Neutrino Oscillations

Lecture III

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Outline of lectures I-IV:

Lecture I
Pedagogical introduction + warm-up exercise

Lecture II
3ν osc. in vacuum and matter: notation and basic math

Lecture III
2ν approximations of phenomenological interest

Lecture IV
Back to 3ν oscillations: Status and Perspectives

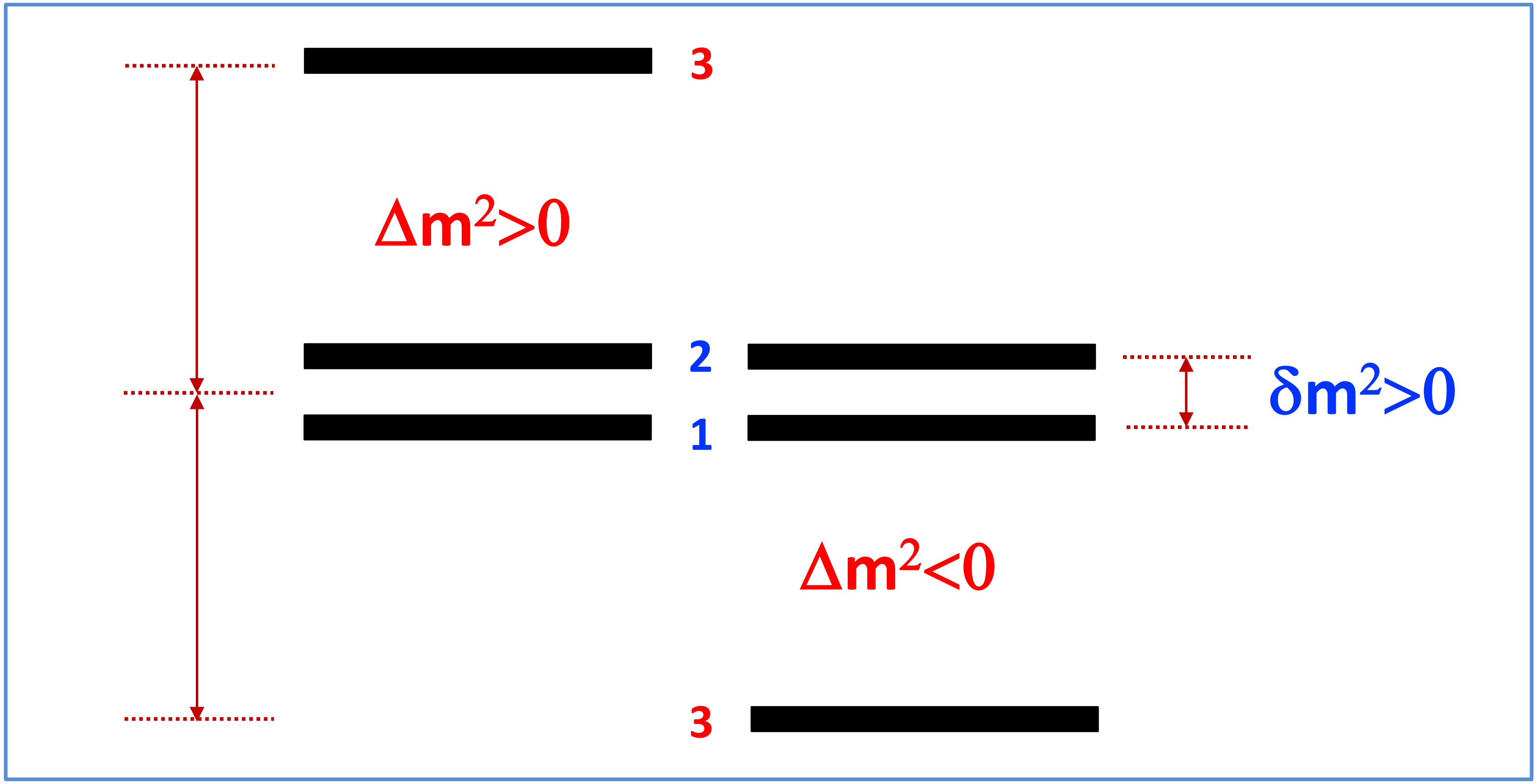
Conventions: PMNS mixing matrix

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{bmatrix} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{bmatrix}$$

$UU^\dagger = 1$ $\quad$ $U \to U^*$ for $\overline{\nu}$ $\quad$ $c_{ij} = \cos\theta_{ij}$ $\quad$ $s_{ij} = \sin\theta_{ij}$

Conventions: Mass-squared spectrum



$$\Delta m^2 = \frac{1}{2}(\Delta m^2_{31} + \Delta m^2_{32}) \begin{cases} > 0 & \text{NO} \\ < 0 & \text{IO} \end{cases}$$

$$\delta m^2 = m^2_2 - m^2_1 > 0$$

The presence of two small dimensionless parameters,

$$\delta m^2 / \Delta m^2 \sim 3 \times 10^{-2}$$

$$\sin^2\theta_{13} \sim 2 \times 10^{-2}$$

allows useful $3\nu \rightarrow 2\nu$ approximations and simplifies the understanding of phenomenology

In particular, experiments so far are sensitive to either δm^2 or Δm^2 (in first approximation)

	Initial flavors	Typical E	Typical L
Long-baseline reactor neutrinos: KamLAND	$\bar{\nu}_e$	few MeV	$O(10^2)$ km
Solar neutrinos: Chlorine, Gallium, Super-K, SNO, Borexino…	ν_e	O(1-10) MeV	1 a.u.

$\Delta m^2 L/4E \gg 1$ → **mainly sensitive to δm^2** (+ averaged Δm^2 oscillations)

	Initial flavors	Typical E	Typical L
Short-baseline (SBL) reactor neutrinos: CHOOZ, Double Chooz, RENO, Daya Bay…	$\bar{\nu}_e$	few MeV	O(1) km
Atmospheric neutrinos: MACRO, MINOS, (Super)-Kamiokande, IceCube…	$\overset{(-)}{\nu}_\mu$ $\overset{(-)}{\nu}_e$	> O(0.1) GeV	$O(10^{1\text{-}4})$ km
Long-baseline (LBL) accelerator neutrinos: K2K, OPERA, T2K, NOvA…	$\overset{(-)}{\nu}_\mu$ mostly	O(1) GeV	$O(10^{2\text{-}3})$ km

$\delta m^2 L/4E \ll 1$ → mainly sensitive to Δm^2

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$\delta m^2 L/4E \ll 1$ → mainly sensitive to Δm^2

What about $A/\Delta m^2$ for these expt's? (matter effects)

SBL reactors:	negligible
LBL accelerators:	small
Atmospheric:	sizeable in principle but … ~decoupled from leading $\nu_\mu \to \nu_\tau$

In first approximation, set both $\delta m^2 = 0$ and $A = 0$ →

Exercise: Dominant Δm^2 oscillations in vacuum

*For experiments with dominant **Δm^2 oscillations,** the probabilities are:*

$$P_{\alpha\alpha} = 1 - 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2)\sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$$P_{\alpha\beta} = 4|U_{\alpha 3}|^2|U_{\beta 3}|^2\sin^2\left(\frac{\Delta m^2 x}{4E}\right) \ , \ \alpha \neq \beta$$

Note that, in this approximation, the probabilities do not depend on:

- CP violation phase δ
- neutrino/antineutrino distinction
- mass ordering = sign(Δm^2)
- mixing angle θ_{12}

This class of experiments ~probes Δm^2 and the mixing matrix elements $|U_{\alpha 3}|^2$ of ν_3 with $\nu_\alpha=(\nu_e, \nu_\mu, \nu_\tau)$

$$\begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{bmatrix}$$

Relevant phenomenological channels:

$$P(\nu_e \to \nu_e) \simeq 1 - \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m^2 L}{4E} \right)$$

$$P(\nu_\mu \to \nu_e) \simeq s_{23}^2 \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m^2 L}{4E} \right)$$

$$P(\nu_\mu \to \nu_\mu) \simeq 1 - 4c_{13}^2 s_{23}^2 (1 - c_{13}^2 s_{23}^2) \sin^2 \left(\frac{\Delta m^2 L}{4E} \right)$$

$$P(\nu_\mu \to \nu_\tau) \simeq c_{13}^4 \sin^2 2\theta_{23} \left(\frac{\Delta m^2 L}{4E} \right)$$

Short-baseline reactor experiments

$$P(\nu_e \to \nu_e) \simeq 1 - \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m^2 L}{4E}\right)$$

$$P(\nu_\mu \to \nu_e) \simeq s_{23}^2 \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m^2 L}{4E}\right)$$

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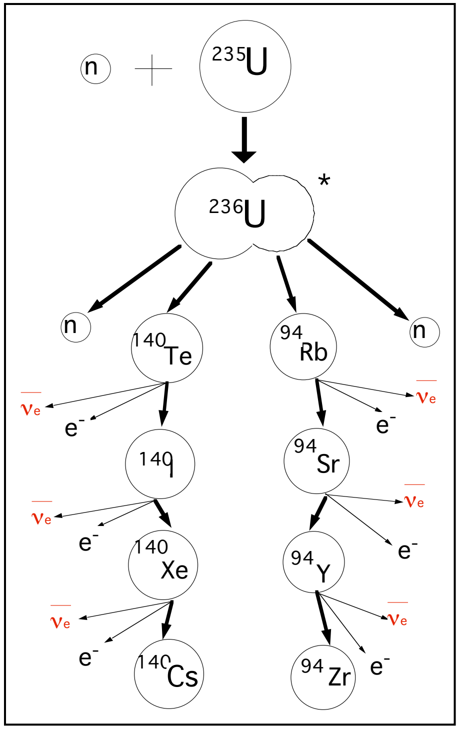
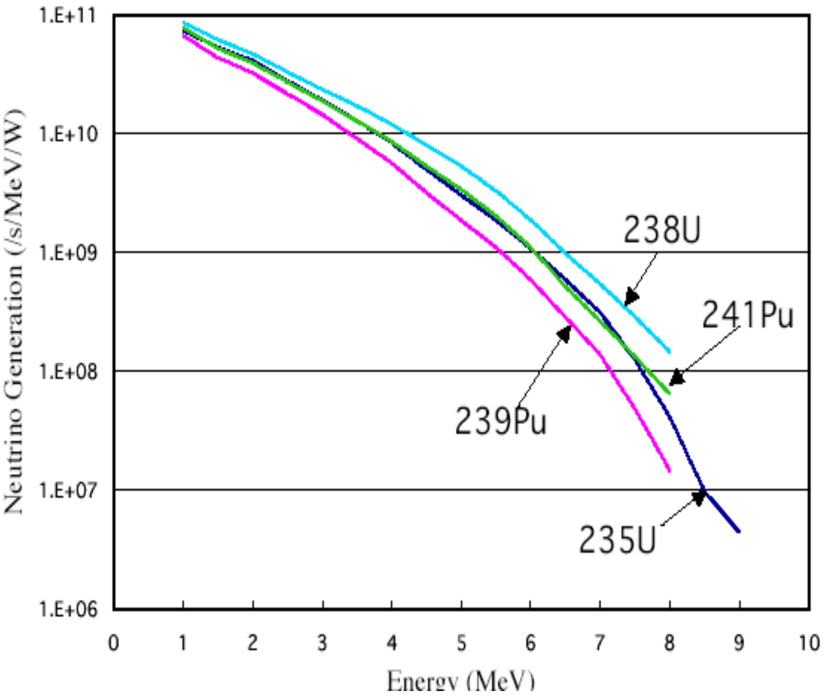
SBL reactor expt's: testing anti-ν_e disappearance

Production: **Intense sources of anti-ν_e (~6×10^{20}/s/reactor)**

Typically, 6 neutron decays to reach stable matter from fission:



~200 MeV per fission / 6 decays: Typical available neutrino energy E~ few MeV

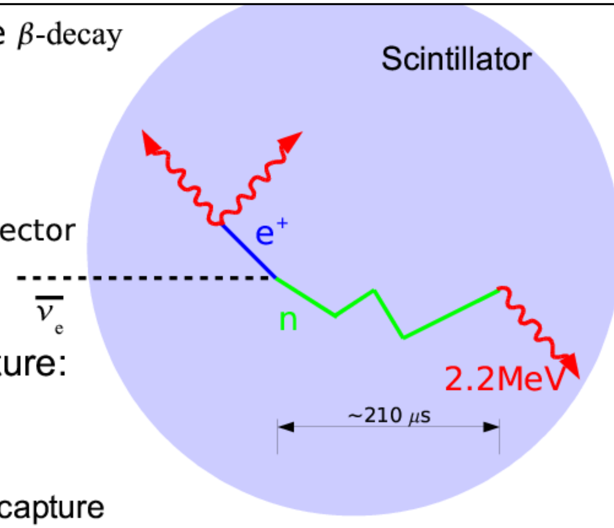


Detection

Reaction Process: inverse β-decay

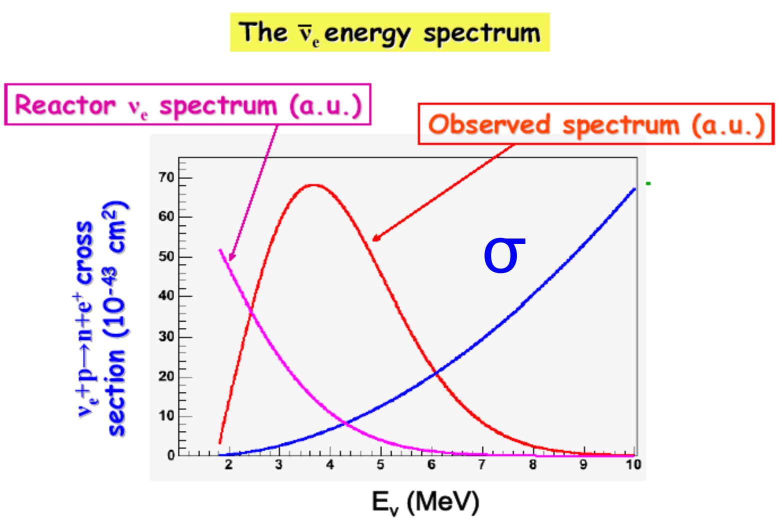
$$\overline{\nu}_e + p \rightarrow e^+ + n$$
$$n + p \rightarrow d + \gamma$$

Scintillator is target and detector

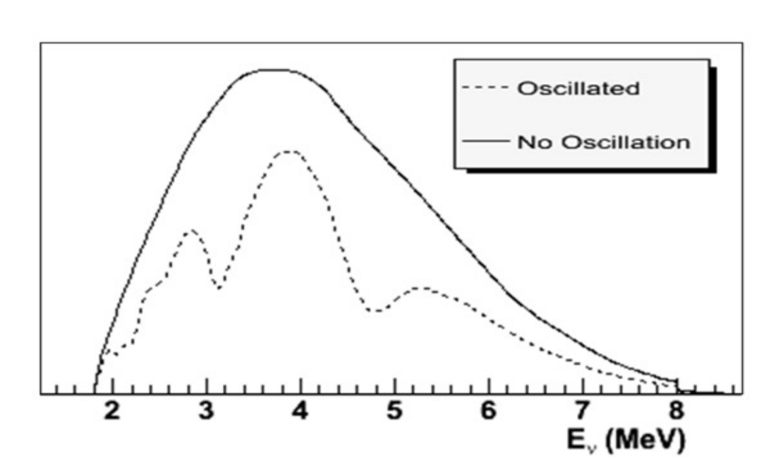


- Distinct two-step signature:
  - prompt event: positron
    $$E_\nu \approx E_{e^+} + 0.8\ MeV$$
  - delayed event: neutron capture after ~210μs
    - 2.2 MeV gamma

Delayed coincidence: good background rejection

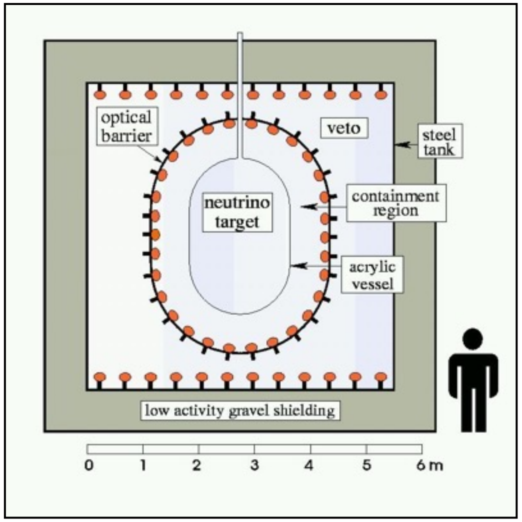


Oscillations → Spectral distortions



The short-baseline reactor experiment CHOOZ (1998+)

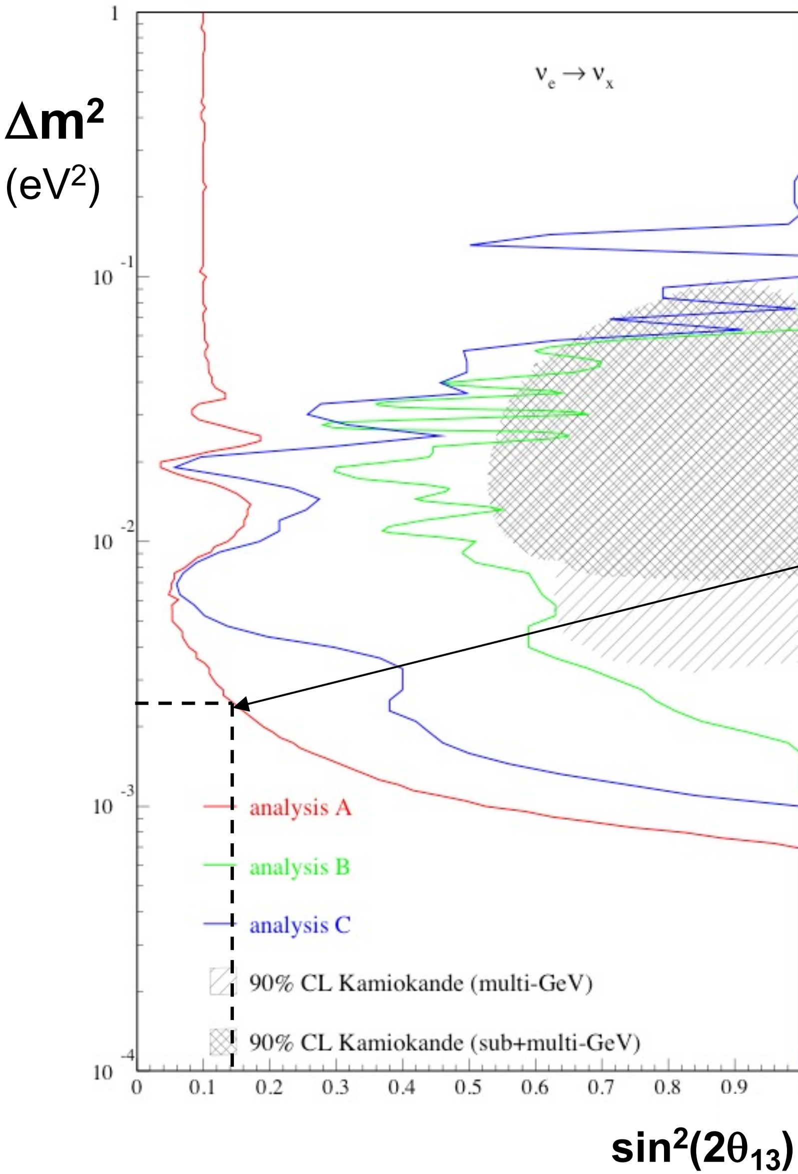
L ~1 km →



No spectral distortion found within uncertainties.
Probably (one of) the most cited negative results ever!

