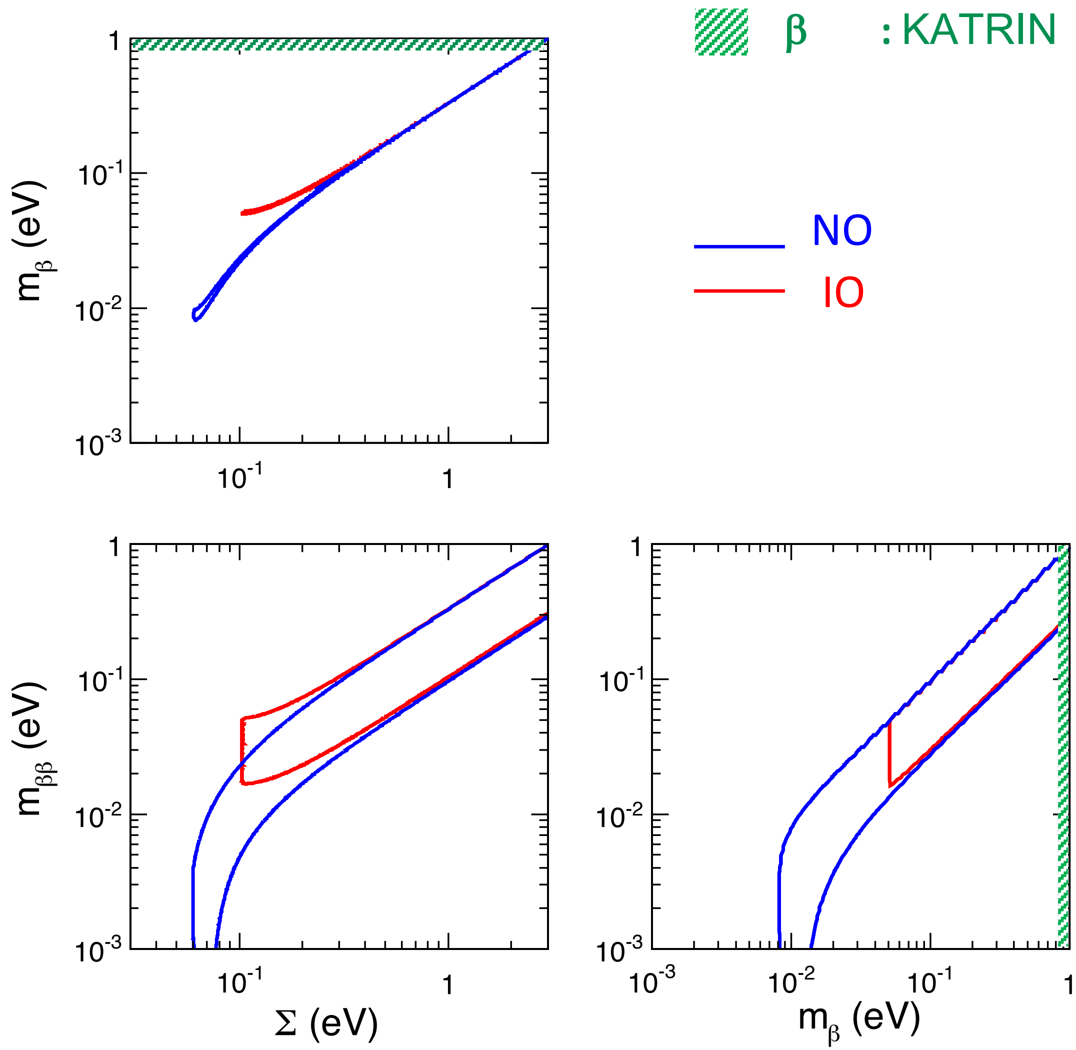
Non-oscillation parameter space (2σ) constrained by oscillations:



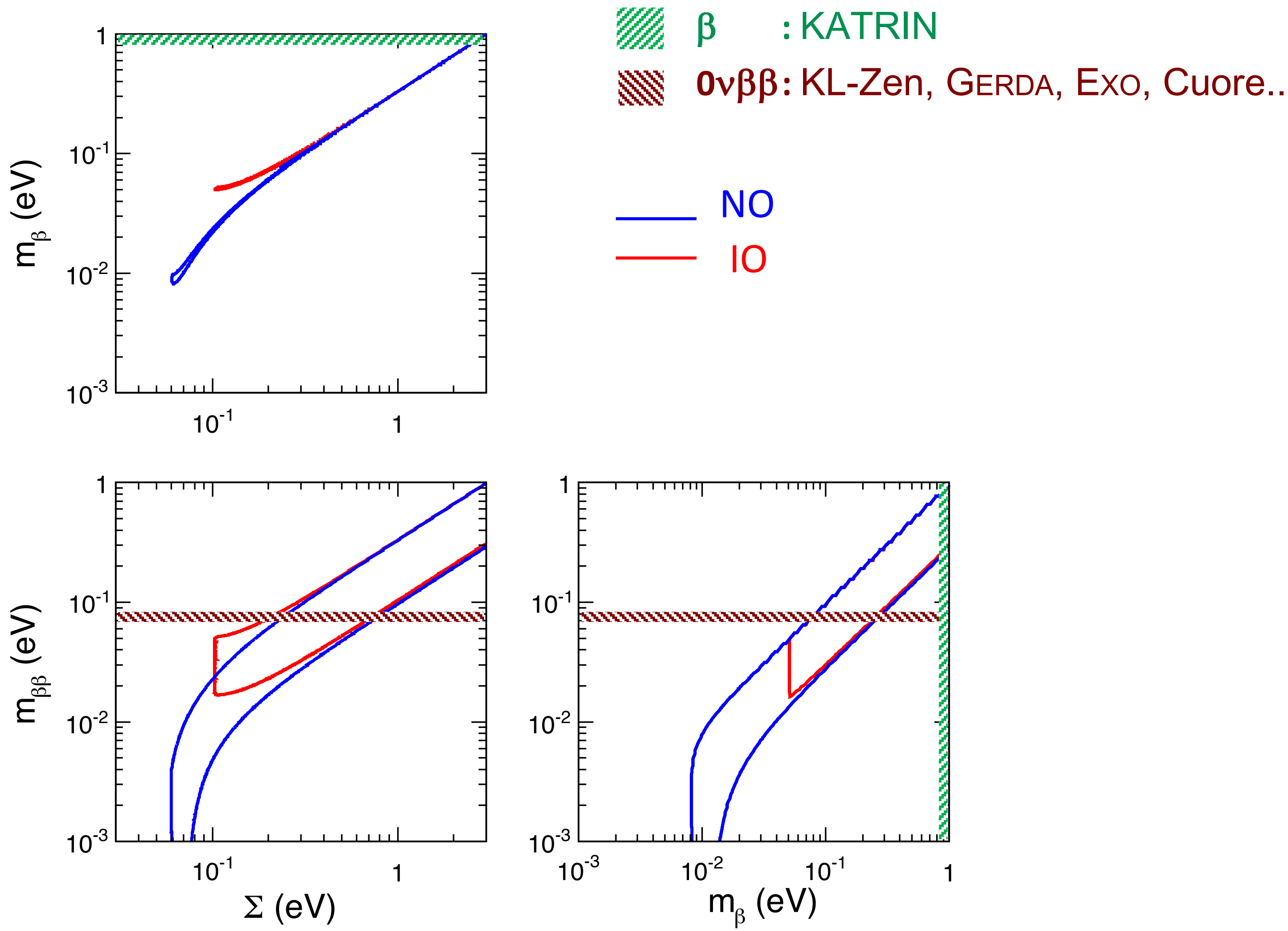
$m_{\beta\beta}$ spread due to Majorana CP phase(s): accessible in principle

(but: no NME errors included here!)

