Neutrino Oscillations

Lecture II

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Outline of lectures I-IV:

Lecture I
Pedagogical introduction + warm-up exercise

Lecture II
3ν osc. in vacuum and matter: notation and basic math

Lecture III
2ν approximations of phenomenological interest

Lecture IV
Back to 3ν oscillations: Status and Perspectives

The "standard" 3ν oscillation framework

Physics facts and mass notation:

- There are three mass states ν_1, ν_2, ν_3 with masses m_1, m_2, m_3
- Neutrino oscillations probe the differences $\Delta E \propto \Delta m^2_{ij}$
- There are only two independent Δm^2_{ij} , say, δm^2 and Δm^2
- Experimentally, very different scales: $\delta m^2 / \Delta m^2 \sim 1/30$
  Difficult to observe both! Current expts sensitive to a dominant one.

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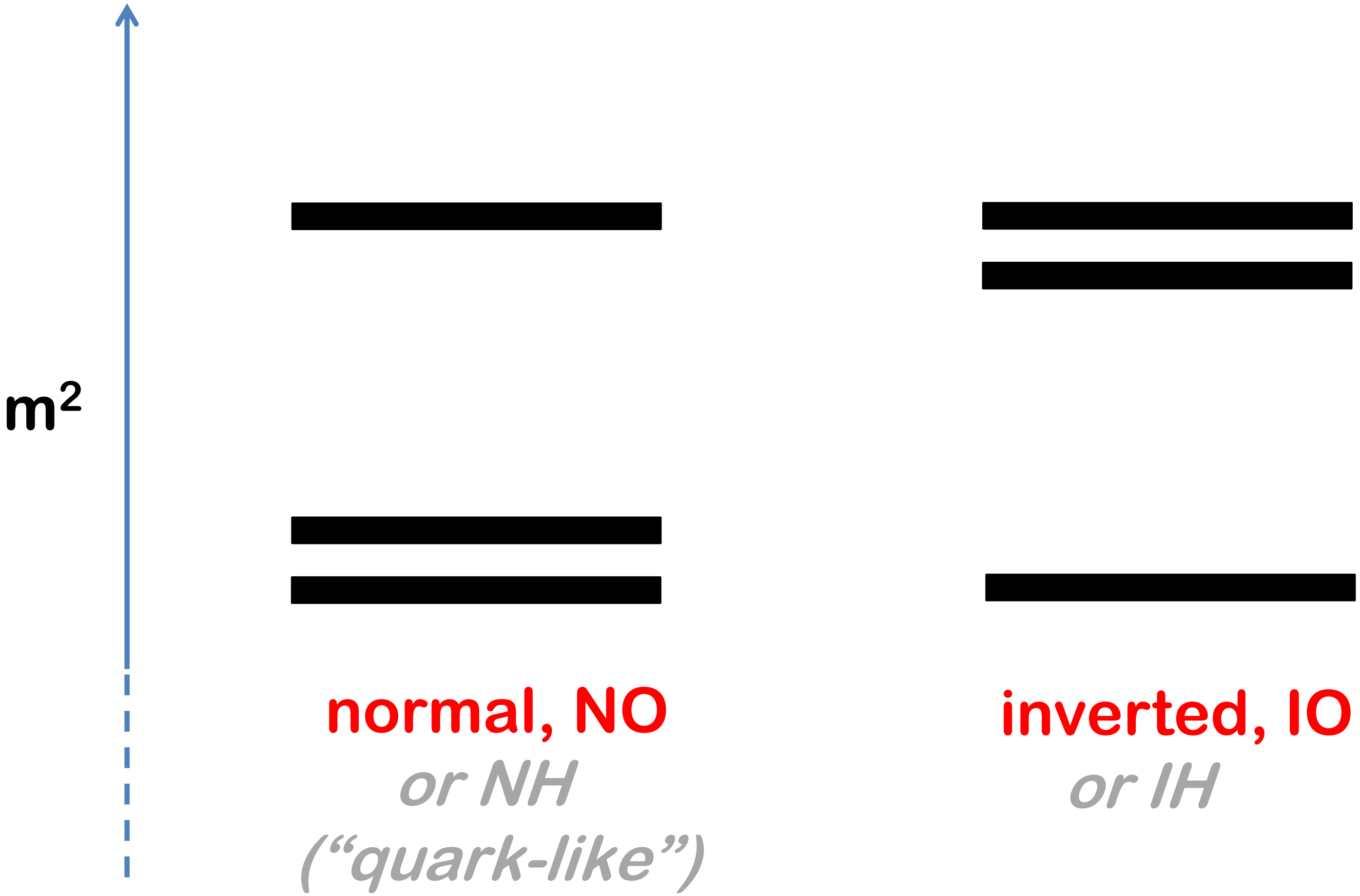
  Difficult to observe both! Current expts sensitive to a dominant one.

$\delta m^2 \simeq 7.5 \times 10^{-5}\ \mathrm{eV}^2$ ← "small" *or "solar"* splitting

$\Delta m^2 \simeq 2.5 \times 10^{-3}\ \mathrm{eV}^2$ ← "large" *or "atmospheric"* splitting

↑ *(somewhat obsolete terms)*

Two possible mass orderings (*or hierarchies*)



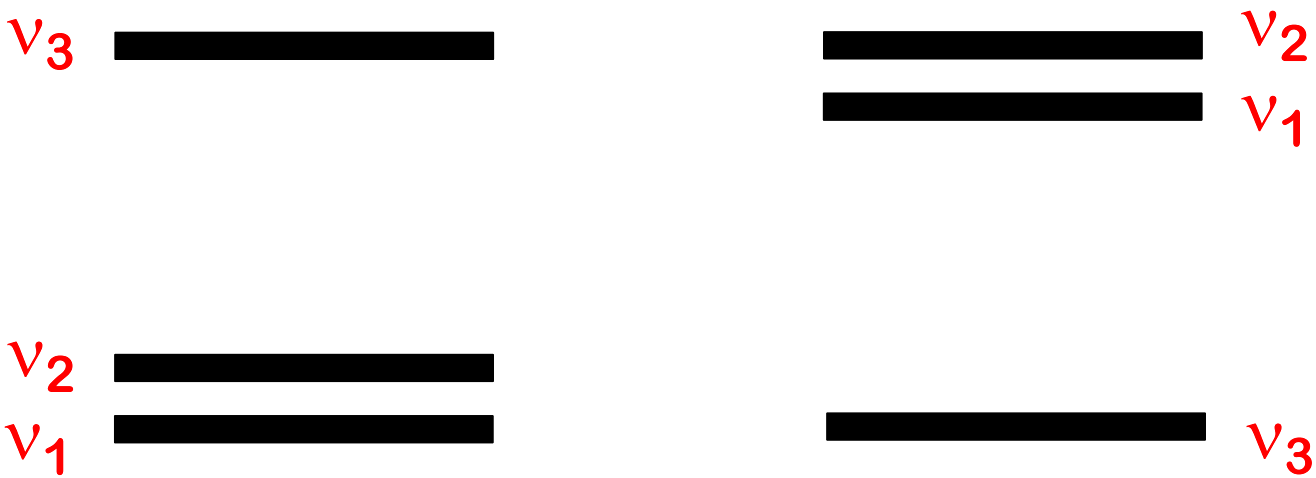
0 ?