First data: Phys. Lett. B 466, 415 (1999) >2000 cites
Final data: Eur. Phys. J. C 27, 331 (2003) >1500 cites

CHOOZ exclusion plot



Interpretation

In our approximation:

$P_{ee} = 1 - \sin^2(2\theta_{13}) \sin^2(\Delta m^2 L/4E_\nu)$

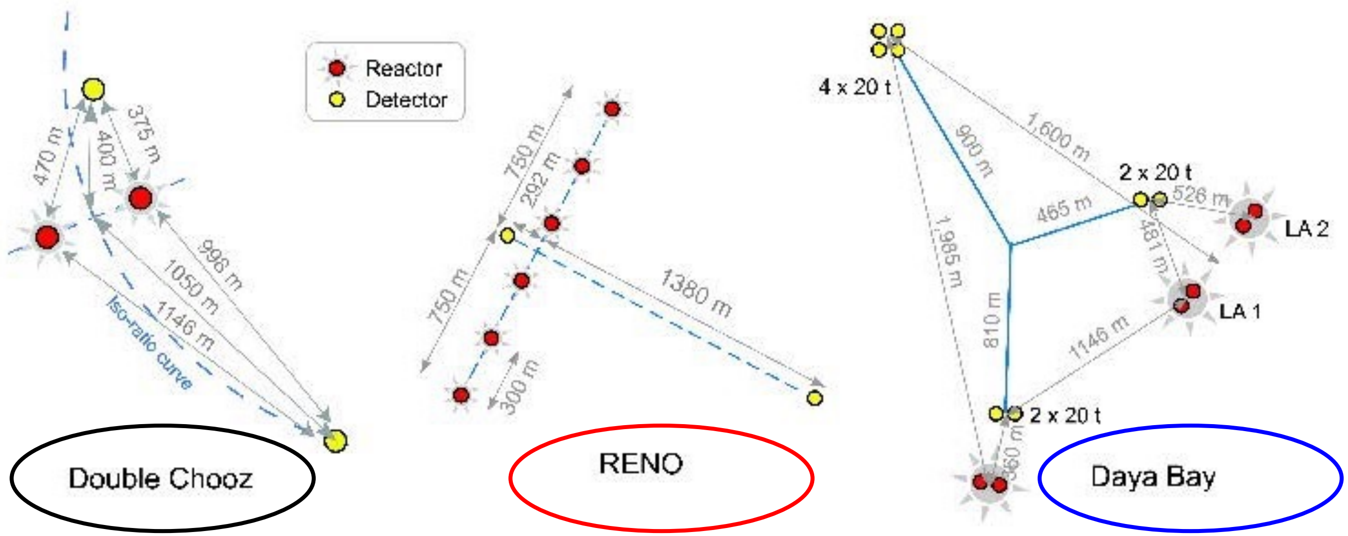
For any value of Δm^2 in the range allowed by atmospheric ν data (next slides), get stringent upper bound on θ_{13}

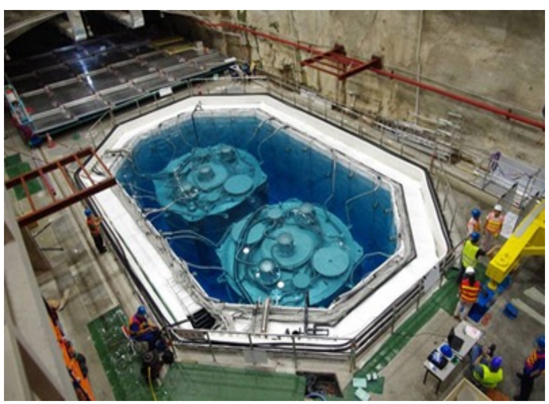
$\sin^2 2\theta_{13} < O(10\%)$
(depending on Δm^2)

[... At that time, nobody could know that ϑ_{13} was just behind the corner: less than a factor of two in sensitivity!]

Need to use a second (close) detector to reduce syst's by far/near ratio →

SBL reactor expts with near & far detectors (ND & FD)

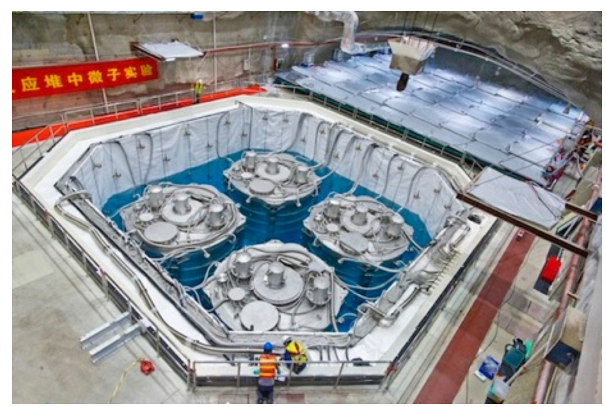




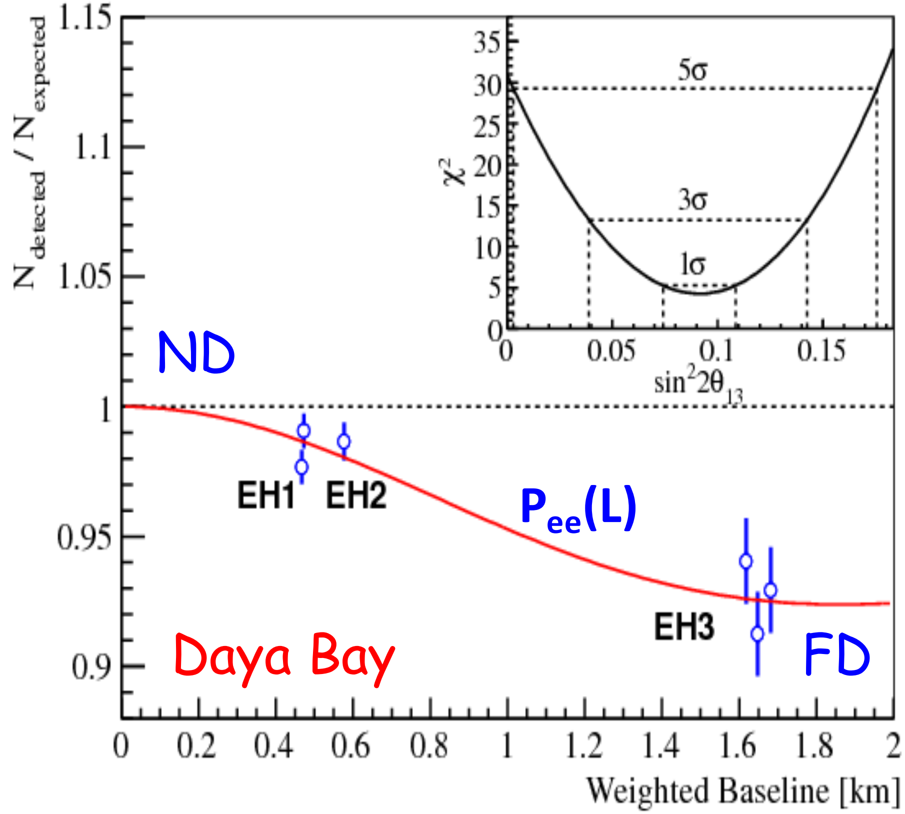
E.g, for Daya Bay:

← ND

FD →



2012: discovery of $\theta_{13}>0$! ($\sin^2\theta_{13}$~0.022 at ~fixed Δm^2)

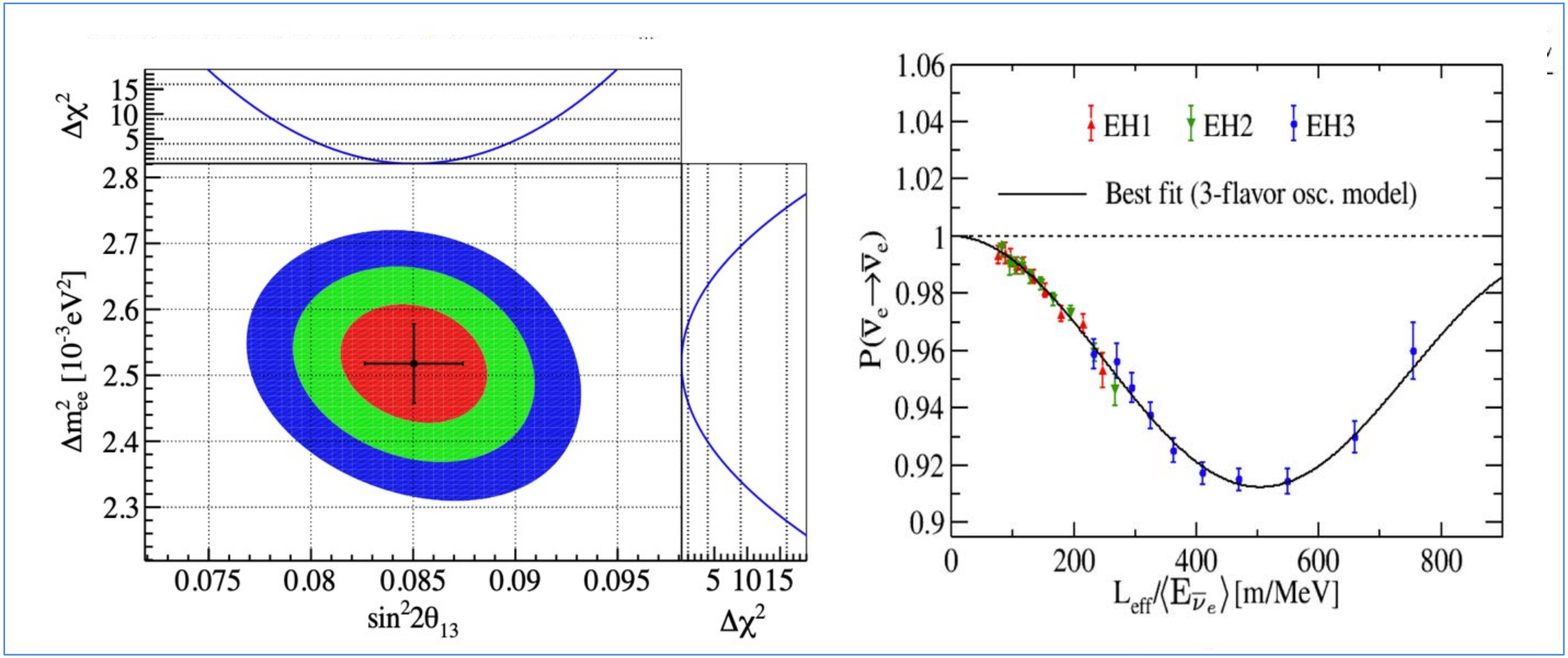


Daya Bay (& RENO): disappearance at FD w.r.t. ~unoscillated at ND

Double Chooz results with FD were also consistent with Daya Bay & RENO.

Interestingly, approximate value of θ_{13} was previously hinted from other data: weaker signals were also coming from other experiments < 2012 (see later).

Latest Daya Bay results (PRL 2023)



↑ **Precise measurement of both Δm^2 ("mass") and θ_{13} ("mixing") oscill. parameters in $\nu_e \rightarrow \nu_e$ channel**

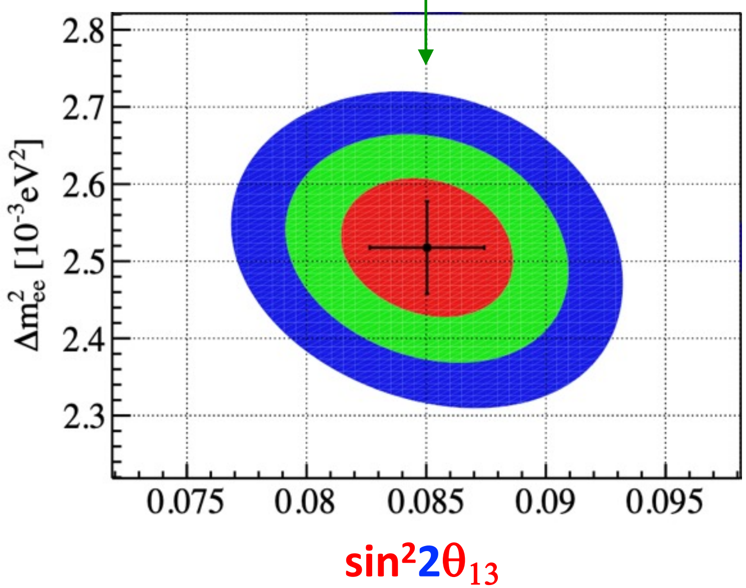
↑ **½ osc. cycle in L/E!**

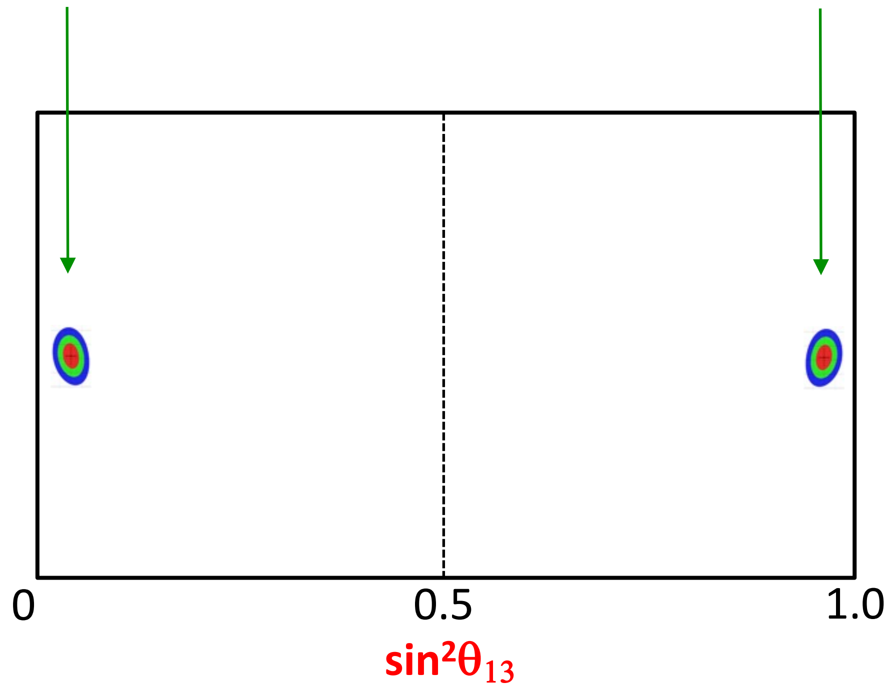
Position of oscillation dip in L/E determines Δm^2, while depth fixes θ_{13}

But… barring prejudices in favor of "small" mixing:

How can one break octant symmetry and tell

$\sin^2 2\theta_{13} = 0.085$ ⇨ $\sin^2\theta_{13} = 0.022$ **from** $\sin^2\theta_{13} = 0.978$?





The allowed octant for θ_{13} (the first) was already clear in the same year of the CHOOZ results (1998), thanks to the discovery of atmospheric neutrino oscillations →

Atmospheric neutrino experiments

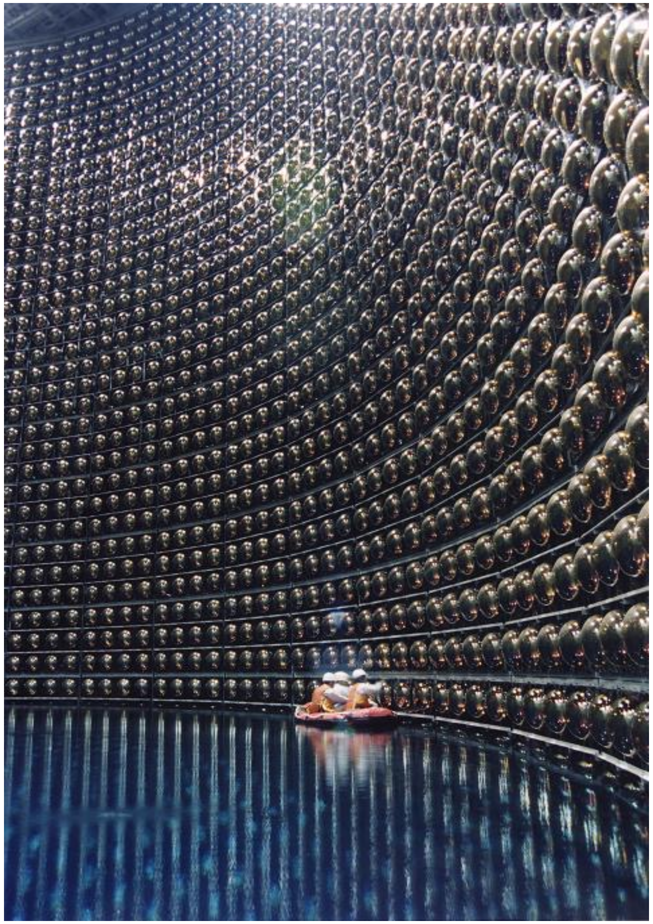
$$P(\nu_e \to \nu_e) \simeq 1 - \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

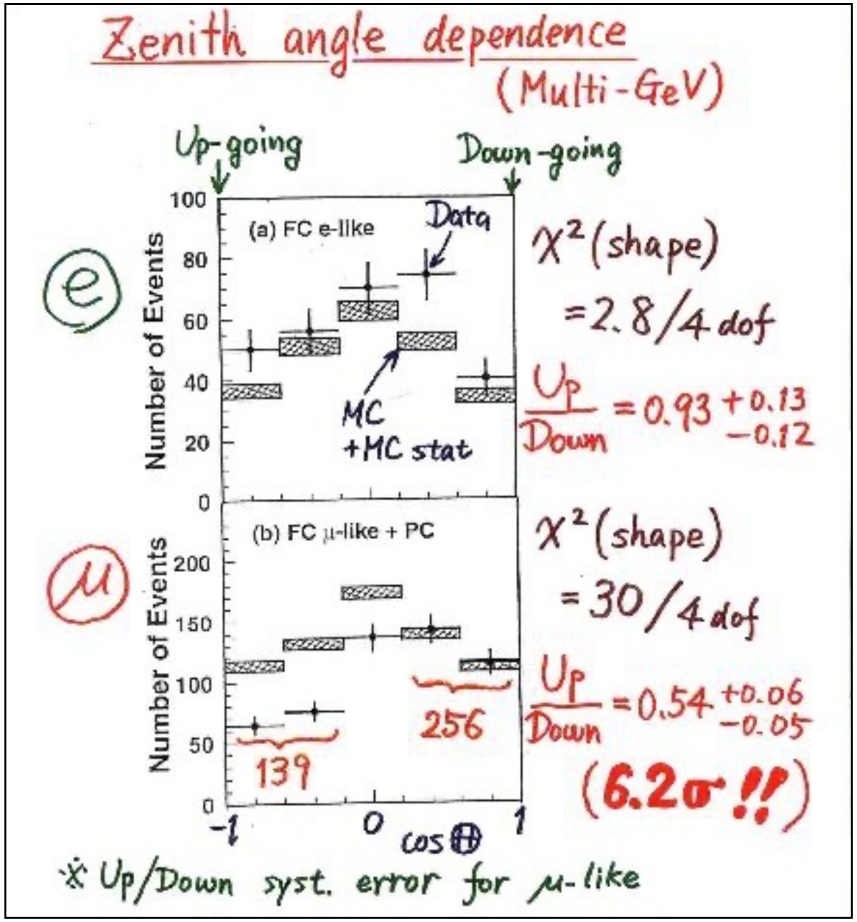
$$P(\nu_\mu \to \nu_e) \simeq s_{23}^2 \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$$P(\nu_\mu \to \nu_\mu) \simeq 1 - 4c_{13}^2 s_{23}^2 (1 - c_{13}^2 s_{23}^2) \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$$P(\nu_\mu \to \nu_\tau) \simeq c_{13}^4 \sin^2 2\theta_{23} \left(\frac{\Delta m^2 L}{4E}\right)$$