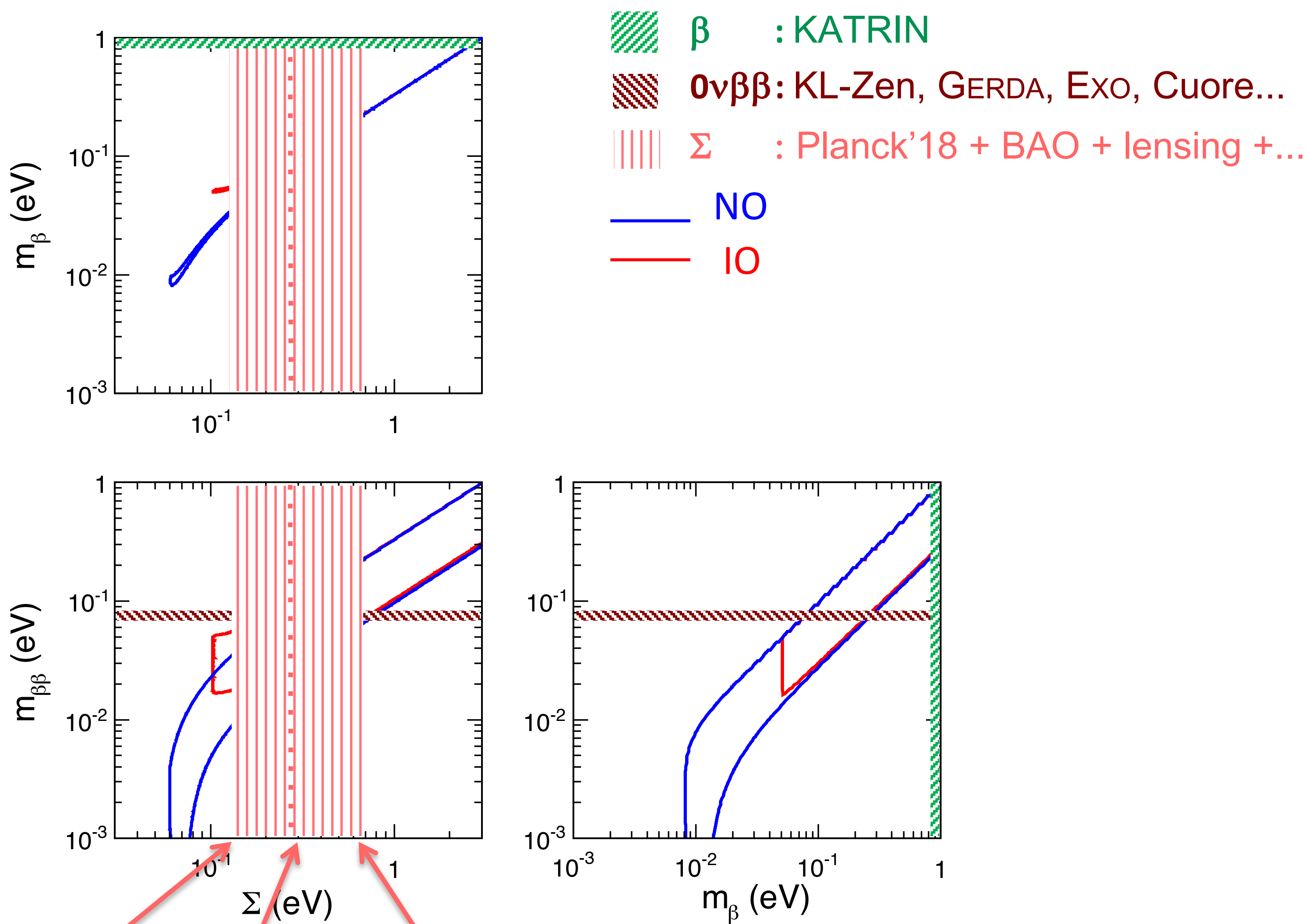
No signal (yet), but upper limits on m_β



…on $m_{\beta\beta}$, starting to cover non-degenerate mass regions (for favorable NME)



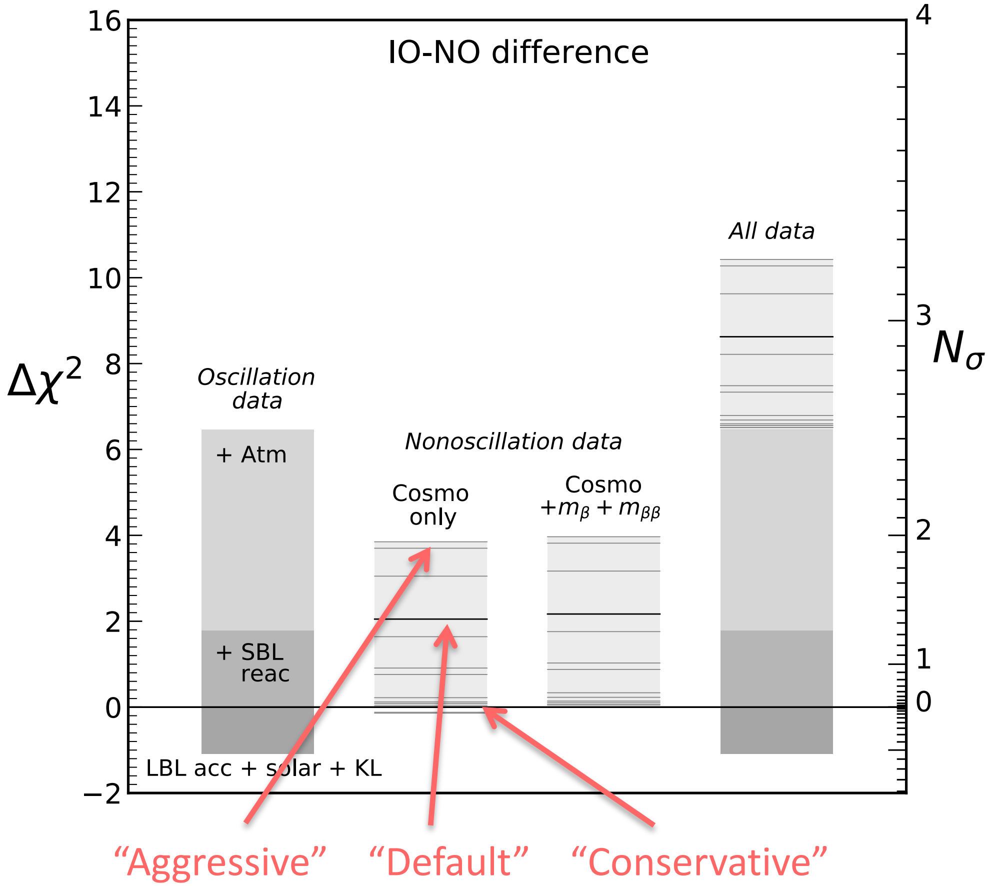
… and on Σ, from a variety of cosmo bounds, with IO "under pressure"



"Aggressive" "Default" "Conservative" cosmological limits

Grand total (oscillations + nonoscillations) for the IO-NO difference

[envelope of conservative, default, aggressive cosmological fits = horizontal lines]

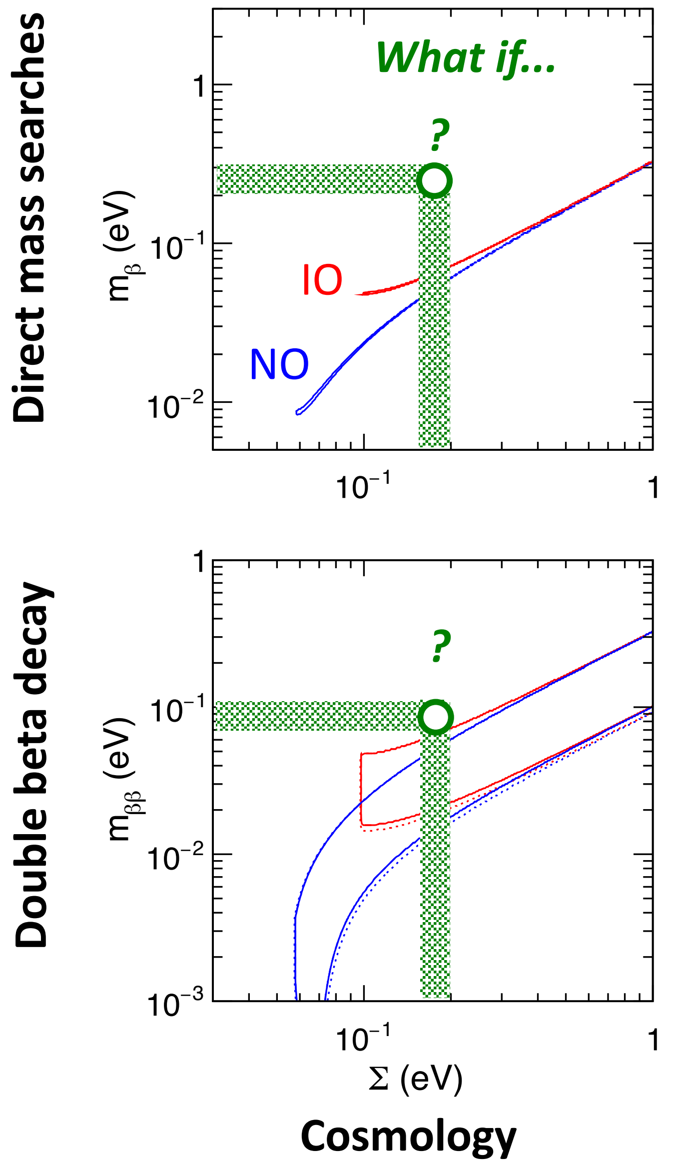


2.5σ (osc) ⊕ up to 0.7σ (nonosc) =

2.5σ – 3.2σ in favor of NO (all 2021 ν data)

Presumably stronger with 2022-2024 data. Progress expected on all fronts!

Surprises might also bring us beyond 3ν and re-shape the field…



Disagreement among future data (barring expt mistakes) **might point towards new possibilities:**

- *Cosmology beyond ΛCDM*
- *Alternative DBD mechanisms*
- *New interactions (NSI)*
- *New neutrino states …*

Main contender in current ν physics:
Light sterile ν at O(1 eV) scale
but… confusing/unconfirmed hints

In any case: generic expectations **for new possible ν mass state(s)**

Summary

3ν knowns:

δm^2, $|\Delta m^2|$, θ_{13}, θ_{12} and θ_{23} (up to octant) → *worldwide precision physics program*

3ν unknowns:

NO/IO, CPV, abs. mass, Majorana/Dirac → *ongoing searches aiming at discoveries*

Be open to:

New ν states and interactions, HE/LE links → *diversity of expt/theo approaches*

Be part of this adventure!