Absolute mass scale still unknown, but
upper limits exist: **m < O(0.1-1) eV**

PDG convention for 3ν masses:

(ν_1,ν_2) = "close" states, with m_2>m_1 always

ν_3 = "lone" state, with m_3>$m_{1,2}$ in NO (m_3 <$m_{1,2}$ in IO)



Our notation for splittings: Define as independent ones

$$\delta m^2 = m_2^2 - m_1^2 > 0$$

$$\Delta m^2 = \frac{1}{2}(\Delta m^2_{31} + \Delta m^2_{32}) \begin{cases} > 0 & \text{NO} \\ < 0 & \text{IO} \end{cases}$$

PDG convention for 3ν mixing:

Three Euler rotations, one being complex

$$\nu_\alpha = U_{\alpha i}\nu_i \qquad \begin{matrix} \alpha = e,\ \mu,\ \tau \\ i = 1,\ 2,\ 3 \end{matrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This rotation ordering happens to be particularly useful for phenomenologically interesting limits

$$\begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} = \begin{bmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{bmatrix}$$

$$UU^\dagger = 1 \qquad U \to U^* \ \text{for}\ \overline{\nu} \qquad c_{ij} = \cos\theta_{ij} \qquad s_{ij} = \sin\theta_{ij}$$

Phase: **$\delta = \delta_{CP}$ = "Dirac" phase**. Governs possible CP violation in oscillations

If ν are Majorana: two additional (relative) **"Majorana" phases ϕ_{21}, ϕ_{31}**

$$U \rightarrow U U_M \ , \ U_M = \mathrm{diag}[1, \ \phi_{21}, \phi_{31}]$$

The Majorana phases are probed in $0\nu\beta\beta$ decay, but not in oscillations.

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Proof: The flavor evolution operator (Lecture I) reads

$$S_f = U S_m U^\dagger \qquad \text{with} \qquad S_m = \mathrm{diag}\left[e^{-i m_i^2 x/2E}\right]$$

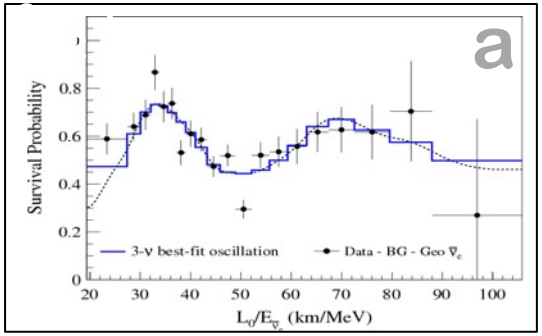
and is unaffected by $U \to U U_M$

$$S_f \to U U_M S_m (U U_M)^\dagger = U (U_M S_m U_M^\dagger) U^\dagger = U S_m U^\dagger = S_f$$

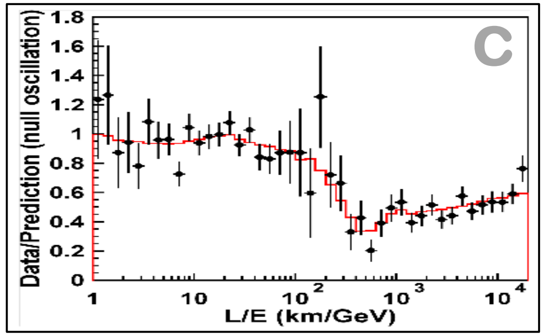
In general, standard 3ν oscillations do not distinguish Dirac/Majorana ν

3ν can explain α → β oscillations seen in vacuum and matter...

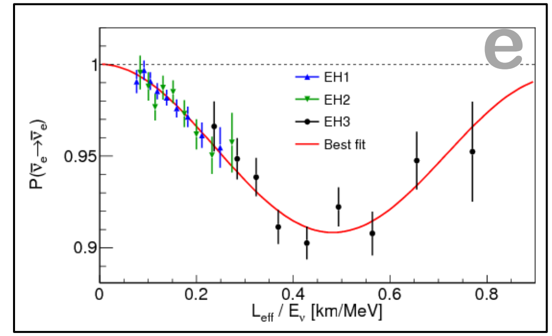
e→e (KamLAND, KL)



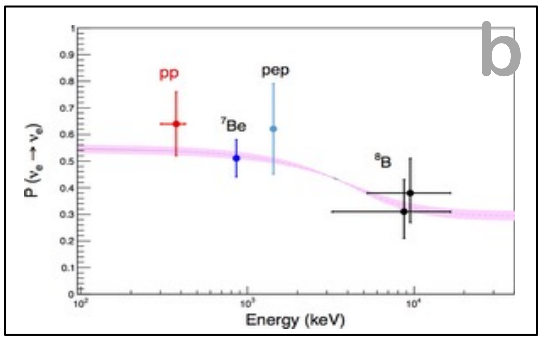
μ→μ (Atmospheric)



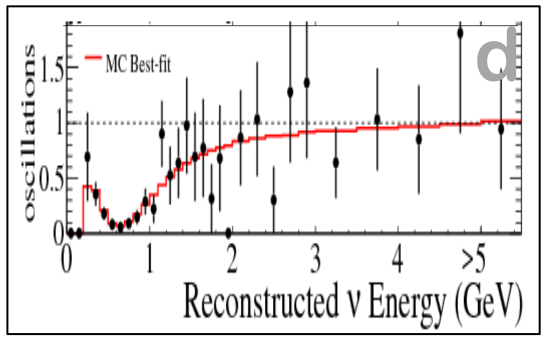
e→e (SBL Reac.)