Atmospheric neutrinos:
The 1998 Super-Kamiokande breakthrough

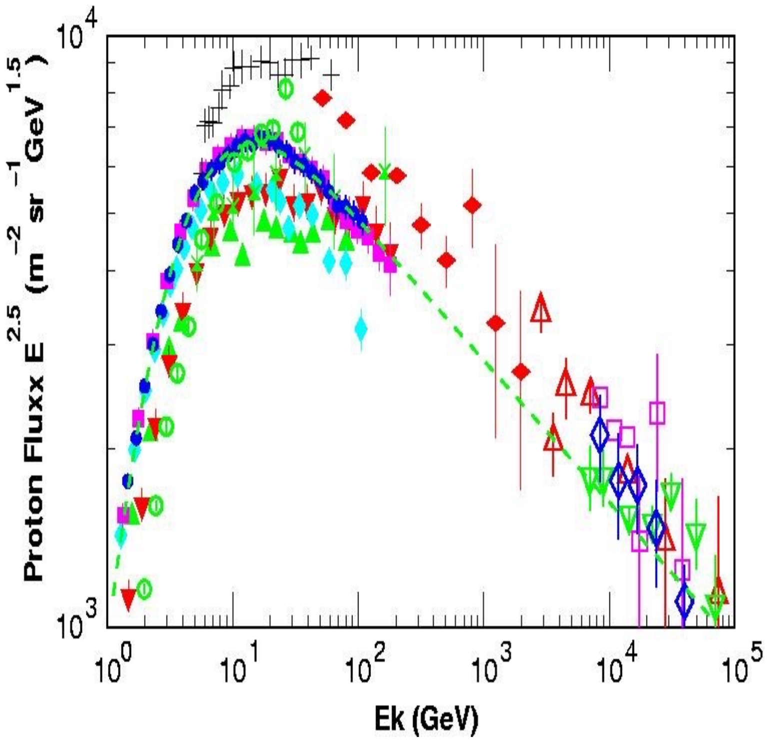


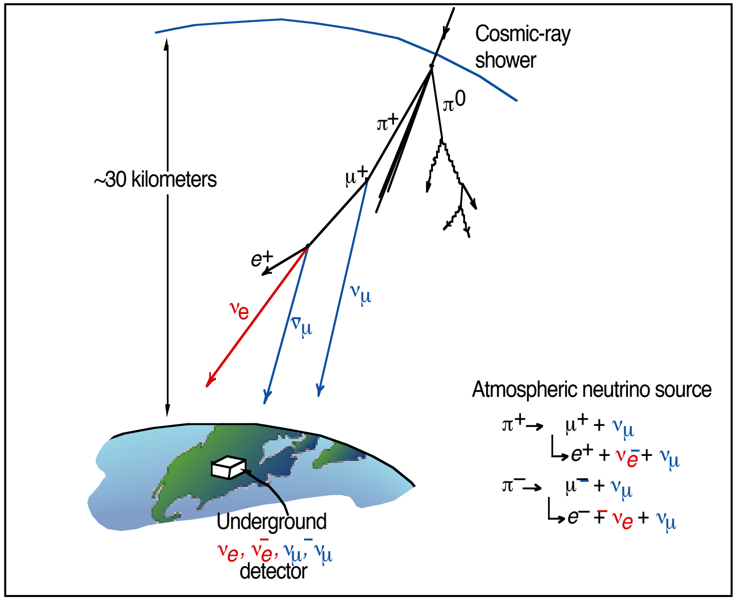


(T. Kajita at Neutrino' 98, Takayama)

Production

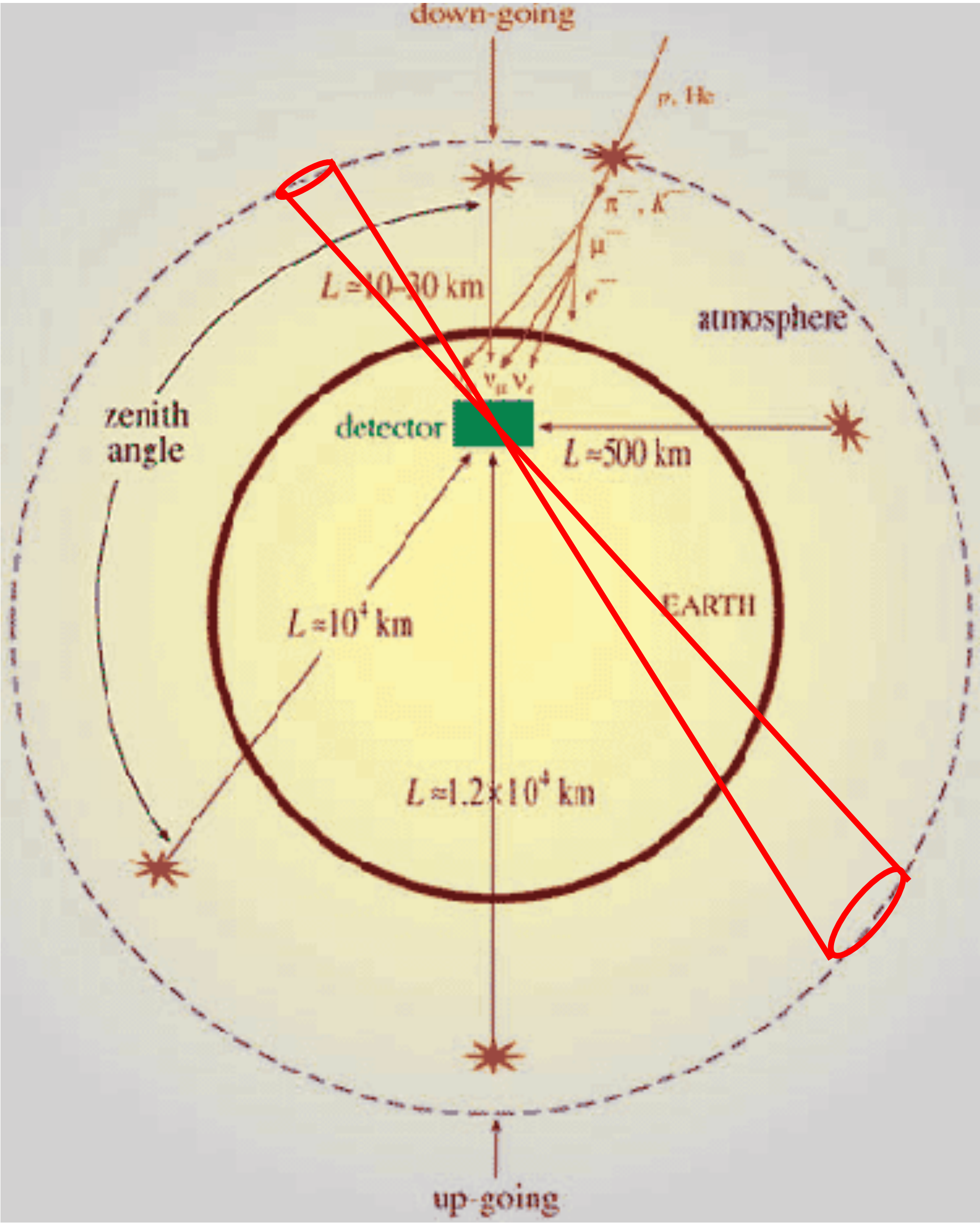
Cosmic rays hitting the atmosphere can generate secondary (anti)neutrinos with electron and muon flavor via meson decays.





Primary flux affected by large normalization uncertainties…

… but (anti)neutrino **flavor ratio (μ/e ~ 2)** robust within few %



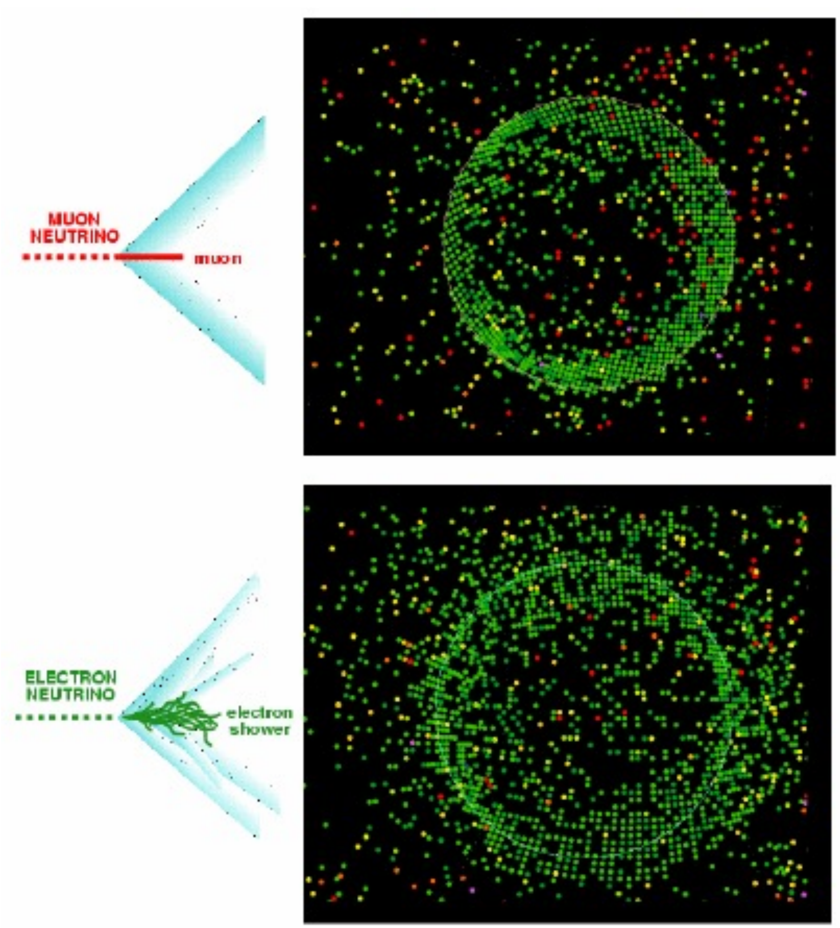
Moreover: same ν flux from opposite solid angles (up-down symmetry)

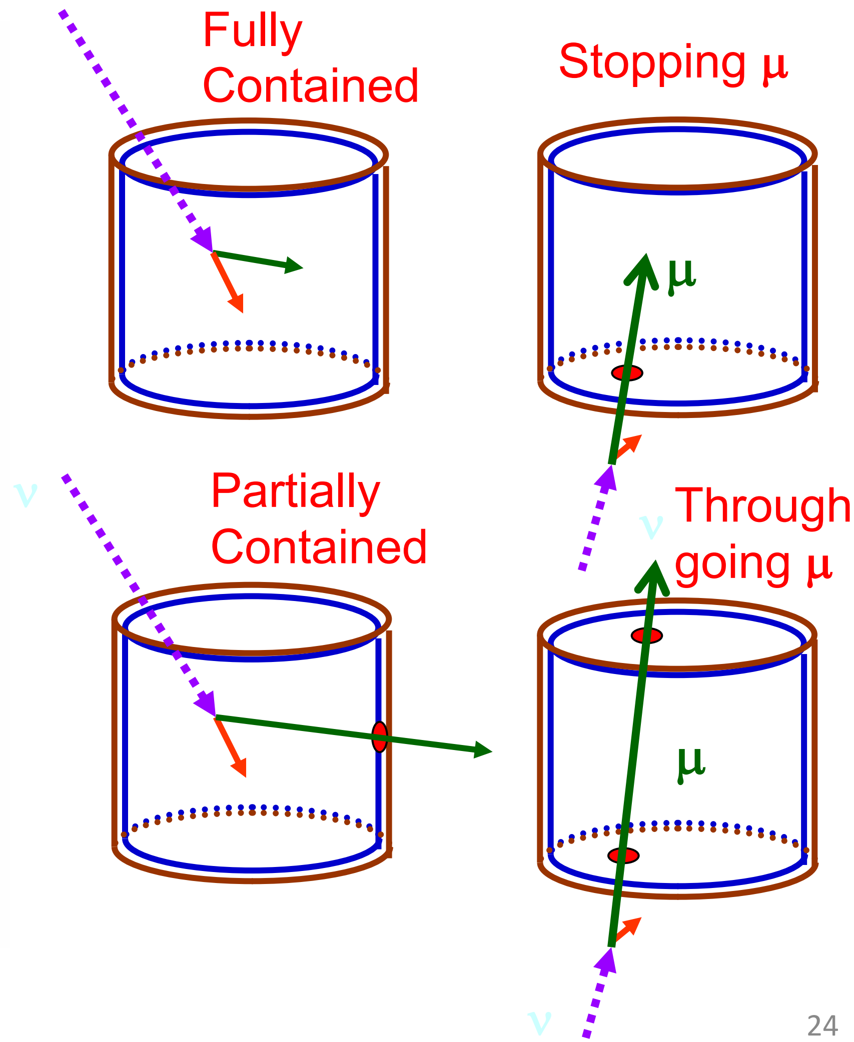
[Flux dilution (~1/r²) is compensated by larger production surface (~r²)]

Should be reflected in symmetry of event zenith spectra, if energy & angle can be reconstructed well enough

Detection in SK

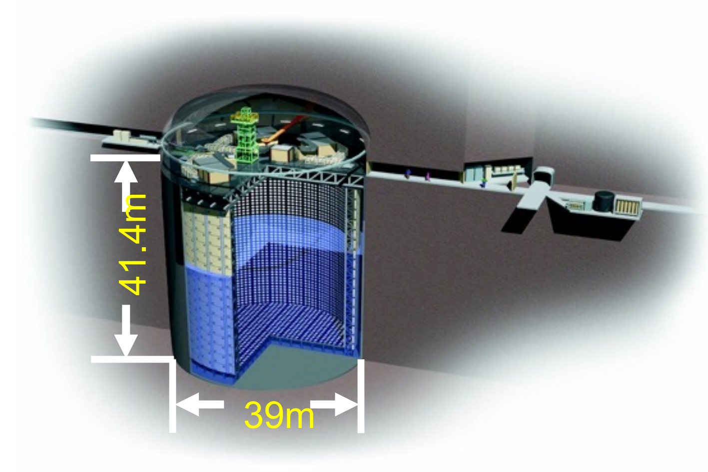
Parent neutrinos detected via CC interactions in the target (water).
Final-state μ and e distinguished by ≠ Cherenkov ring sharpness.
(But: no charge discrimination, no τ event reconstruction). Topologies:

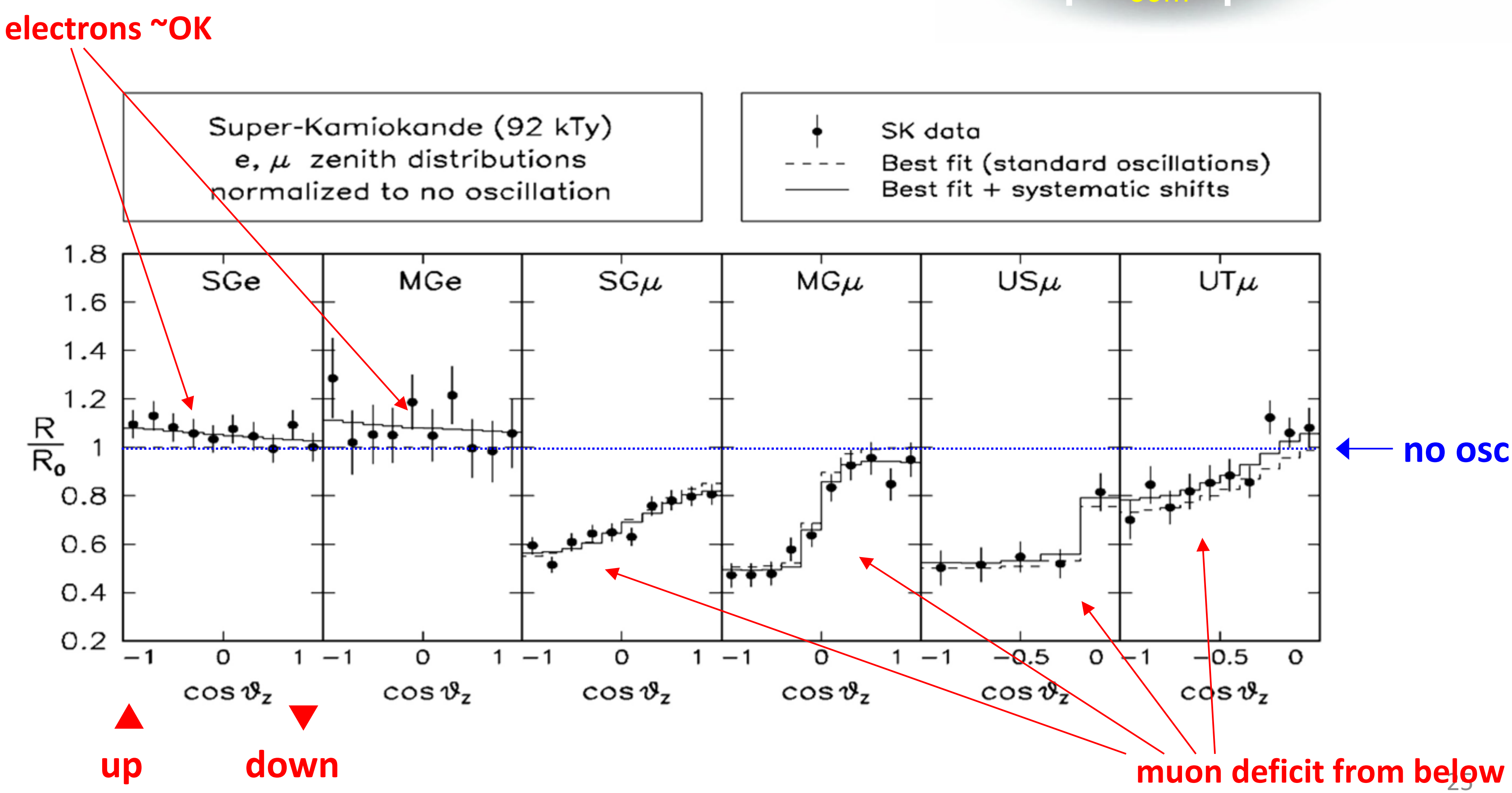




Early results - SK zenith distributions

SGe	Sub-GeV electrons
MGe	Multi-GeV electrons
SGμ	Sub-GeV muons
MGμ	Multi-GeV muons
USμ	Upward Stopping muons
UTμ	Upward Through-going muons





Observations over several decades in L/E:

ν_μ induced events: large disappearance from below

ν_e induced events: almost as expected

Interpretation via oscillations: $P_{\mu\mu} < 1$ and $P_{\mu e} \sim 0$

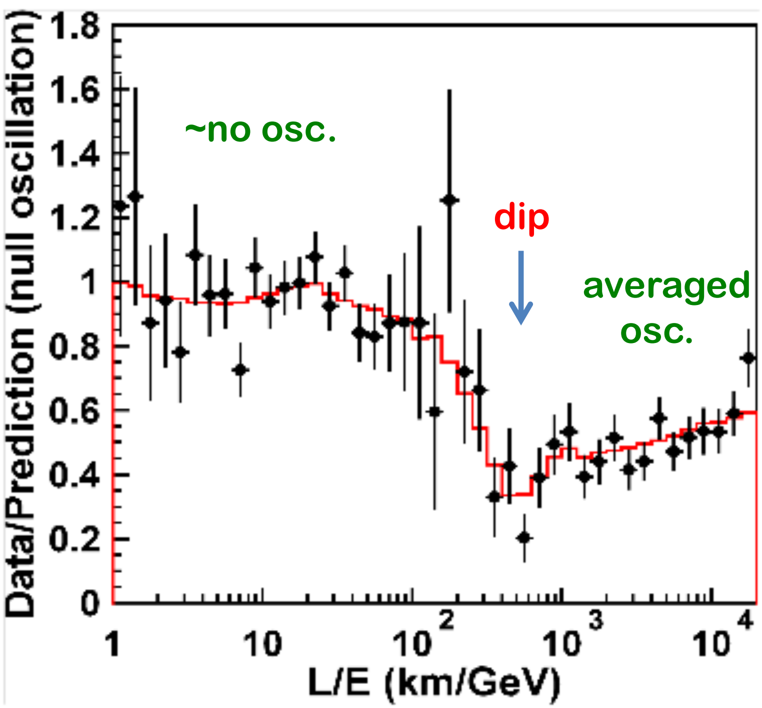
$$
\begin{aligned}
P(\nu_e \to \nu_e) &\simeq 1 - \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \\
P(\nu_\mu \to \nu_e) &\simeq s_{23}^2 \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \\
P(\nu_\mu \to \nu_\mu) &\simeq 1 - 4c_{13}^2 s_{23}^2 (1 - c_{13}^2 s_{23}^2) \sin^2\left(\frac{\Delta m^2 L}{4E}\right) \\
P(\nu_\mu \to \nu_\tau) &\simeq c_{13}^4 \sin^2 2\theta_{23} \left(\frac{\Delta m^2 L}{4E}\right)
\end{aligned}
$$

→ Need θ_{13} **zero or small** (consistent with CHOOZ in 1st octant of θ_{13})
→ Need θ_{23} **sizeable**, around **$\sin^2\theta_{23} \sim 0.5$**
→ Dominant $\nu_\mu \to \nu_\tau$ **oscillation** channel with **$\Delta m^2 \sim 2.5 \times 10^{-3}$ eV^2**
→ Small role of ν_e and of matter effects (if any)