While answering old questions… → *new questions will emerge!*

ν . . . ?

Epilogue

An old Latin saying:

Nomen [est] Omen

"Name [is] Destiny"

Neutrino – What's the root of this name?

Language	Word tree	…*Some branches*	Meaning
Physics (Fermi 1934)	NEUTR-INO		Little neutral one
Italian	NEUTRO		Neutral
Latin	NE-UTER		Not either; neutral
Latin	UTER		Either
Greek	↑	*OUDETEROS*	*Neutral*
Old High German	↑ ↗	*HWEDAR*	*Which of two; whether*
Phonetic change/loss	[K]UOTER[US]		Which of the two?
Ionic Greek	KOTEROS		Which of the two?
Sanskrit	KATARAS		Which of the two?

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Sanskrit	KATARAS		Which of the two?
Latin	↑	*QUANTUS*	*How much?*
Sanskrit	↑	*KATAMAS*	*Which out of many?*
Sanskrit	↑	*KATHA*	*How?*
Sanskrit	↑ ↗	*KAS*	*Who?*
Indo-European root	KA *or* KWA		Interrogative base

The root of the name [neutrino] ... is a [kwa]stion

Language	Word tree	*...Some branches*	Meaning
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Indo-European root	**KA** *or* **KWA**		**Interrogative base**

If "name is destiny," then …

Neutrino's destiny is to raise questions!

Thank you for your attention

Finanziato dall'Unione europea NextGenerationEU
Ministero dell'Università e della Ricerca
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End of Lectures

Acknowledgments of support by:

INFN Bologna & Bari
PRIN 2022 "PANTHEON"

Solutions to exercises for Lecture IV: extra slides →

Summary of known and unknown 3ν oscillation parameters, ~2021

TABLE I: Global 3ν analysis of oscillation parameters: best-fit values and allowed ranges at $N_\sigma = 1$, 2 and 3, for either NO or IO, including all data. The latter column shows the formal "1σ fractional accuracy" for each parameter, defined as 1/6 of the 3σ range, divided by the best-fit value and expressed in percent. We recall that $\Delta m^2 = m_3^2 - (m_1^2 + m_2^2)/2$ and that $\delta \in [0, 2\pi]$ (cyclic). The last row reports the difference between the χ^2 minima in IO and NO.

Parameter	Ordering	Best fit	1σ range	2σ range	3σ range	"1σ" (%)
$\delta m^2/10^{-5}$ eV^2	NO, IO	7.36	7.21 – 7.52	7.06 – 7.71	6.93 – 7.93	2.3
$\sin^2\theta_{12}/10^{-1}$	NO, IO	3.03	2.90 – 3.16	2.77 – 3.30	2.63 – 3.45	4.5
$\|\Delta m^2\|/10^{-3}$ eV^2	NO	2.485	2.454 – 2.508	2.427 – 2.537	2.401 – 2.565	1.1
	IO	2.455	2.430 – 2.485	2.403 – 2.513	2.376 – 2.541	1.1
$\sin^2\theta_{13}/10^{-2}$	NO	2.23	2.17 – 2.30	2.11 – 2.37	2.04 – 2.44	3.0
	IO	2.23	2.17 – 2.29	2.10 – 2.38	2.03 – 2.45	3.1
$\sin^2\theta_{23}/10^{-1}$	NO	4.55	4.40 – 4.73	4.27 – 5.81	4.16 – 5.99	6.7
	IO	5.69	5.48 – 5.82	4.30 – 5.94	4.17 – 6.06	5.5
δ/π	NO	1.24	1.11 – 1.42	0.94 – 1.74	0.77 – 1.97	16
	IO	1.52	1.37 – 1.66	1.22 – 1.78	1.07 – 1.90	9
$\Delta\chi^2_{\mathrm{IO-NO}}$	IO−NO	+6.5				

hep-ph 2107.00532

Relevant probabilities in JUNO and DUNE

- Two advanced exercises on ν oscillation unknowns:
  $\pm\Delta m^2$ (mass ordering), δ_{CP} (CP viol.), $\theta_{23}-\pi/4$ (θ_{23} octant)
- (1) Calculation of $P_{ee}(\delta m^2, \Delta m^2, \theta_{12}, \theta_{13})$ in JUNO (in vacuum)
  → probes $\pm\Delta m^2$.
- (2) Calculation of $P_{\mu e}(\delta m^2, \Delta m^2, \theta_{12}, \theta_{13}, \theta_{23}, \delta)$ in DUNE (matter)
  → probes $\pm\Delta m^2$, δ, θ_{23} octant

As we have seen, three unknowns in the current 3ν framework are related to:

- Mass ordering $\pm\Delta m^2$: NO or IO ?
- CP violation phase δ: $\delta = 0, \pi$, or $\sin\delta \neq 0$?
- θ_{23} octant ambiguity: $\theta_{23} \gtrless \pi/4$?

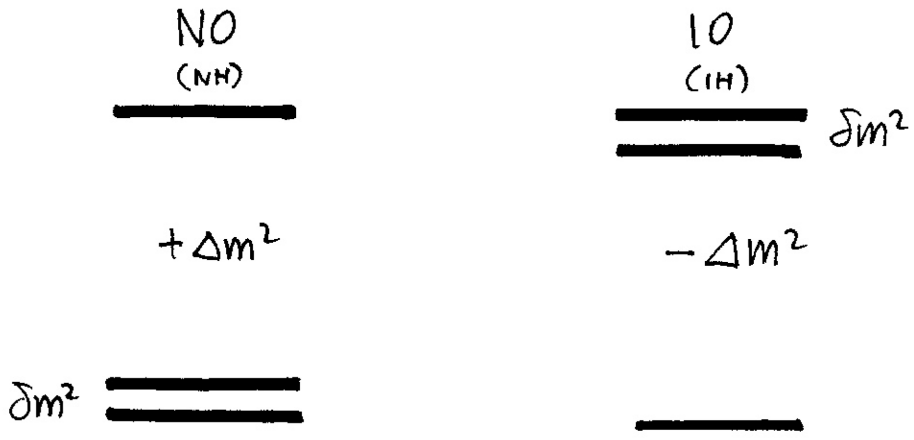
We shall work out two advanced exercises, in order to calculate two oscillations probabilities relevant for experiments in construction:

JUNO (reactor) $\quad P_{ee} = P_{ee}(\delta m^2, \Delta m^2, \theta_{12}, \theta_{13}) \longrightarrow$ probes $\pm\Delta m^2$

DUNE (acceler.) $\quad P_{\mu e} = P_{\mu e}(\delta m^2, \Delta m^2, \theta_{12}, \theta_{13}, \theta_{23}, \delta)$

$\longrightarrow$ probes $\begin{cases} \pm\Delta m^2 \\ \theta_{23} - \pi/4 \\ \delta_{CP} \end{cases}$

Mass ordering:



- How to probe ν mass ordering ("hierarchy") in ν oscill. searches?

- In oscillations, need to observe <u>interference</u> of oscillations driven by $\pm\Delta m^2$ with oscillations driven by another quantity Q with known sign. Three options:

$\pm\Delta m^2$ vs	$Q = \delta m^2$	← medium baseline reactors (eg JUNO)
$\pm\Delta m^2$ vs	$Q = 2\sqrt{2}\, G_F N_e E$	← atmosph. or acceler. ν (e.g. DUNE)
$\pm\Delta m^2$ vs	$Q = 2\sqrt{2}\, G_F N_\nu E$	← collective effects in SNe (difficult!)

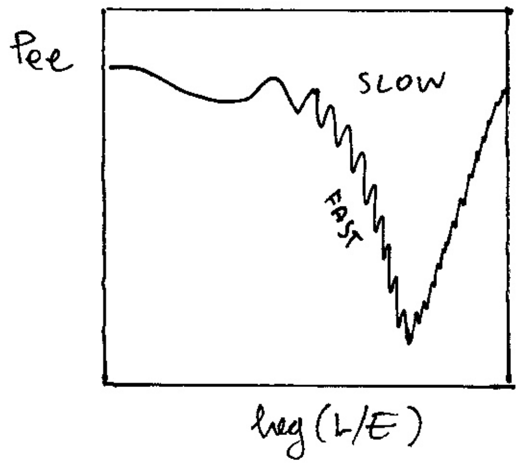
(The second option is already favoring NO at the level of $\sim 2 \div 3\sigma$ with current data from atmospheric and LBL accelerator oscillation searches)

JUNO (reactor neutrino experiment at medium baseline, $E \sim$ few MeV, $L \sim 50$ km)

- SBL reactor experiments ($L \sim 1$ km) $\rightarrow$ sensitive to $|\Delta m^2|$
- KamLAND react. expt. ($L \sim \mathcal{O}(100)$ km) $\rightarrow$ " " δm^2
- Medium-baseline reactor experiments ($L \sim 50$ km) $\rightarrow$ sensitive to δm^2, $\pm\Delta m^2$

<u>JUNO : in construction in China</u>

— Sensitive to both "slow" oscillations
(driven by $\delta m^2, \theta_{12}$)
and "fast" oscillations
(driven by $\Delta m^2, \theta_{13}$)



⇨ Need very high E resolution!