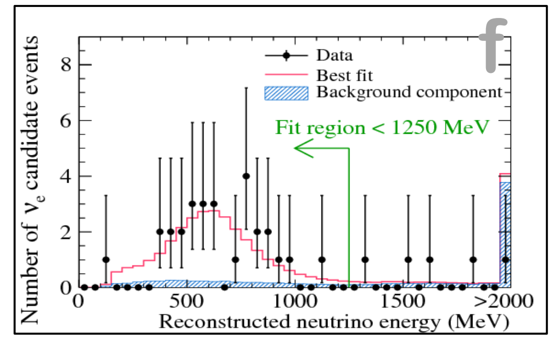
e→e (Solar)



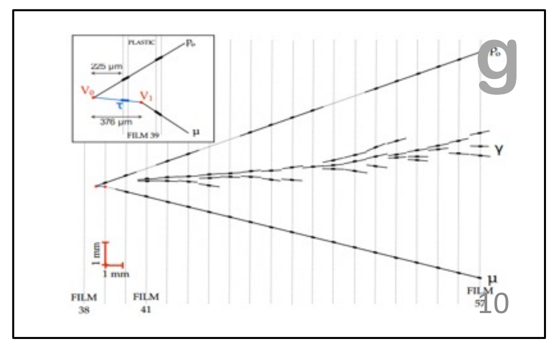
μ→μ (LBL Accel)



μ→e (LBL Accel)



μ→τ (OPERA, SK, DC)

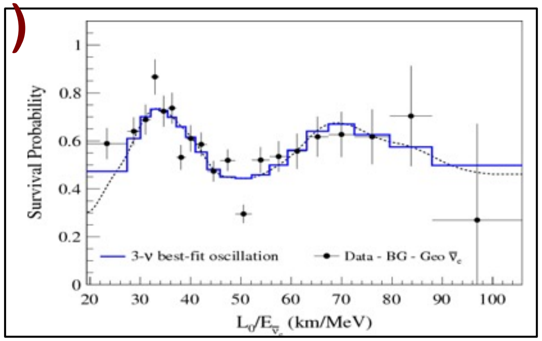


LBL = Long baseline (few x 100 km); **SBL = short baseline** (~1 km)

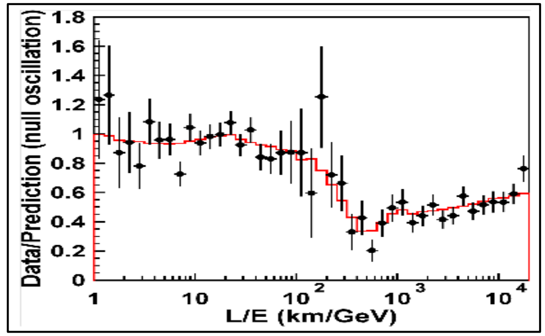
(a) KamLAND reactor [plot]; (b) Borexino [plot], Homestake, Super-K, SAGE, GALLEX/GNO, SNO; (c) Super-K atmosph. [plot], DeepCore, MACRO, MINOS etc.; (d) T2K (plot), NOvA, MINOS, K2K LBL accel.; (e) Daya Bay [plot], RENO, Double Chooz SBL reactor; (f) T2K [plot], MINOS, NOvA LBL accel.; (g) OPERA [plot] LBL accel., Super-K and IC-CD atmospheric.

...with dominant parameters (Lecture III):

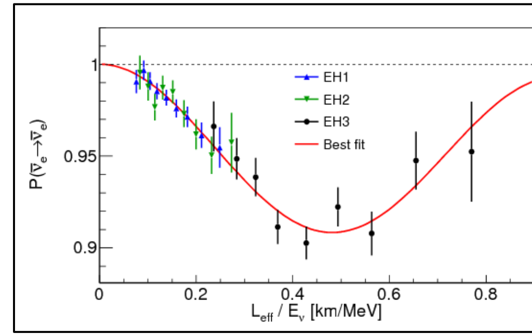
$e \to e$ $(\delta m^2, \theta_{12})$



$\mu \to \mu$ $(\Delta m^2, \theta_{23})$



$e \to e$ $(\Delta m^2, \theta_{13})$



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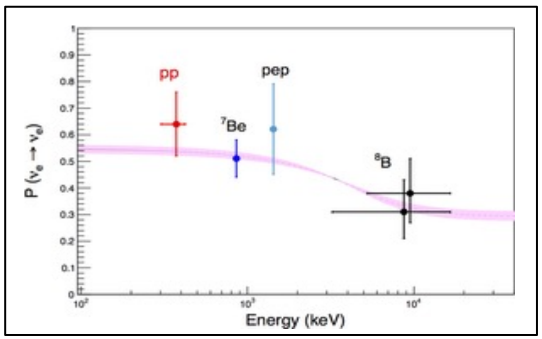
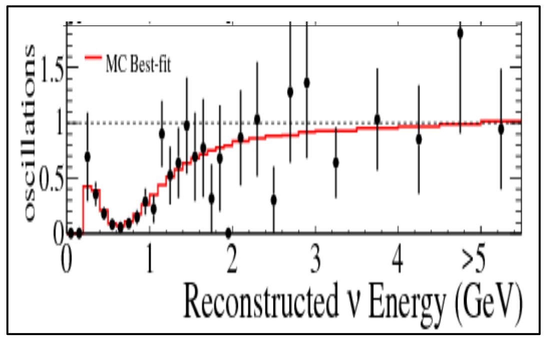
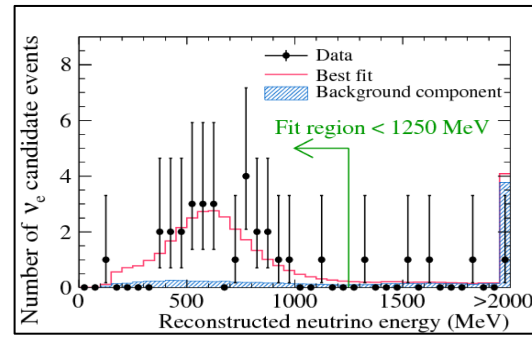


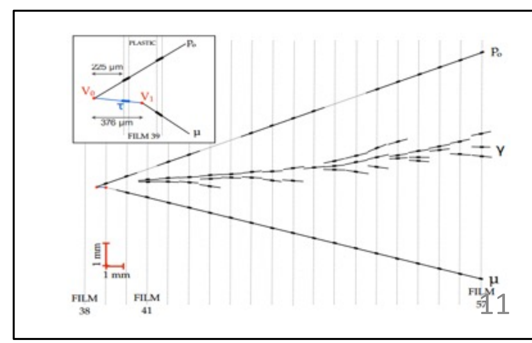
$\mu \to \mu$ $(\Delta m^2, \theta_{23})$



$\mu \to e$ $(\Delta m^2, \theta_{13}, \theta_{23})$



$\mu \to \tau$ $(\Delta m^2, \theta_{23})$



Established so far:

δm^2 $|\Delta m^2|$ θ_{12} θ_{23} θ_{13}

δm^2	$\sim 7 \times 10^{-5}$ eV^2	← "small" splitting
$\|\Delta m^2\|$	$\sim 2 \times 10^{-3}$ eV^2	← "large" splitting
$\sin^2\theta_{12}$	~ 0.3	← "large" 12 mixing
$\sin^2\theta_{23}$	~ 0.5	← "nearly maximal" 23 mixing
$\sin^2\theta_{13}$	~ 0.02	← "small" 13 mixing

We shall now deal with

3ν flavor evolution

in vacuum and in matter

δm^2	$\sim 7 \times 10^{-5}\ \mathrm{eV}^2$
$\lvert\Delta m^2\rvert$	$\sim 2 \times 10^{-3}\ \mathrm{eV}^2$
$\sin^2\theta_{12}$	~ 0.3
$\sin^2\theta_{23}$	~ 0.5
$\sin^2\theta_{13}$	~ 0.02

The presence of two small dimensionless parameters,

$$\delta m^2 / \Delta m^2 \sim 3 \times 10^{-2}$$

$$\sin^2\theta_{13} \sim 2 \times 10^{-2}$$

will allow useful $3\nu \rightarrow 2\nu$ approximations and simplify the understanding of phenomenology (next Lecture)

We shall now deal with

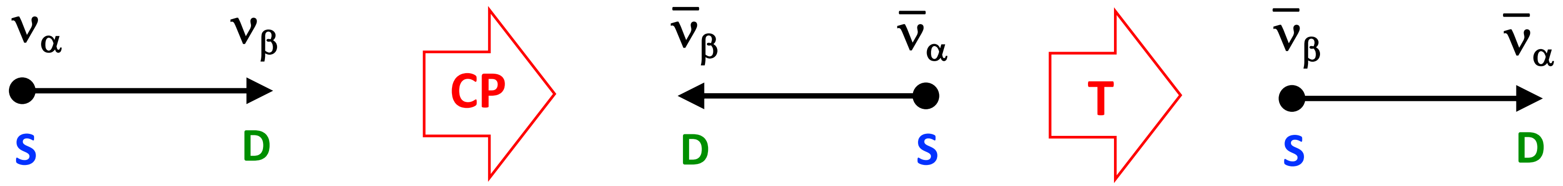
3ν flavor evolution

in vacuum and in matter

Three-neutrino flavor evolution: CP, T symmetries *(intuitive approach)*

C = charge conjugation (particle-antiparticle exchange)
P = parity (space reversal)
T = time reversal

Action of **CP** and **T** on $\nu_\alpha \rightarrow \nu_\beta$ oscillations from source **S** to detector **D**:



(CP violation: one of the Sakharov conditions to generate matter-antimatter asymmetry in the Universe)

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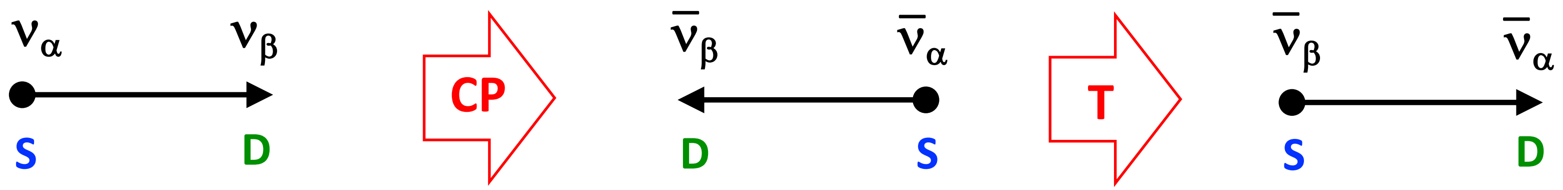
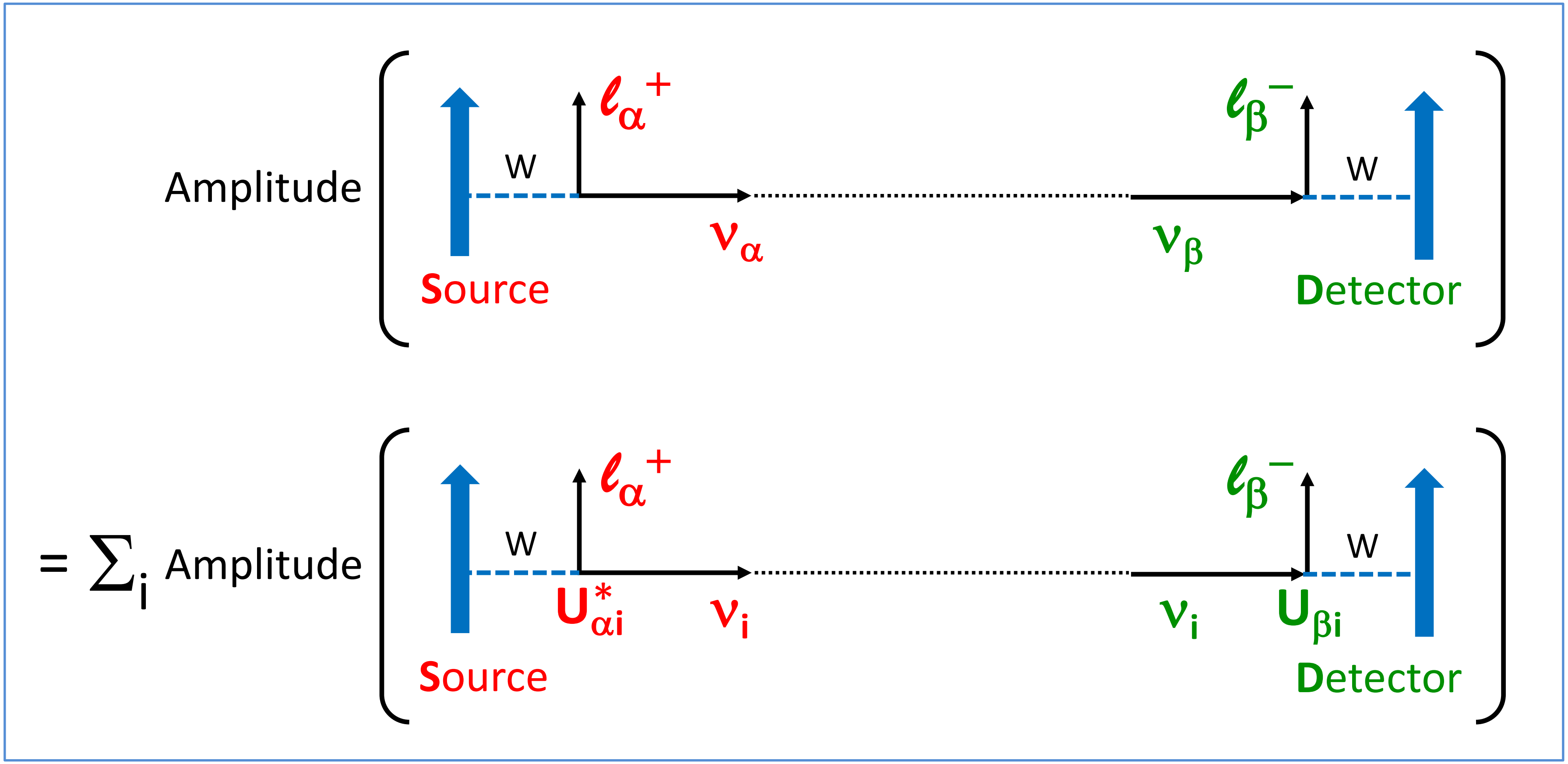