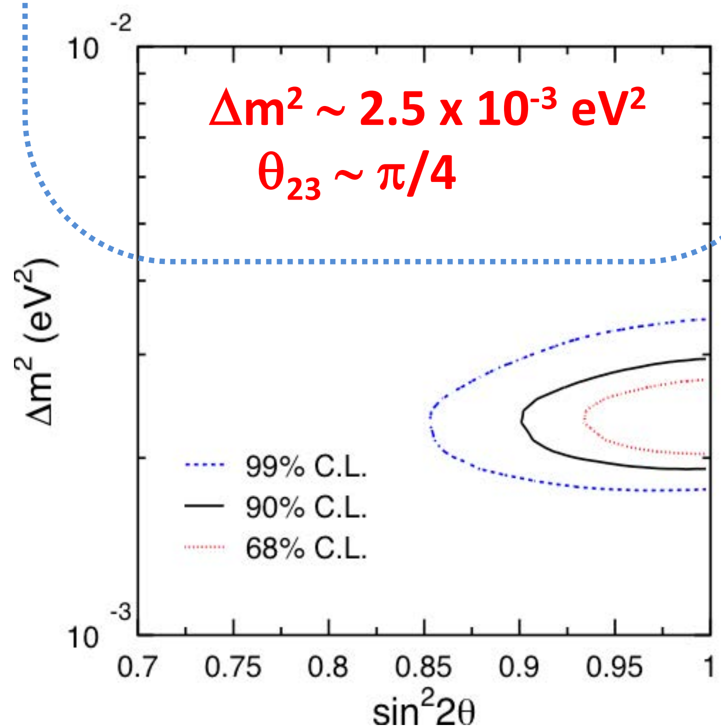
Results were consistent with other atmospheric experiments using different techniques (MACRO, Soudan2) but with lower statistics

+ Dedicated L/E analysis in SK to "see" half-period of oscillations

1st oscillation dip still visible despite large L & E smearing



Strong constraints on the parameters (Δm^2, θ_{23})



In recent years: also consistent with high-stat. IceCube atm. Data

+ statistical evidence for ν_τ appearance

+ sensitivity to subleading effects

Considering all oscillation channels, current atmospheric ν data are consistent with small **nonzero θ_{13}** in this approximation…

$$P(\nu_e \to \nu_e) \simeq 1 - \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$$P(\nu_\mu \to \nu_e) \simeq s_{23}^2 \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$$P(\nu_\mu \to \nu_\mu) \simeq 1 - 4c_{13}^2 s_{23}^2 (1 - c_{13}^2 s_{23}^2) \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$$P(\nu_\mu \to \nu_\tau) \simeq c_{13}^4 \sin^2 2\theta_{23} \left(\frac{\Delta m^2 L}{4E}\right)$$

… and actually have some sensitivivity to 3ν + matter effects beyond this approximation, testing

δ, NO/IO *(Lecture IV)*

Long-baseline accelerator experiments

"Reproducing atmospheric ν_μ physics" in controlled conditions

$$P(\nu_e \to \nu_e) \simeq 1 - \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$$P(\nu_\mu \to \nu_e) \simeq s_{23}^2 \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

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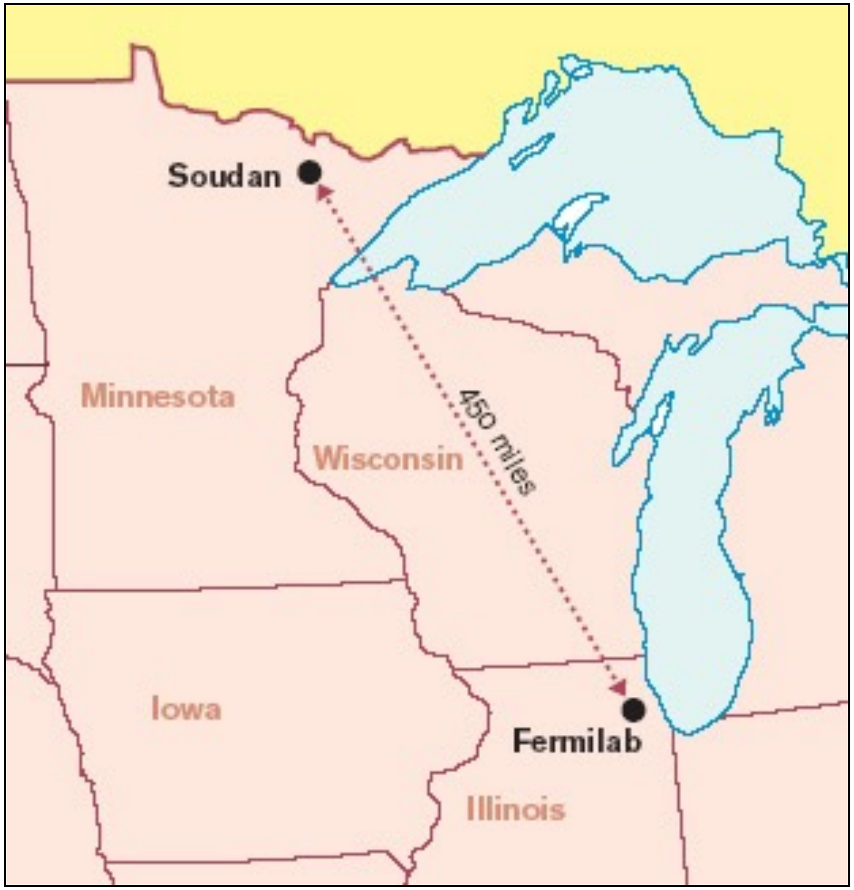
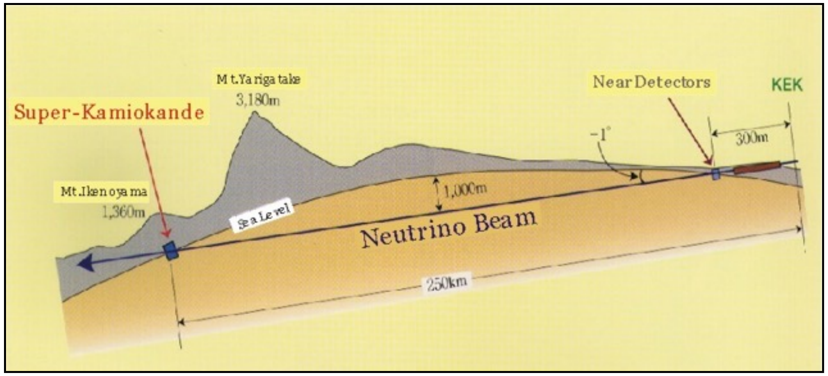
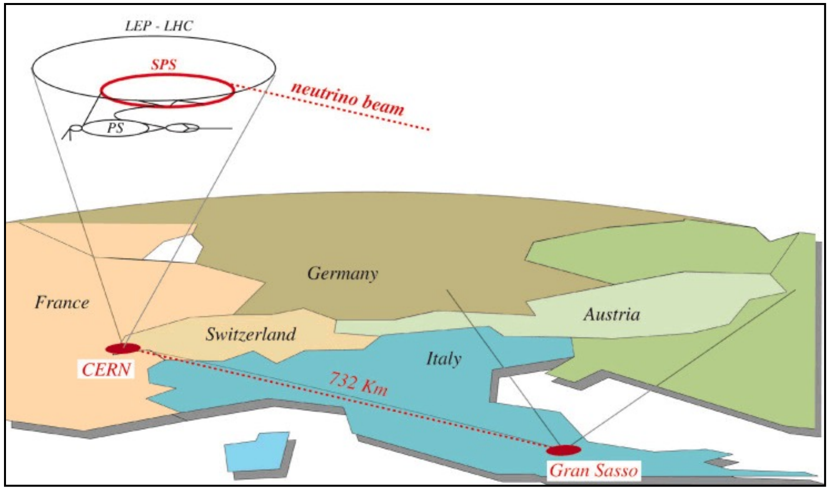
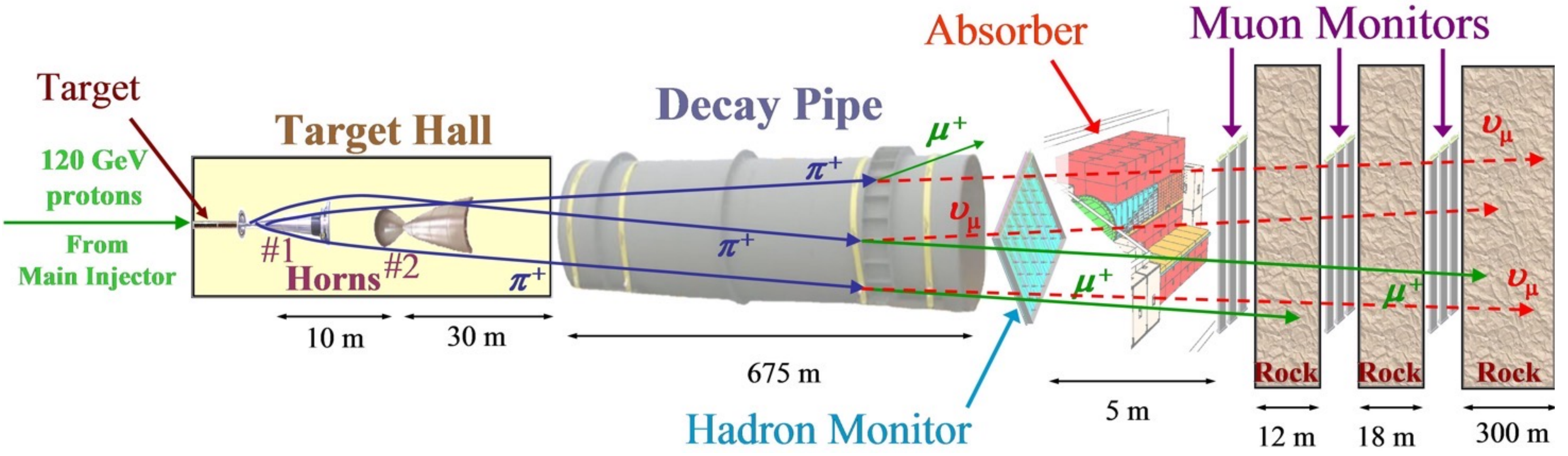
Long-baseline neutrino experiments
K2K, T2K (JP) , MINOS, NOvA (US), OPERA (CERN)





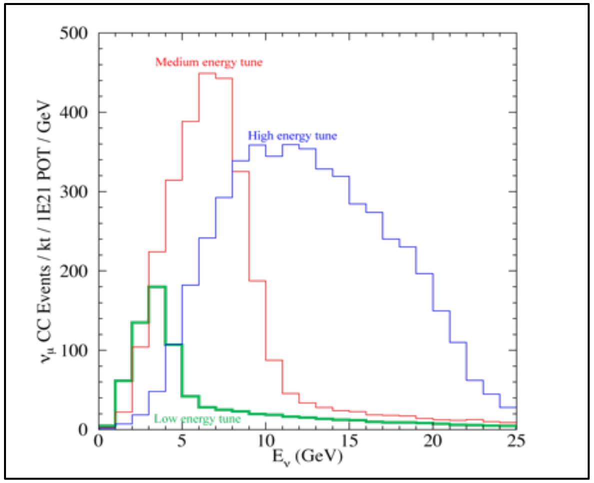


Production (e.g., MINOS)



π decay: ν energy is only function of νπ angle and π energy

Spectra:

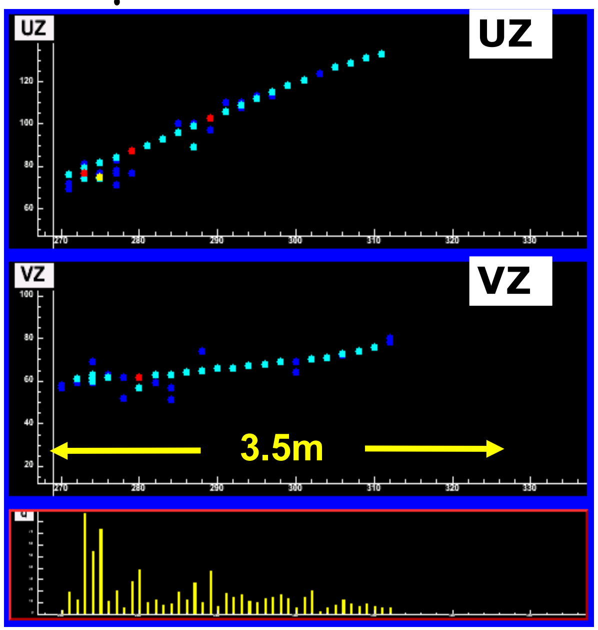


(Far) Detection

K2K, **T2K**: Cherenkov technique in SK

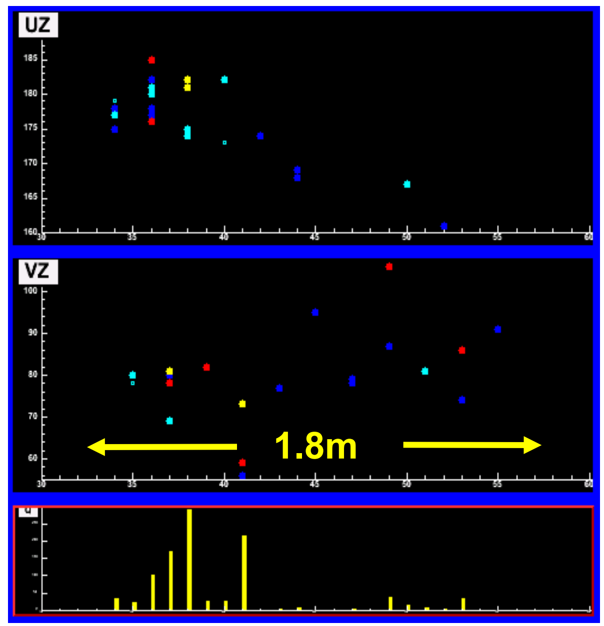
MINOS, **NOvA**: Scintillator detectors

ν_μ **CC Event**



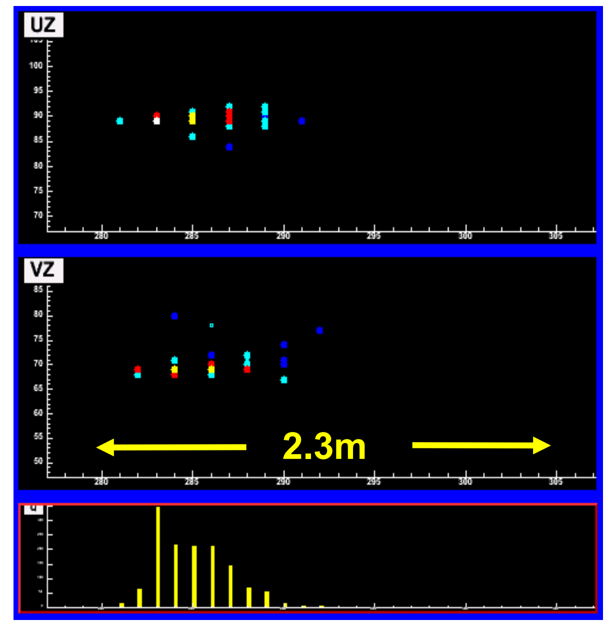
- Long muon track + hadronic activity at vertex

NC Event



- Short showering event, often diffuse

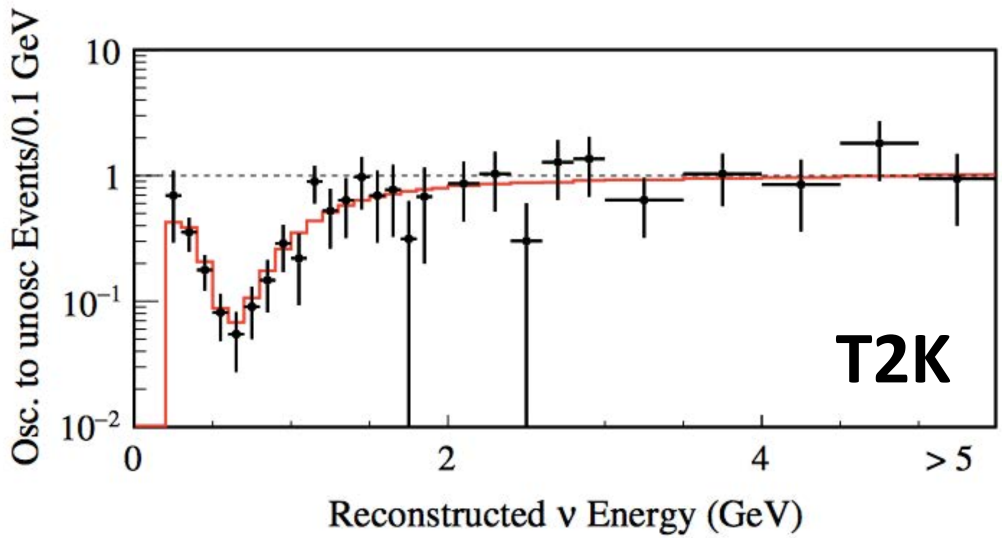
ν_e **CC Event**

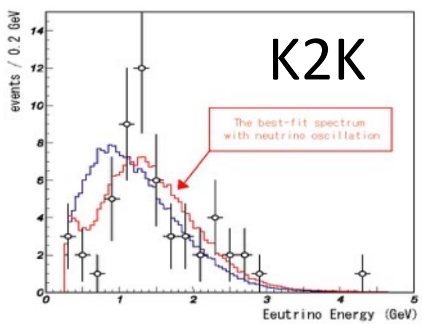


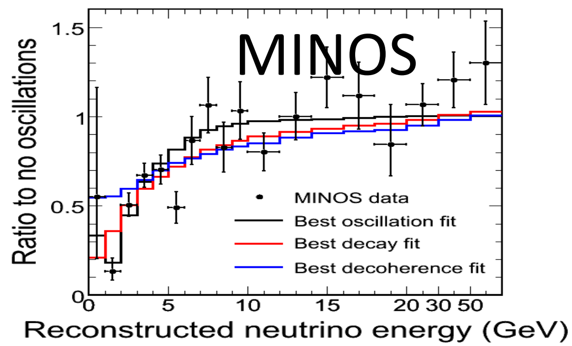
- Short event with typical EM shower profile

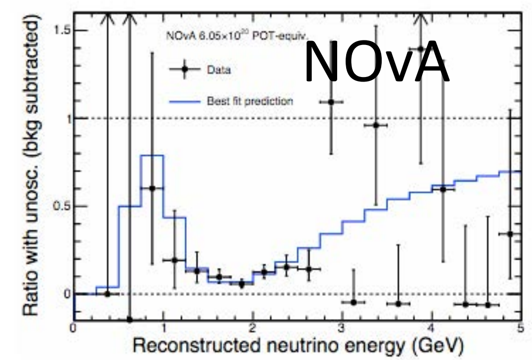
K2K, MINOS, T2K, NOvA supplemented by near detectors to constrain neutrino cross sections and to measure $P_{\mu\mu}$

Early oscillation results in muon neutrino disappearance mode, $P_{\mu\mu}$









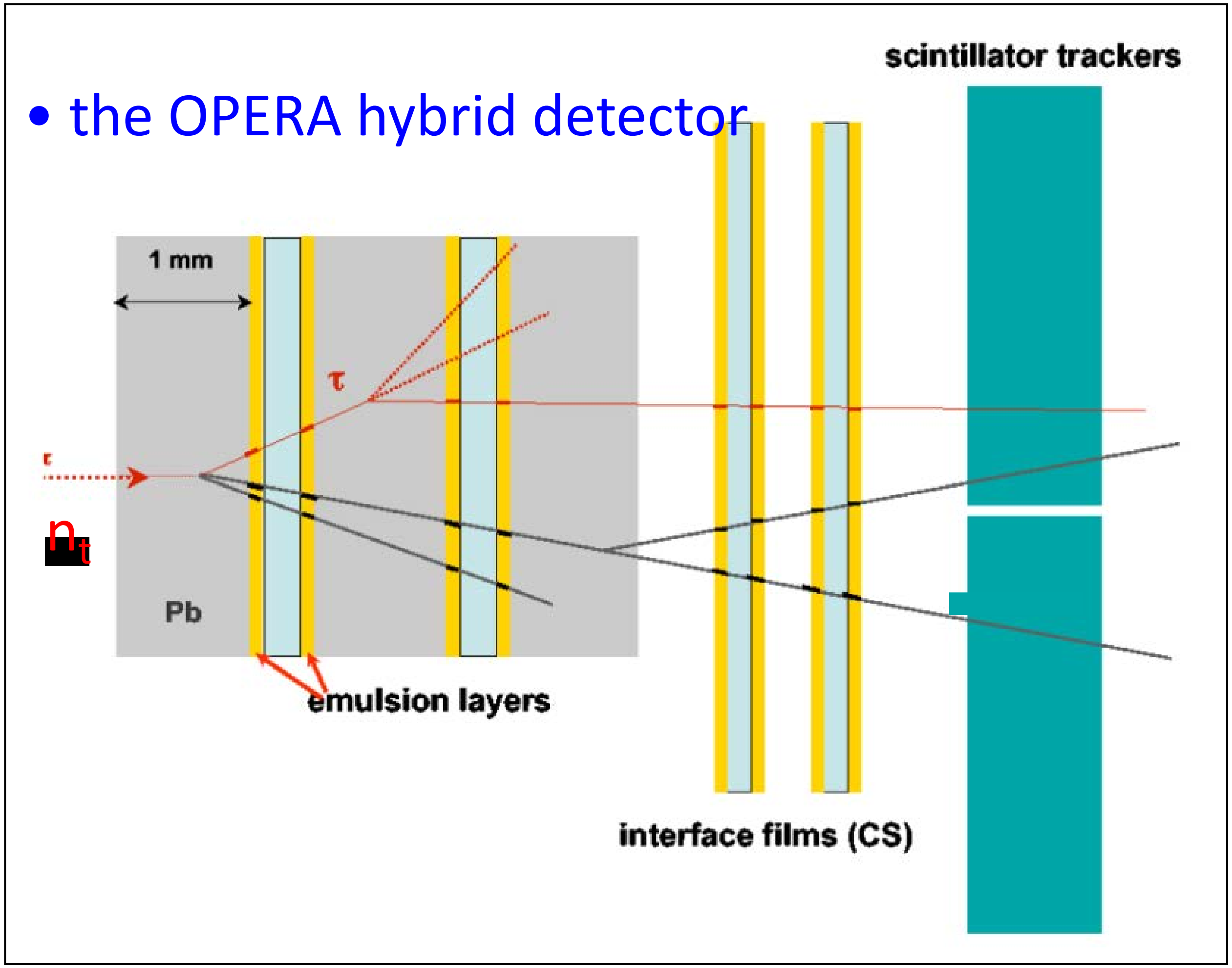
1st oscillation dip observed in energy spectrum

(equivalent to L/E spectrum since L is fixed).