— "Fast" oscillation peaks displaced by $\pm\Delta m^2$ vs δm^2

- Most of the oscillation physics for JUNO is embedded in Eq. (2.1) of 1507.05613 (Neutrino Physics with JUNO, ~800 cites)

- This equation shows the survival probability $P(\bar{\nu}_e \to \bar{\nu}_e)$:

$$P_{ee} = P_{ee}(\delta m^2, \Delta m^2, \theta_{12}, \theta_{13})$$

in vacuum ($N_e = 0$).

- The equation is worked out in the following pages.

- Corrections must be applied to account for small matter effects (omitted)

Exercise : P_{ee} in vacuum, general 3ν case

Calculate P_{ee} in vacuum in the general 3ν case, in terms of θ_{ij}, δm^2, $\pm\Delta m^2$ and prove that it is not invariant under change of hierarchy : $+\Delta m^2 \to -\Delta m^2$.
(This implies that precision reactor experiments may be sensitive to the hierarchy)

Solution — Let us consider normal hierarchy ($+\Delta m^2$) for definiteness. Then:

$m_2^2 - m_1^2 = \delta m^2$; $m_3^2 - m_2^2 = \Delta m^2 - \delta m^2/2$; $m_3^2 - m_1^2 = \Delta m^2 + \delta m^2/2$ $\left(\Delta m^2 \overset{\text{def}}{=} \frac{1}{2}(\Delta m^2_{31} + \Delta m^2_{32})\right)$

$$\mathrm{Im}(J_{ee}^{ij}) = 0; \quad \mathrm{Re}(J_{ee}^{ij}) = |U_{ei}|^2 |U_{ej}|^2 = \begin{cases} s_{12}^2 c_{12}^2 c_{13}^4 & ij = 12 \\ s_{12}^2 s_{13}^2 c_{13}^2 & ij = 23 \\ c_{12}^2 s_{13}^2 c_{13}^2 & ij = 13 \end{cases} \quad ; \text{ then:}$$

$$\begin{aligned} P_{ee}^{3\nu} = 1 &- \cos^4\theta_{13} \sin^2 2\theta_{12} \sin^2\left(\frac{\delta m^2 x}{4E}\right) \\ &- \sin^2\theta_{12} \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 - \frac{\delta m^2}{2}}{4E} x\right) \\ &- \cos^2\theta_{12} \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 + \frac{\delta m^2}{2}}{4E} x\right) \end{aligned}$$

Note that the above $P_{ee}^{3\nu}$ is not invariant under the replacement $\Delta m^2 \to -\Delta m^2$. It would be so only for $\theta_{12} = \pi/4$ (i.e. $\sin^2\theta_{12} = \frac{1}{2} = \cos^2\theta_{12}$) which, however, is experimentally excluded ($\sin^2\theta_{12} \simeq 0.3 < 1/2$).

Exercise: P_{ee} in vacuum – general case in alternative formulation

Prove that $P_{ee}^{3\nu}$ can be recast in the following form:

$$P_{ee}^{3\nu} = c_{13}^4 P_{ee}^{2\nu} + s_{13}^4 + 2 s_{13}^2 c_{13}^2 \sqrt{P_{ee}^{2\nu}} \cos\left(\frac{\Delta m_{ee}^2 x}{2E} \pm \varphi\right) \quad \begin{pmatrix} + = NH \\ - = IH \end{pmatrix}$$

where $P_{ee}^{2\nu} = 1 - \sin^2 2\theta_{12} \sin^2\left(\frac{\delta m^2 x}{4E}\right)$

and $\Delta m_{ee}^2 = c_{12}^2 \Delta m_{31}^2 + s_{12}^2 \Delta m_{32}^2 = \Delta m^2 \pm \frac{1}{2}(c_{12}^2 - s_{12}^2)\,\delta m^2$

with
$$\begin{cases} \cos\varphi = \left[c_{12}^2 \cos(2 s_{12}^2 \Delta_{21}) + s_{12}^2 \cos(2 c_{12}^2 \Delta_{21})\right] / \sqrt{P_{ee}^{2\nu}} \\ \sin\varphi = \left[c_{12}^2 \sin(2 s_{12}^2 \Delta_{21}) - s_{12}^2 \sin(2 c_{12}^2 \Delta_{21})\right] / \sqrt{P_{ee}^{2\nu}} \end{cases} \quad \leftarrow \Delta_{21} = \frac{\delta m^2 x}{4E}$$

This formulation of $P_{ee}^{3\nu}$ emphasizes the physical effect of the mass hierarchy, namely, the fact that NH (IH) induces an advancement (retardation) of phase φ, with respect to the dominant "phase" induced by the effective mass parameter Δm_{ee}^2. It is particularly useful in the discussion of future medium-baseline reactor experiments sensitive to the hierarchy, see e.g. eq.(2.1) of arXiv 1507.05613.

Solution –

Assume NH for the moment. (For IH, just flip the relative sign of Δm^2 and δm^2).

Definitions: $\Delta m^2_{ij} = m_i^2 - m_j^2$; $\Delta_{ij} = \Delta m^2_{ij}\, x/4E$

$\Delta m^2_{ee} = c^2_{12}\,\Delta m^2_{31} + s^2_{12}\,\Delta m^2_{32}$; $\Delta_{ee} = \Delta m^2_{ee}\, x/4E$

$$\rightarrow \begin{cases} \Delta m^2_{31} = \Delta m^2_{ee} + s^2_{12}\,\delta m^2 = \Delta m^2 + \delta m^2/2 \\ \Delta m^2_{32} = \Delta m^2_{ee} - c^2_{12}\,\delta m^2 = \Delta m^2 - \delta m^2/2 \\ \Delta m^2 = \Delta m^2_{ee} - \frac{1}{2}(c^2_{12} - s^2_{12})\,\delta m^2 \end{cases}$$

Then the $P^{3\nu}_{ee}$ obtained in the previous exercise can be re-written as:

$$P^{3\nu}_{ee} = 1 - c^4_{13}(1 - P^{2\nu}_{ee}) - \sin^2 2\theta_{13}\left[s^2_{12}\sin^2(\Delta_{ee} - c^2_{12}\Delta_{21}) + c^2_{12}\sin^2(\Delta_{ee} + s^2_{12}\Delta_{21})\right]$$

$$= 1 - c^4_{13} + c^4_{13}P^{2\nu}_{ee} + \frac{1}{2}\sin^2 2\theta_{13}\left[s^2_{12}\cos(2\Delta_{ee} - 2c^2_{12}\Delta_{21}) + c^2_{12}\cos(2\Delta_{ee} + 2s^2_{12}\Delta_{21}) - 1\right]$$

$$= c^4_{13}P^{2\nu}_{ee} + s^4_{13} + 2s^2_{13}c^2_{13}\left[s^2_{12}\cos(2\Delta_{ee} - 2c^2_{12}\Delta_{21}) + c^2_{12}\cos(2\Delta_{ee} + 2s^2_{12}\Delta_{21})\right]$$

Let us recast the last term in [...] in the following form:

$$s^2_{12}\cos(2\Delta_{ee} - 2c^2_{12}\Delta_{21}) + c^2_{12}\cos(2\Delta_{ee} + 2s^2_{12}\Delta_{21}) = \eta\cos(2\Delta_{ee} + \varphi)$$

with the amplitude η and phase φ to be determined.

In other words, we are summing two "waves" with oscillating phases slightly different from $2\Delta_{ee}$, into a "single wave" with a phase which is also slightly different from $2\Delta_{ee}$.