If **CP** invariance holds, then $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)$ $\Leftarrow$ $(\nu \leftrightarrow \bar{\nu})$

If **T** invariance holds, then $\begin{cases} P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha) \\ P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \end{cases}$ $\Leftarrow$ $(\alpha \leftrightarrow \beta)$

If **CPT** invariance holds, then $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha)$ $\Leftarrow$ $(\nu \leftrightarrow \bar{\nu}) \oplus (\alpha \leftrightarrow \beta)$

From Lecture I:

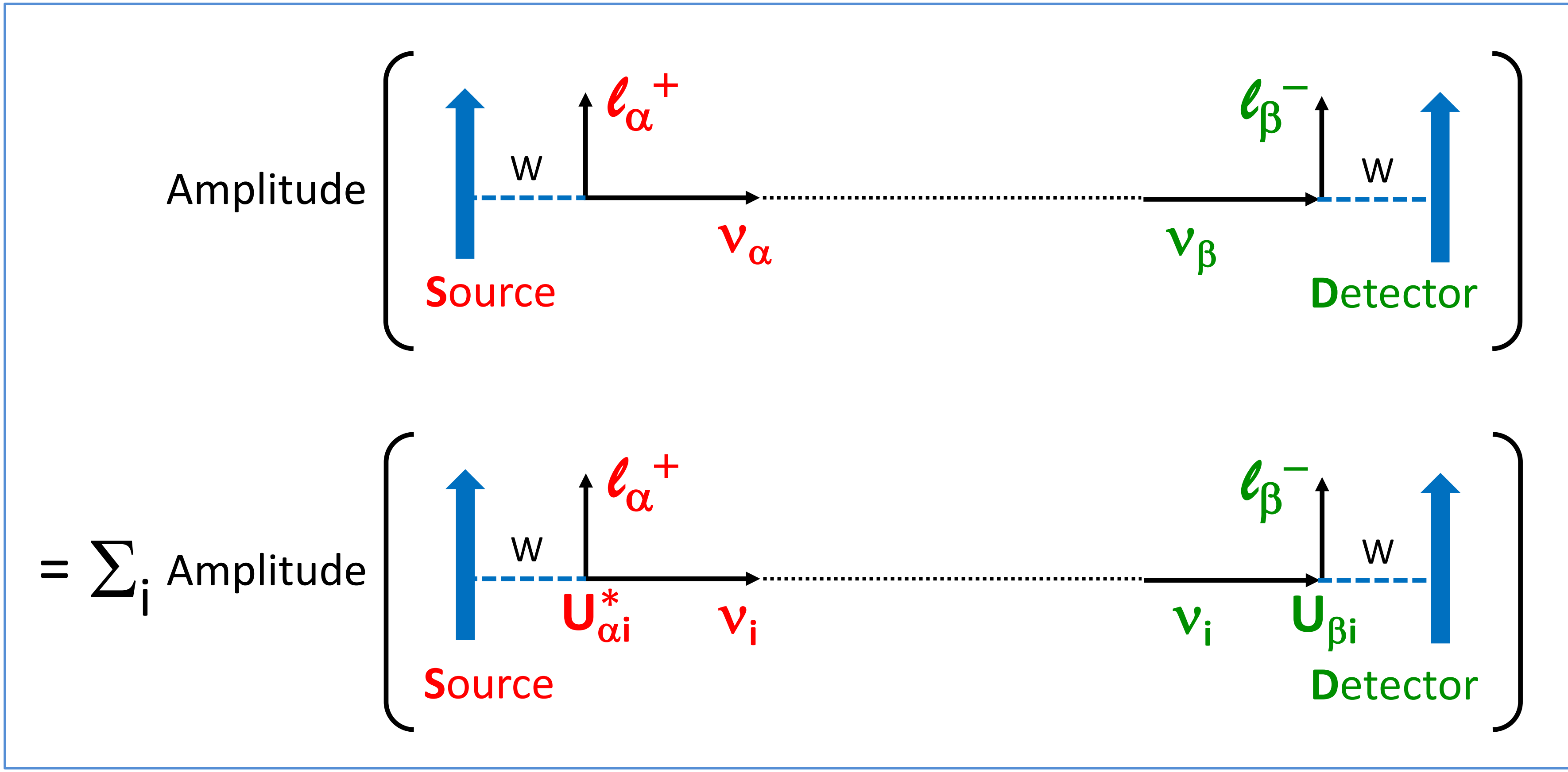


for $\nu \rightarrow \bar{\nu}$

$\ell_\alpha^\pm \rightarrow \ell_\alpha^\mp$

$U \rightarrow U^*$

From Lecture I:



Also, swapping initial/final flavors is equivalent to

$$U \rightarrow U^*$$

for $\nu \rightarrow \bar{\nu}$

$$\ell_\alpha^\pm \rightarrow \ell_\alpha^\mp$$

$$U \rightarrow U^*$$

$$(\nu \to \bar{\nu}) \Leftrightarrow U \to U^*$$
$$(\alpha \leftrightarrow \beta) \Leftrightarrow U \to U^*$$

If **CP** invariance holds, then $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) \quad \Leftarrow \quad (\nu \leftrightarrow \bar{\nu})$

If **T** invariance holds, then $\begin{cases} P(\nu_\alpha \to \nu_\beta) = P(\nu_\beta \to \nu_\alpha) \\ P(\bar{\nu}_\alpha \to \bar{\nu}_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \end{cases} \quad \Leftarrow \quad (\alpha \leftrightarrow \beta)$

If **CPT** invariance holds, then $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha) \quad \Leftarrow \quad (\nu \leftrightarrow \bar{\nu}) \oplus (\alpha \leftrightarrow \beta)$

CP and **T** invariance hold iff **U real** ($\sin\delta = 0$)
CPT invariance holds for **any U** (as it should)