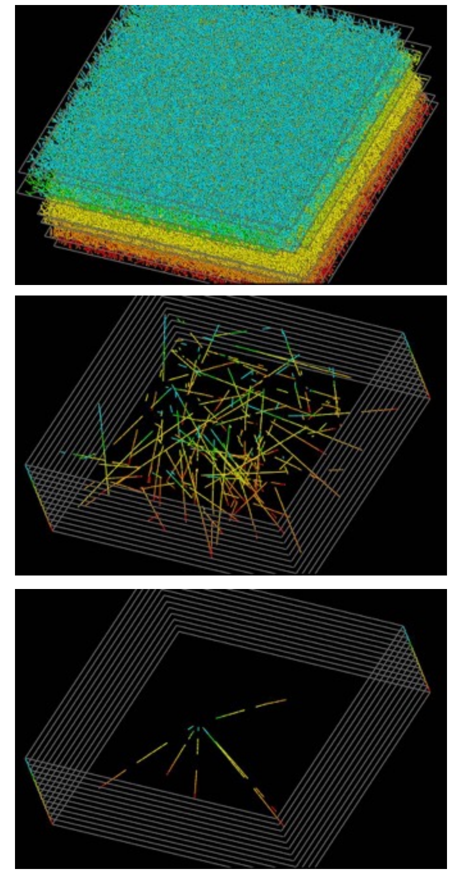
[Exotic explanations without dip (decay, decoherence) excluded]

Testing dominant $\nu_\mu \rightarrow \nu_\tau$ oscillations via **direct τ appearance**: **OPERA**

- the OPERA hybrid detector



Finding needles in a haystack…



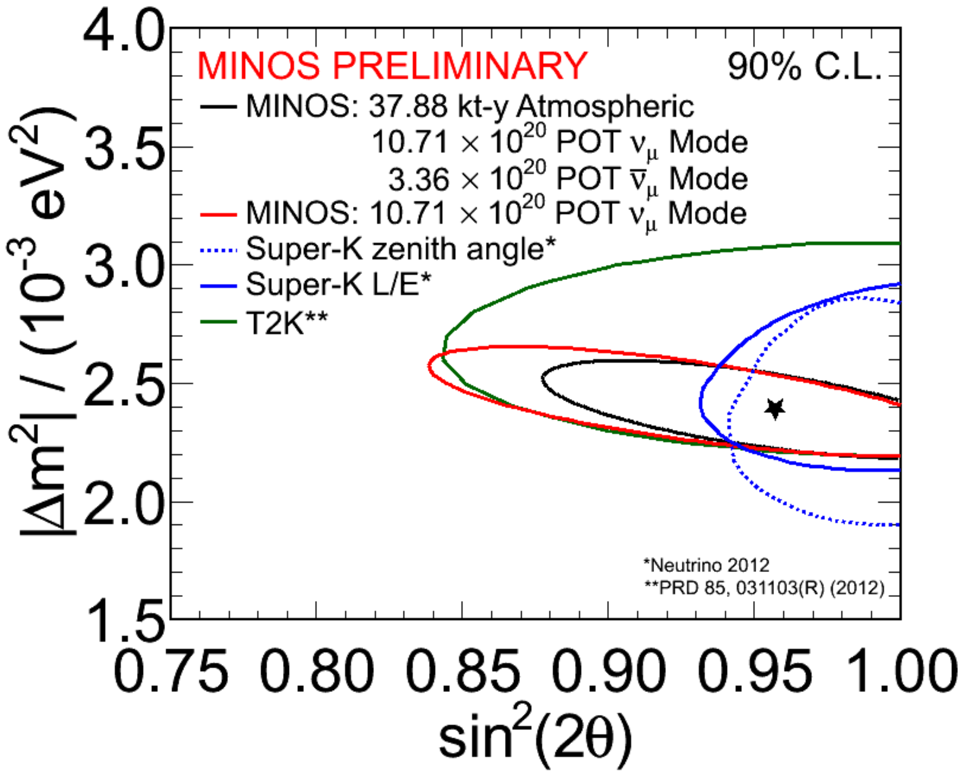
10 "τ needles" found! (consistent with expected signal)

Interpretation of LBL disappearance data

Dip position and depth determine Δm^2 and θ_{23}

Osc. parameters consistent among atm and LBL experiments

Old-fashioned way to present constraints in terms of $2\theta_{23}$:

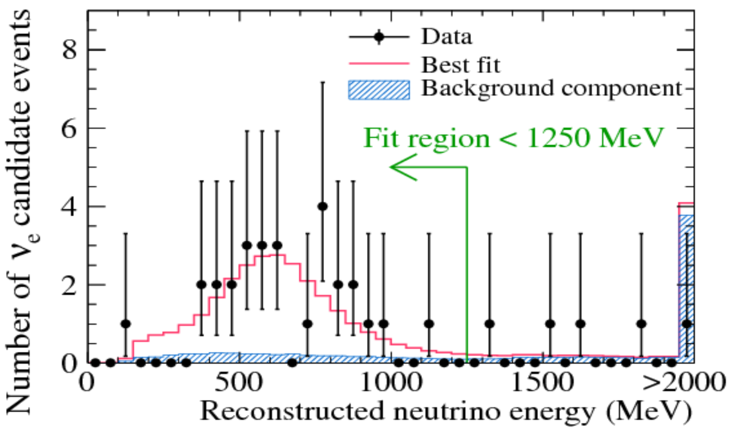


The format of such "2ν" plots is, however, obsolete...

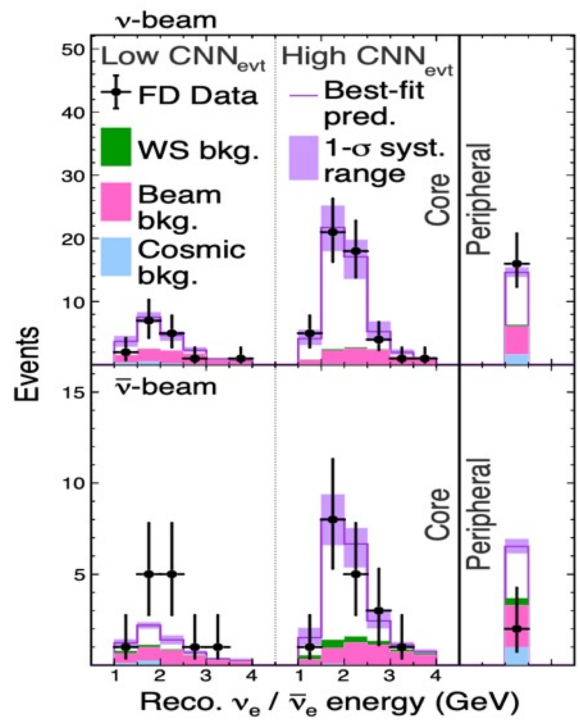
In particular, we know that $\theta_{13}>0$ from SBL reactors:

What about $\mu \rightarrow e$ flavor appearance in LBL experiments?

→ Observed in T2K & NOvA; e-like event rate consistent with reactors' θ_{13} !



e.g., T2K



e.g., NOvA

(Both neutrino and antineutrino channels tested)

In particular, we know that $\theta_{13}>0$ from SBL reactors:

What about $\mu \rightarrow e$ flavor appearance in LBL experiments?

→ Observed in T2K & NOvA; e-like event rate consistent with reactors' θ_{13} !

For $\theta_{13}>0$, relevant appear./disapp. probabilities are θ_{23}–octant asymmetric,

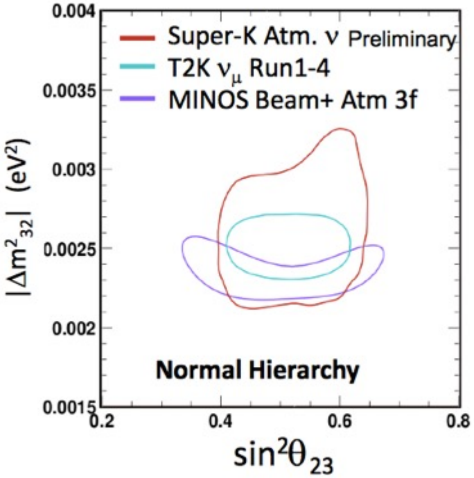
$$P(\nu_\mu \to \nu_e) \simeq s^2_{23} \sin^2 2\theta_{13} \sin^2 \left(\frac{\Delta m^2 L}{4E}\right)$$

$$P(\nu_\mu \to \nu_\mu) \simeq 1 - 4c^2_{13} s^2_{23} (1 - c^2_{13} s^2_{23}) \sin^2 \left(\frac{\Delta m^2 L}{4E}\right)$$

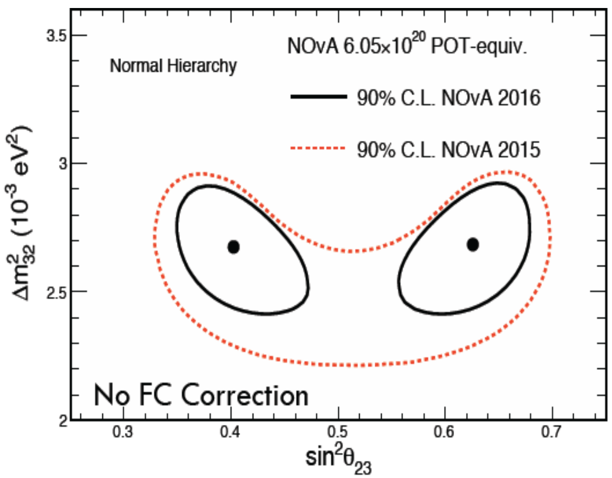
$\sin^2 2\theta_{23} \rightarrow \sin^2\theta_{23}$!

Examples of early (slightly asym.) ATM+LBL plots in terms of $\sin^2\theta_{23}$

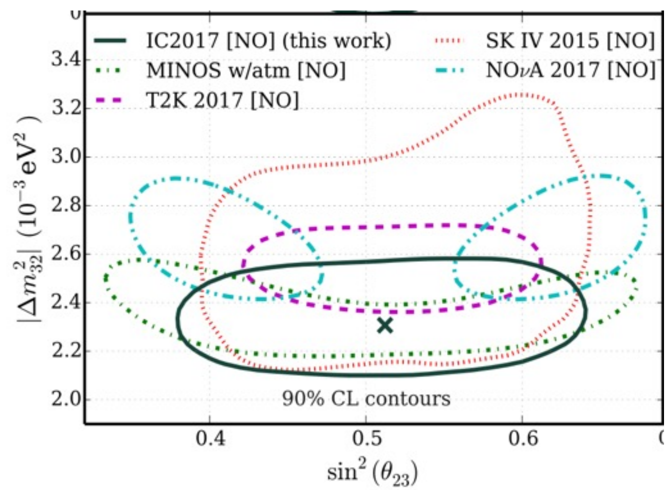
SK, T2K, MINOS 2015



NOvA 2016

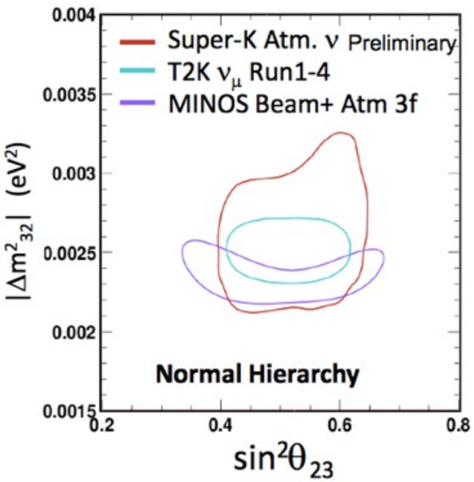


IceCube +all, 2017

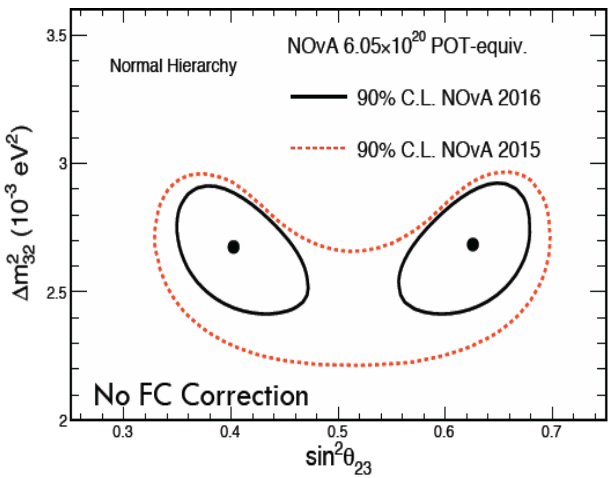


Examples of early (slightly asym.) ATM+LBL plots in terms of $\sin^2\theta_{23}$

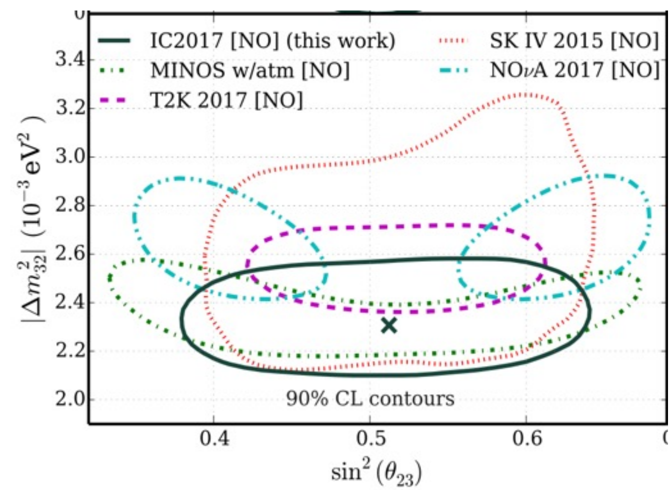
SK, T2K, MINOS 2015



NOvA 2016



IceCube +all, 2017



Not well established how close is θ_{23} to $\pi/4$ (maximal mixing).
If nonmaximal: first or second octant? → "octant ambiguity"

Current frontier in LBL/Atm oscillation searches: probe subleading effects related to θ_{23} octant, matter, NO/IO, δ_{CP}, δm^2, θ_{12}, ...

(Lecture IV)

Let us now discuss

Oscillation searches mainly sensitive to δm^2

	Initial flavors	Typical E	Typical L
Long-baseline reactor neutrinos: **KamLAND**	$\bar{\nu}_e$	few MeV	$O(10^2)$ km
Solar neutrinos: Chlorine, Gallium, Super-K, SNO, Borexino…	ν_e	O(1-10) MeV	1 a.u.

Δm^2 L/4E $\gg$ 1 → mainly sensitive to δm^2 (+ averaged Δm^2 oscillations)

$A/\delta m^2$ (matter effects) **negligible for KamLAND but not for solar neutrinos**

Exercise: Dominant δm^2 oscillations in vacuum with averaged Δm^2

For $\delta m^2 \neq 0$ and $\Delta m^2 = \infty$, the e-flavor survival probability in vacuum is:

$$P_{ee} \simeq \cos^4\theta_{13}\left[1 - \sin^2 2\theta_{12}\sin^2\left(\frac{\delta m^2 x}{4E}\right)\right] + \sin^4\theta_{13}$$

Applicable to the KamLAND experiment.

Note that this e-flavor survival probability does not depend on θ_{23}.

(**For E ~O(MeV), below μ and τ production via CC, one probes only ν_e disappearance and P_{ee}**)

Exercise: Dominant δm² oscillations in vacuum with averaged Δm²

For **$\delta m^2 \neq 0$** and **$\Delta m^2 = \infty$**, the e-flavor survival probability in vacuum is:

$$P_{ee} \simeq \cos^4\theta_{13}\left[1 - \sin^2 2\theta_{12}\sin^2\left(\frac{\delta m^2 x}{4E}\right)\right] + \sin^4\theta_{13}$$

Applicable to the KamLAND experiment.

Note that this e-flavor survival probability does not depend on θ_{23}.

Note that this probability is of the form:

$$P_{ee}^{3\nu} \simeq \boxed{c_{13}^4}\, P_{ee}^{2\nu}(\delta m^2,\, \theta_{12}) + \boxed{s_{13}^4}$$

This form holds also in matter for solar ν (proof omitted)

→ **Both KamLAND and solar ν probe δm^2 and the mixing matrix elements $|U_{ei}|^2$ of ν_e with $\nu_i=(\nu_1, \nu_2, \nu_3)$**

$$\begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{bmatrix}$$

Next task: calculate P_{ee} in matter for solar neutrinos.

Steps: (1) find the effective 2ν oscillation parameters
(2) apply to them the adiabatic evolution
(3) apply the previous $2\nu \rightarrow 3\nu$ correction

Exercise: Relation between 2ν oscillation parameters in matter and vacuum

The effective 2ν parameters in matter can be written as:

$$\sin 2\tilde{\theta}_{12} = \frac{\sin 2\theta_{12}}{\sqrt{\left(\cos\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}} \qquad \delta\tilde{m}^2 = \delta m^2 \frac{\sin 2\theta_{12}}{\sin 2\tilde{\theta}_{12}}$$

where $A = 2\sqrt{2}\, G_F N_e E$ for neutrinos ($A \rightarrow -A$ for antineutrinos)

Note that:

- effective mixing breaks the vacuum octant symmetry $\theta_{12} \rightarrow \pi/2 - \theta_{12}$
- for $A/\delta m^2 \ll 1$ *(vacuum-dominated)* it is $\tilde{\theta}_{12} \simeq \theta_{12}$
- for $A/\delta m^2 \gg 1$ *(matter-dominated)* it is $\tilde{\theta}_{12} \simeq \pi/2$
- mixing angle is resonant for $\cos\theta_{12} \sim A/\delta m^2$ (MSW resonance)

Exercise: Adiabatic 2ν transition probability for solar neutrinos

For a solar neutrino produced at $\mathbf{x_i}$ and reaching vacuum:

$$P_{ee}^{2\nu} = \cos^2\tilde{\theta}_{12}(x_i)\cos^2\theta_{12} + \sin^2\tilde{\theta}_{12}(x_i)\sin^2\theta_{12}$$

(averaging out many oscillations along propagation)

This equation contains most of the relevant physics, up to subleading $2\nu \rightarrow 3\nu$ corrections (previously noted) and Earth matter effects (day-night differences).