In order to fulfill the previous eq. in (η, φ) it must be:

$$s_{12}^2 \cos(2\Delta_{ee}) \cos(2c_{12}^2 \Delta_{21}) + s_{12}^2 \sin(2\Delta_{ee}) \sin(2c_{12}^2 \Delta_{21})$$
$$+ c_{12}^2 \cos(2\Delta_{ee}) \cos(2s_{12}^2 \Delta_{21}) - c_{12}^2 \sin(2\Delta_{ee}) \sin(2s_{12}^2 \Delta_{21})$$
$$= \eta \cos(2\Delta_{ee}) \cos\varphi - \eta \sin(2\Delta_{ee}) \sin\varphi$$

and thus:

$$\cos(2\Delta_{ee}) \left[s_{12}^2 \cos(2c_{12}^2 \Delta_{21}) + c_{12}^2 \cos(2s_{12}^2 \Delta_{21}) - \eta \cos\varphi \right]$$
$$= \sin(2\Delta_{ee}) \left[-s_{12}^2 \sin(2c_{12}^2 \Delta_{21}) + c_{12}^2 \sin(2s_{12}^2 \Delta_{21}) - \eta \sin\varphi \right]$$

which, in general, can be solved only if the terms in [...] are both vanishing:

$$\rightarrow \begin{cases} s_{12}^2 \cos(2c_{12}^2 \Delta_{21}) + c_{12}^2 (2s_{12}^2 \Delta_{21}) = \eta \cos\varphi \\ s_{12}^2 \sin(2c_{12}^2 \Delta_{21}) - c_{12}^2 (2s_{12}^2 \Delta_{21}) = -\eta \sin\varphi \end{cases}$$

If we square and sum, we get:

$$\eta^2 = s_{12}^4 + c_{12}^4 + 2s_{12}^2 c_{12}^2 \left[\cos(2\Delta_{12} c_{12}^2) \cos(2\Delta_{12} s_{12}^2) - \sin(2s_{12}^2 \Delta_{21}) \sin(2\Delta_{21} c_{12}^2) \right]$$

where the term in [...] can be simplified by noticing that:

$$\sin^2(\Delta_{21}) = \sin^2(\Delta_{21}(c_{12}^2 + s_{12}^2)) = \frac{1}{2}\left(1 - \cos 2(\Delta_{21}(c_{12}^2 + s_{12}^2))\right)$$
$$= \frac{1}{2} - \frac{1}{2}\left[\cos(2\Delta_{21} c_{12}^2) \cos(2\Delta_{21} s_{12}^2) - \sin(2\Delta_{21} c_{12}^2) \sin(2\Delta_{21} s_{12}^2) \right]$$

$$\rightarrow \eta^2 = s_{12}^4 + c_{12}^4 + 4s_{12}^2 c_{12}^2 \left[\frac{1}{2} - \sin^2(\Delta_{21}) \right]$$
$$= 1 - 4s_{12}^2 c_{12}^2 \sin^2(\Delta_{21}) \equiv P_{ee}^{2\nu}$$

Therefore, it is also:

$$\begin{cases} \cos\varphi = \dfrac{1}{\sqrt{P_{ee}^{2\nu}}} \left(c_{12}^2 \cos(2 s_{12}^2 \Delta_{21}) + s_{12}^2 \cos(2 c_{12}^2 \Delta_{21}) \right) \\ \sin\varphi = \dfrac{1}{\sqrt{P_{ee}^{2\nu}}} \left(c_{12}^2 \sin(2 s_{12}^2 \Delta_{21}) - s_{12}^2 \sin(2 c_{12}^2 \Delta_{21}) \right) \end{cases}$$

which completes the proof.

For IH: $\varphi \to -\varphi$.

LBL accelerators : appearance channel

- The appearance channel $\overset{(-)}{\nu}_\mu \to \overset{(-)}{\nu}_e$ in LBL accelerators (as well as $\overset{(-)}{\nu}_e \to \overset{(-)}{\nu}_\mu$) contains very rich – although entangled – information, being sensitive to all the oscillation parameters <u>and</u> to matter effects.

- Most of the physics is embedded in the oscillation probability $P(\nu_\mu \to \nu_e)$

$$P_{\mu e} = P_{\mu e}(\delta m^2, \Delta m^2, \theta_{12}, \theta_{13}, \theta_{23}, \delta) \quad \text{in matter},$$

where one can assume ~ constant density with good approximation

- In the following, we shall derive an approximate analytical expression for $P_{\mu e}$, widely quoted in the literature. See, e.g. Eq. (3.5) in 1512.06148 (Physics Program for DUNE, ~800 cites) ; DUNE: $L \sim 1{,}300$ km, $E \sim$ few GeV.

- An important result is that $P_{\mu e}$ can be cast in the form:

$$P_{\mu e} = A \cos\delta + B \sin\delta + C$$

and similarly for $\overline{P}_{\mu e}$ with $\overline{\nu}$ (with different coefficients $\overline{A}, \overline{B}, \overline{C}$).

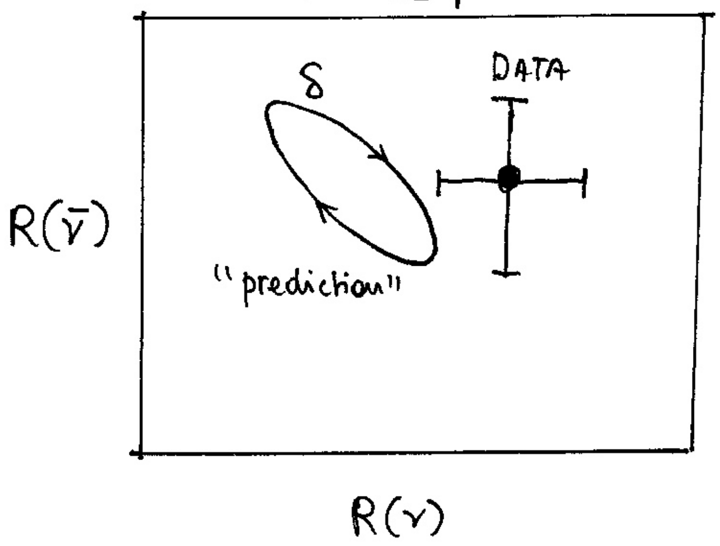
- Two equations of the kind
$$P_{\mu e} = A\cos\delta + B\sin\delta + C$$
$$\bar{P}_{\mu e} = \bar{A}\cos\delta + \bar{B}\sin\delta + \bar{C}$$
define an ellipse in the $(P, \bar{P})$ plane → "bi-probability" plots.

- In terms of event rates, $R \sim \int \phi P \sigma \epsilon$, one gets the so-called "bi-rate" plots $(R, \bar{R})$ for $\nu/\bar{\nu}$ events.

- Coefficients $\overset{(-)}{A}, \overset{(-)}{B}, \overset{(-)}{C}$ depend on $\pm\Delta m^2$, δ_{CP}, $\theta_{23} - \pi/4$

→ Comparison of ellipses with expt. data help to constrain such unknown parameters:

Bi-rate plot



"Predictions" (ellipses) move around and change dimensions + slope as a function of the unknown parameters.

Current results not yet converging (T2K vs NOVA)

See eg. 2107.00532

- In the following pages we shall work out the equation relevant for $P_{e\mu}$ ($P_{\mu e} = P_{e\mu}(\delta \to -\delta)$) in LBL accelerator expts., valid for constant density, at 2nd order in the small parameters δm^2 and θ_{13}.

- First we discuss some general reduction tools for 3ν probabilities.

- Then we apply further approximate reductions of the kind $3\nu = 2\nu \oplus 1\nu$

- We also show different (but equivalent) forms of $P_{e\mu}$ in the literature

3ν oscillations in matter: general reduction tools

- 3ν Hamiltonian in flavor basis for generic $N_e(x)$ density profile:

$\tilde{H} = \frac{1}{2E} U \mathcal{M}^2 U^+ + V$ where

$\mathcal{M}^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$ and

$U = O_{23} \Gamma_\delta O_{13} \Gamma_\delta^+ O_{12}$ with $\Gamma_\delta = \mathrm{diag}(1, 1, e^{i\delta})$ and $O_{ij}^+ = O_{ij}^T$

$V(x) = \mathrm{diag}(\sqrt{2}\, G_F N_e(x), 0, 0)$

- It is easy to prove that:

$(O_{23}\Gamma_\delta)^+ V (O_{23}\Gamma_\delta) = V$

$\Gamma_\delta^+ O_{12} \mathcal{M}^2 O_{12}^T \Gamma_\delta = O_{12} \mathcal{M}^2 O_{12}^T$

- Let's go from the flavor basis to a new "primed flavor basis" defined as:

$$\begin{bmatrix} (\nu^e)' \\ (\nu^\mu)' \\ (\nu^\tau)' \end{bmatrix} = (O_{23}\Gamma_\delta)^+ \begin{bmatrix} \nu^e \\ \nu^\mu \\ \nu^\tau \end{bmatrix} \leftarrow \text{components}, \quad \text{with } O_{23}\Gamma_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} e^{i\delta} \\ 0 & -s_{23} & c_{23} e^{i\delta} \end{pmatrix}$$

(note: $(\nu^e)' = (\nu^e)$)

- Hamiltonian in the primed basis:

$\tilde{H}' = (O_{23}\Gamma_\delta)^+ \tilde{H} (O_{23}\Gamma_\delta) = O_{13} O_{12} \frac{\mathcal{M}^2}{2E} (O_{13} O_{12})^T + V$

Such $\tilde{H}'$ does not depend on δ (and is thus real symmetric), nor on θ_{23}.

It is thus simpler to find the evolution operator $\tilde{S}'$ in the primed basis.