Exercise: General 3ν oscillations in vacuum

The oscillation probability in vacuum can be written as:

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} x}{4E}\right) - 2\sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij} x}{2E}\right)$$

where:

$$\Delta m^2_{ij} = m^2_i - m^2_j$$

$$J^{ij}_{\alpha\beta} = U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}$$

This formula applyes to any flavor α and β:

$\alpha \neq \beta$ → appearance channel, appearance probability
$\alpha = \beta$ → disappearance channel, survival probability

Despite its simplicity, it contains a lot of physics…

CP properties:

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J_{\alpha\beta}^{ij} \sin^2\left(\frac{\Delta m_{ij}^2 x}{4E}\right) - 2\sum_{i<j} \mathrm{Im}\, J_{\alpha\beta}^{ij} \sin\left(\frac{\Delta m_{ij}^2 x}{2E}\right)$$

CP-conserving part P^{CP}
not changing for U↔U*

CP-violating part P^{CPV}
changes sign for U↔U*

Re

Im

$$J_{\alpha\beta}^{ij} = U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j}$$

For $\alpha = \beta$: $\mathrm{Im}(J_{\alpha\beta}^{ij}) = 0$ → **CP violation** must be probed in appearance, α≠β

Exercise: J = Jarlskog invariant

Using the previous PDG convention for the mixing matrix, it's easy to find that:

$$J = \mathrm{Im}(J_{e\mu}^{12}) = \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta$$

Then, using the unitarity of U, for $\alpha \neq \beta$ it is:

$$\mathrm{Im}(J_{\alpha\beta}^{ij}) = \begin{cases} +J & \text{for } (\alpha,\beta) = (e,\mu),\ (\mu,\tau),\ (\tau,e) \quad \leftarrow \text{flavor cyclic} \\ +J & \text{for } (i,j) = (1,2),\ (2,3),\ (3,1) \quad \leftarrow \text{generation cyclic} \\ -J & \text{otherwise} \end{cases}$$

Note that J changes sign from neutrinos to antineutrinos in the same channel

EPS Prize 2023 to Cecilia Jarlskog:
for the discovery of an invariant measure of CP violation in both quark and lepton sectors
EPS Prize 2023 also to … wait for a few more slides!

Exercise: P^{CPV} in product form

Using J, the term P^{CPV} can be written as

$$P_{\alpha\beta}^{\mathrm{CPV}} = \pm 8J \sin\left(\frac{\Delta m_{12}^2 x}{4E}\right) \sin\left(\frac{\Delta m_{23}^2 x}{4E}\right) \sin\left(\frac{\Delta m_{31}^2 x}{4E}\right)$$

Summary of conditions to have CP violation

$$J = \mathrm{Im}(J^{12}_{e\mu}) = \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta$$

$$P^{\mathrm{CPV}}_{\alpha\beta} = \pm 8J\sin\left(\frac{\Delta m^2_{12}x}{4E}\right)\sin\left(\frac{\Delta m^2_{23}x}{4E}\right)\sin\left(\frac{\Delta m^2_{31}x}{4E}\right)$$

⇩

$\delta \neq 0, \pi$ ← U must be complex, $\sin\delta \neq 0$

$\alpha \neq \beta$ ← Need appearance experiments

$\theta_{ij} \neq 0$ ← All mixing angles should be ≠0

$\Delta m^2_{ij} \neq 0$ ← Need some sensitivity to both δm^2 and Δm^2

CP-violating oscillations involve all the mass-mixing parameters

(genuine 3ν phenomenon, experimentally challenging!)

Summary of conditions to have CP violation

$$J = \mathrm{Im}(J_{e\mu}^{12}) = \frac{1}{8}\sin 2\theta_{12}\sin 2\theta_{23}\sin 2\theta_{13}\cos\theta_{13}\sin\delta$$

2012: θ_{13} is nonzero!

2023 EPS High Energy and Particle Physics Prize is awarded to

Cecilia Jarlskog for the discovery of an invariant measure of CP violation in both quark and lepton sectors; and to the

Daya Bay and RENO collaborations for the observation of short-baseline reactor electron-antineutrino disappearance, providing the first determination of the neutrino mixing angle θ_{13}, which paves the way *for* the detection of CP violation in the lepton sector.

At present: only hints of leptonic CPV… (Lecture IV)

In principle… All the oscill. parameters + CP phase + NO/IO might be determined by precise measurements of $P_{\alpha\beta}$ for selected channels and L/E:

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} x}{4E}\right) - 2\sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij} x}{2E}\right)$$

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In practice… We never measure $P_{\alpha\beta}$ but event rates R, and do it only for experimentally feasible oscillation channels and L/E:

$$\mathbf{R}_\beta \sim \int \mathbf{\Phi}_\alpha \otimes \mathbf{P}_{\alpha\beta} \otimes \boldsymbol{\sigma}_\beta \otimes \epsilon_\beta$$

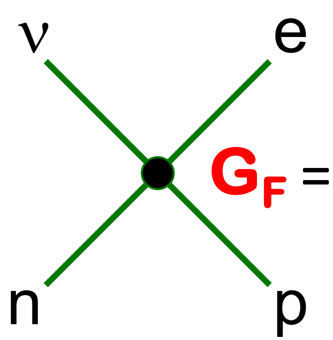
Observable event rate	**Source flux (production)**	**Propagation (flavor change)**	**Interaction and detection**

→ **each ingredient of R is a research field of its own**

→ **need to account for a vast ν phenomenology**

→ **must consider realistic propagation, e.g., in matter**

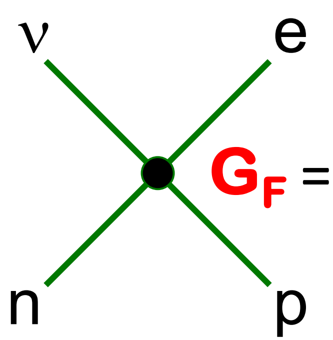
Three-neutrino flavor evolution in matter *(intuitive approach)*



$G_F = 1.166 \times 10^{-5}\ \text{GeV}^{-2}$ → Sets the energy scale $\sqrt{(1/G_F)} \sim O(\text{few } 10^2)$ GeV of weak interactions

Amplitude squared → Neutrino cross section $\propto G_F^2$

Three-neutrino flavor evolution in matter *(intuitive approach)*



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Amplitude squared → Neutrino cross section $\propto G_F^2$

During propagation, **absorption processes** $\propto G_F^2$ are negligible, except at very high energy. E.g., Earth starts to be opaque to ν for E > O (10) TeV.