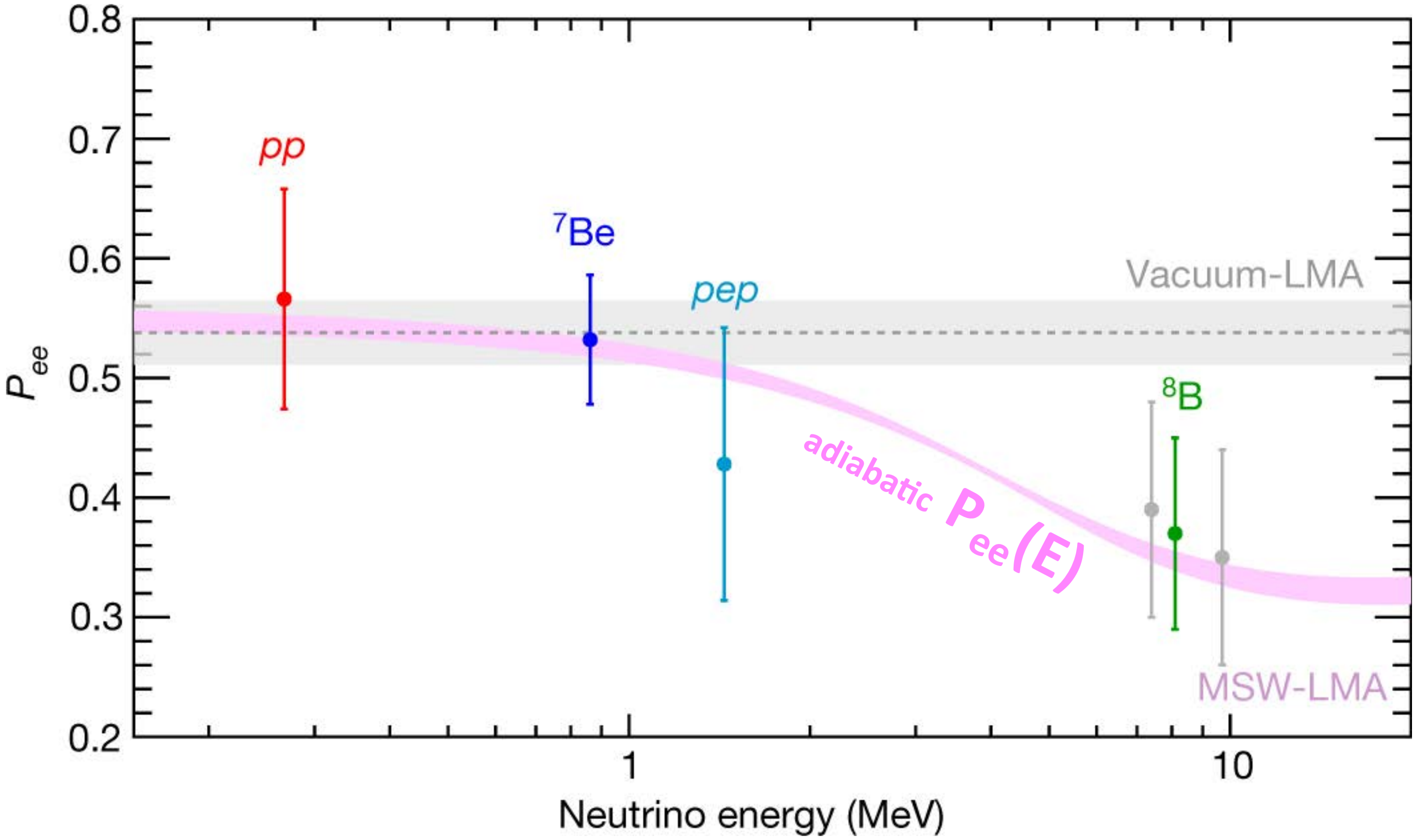
Relevant limits:

Low E → $A/\delta m^2 \ll 1$ *(vacuum-dom.)* → $P_{ee} \simeq 1 - \frac{1}{2}\sin^2 2\theta_{12}$ **(octant sym)**

High E → $A/\delta m^2 \gg 1$ *(matter-dom.)* → $P_{ee} \simeq \sin^2\theta_{12}$ **(octant asym)**

At intermediate energies, $A/\delta m^2 \sim O(1)$ → ***get info on δm^2***

Iconic results from the Borexino experiment at Gran Sasso:

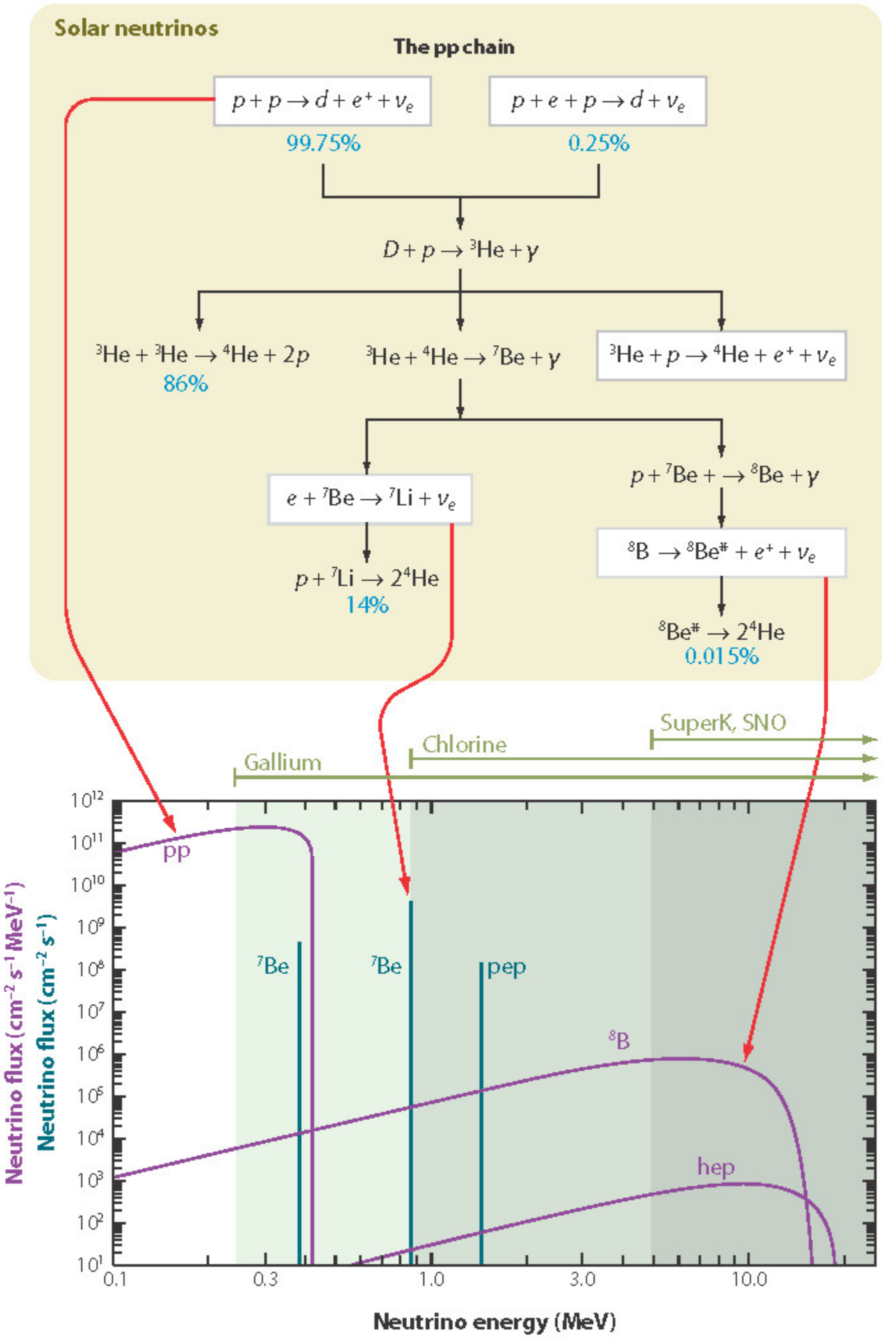


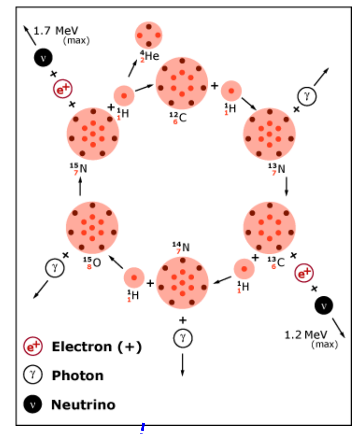
Monotonic transition, not periodic oscillation! Given the ***solar core density****, the* ***vacuum-matter transition*** *occurs in the middle of the solar* ν *spectrum, allowing us to* ***measure*** δm^2 ***and*** θ_{12} ***(+ its octant):*** *a lucky "anthropic" coincidence…*

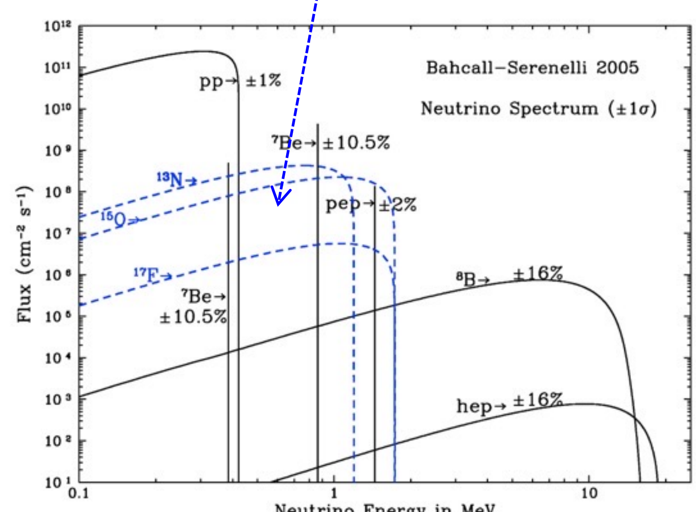
More about solar neutrino and KamLAND reactor results→

Solar neutrinos: Production

pp (+CNO) cycle







Detection

Radiochemical: count the decays of unstable final-state nuclei. (low energy threshold, but energy and time info lost/integrated)

$^{37}Cl + \nu_e \rightarrow {}^{37}Ar + e$ (CC) Homestake

$^{71}Ga + \nu_e \rightarrow {}^{71}Ge + e^-$ (CC) GALLEX/GNO, SAGE

Elastic scattering: events detected in real time with either "high" threshold (Č, directional) or "low" threshold (Scintillators)

$\nu_x + e^- \rightarrow \nu_x + e^-$ (NC,CC) SK, SNO, Borexino

Interactions on Deuterium: CC events detected in real time; NC events separated statistically + using neutron counters.

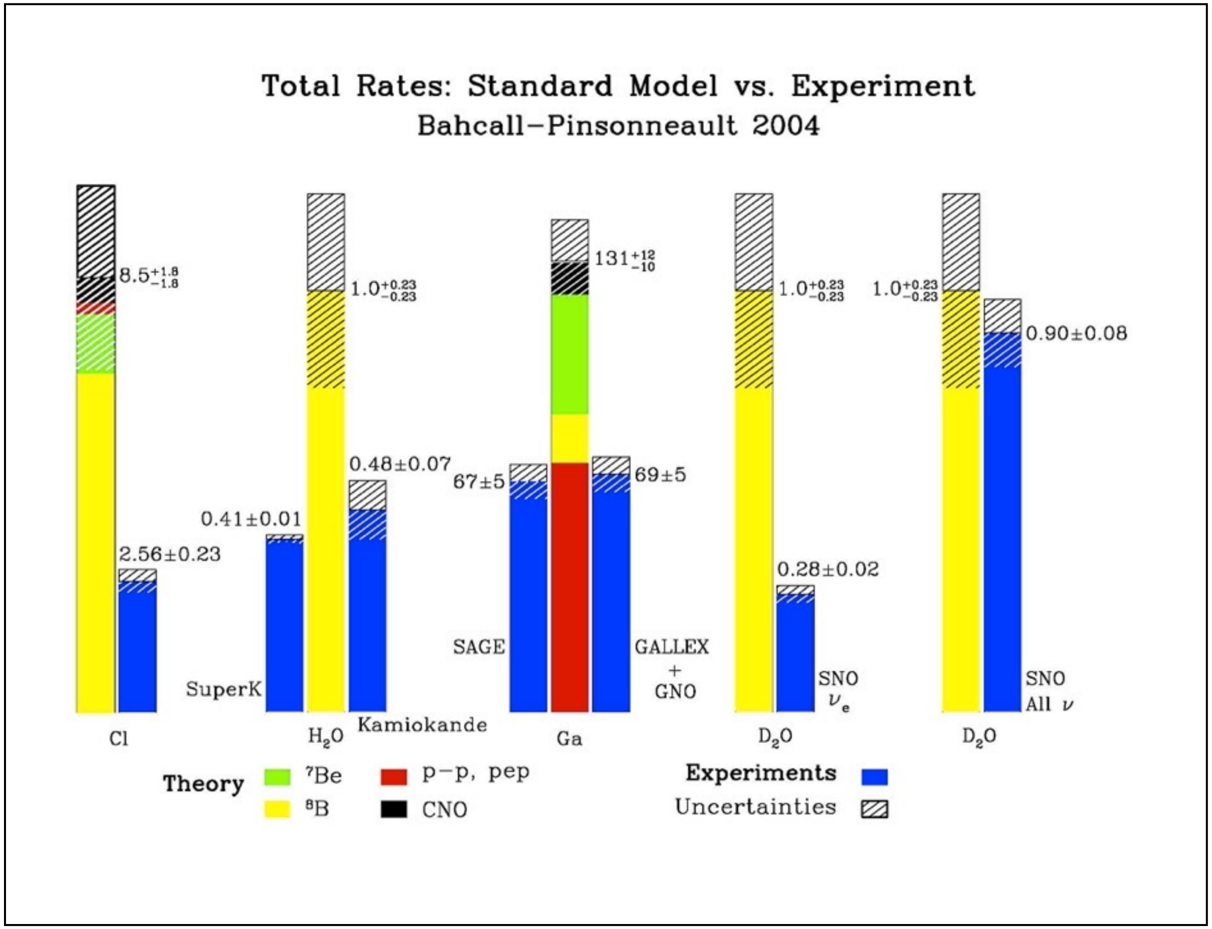
$\nu_e + d \rightarrow p + p + e^-$ (CC)

$\nu_x + d \rightarrow p + n + \nu_x$ (NC)

SNO (Sudbury Neutrino Observatory)

The "solar neutrino problem"

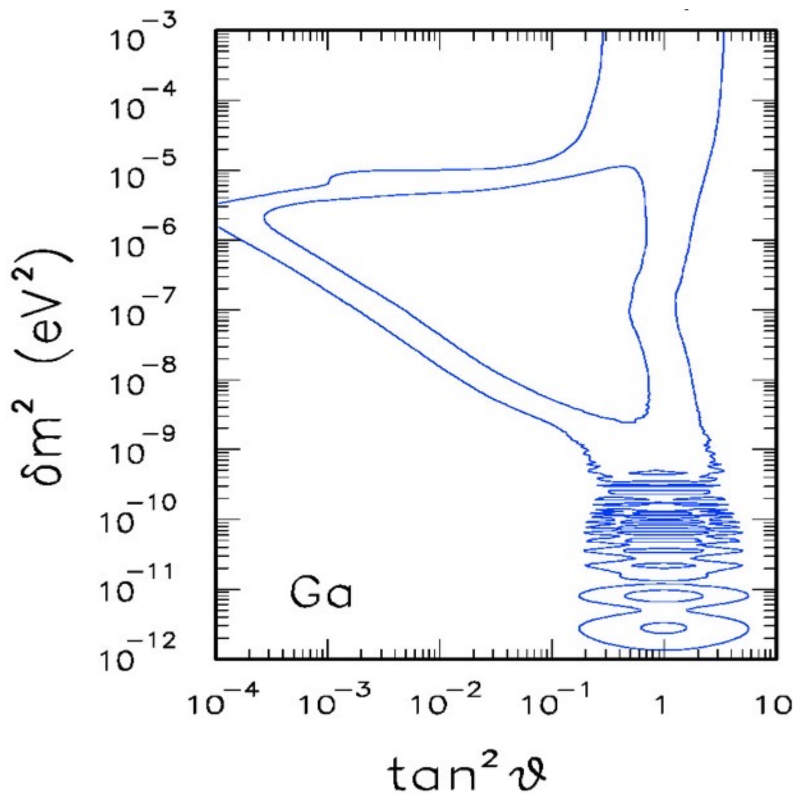
All CC-sensitive results indicated a ν_e deficit…



…as compared to solar model expectations

Interpretation

In the “past millennium”: solar neutrino oscillations? Maybe, but…
- large uncertainties in the parameter space and/or the solar model
- no unmistakable evidence for flavor transitions (“smoking gun”)



E.g., in Gallium expts:

“matter” (MSW) solutions

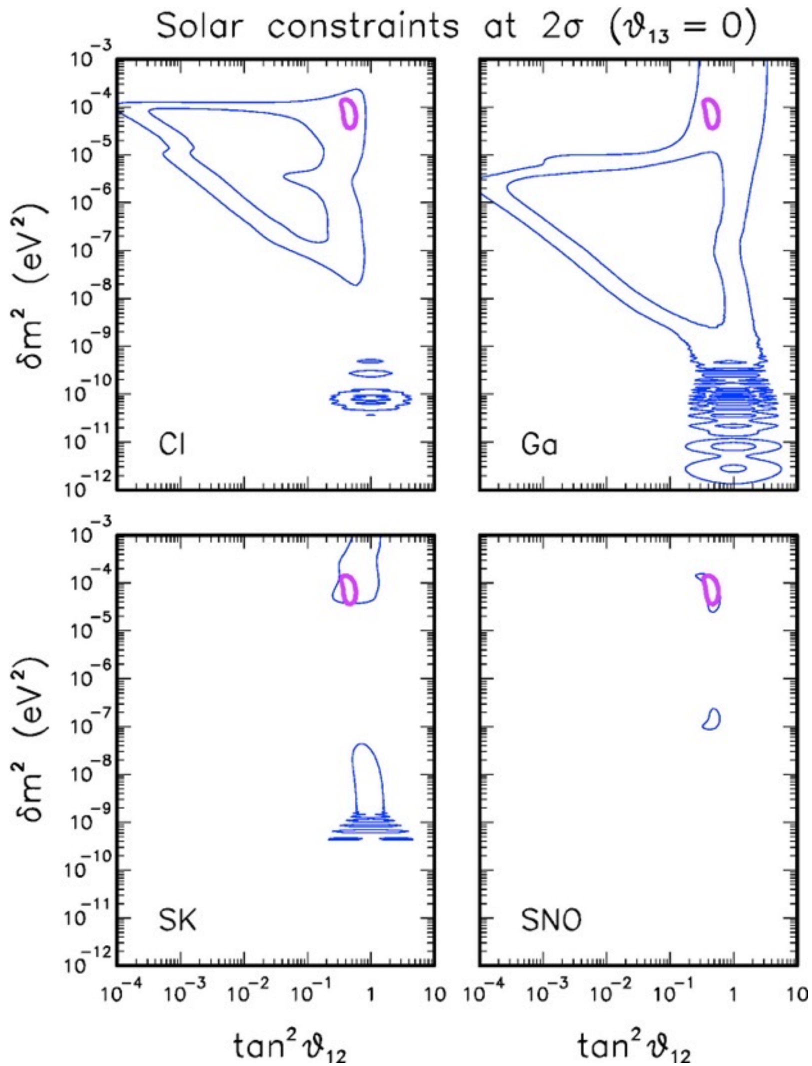
“just-so” vacuum sol.’s

“small” mixing

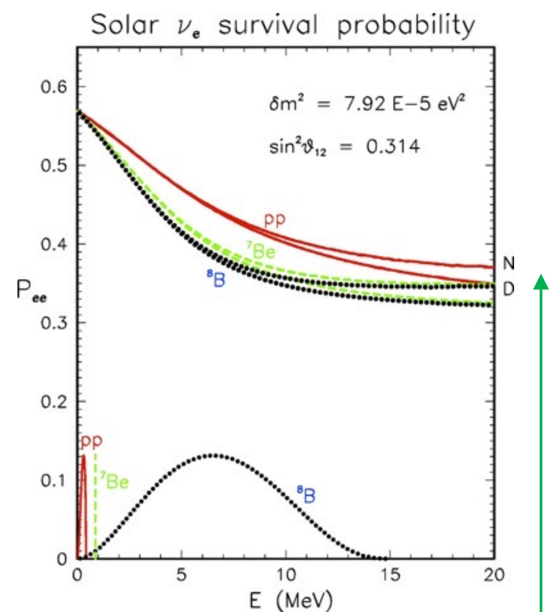
“large” mixing

+ many “exotic” or non-oscillatory solutions…

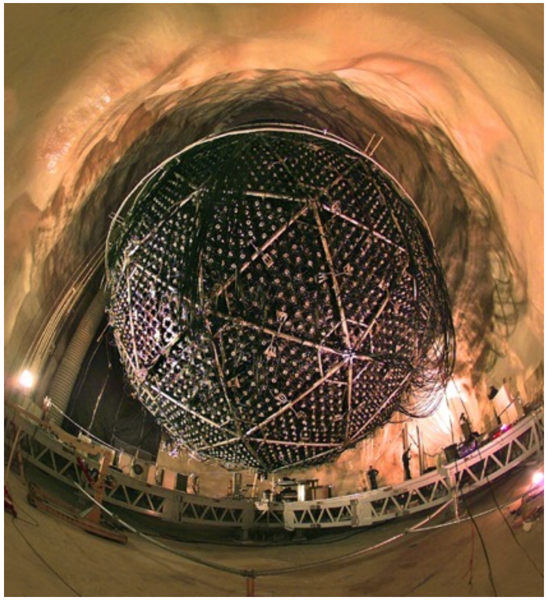
But, in 2002 ("annus mirabilis"), one global solution was finally singled out by combination of all solar data ("large mixing angle" or **LMA**).