- Given $\tilde{S}'$ in the primed basis, the evolution operator $\tilde{S}$ in flavor basis is:
$\tilde{S}(x_f, x_i) = (O_{23}\Gamma_\delta)\,\tilde{S}'(x_f, x_i)\,(O_{23}\Gamma_\delta)^\dagger$

- In terms of matrix components:
if $\tilde{S}' = \begin{pmatrix} \tilde{S}'_{ee} & \tilde{S}'_{e\mu} & \tilde{S}'_{e\tau} \\ \tilde{S}'_{\mu e} & \tilde{S}'_{\mu\mu} & \tilde{S}'_{\mu\tau} \\ \tilde{S}'_{\tau e} & \tilde{S}'_{\tau\mu} & \tilde{S}'_{\tau\tau} \end{pmatrix}$, then $\tilde{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{i\delta} \\ 0 & -s_{23} & c_{23}e^{i\delta} \end{pmatrix} \tilde{S}' \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23}e^{-i\delta} & c_{23}e^{-i\delta} \end{pmatrix}$

$$\tilde{S}_{ee} = \tilde{S}'_{ee}$$
$$\tilde{S}_{\mu e} = \tilde{S}'_{\mu e}c_{23} + \tilde{S}'_{\tau e}s_{23}e^{i\delta}$$
$$\tilde{S}_{\tau e} = -\tilde{S}'_{\mu e}s_{23} + \tilde{S}'_{\tau e}c_{23}e^{i\delta}$$
$$\tilde{S}_{\mu\mu} = \tilde{S}'_{\mu\mu}c^2_{23} + \tilde{S}'_{\mu\tau}c_{23}s_{23}e^{-i\delta} + \tilde{S}'_{\tau\mu}c_{23}s_{23}e^{i\delta} + \tilde{S}'_{\tau\tau}s^2_{23}$$
$$\tilde{S}_{\tau\mu} = -\tilde{S}'_{\mu\mu}c_{23}s_{23} - \tilde{S}'_{\mu\tau}s^2_{23}e^{-i\delta} + \tilde{S}'_{\tau\mu}c^2_{23}e^{i\delta} + \tilde{S}'_{\tau\tau}c_{23}s_{23}$$
$$\tilde{S}_{\tau\tau} = \tilde{S}'_{\mu\mu}s^2_{23} - \tilde{S}'_{\mu\tau}c_{23}s_{23} - \tilde{S}'_{\tau\mu}c_{23}s_{23}e^{i\delta} + \tilde{S}'_{\tau\tau}c^2_{23}$$

with $\tilde{S}_{e\mu}$, $\tilde{S}_{e\tau}$, $\tilde{S}_{\mu\tau}$ obtained by $\tilde{S}'_{\alpha\beta} \leftrightarrow \tilde{S}'_{\beta\alpha}$ <u>and</u> $\delta \leftrightarrow -\delta$

- In general, $\tilde{S}'_{\alpha\beta} \neq \tilde{S}'_{\beta\alpha}$, even if $\tilde{H}'_{\alpha\beta} = \tilde{H}'_{\beta\alpha}$ (real symmetric).
Indeed, let us divide a generic $N_e(x)$ profile into steps $\{\Delta x_i\}_{i=1,\dots,N}$ with constant N_e in each step. Then:
$\tilde{S}' = e^{-i\tilde{H}'_N\Delta x_N}\, e^{-i\tilde{H}'_{N-1}\Delta x_{N-1}} \cdots e^{-i\tilde{H}'_2\Delta x_2}\, e^{-i\tilde{H}'_1\Delta x_1}$.

Then, although $(H'_i)^T = H'_i$, the transpose of $\tilde{S}'$ is not equal to $\tilde{S}'$, since the ordering of the steps is reversed from $1, \dots, N$ to $N, \dots, 1$ ("reversed" profile): $(\tilde{S}')^T = e^{-i\tilde{H}_1 \Delta x_1} \dots e^{-i\tilde{H}_N \Delta x_N}$. In other words:

$\tilde{S}'_{\alpha\beta}$ [direct profile] $=$ $\tilde{S}'_{\beta\alpha}$ [reverse profile] $\neq$ $\tilde{S}'_{\beta\alpha}$ [direct profile].

Only if the direct and reverse profile are symmetrical (= coincide upon reflection) then it is: $\tilde{S}'_{\alpha\beta}$ [symmetric profile] $=$ $\tilde{S}'_{\beta\alpha}$ [symmetric profile].

This is true, in particular, for <u>constant density</u> N_e:

$$\tilde{S}'_{\alpha\beta} = \tilde{S}'_{\beta\alpha} \quad \text{for } N_e = \text{const.}$$

In this case: $\tilde{S}_{\alpha\beta} = \tilde{S}_{\beta\alpha}(\delta \to -\delta)$ and $P_{\alpha\beta} = P_{\beta\alpha}(\delta \to -\delta)$

since $P_{\alpha\beta} = |\tilde{S}_{\beta\alpha}|^2 = P(\nu_\alpha \to \nu_\beta)$.

- Further reductions come from some peculiar symmetries of $\tilde{S}_{\alpha\beta}$ under the substitutions $s_{23} \to \pm c_{23}$ <u>and</u> $c_{23} \to \mp s_{23}$:

$$\tilde{S}_{\tau e} = \pm \tilde{S}_{\mu e}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \quad \Rightarrow \quad P_{e\tau} = P_{e\mu}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \stackrel{\text{def}}{=} P'_{e\mu}$$

$$\tilde{S}_{\mu\tau} = \mp \tilde{S}_{\tau\mu}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \quad \Rightarrow \quad P_{\tau\mu} = P_{\mu\tau}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \stackrel{\text{def}}{=} P'_{\mu\tau}$$

$$\tilde{S}_{\mu\mu} = \pm \tilde{S}_{\tau\tau}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \quad \Rightarrow \quad P_{\mu\mu} = P_{\tau\tau}\Big|_{\substack{s_{23} \to \pm c_{23} \\ c_{23} \to \mp s_{23}}} \stackrel{\text{def}}{=} P'_{\tau\tau}$$

The previous relations, together with the unitarity of $P_{\alpha\beta}$, allow to express all the probabilities in terms of just two, e.g., $P_{e\mu}$ and $P_{\mu\tau}$, and their transformed $P'_{e\mu}\Big|_{\substack{s_{23}\to\pm c_{23}\\ c_{23}\to\mp s_{23}}}$ and $P'_{\mu\tau}\Big|_{\substack{s_{23}\to\pm c_{23}\\ c_{23}\to\mp s_{23}}}$

[It is equivalent to choose the upper or the lower substitution.]

Explicitly:

$$P_{ee} = 1 - P_{e\mu} - P_{e\tau} = 1 - P_{e\mu} - P'_{e\mu}$$

$$P_{e\tau} = P'_{e\mu}$$

$$P_{\mu e} = 1 - P_{\mu\mu} - P_{\mu\tau} = 1 - P_{\mu\mu} - P_{e\mu} + P_{e\mu} - P_{\mu\tau} = P_{e\mu} + P_{\tau\mu} - P_{\mu\tau} = P_{e\mu} + P'_{\mu\tau} - P_{\mu\tau}$$

$$P_{\mu\mu} = 1 - P_{e\mu} - P_{\tau\mu} = 1 - P_{e\mu} - P'_{\mu\tau}$$

$$P_{\tau\mu} = P'_{\mu\tau}$$

$$P_{\tau\tau} = 1 - P_{e\tau} - P_{\mu\tau} = 1 - P'_{e\mu} - P_{\mu\tau}$$

We also have: $P(\bar\nu_\alpha \to \bar\nu_\beta) = P(\nu_\alpha \to \nu_\beta \mid V \to -V;\ \delta \to -\delta)$

and, in constant density: $P_{\alpha\beta} = P_{\beta\alpha}(\delta \to -\delta)$.

Such relations allow to reduce the calculation of $P_{\alpha\beta}(\nu, \bar\nu)$ to a few independent probabilities. In the following we shall calculate one of them, $P_{\alpha\beta}$ for $(\alpha\beta) = (e\mu)$, at second order in the small parameters δm^2 and θ_{13}.

Calculation of $P_{e\mu}$ in constant density and 2nd order in δm^2, θ_{13}

- The so-called "golden channel" $\nu_e \to \nu_\mu$ is particularly important in the context of future long-baseline accelerator experiments ($E \gtrsim 1$ GeV). In this context:
- Let us show that, for constant density N_e, and at 2nd order in the small parameters δm^2 and θ_{13}, $P_{e\mu} = P(\nu_e \to \nu_\mu)$ takes the form:

$$\begin{cases} P_{e\mu} = X \sin^2 2\theta_{13} + Y \sin 2\theta_{13} \cos\left(\delta - \dfrac{\Delta m^2 x}{4E}\right) + Z, \quad \text{where:} \\ X = \sin^2\theta_{23} \left(\dfrac{\Delta m^2}{A - \Delta m^2}\right)^2 \sin^2\left(\dfrac{A - \Delta m^2}{4E} x\right) \\ Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\dfrac{\delta m^2}{A}\right)\left(\dfrac{\Delta m^2}{A - \Delta m^2}\right) \sin\left(\dfrac{Ax}{4E}\right) \sin\left(\dfrac{A - \Delta m^2}{4E} x\right) \\ Z = \cos^2\theta_{23} \sin^2 2\theta_{12} \left(\dfrac{\delta m^2}{A}\right)^2 \sin^2\left(\dfrac{Ax}{4E}\right) \end{cases}$$

(Note that, sometimes, a further $\cos\theta_{13}$ factor is inserted in Y. This, however, is irrelevant at the stated 2nd order approximation.)