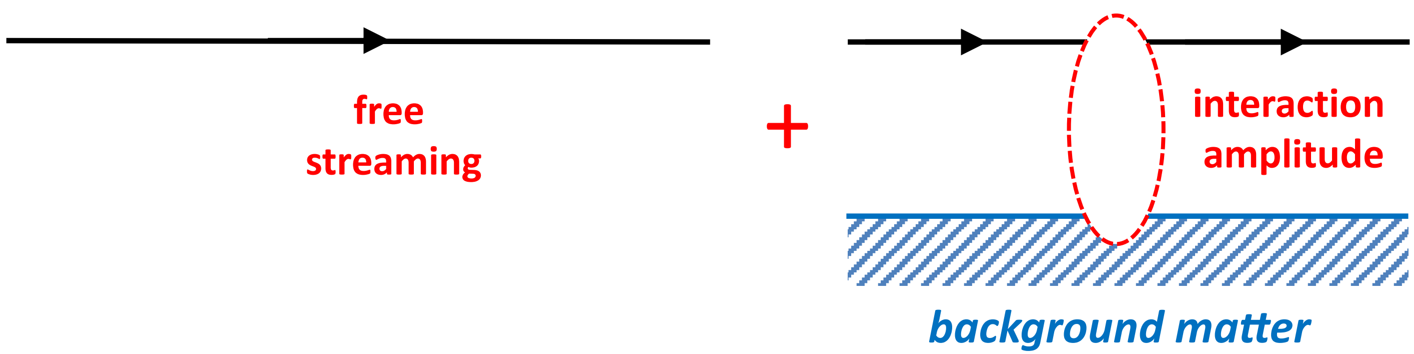
But: Are also **scattering amplitudes** $\propto G_F$ irrelevant for ν flavor evolution? Not necessarily, if the process occurs along the direction of propagation!

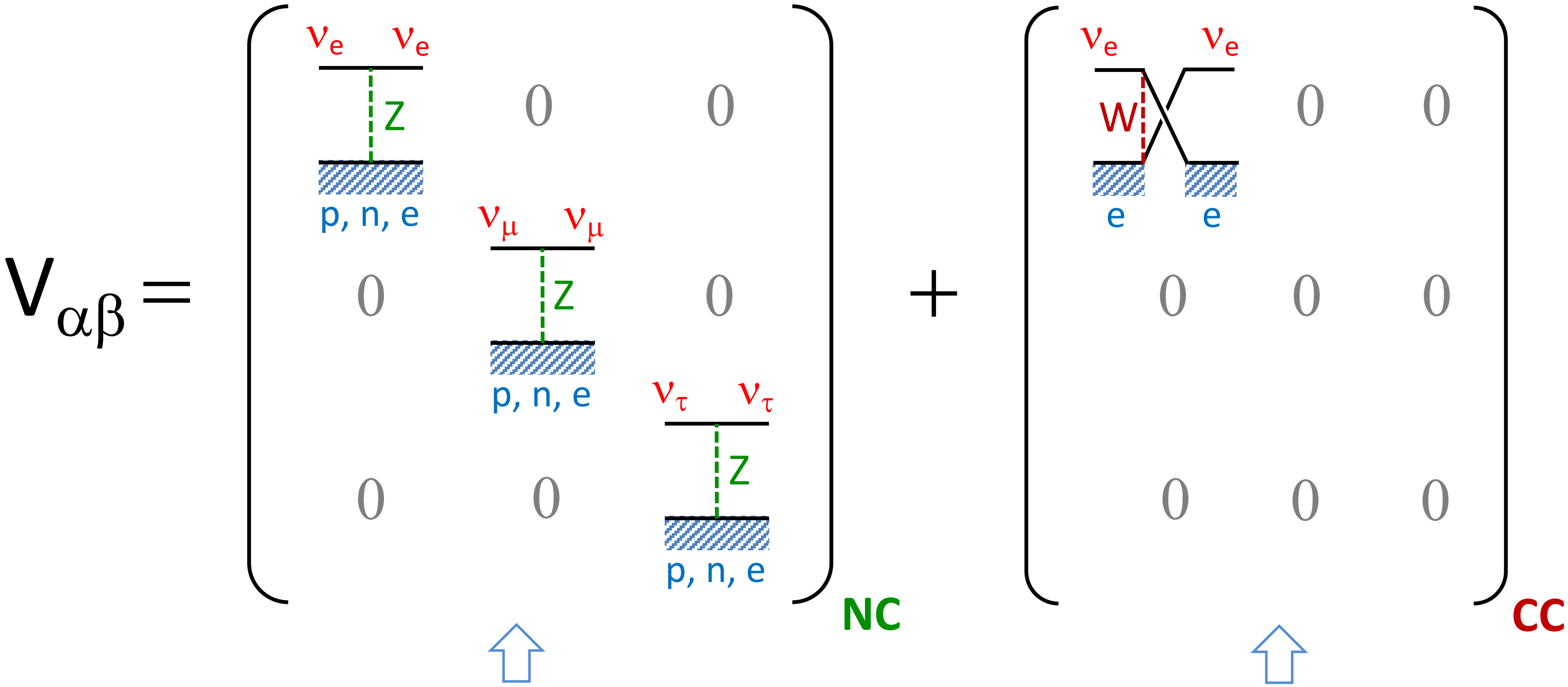
It was first realized by Wolfenstein, and later elaborated by Mykheev and Smirnov (**MSW**), that neutrinos travelling in a fermion background receive a contribution to coherent forward scattering (i.e., along the same direction of propagation) in the form of a tiny interaction energy $\mathbf{V}_{\alpha\beta}$

3ν Hamiltonian in matter:

vacuum (kinematics) | matter (dynamics)