For LMA parameters, evolution is **adiabatic** in solar matter.



In the Earth: small day/night (D/N) effects, seen at ~3sigma.



Crucial role played by SNO in 2002

In deuterium one can separate CC events (counting only ν_e) from NC events (counting ν_e, ν_μ, ν_τ), and double check via Elast. Scatt. events (due to both NC and CC):

$$\begin{aligned} \mathrm{CC}: &\quad \nu_e + d \to p + p + e \\ \mathrm{NC}: &\quad \nu_{e,\mu,\tau} + d \to p + n + \nu_{e,\mu,\tau} \\ \mathrm{ES}: &\quad \nu_{e,\mu,\tau} + e \to e + \nu_{e,\mu,\tau} \end{aligned}$$

$$\frac{\mathrm{CC}}{\mathrm{NC}} \sim \frac{\phi(\nu_e)}{\phi(\nu_e) + \phi(\nu_{\mu,\tau})}$$

thus: $$\frac{\mathrm{CC}}{\mathrm{NC}} < 1 \;\Rightarrow\; \phi(\nu_{\mu,\tau}) > 0 \;\Rightarrow\; \nu_e \to \nu_{\mu,\tau}$$

CC/NC ~ 0.3 < 1

"Smoking gun" of flavor change, indep. of solar model (confirmed!)

CC/NC ~ P_{ee} ~ $\sin^2\theta_{12}$ (LMA) **~ 0.3 < ½**

Evidence of: mixing angle in first octant + matter effects.

SK atmospheric + SNO solar = Nobel Prize 2015!

"...for the discovery of neutrino oscillations, which shows that neutrinos have mass"

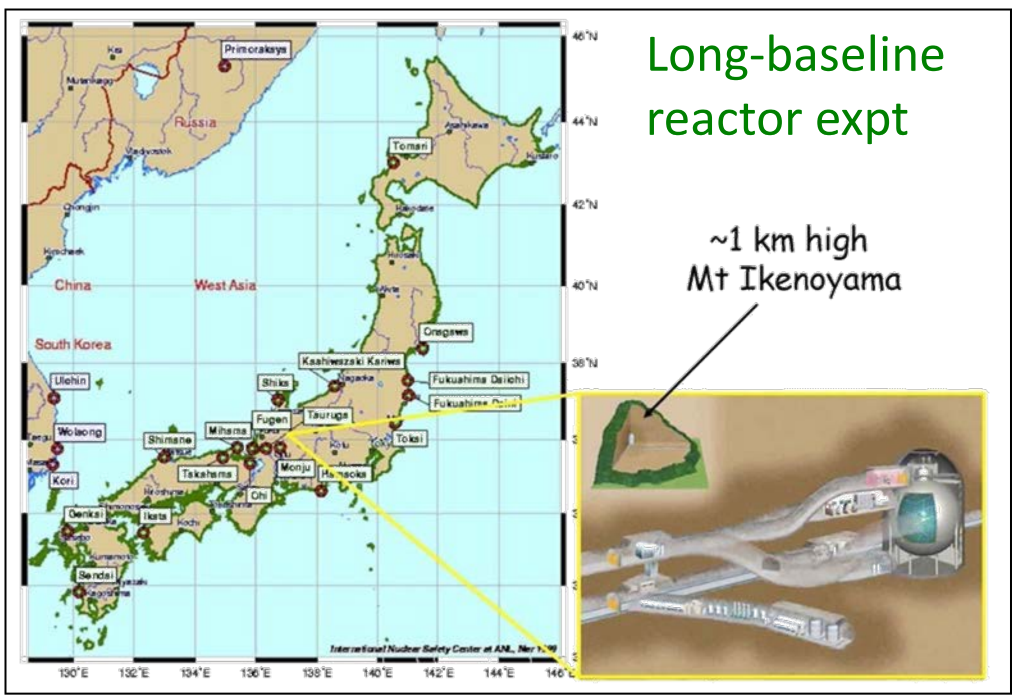
Takaaki Kajita **Art McDonald**

Also in 2002… **KamLAND**: **1000 ton mineral oil detector,** "surrounded" by nuclear reactors producing anti-ν_e. Characteristics:

$A/\delta m^2 << 1$ in Earth crust
(vacuum approxim. OK)
L~100-200 km
E_ν~ few MeV

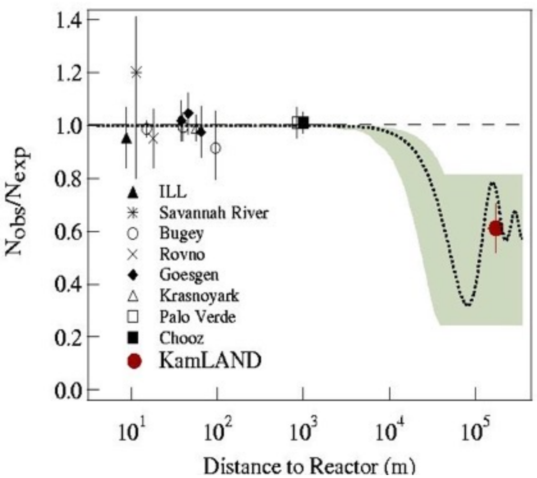
⇨

With previous ($\delta m^2,\theta_{12}$) parameters it is **($\delta m^2 L/4E$)~O(1)** and reactor neutrinos should oscillate with large amplitude (large θ_{12})

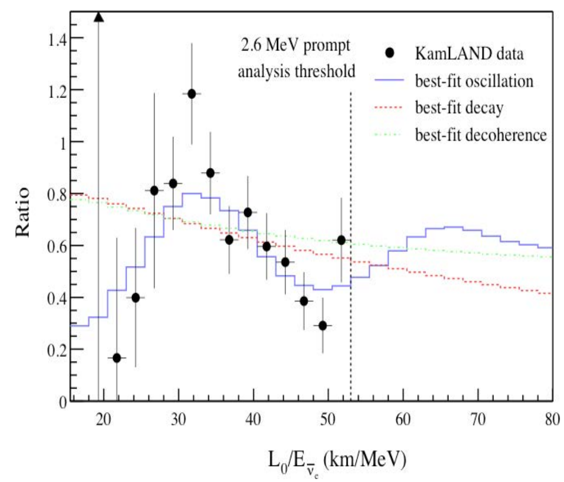


KamLAND results

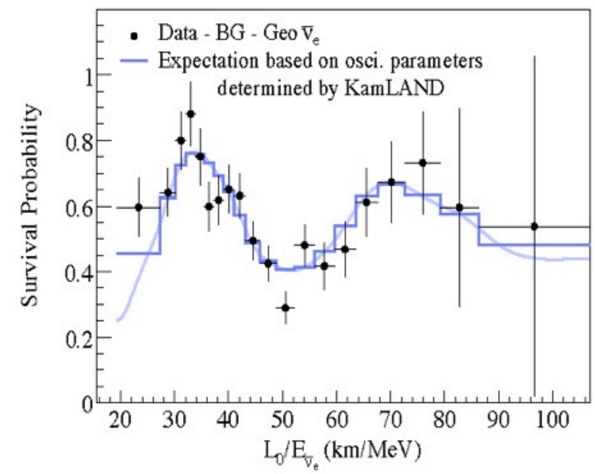
2002: electron flavor disappearance observed



2004: half-period of oscillation observed



2007+: one period of oscillation observed



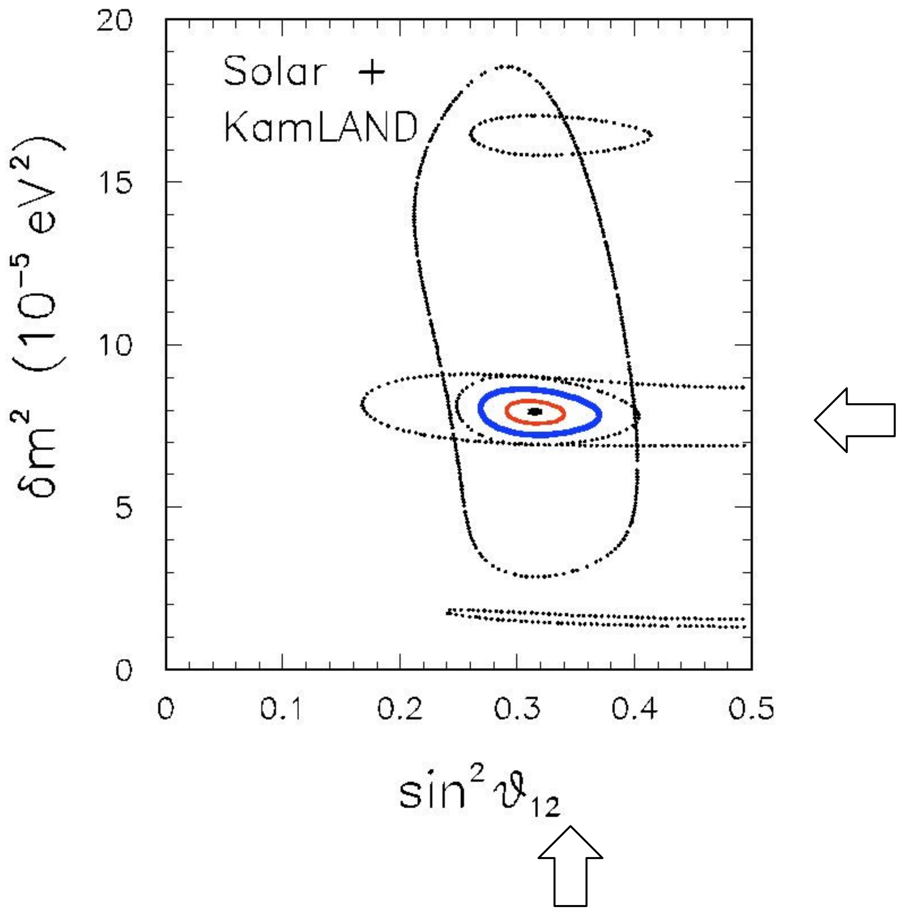
Direct observation of δm^2 oscillations!

(get precise δm^2 value from dip & peak positions)

At the right place (L~O(10^2) km accidentally for reactors around Kamioka) and at the right time (before Fukushima accident): anthropic coincidence...

Interpretation in terms of 2ν oscillations (θ_{13}~0)

(δm^2, θ_{12}) bounds: complementarity of solar/KL neutrinos



KamLAND

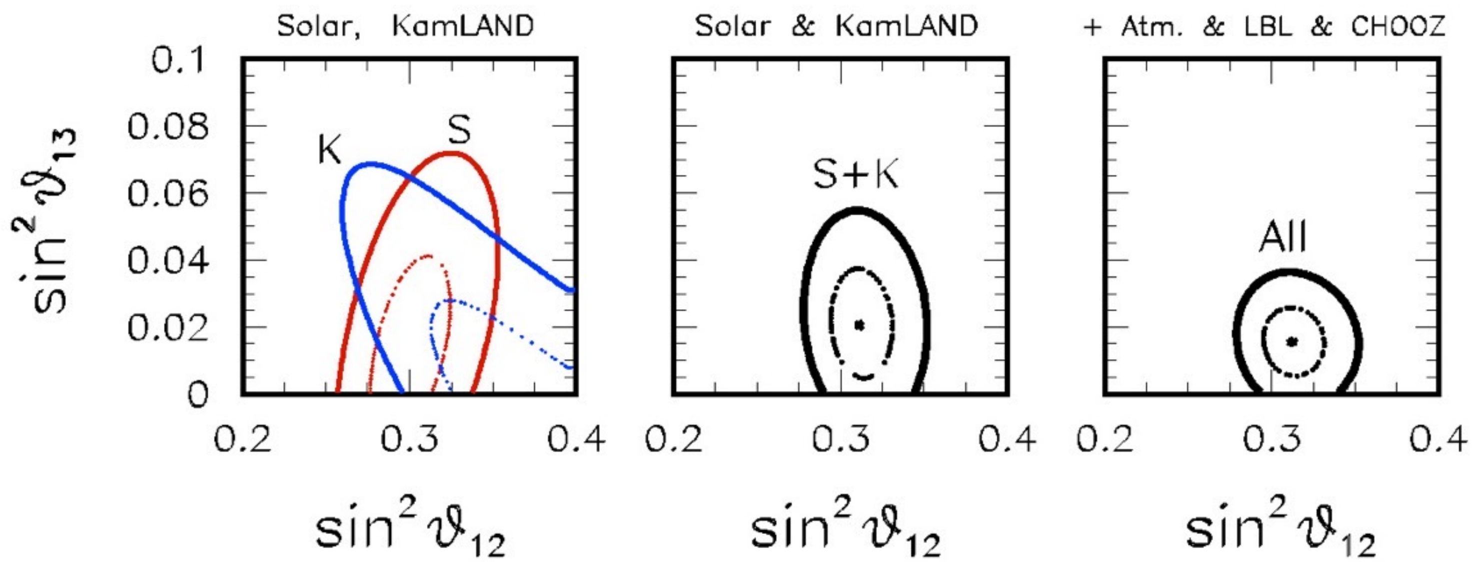
Solar

Note:
KamLAND solution is octant-symmetric (not shown). Could not exclude $\theta_{12} > \pi/4$

More refined (3ν) interpretation

Go beyond dominant 3ν oscillations. Include subleading θ_{13} effects in solar+KamLAND combination (as well as other data).

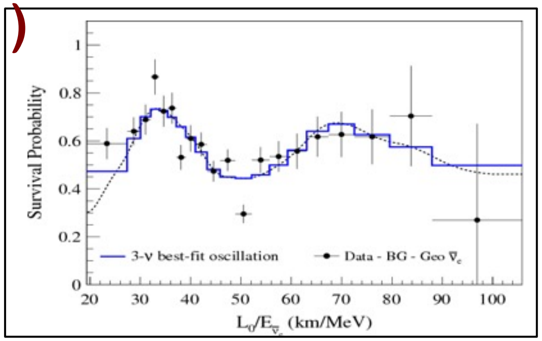
Interesting hints for $\theta_{13} > 0$ emerged as early as 2008...
corroborated by T2K $\nu_\mu \rightarrow \nu_e$ in 2011 ... established by reactors in 2012!



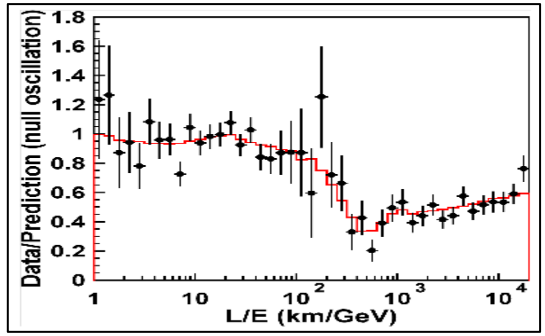
$$P_{ee}^{3\nu} = c_{13}^4 \, P_{ee}^{2\nu}(\delta m^2, \theta_{12}) + s_{13}^4$$

Recap: dominant parameters in past/current experiments

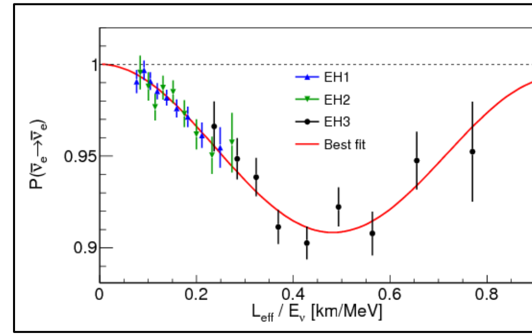
$e \to e$ (δm^2 , θ_{12})



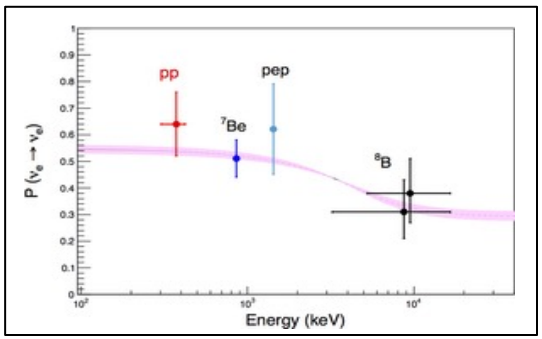
$\mu \to \mu$ (Δm^2 , θ_{23})



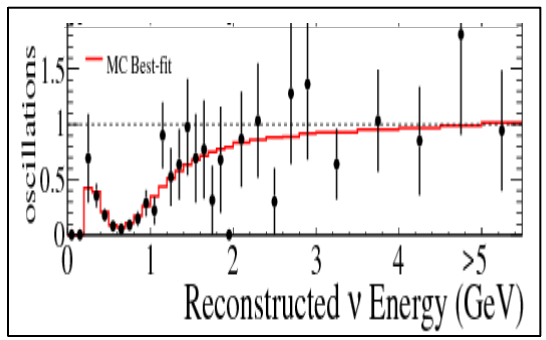
$e \to e$ (Δm^2 , θ_{13})



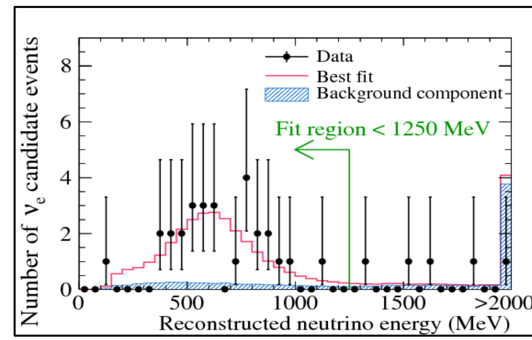
$e \to e$ (δm^2 , θ_{12})



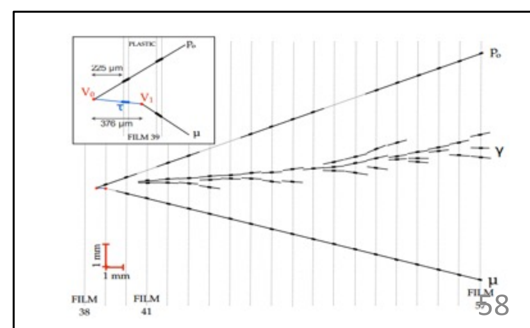
$\mu \to \mu$ (Δm^2 , θ_{23})



$\mu \to e$ (Δm^2 , θ_{13} , θ_{23})



$\mu \to \tau$ (Δm^2 , θ_{23})



Established so far:

δm^2 $|\Delta m^2|$ θ_{12} θ_{23} θ_{13}

Each probed by at least two different classes of experiments!

Recap: 3ν status

5 knowns:

- $\delta m^2 \sim 7 \times 10^{-5}$ eV^2
- $|\Delta m^2| \sim 2 \times 10^{-3}$ eV^2
- $\sin^2\theta_{12} \sim 0.3$
- $\sin^2\theta_{23} \sim 0.5$
- $\sin^2\theta_{13} \sim 0.02$

Oscillations

Non-oscillat.