- Note that, given $P_{e\mu} = P(\nu_e \to \nu_\mu)$ one can get:

$$P(\bar\nu_e \to \bar\nu_\mu) = P(\nu_e \to \nu_\mu \mid A \to -A,\ \delta \to -\delta)$$

$$P(\nu_\mu \to \nu_e) = P(\nu_e \to \nu_\mu \mid \delta \to -\delta)$$

$$P(\nu_e \to \nu_\tau) = P(\nu_e \to \nu_\mu)\Big|_{\substack{c_{23} \to \mp s_{23} \\ s_{23} \to \pm c_{23}}}$$

while changing mass hierarchy is equivalent to $\Delta m^2 \to -\Delta m^2$.

- We start the calculation by reminding that:

$P(\nu_e \to \nu_\mu) = P_{e\mu} = |\tilde{S}_{\mu e}|^2$ with $\tilde{S}_{\mu e} = \tilde{S}'_{\mu e} c_{23} + \tilde{S}'_{\tau e} s_{23} e^{i\delta}$, so that:

$P_{e\mu} = |\tilde{S}'_{\mu e} c_{23} + \tilde{S}'_{\tau e} s_{23} e^{i\delta}|^2 = A_{e\mu} \cos\delta + B_{e\mu} \sin\delta + C_{e\mu}$ with:

$$A_{e\mu} = 2\,\mathrm{Re}\,[\tilde{S}'^{*}_{\mu e} \tilde{S}'_{\tau e}]\, c_{23} s_{23}$$
$$B_{e\mu} = -2\,\mathrm{Im}\,[\tilde{S}'^{*}_{\mu e} \tilde{S}'_{\tau e}]\, c_{23} s_{23}$$
$$C_{e\mu} = |\tilde{S}'_{\mu e}|^2 c^2_{23} + |\tilde{S}'_{\tau e}|^2 s^2_{23}$$

- The next "trick" is to reduce the evolution from 3ν to approximately $2\nu \oplus 1\nu$, by exploiting the expansion in two small parameters: δm^2 and θ_{13} (or better s_{13}). A term T will be called "of 1st order" if proportional to s_{13} or δm^2:

$T \sim O_1$ if $T \propto s_{13}$ or $T \propto \delta m^2$. Analogously:

$T \sim O_2$ if $T \propto s^2_{13}$ or $T \propto (\delta m^2)^2$ or $T \propto s_{13}\, \delta m^2$ etc.

- We shall show that $\tilde{S}'_{\mu e} \sim O_1$ and $\tilde{S}'_{\tau e} \sim O_1$. Therefore, since $P_{e\mu}$ is quadratic in $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$, it is $P_{e\mu} \sim O_2$ as desired.

- Let's remind that, in primed basis and for normal hierarchy:

$$\tilde{H}' = O_{13} O_{12} \frac{\mathcal{M}^2}{2E} (O_{13} O_{12})^T + V$$

$$\mathcal{M}^2 = \mathrm{diag}\left(-\frac{\delta m^2}{2}, +\frac{\delta m^2}{2}, \Delta m^2\right) \qquad \leftarrow \text{up to terms} \propto \mathbb{1}$$

$$V = \mathrm{diag}\,(\sqrt{2} G_F N_e, 0, 0)$$

- In the primed basis, the evolution decouples as $3\nu = 2\nu \oplus 1\nu$ in two limits:

$s_{13} \to 0 \quad \Rightarrow \quad O_{13} = \mathbb{1}$

$\delta m^2 \to 0 \quad \Rightarrow \quad O_{12}\, \mathcal{M}^2 O_{12}^T = \mathcal{M}^2$

It is then convenient to define:

$\tilde{H}^l = \lim_{s_{13}\to 0} \tilde{H}'$

$\tilde{H}^h = \lim_{\delta m^2 \to 0} \tilde{H}'$

(← "l" an "h" refer to "low" and "high" 2ν subcases in literature jargon.)

and to study the evolution operator components $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$ in $\tilde{H}^l$ and $\tilde{H}^h$. The task is simpler since both $\tilde{H}^l$ and $\tilde{H}^h$ have only 1 nontrivial 2x2 submatrix.

- $\tilde{H}^l$ in primed basis ($s_{13} \to 0$ limit):

$$\tilde{H}^l = \lim_{s_{13}\to 0} \tilde{H}' = \frac{1}{2E}\left[O_{12}\begin{pmatrix} -\delta m^2/2 & & \\ & +\delta m^2/2 & \\ & & \Delta m^2 \end{pmatrix} O_{12}^T + \begin{pmatrix} A & & \\ & 0 & \\ & & 0 \end{pmatrix}\right]$$

$$= \frac{A}{4E}\mathbb{1} + \frac{1}{4E}\begin{bmatrix} A - \cos 2\theta_{12}\,\delta m^2 & \sin 2\theta_{12}\,\delta m^2 & 0 \\ \sin 2\theta_{12}\,\delta m^2 & \cos 2\theta_{12}\,\delta m^2 - A & 0 \\ 0 & 0 & 2\Delta m^2 - A \end{bmatrix}.$$ Given this structure:

In the primed basis, for $s_{13} \to 0$, the (e, μ') flavors evolve separately from the τ' one →

$\tilde{S}^l_{\tau e} = \lim_{s_{13}\to 0} \tilde{S}'_{\tau e} = 0$ (i.e., no $\nu_e \to \nu'_\tau$ transitions), thus:

$\tilde{S}'_{\tau e} = \mathcal{O}(s_{13}) = \mathcal{O}_1$ at least.

Instead, $\tilde{S}^l_{\mu e}$ is nonzero. From the 2ν case in matter (already worked out) we get:

$\tilde{S}^l_{\mu e} = e^{-i\frac{A}{4E}x}\left[-i \sin 2\tilde{\theta}_{12} \sin\left(\frac{\delta \tilde{m}^2 x}{4E}\right)\right]$ by exponentiation of $\tilde{H}^l$, with:

$\sin 2\tilde{\theta}_{12} = \sin 2\theta_{12} / \sqrt{(\cos 2\theta_{12} - A/\delta m^2)^2 + \sin^2 2\theta_{12}}$ and $\delta\tilde{m}^2 = \delta m^2 \sin 2\theta_{12} / \sin 2\tilde{\theta}_{12}$, implying:
$\tilde{S}^{\ell}_{\mu e} = \mathcal{O}(\delta m^2) = O_1$

- $\tilde{H}^h$ in primed basis ($\delta m^2 \to 0$ limit):

$$\tilde{H}^h = \lim_{\delta m^2 \to 0} \tilde{H}' = \frac{1}{2E}\left(O_{13} \begin{bmatrix} 0 & & \\ & 0 & \\ & & \Delta m^2 \end{bmatrix} O_{13}^T + \begin{bmatrix} A & & \\ & 0 & \\ & & 0 \end{bmatrix} \right)$$

$$= \left(\frac{\Delta m^2}{4E} + \frac{A}{4E} \right) \mathbb{1} + \frac{1}{4E} \begin{bmatrix} A - \cos 2\theta_{13} \Delta m^2 & 0 & \sin 2\theta_{13} \Delta m^2 \\ 0 & -\Delta m^2 - A & 0 \\ \sin 2\theta_{13} \Delta m^2 & 0 & \cos 2\theta_{13} \Delta m^2 - A \end{bmatrix}.$$ Given this structure:

In the primed basis, for $\delta m^2 \to 0$, the (e, τ') flavors evolve separately from the μ' one $\to$
$\tilde{S}^h_{\mu e} = \lim_{\delta m^2 \to 0} \tilde{S}'_{\mu e} = 0$ (no $\nu_e \to \nu_{\mu'}$ transitions), thus:

$\tilde{S}'_{\mu e} = \mathcal{O}(\delta m^2) = O_1$ at least.

Instead, $\tilde{S}^h_{\tau e}$ is nonzero. From the 2ν case in matter (already worked out) we get:
$\tilde{S}^h_{\tau e} = e^{-i\frac{A}{4E}x} e^{-i\frac{\Delta m^2 x}{4E}} \left[-i \sin 2\tilde{\theta}_{13} \sin\left(\frac{\Delta\tilde{m}^2 x}{4E} \right) \right]$ by exponentiation of $\tilde{H}^h$, with:

$\sin 2\tilde{\theta}_{13} = \sin 2\theta_{13} / \sqrt{(\cos 2\theta_{13} - A/\Delta m^2)^2 + \sin^2 2\theta_{13}}$ and $\Delta\tilde{m}^2 = \Delta m^2 \sin 2\theta_{13} / \sin 2\tilde{\theta}_{13}$, implying:
$\tilde{S}^h_{\tau e} = \mathcal{O}(s_{13}) = O_1$

- Summarizing, at O_1 we have that:

$$S'_{\mu e} = \mathcal{O}(\delta m^2) \simeq \tilde{S}^{\ell}_{\mu e} = e^{-i\frac{A}{4E}x} \left[-i \sin 2\tilde{\theta}_{12} \sin\left(\frac{\delta\tilde{m}^2 x}{4E} \right) \right]$$

$$S'_{\tau e} = \mathcal{O}(s_{13}) \simeq \tilde{S}^h_{\tau e} = e^{-i\frac{A}{4E}x} e^{-i\frac{\Delta m^2}{4E}x} \left[-i \sin 2\tilde{\theta}_{13} \sin\left(\frac{\Delta\tilde{m}^2 x}{4E} \right) \right]$$

- One can drop the overall phase $e^{-i\frac{A}{4E}x}$ and get:

$$\tilde{S}'_{\mu e} = \left[-i\sin 2\tilde{\theta}_{12}\sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right)\right] + O_2$$

$$\tilde{S}'_{\tau e} = e^{-i\frac{\Delta m^2}{4E}x}\left[-i\sin 2\tilde{\theta}_{13}\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\right] + O_2$$

which provide all that is needed to get $P_{e\mu}$ as a quadratic form in $\tilde{S}'_{\mu e}$ and $\tilde{S}'_{\tau e}$. Indeed:

- $$A_{e\mu} = 2\,\mathrm{Re}\left[\tilde{S}'^{*}_{\mu e}\tilde{S}'_{\tau e}\right]c_{23}s_{23} = \sin 2\tilde{\theta}_{12}\sin 2\tilde{\theta}_{13}\sin 2\theta_{23}\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right)\cos\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)$$

$$B_{e\mu} = -2\,\mathrm{Im}\left[\tilde{S}'^{*}_{\mu e}\tilde{S}'_{\tau e}\right]c_{23}s_{23} = \sin 2\tilde{\theta}_{12}\sin 2\tilde{\theta}_{13}\sin 2\theta_{23}\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)\sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right)\sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)$$

$$C_{e\mu} = |\tilde{S}'_{\mu e}|^2 c_{23}^2 + |\tilde{S}'_{\tau e}|\, s_{23}^2 = \cos^2\theta_{23}\sin^2 2\tilde{\theta}_{12}\sin^2\left(\frac{\delta\tilde{m}^2 x}{4E}\right) + \sin^2\theta_{23}\sin^2 2\tilde{\theta}_{13}\sin^2\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)$$

$$P_{e\mu} = A_{e\mu}\cos\delta + B_{e\mu}\sin\delta + C_{e\mu}$$

- The solution will now be further reduced by a proper organization of terms, as well as by an expansion in the small parameter:

$$\frac{\delta m^2}{A} = \frac{\delta m^2}{2\sqrt{2}G_F N_e E} \ll 1$$ (valid for $E \gtrsim 1$ GeV and N_e in the crust or mantle).

In particular, in this "high-energy approximation" (useful for accelerator neutrino experiments) one can express the matter parameters $\delta\tilde{m}^2$, $\Delta\tilde{m}^2$, $\tilde{\theta}_{12}$ and $\tilde{\theta}_{13}$ in terms of the vacuum parameters δm^2, Δm^2, θ_{12} and θ_{13} (together with an expansion in the small parameter s_{13}).

- For $\delta m^2/A \ll 1$:

$$\sin 2\tilde{\theta}_{12} = \sin 2\theta_{12} \Big/ \left[(\cos 2\theta_{12} - A/\delta m^2)^2 + \sin^2 2\theta_{12} \right]^{1/2}$$

$$= \sin 2\theta_{12} \Big/ \left[\cos^2 2\theta_{12} - \frac{2A}{\delta m^2} \cos 2\theta_{12} + \left(\frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12} \right]^{\frac{1}{2}}$$

$$= \sin 2\theta_{12} \Big/ \left[\left(\frac{A}{\delta m^2}\right)^2 \left(1 - 2\frac{\delta m^2}{A} \cos 2\theta_{12} + \cdots\right) \right]^{1/2}$$

$$\simeq \sin 2\theta_{12} \Big/ \left[\frac{|A|}{\delta m^2} \left(1 - \frac{\delta m^2}{A} \cos 2\theta_{12}\right) \right] \simeq \sin 2\theta_{12} \frac{\delta m^2}{|A|} + O_2$$

$$\delta m^2 / \delta \tilde{m}^2 = \sin 2\tilde{\theta}_{12} / \sin 2\theta_{12}$$

$$= \delta m^2 / |A| + O_2 \quad \Rightarrow \delta \tilde{m}^2 = |A| + O_2 \text{, thus:}$$

$$\sin\left(\frac{\delta \tilde{m}^2 x}{4E}\right) \simeq \sin\left(\frac{|A| x}{4E}\right) + O_2$$

- For $s_{13} \ll 1$:

$$\sin 2\tilde{\theta}_{13} = \sin 2\theta_{13} \Big/ \left[(\cos 2\theta_{13} - A/\Delta m^2)^2 + \sin^2 2\theta_{13} \right]^{1/2}$$

$$\simeq \sin 2\theta_{13} \Big/ \left[\left(1 - \frac{A}{\Delta m^2}\right)^2 \right]^{1/2} + O_2 = \sin 2\theta_{13} \Big/ \left| 1 - \frac{A}{\Delta m^2} \right| + O_2$$

$$\sin 2\tilde{\theta}_{13} = \left| \frac{\Delta m^2}{\Delta m^2 - A} \right| \sin 2\theta_{13} + O_2$$

$$\Delta \tilde{m}^2 = \Delta m^2 \sin 2\theta_{13} / \sin 2\tilde{\theta}_{13} \simeq \Delta m^2 \left| \frac{\Delta m^2 - A}{\Delta m^2} \right|$$

- We have then:

$$A_{e\mu} \simeq \sin 2\theta_{12} \left(\frac{\delta m^2}{|A|}\right) \sin 2\theta_{13} \left|\frac{\Delta m^2}{\Delta m^2 - A}\right| \sin 2\theta_{23} \sin\left(\frac{|A| x}{4E}\right) \sin\left(\Delta m^2 \left|\frac{\Delta m^2 - A}{\Delta m^2}\right| \frac{x}{4E}\right) \cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} \simeq \sin 2\theta_{12} \left(\frac{\delta m^2}{|A|}\right) \sin 2\theta_{13} \left|\frac{\Delta m^2}{\Delta m^2 - A}\right| \sin 2\theta_{23} \sin\left(\frac{|A| x}{4E}\right) \sin\left(\Delta m^2 \left|\frac{\Delta m^2 - A}{\Delta m^2}\right| \frac{x}{4E}\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} \simeq \cos^2\theta_{23} \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{Ax}{4E}\right) + \sin^2\theta_{23} \sin^2 2\theta_{13} \left(\frac{\Delta m^2}{\Delta m^2 - A}\right)^2 \sin^2\left(\frac{|\Delta m^2 - A|}{4E} x\right)$$

- Absolute values can be eliminated by inspection of all relevant $\pm$ cases.
  E.g.: by changing sign of $(\Delta m^2 - A)$: $A_{e\mu}$, $B_{e\mu}$ and $C_{e\mu}$ do not change.
  By changing sign of Δm^2: only $A_{e\mu}$ changes. Etc...

  Then we have:

$$A_{e\mu} \simeq \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right) \frac{\Delta m^2}{A - \Delta m^2} \sin\left(\frac{Ax}{4E}\right) \sin\left(\frac{A - \Delta m^2}{4E} x\right) \cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} \simeq \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right) \frac{\Delta m^2}{A - \Delta m^2} \sin\left(\frac{Ax}{4E}\right) \sin\left(\frac{A - \Delta m^2}{4E} x\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} \simeq \cos^2\theta_{23} \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{Ax}{4E}\right) + \sin^2\theta_{23} \sin^2 2\theta_{13} \left(\frac{\Delta m^2}{\Delta m^2 - A}\right)^2 \sin^2\left(\frac{\Delta m^2 - A}{4E} x\right)$$

$$P_{e\mu} = A_{e\mu} \cos\delta + B_{e\mu} \sin\delta + C_{e\mu}$$

- Finally, the terms in $P_{e\mu}$ can be organized as:

$$P_{e\mu} = X \sin^2 2\theta_{13} + Y \sin 2\theta_{13} \cos\left(\delta - \frac{\Delta m^2 x}{4E}\right) + Z \quad \text{where}$$

$$X = \sin^2\theta_{23} \left(\frac{\Delta m^2}{A - \Delta m^2}\right)^2 \sin^2\left(\frac{A - \Delta m^2}{4E} x\right)$$

$$Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right)\left(\frac{\Delta m^2}{A - \Delta m^2}\right) \sin\left(\frac{Ax}{4E}\right) \sin\left(\frac{A - \Delta m^2}{4E} x\right)$$

$$Z = \cos^2\theta_{23} \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{Ax}{4E}\right)$$

as desired

- In the literature, one can also find the following way to organize terms:

$$P_{e\mu} = x^2 f^2 + 2xy fg \cos(\Delta - \delta) + y^2 g^2 \quad , \text{where}$$

$$x = \sin\theta_{23} \sin 2\theta_{13}$$

$$y = \frac{\delta m^2}{\Delta m^2} \cos\theta_{23} \sin 2\theta_{12}$$

$$\Delta = \Delta m^2 x / 4E$$

$$f = \sin\left(\frac{\Delta m^2 - A}{4E} x\right) \frac{\Delta m^2}{\Delta m^2 - A}$$

$$g = \sin\left(\frac{Ax}{4E}\right) \frac{\Delta m^2}{A}$$