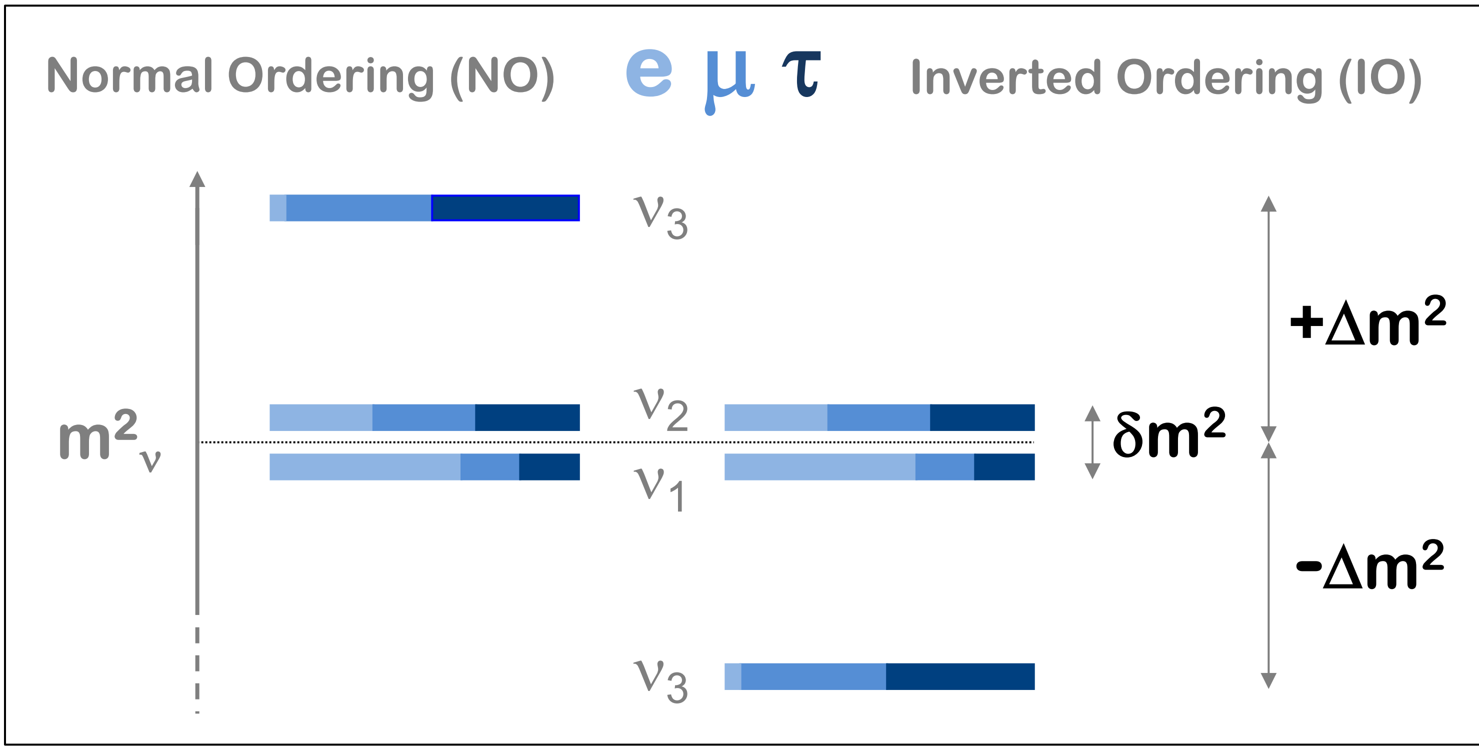
5 unknowns:

Oscillations:

- δ **CPV** Dirac phase
- **sign(Δm^2)** → NO/IO
- θ_{23} **octant** degeneracy

Non-oscillat.:

- **absolute mass** scale
- **Dirac/Majorana** nature



$$|U_{ei}|^2 + |U_{\mu i}|^2 + |U_{\tau i}|^2 = 1$$

End of Lecture III

Solutions to exercises: extra slides →

Exercise: Dominant Δm^2 oscillations in vacuum

From the general 3ν formula in vacuum (take $\delta m^2 = 0$):

$\alpha = \beta$:

$$P_{\alpha\alpha} = 1 - 4\,\mathrm{Re}\left(J^{13}_{\alpha\alpha} + J^{23}_{\alpha\alpha}\right)\sin^2\left(\frac{\Delta m^2 x}{4E}\right) - 2\,\mathrm{Im}\left(J^{13}_{\alpha\alpha} + J^{23}_{\alpha\alpha}\right)\sin\left(\frac{\Delta m^2 x}{2E}\right)$$

$$J^{13}_{\alpha\alpha} + J^{23}_{\alpha\alpha} = |U_{\alpha 1}|^2|U_{\alpha 3}|^2 + |U_{\alpha 2}|^2|U_{\alpha 3}|^2 = |U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2)$$ with null imaginary part.

$$P_{\alpha\alpha} = 1 - 4|U_{\alpha 3}|^2(1 - |U_{\alpha 3}|^2)\sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

$\alpha \neq \beta$:

$$P_{\alpha\beta} = -4\,\mathrm{Re}\left(J^{13}_{\alpha\beta} + J^{23}_{\alpha\beta}\right)\sin^2\left(\frac{\Delta m^2 x}{4E}\right) - 2\,\mathrm{Im}\left(J^{13}_{\alpha\beta} + J^{23}_{\alpha\beta}\right)\sin\left(\frac{\Delta m^2 x}{2E}\right)$$

$$\begin{aligned}
J^{13}_{\alpha\beta} + J^{23}_{\alpha\beta} &= U_{\alpha 1}U^*_{\beta 1}U^*_{\alpha 3}U_{\beta 3} + U_{\alpha 2}U^*_{\beta 2}U^*_{\alpha 3}U_{\beta 3} \\
&= U^*_{\alpha 3}U_{\beta 3}\left(U_{\alpha 1}U^*_{\beta 1} + U_{\alpha 2}U^*_{\beta 2}\right) \\
&= U^*_{\alpha 3}U_{\beta 3}\left(-U_{\alpha 3}U^*_{\beta 3}\right) \qquad \leftarrow \text{unitarity of } U \\
&= -|U_{\alpha 3}|^2|U_{\beta 3}|^2 \qquad \text{with null imaginary part.}
\end{aligned}$$

$$P_{\alpha\beta} = 4|U_{\alpha 3}|^2|U_{\beta 3}|^2\sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

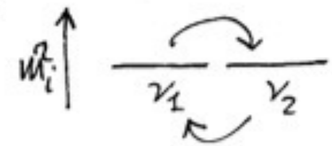
The invariance of such $P_{\alpha\alpha}$, $P_{\alpha\beta}$ under $U \to U^*$ makes them insensitive to δ_{CP} and to $\nu/\bar{\nu}$.

Also, notice invariance under $+\Delta m^2 \to -\Delta m^2$: no sensitivity to NO/IO.

The lack of sensitivity to θ_{12} depends on the approximation $\delta m^2 = m_2^2 - m_1^2 = 0$:

If two states (ν_1, ν_2) are degenerate, then a rotation $\begin{pmatrix} \nu_1' \\ \nu_2' \end{pmatrix} = \begin{pmatrix} R \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$

is physically unobservable :



since it gives again
two degenerate states.

More precisely, remind that

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = R_{23} \cdot R_{13} \cdot R_{12} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \qquad \text{where} \qquad R_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(R_{23}: θ_{23} rotation; R_{13}: θ_{13} rotation; R_{12}: θ_{12} rotation)

Then redifine the degenerate states as: $\begin{pmatrix} \nu_1' \\ \nu_2' \end{pmatrix} = \begin{pmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$

The physics is the same, but θ_{12} has disappeared from the mixing matrix:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = R_{23}(\theta_{23})\, R_{13}(\theta_{13}) \cdot \begin{pmatrix} \nu_1' \\ \nu_2' \\ \nu_3 \end{pmatrix}$$

Exercise: Dominant δm^2 oscillations in vacuum

- For P_{ee} (disappearance): $P^{CPV} = 0$
- For $\Delta m^2 x/4E \gg 1$, oscillations are averaged: $\sin^2\left(\frac{\Delta m^2_{31} x}{4E}\right) \simeq \frac{1}{2} \simeq \sin^2\left(\frac{\Delta m^2_{32} x}{4E}\right)$
- Thus, $P_{ee} = 1 - 4\,\mathrm{Re}\,(J^{12}_{ee}) \sin^2\left(\frac{\delta m^2 x}{4E}\right) - 2\,\mathrm{Re}\,(J^{13}_{ee} + J^{23}_{ee})$

$$= 1 - 4|U_{e1}|^2|U_{e2}|^2 \sin^2\left(\frac{\delta m^2 x}{4E}\right) - 2|U_{e3}|^2\left(|U_{e1}|^2 + |U_{e2}|^2\right)$$

$$= 1 - 4\,c^4_{13}\,s^2_{12}\,c^2_{12}\,\sin^2\left(\frac{\delta m^2 x}{4E}\right) - 2\,s^2_{13}\,c^2_{13} \qquad \leftarrow 1 = (c^2_{13} + s^2_{13})^2$$

$$= c^4_{13} + s^4_{13} - 4\,c^4_{13}\,s^2_{12}\,c^2_{12}\,\sin^2\left(\frac{\delta m^2 x}{4E}\right)$$

$$= c^4_{13}\,P^{2\nu}_{ee} + s^4_{13}$$

where $P^{2\nu}_{ee} = 1 - \sin^2 2\theta_{12} \sin^2\left(\frac{\delta m^2 x}{4E}\right)$ is the $\theta_{13} \to 0$ limit.

- Note that, below μ and τ threshold for production, the flavors ν_μ and ν_τ are not observable: can "rotate" $(\nu_\mu, \nu_\tau) \to (\nu_x, \nu_y)$ leaving ν_e unaltered.

Remind that:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} R(\theta_{13})\, R(\theta_{12}) \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \quad ; \text{ define } \begin{pmatrix} \nu_x \\ \nu_y \end{pmatrix} = \begin{pmatrix} c_{23} & -s_{23} \\ s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \nu_\tau \end{pmatrix}$$

$$\to \begin{pmatrix} \nu_e \\ \nu_x \\ \nu_y \end{pmatrix} = R(\theta_{13})\, R(\theta_{12}) \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

; the physics is the same but θ_{23} has disappeared.

Exercise: 2ν oscillation parameters in matter

- 2ν Hamiltonian in flavor basis for the $(\delta m^2, \theta_{12})$ subsystem:

$$\tilde{H} = U\begin{pmatrix} \frac{m_1^2}{2E} & 0 \\ 0 & \frac{m_2^2}{2E} \end{pmatrix}U^T + \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2E}\left[\begin{pmatrix} c_{12} & s_{12} \\ -s_{12} & c_{12} \end{pmatrix}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}\begin{pmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right]$$

- Since any part $\propto \mathbb{1}$ is unobservable, it is convenient to make $\tilde{H}$ traceless:

$$\tilde{H} = \frac{1}{4E}\begin{bmatrix} A - \cos 2\theta_{12}\,\delta m^2 & \sin 2\theta_{12}\,\delta m^2 \\ \sin 2\theta_{12}\,\delta m^2 & -A + \cos 2\theta_{12}\,\delta m^2 \end{bmatrix} \qquad A = 2\sqrt{2}\,G_F N_e E$$

- Eigenvalues: $\pm \frac{\delta\tilde{m}^2}{4E}$, where $\delta\tilde{m}^2 = \delta m^2 \sqrt{\left(\cos 2\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}$

- Diagonalizing rotation:

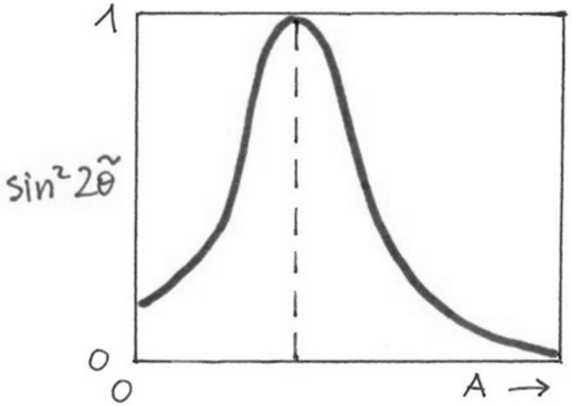
$$\tilde{H} = \begin{pmatrix} \cos\tilde{\theta}_{12} & +\sin\tilde{\theta}_{12} \\ -\sin\tilde{\theta}_{12} & \cos\tilde{\theta}_{12} \end{pmatrix}\begin{pmatrix} -\frac{\delta\tilde{m}^2}{4E} & 0 \\ 0 & +\frac{\delta\tilde{m}^2}{4E} \end{pmatrix}\begin{pmatrix} \cos\tilde{\theta}_{12} & -\sin\tilde{\theta}_{12} \\ \sin\tilde{\theta}_{12} & \cos\tilde{\theta}_{12} \end{pmatrix} = \tilde{U}(\cdot)\tilde{U}^T$$

where

$$\sin 2\tilde{\theta}_{12} = \frac{\sin 2\theta_{12}}{\sqrt{\left(\cos 2\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}}, \qquad \cos 2\tilde{\theta}_{12} = \frac{\cos 2\theta_{12} - A/\delta m^2}{\sqrt{\left(\cos 2\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}}$$

- Hence, $\delta\tilde{m}^2 = \delta m^2 \sin 2\theta_{12} / \sin 2\tilde{\theta}_{12}$

Comments: ($\theta_{12} \equiv \theta$ for simplicity)

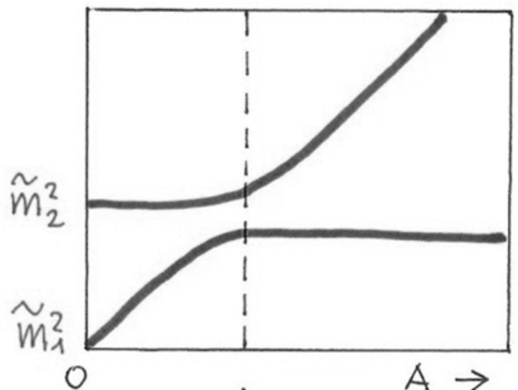


Mykheev-Smirnov-Wolfenstein (MSW) resonance:

For $A/\delta m^2 > 0$, the effective parameters have a resonant behavior around:

$$\frac{A}{\delta m^2} \simeq \cos 2\theta$$

(only for ν: no resonance for $\bar{\nu}$, since $A<0$ for $\bar{\nu}$)



$$\frac{A}{\delta m^2} \simeq \cos 2\theta$$

($A = 2\sqrt{2}\, G_F N_e E$)

Limiting cases:

$A/\delta m^2 \ll 1$: $(\delta\tilde{m}^2, \tilde{\theta}) \simeq (\delta m^2, \theta)$ ← vacuum-like

$A/\delta m^2 \simeq \cos 2\theta$: $(\delta\tilde{m}^2, \tilde{\theta}) \simeq (\delta m^2 \sin 2\theta, \pi/4)$ ← reson.

$A/\delta m^2 \gg 1$: $(\delta\tilde{m}^2, \tilde{\theta}) \simeq (A, \pi/2)$ ← matter dominance

Confirms expectations of large matter effects for $A/\delta m^2 \sim \mathcal{O}(1)$.

- For constant matter $(dA/dx = 0)$ the evolution operator is obtained by exponentiating $\tilde{H}$:

$$\tilde{S} = e^{-i\tilde{H}x} = \tilde{U}\begin{pmatrix} e^{i\delta\hat{m}^2 x/4E} & \\ & e^{-i\delta\hat{m}^2 x/4E} \end{pmatrix}\tilde{U}^T$$

$$= \cos\left(\frac{\delta\tilde{m}^2 x}{4E}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right)\begin{pmatrix} -\cos 2\tilde{\theta}_{12} & \sin 2\tilde{\theta}_{12} \\ \sin 2\tilde{\theta}_{12} & \cos 2\tilde{\theta}_{12} \end{pmatrix}$$

- By squaring the diagonal elements of $\tilde{S}$ one gets the survival probability:

$$P_{ee} = |\tilde{S}_{diag}|^2 = 1 - \sin^2 2\tilde{\theta}_{12} \sin^2\left(\frac{\delta\tilde{m}^2 x}{4E}\right)$$

- By squaring the off-diagonal elements of $\tilde{S}$ one gets the complementary flavor transition probability $P(\nu_e \to \nu_x) = 1 - P_{ee}$

- Note that, for $\delta m^2 \simeq 7.5 \times 10^{-5}\,eV^2$: $\frac{A}{\delta m^2} \sim 0.2\left(\frac{E}{MeV}\right)$ at the Sun center (where $N_e \sim 10^2\, mol/cm^3$). Therefore, $A/\delta m^2 \gtrsim 1$ for $E \gtrsim$ few MeV solar ν.

$\rightarrow$ Expect a transition to large matter effects around $E \sim$ few MeV

.... but: must work out P_{ee} for $A \neq$ constant!

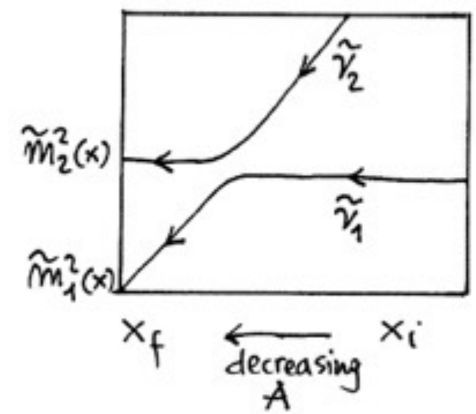
In this context, useful to note that solar ν's oscillate many times from Sun to Earth.

Exercise: Adiabatic 2ν transition probability (solar ν)

- For a slowly changing $A = A(x)$, the mass eigenstates $\nu_1(x)$ and $\nu_2(x)$ evolve independently (no "level crossing") from x_i to x_f ($x = x_f - x_i$) :

$$i\frac{d}{dx}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\end{pmatrix} = \frac{1}{2E}\begin{pmatrix}\tilde{m}_1^2(x) & 0\\ 0 & \tilde{m}_2^2(x)\end{pmatrix}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\end{pmatrix}$$

- Pictorially, they follow the corresponding eigenvalues in matter:



... and just acquire phases during evolution:

$$\nu_1(x_f) = e^{-i\int_{x_i}^{x_f}\frac{\tilde{m}_1^2(x)}{2E}dx}\,\nu_1(x_i) = e^{-i\varphi_1}\,\nu_1(x_i)$$

$$\nu_2(x_f) = e^{-i\int_{x_i}^{x_f}\frac{\tilde{m}_2^2(x)}{2E}dx}\,\nu_2(x_i) = e^{-i\varphi_2}\,\nu_2(x_i)$$

- Relative phase is $e^{-i\varphi} = e^{-i(\varphi_2-\varphi_1)} = e^{-i\int_{x_i}^{x_f}\frac{\delta\tilde{m}^2(x)}{2E}dx}$

(Along the path to the Earth it oscillates many times).

- We can write the initial and final flavor states (ν_e, ν_y), where ν_y is any combination of ν_μ and ν_τ, as:

$$\begin{pmatrix} \nu_e(x_i) \\ \nu_y(x_i) \end{pmatrix} = \begin{pmatrix} \cos\tilde{\theta}_i & \sin\tilde{\theta}_i \\ -\sin\tilde{\theta}_i & \cos\tilde{\theta}_i \end{pmatrix} \begin{pmatrix} \tilde{\nu}_1(x_f) \\ \tilde{\nu}_2(x_f) \end{pmatrix} , \quad \begin{pmatrix} \nu_e(x_f) \\ \nu_y(x_f) \end{pmatrix} = \begin{pmatrix} \cos\tilde{\theta}_f & \sin\tilde{\theta}_f \\ -\sin\tilde{\theta}_f & \cos\tilde{\theta}_f \end{pmatrix} \begin{pmatrix} \tilde{\nu}_1(x_f) \\ \tilde{\nu}_2(x_f) \end{pmatrix}$$

where $\tilde{\theta}_i = \tilde{\theta}_{12}(x_i)$, $\tilde{\theta}_f = \tilde{\theta}_{12}(x_f)$

- Altogether:

$$\begin{pmatrix} \nu_e(x_f) \\ \nu_y(x_f) \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\tilde{\theta}_f & \sin\tilde{\theta}_f \\ -\sin\tilde{\theta}_f & \cos\tilde{\theta}_f \end{pmatrix} \begin{pmatrix} e^{-i\varphi_1} & 0 \\ 0 & e^{-i\varphi_2} \end{pmatrix} \begin{pmatrix} \cos\tilde{\theta}_i & -\sin\tilde{\theta}_i \\ +\sin\tilde{\theta}_i & \cos\tilde{\theta}_i \end{pmatrix}}_{\text{This is the evolution operator } \tilde{S}_f \text{ in flavor basis!}} \begin{pmatrix} \nu_e(x_i) \\ \nu_y(x_i) \end{pmatrix}$$

- Take the element "ee" of $\tilde{S}_f$, square, average out the interference term and get:

$$P_{ee} = |\tilde{S}_{f,ee}|^2 = |\cos\tilde{\theta}_f \cos\tilde{\theta}_i + e^{-i\varphi} \sin\tilde{\theta}_f \sin\tilde{\theta}_i|^2 \simeq \cos^2\tilde{\theta}_f \cos^2\tilde{\theta}_i + \sin^2\tilde{\theta}_f \sin^2\tilde{\theta}_i$$

- At the exit from the Sun, the density vanishes and $\tilde{\theta}_f \to \theta_{12}$ (vacuum)

$$P_{ee} \cong \cos^2\tilde{\theta}_i \cos^2\theta_{12} + \sin^2\tilde{\theta}_i \sin^2\theta_{12}$$