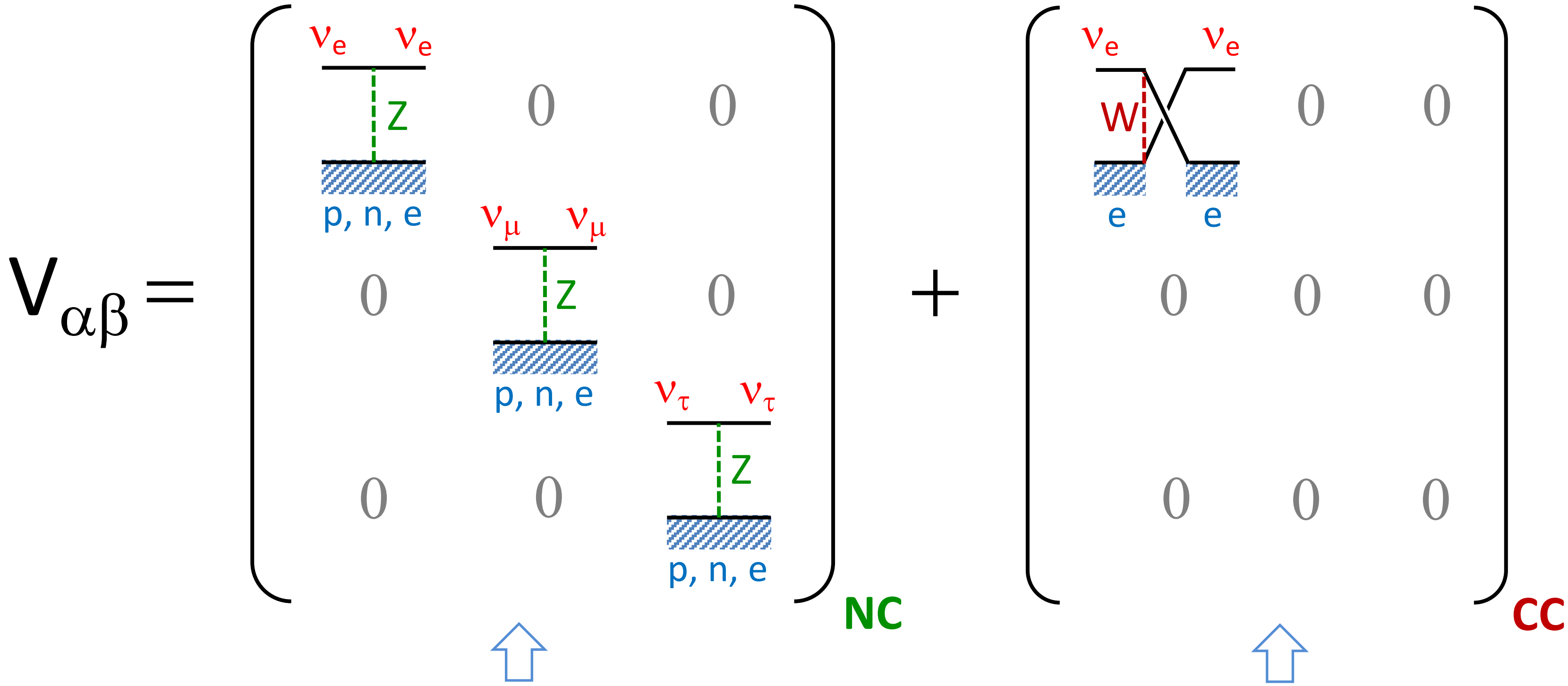
$$H_f = \frac{1}{2E} U \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} V_{ee} & V_{e\mu} & V_{e\tau} \\ V_{\mu e} & V_{\mu\mu} & V_{\mu\tau} \\ V_{\tau e} & V_{\tau\mu} & V_{\tau\tau} \end{pmatrix}$$





proportional to unity
→ unobservable

observable in
ν_e-related oscill.



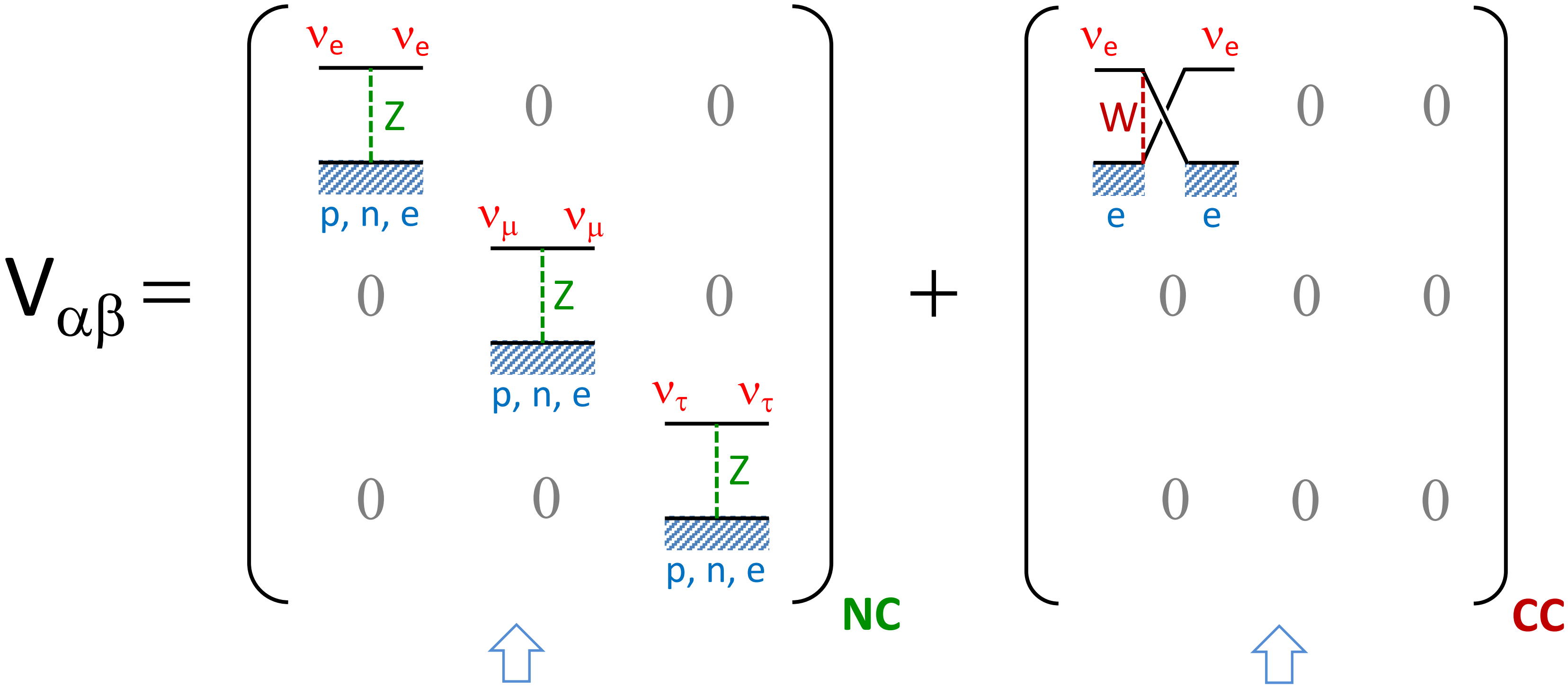
proportional to unity
→ unobservable

observable in
ν_e-related oscill.

Relevant term is the extra "*ee*" energy (ν potential) **$V_{CC} \propto G_F$**

[No μ,τ in ordinary matter → no "$\mu\mu$" or "$\tau\tau$" CC term]

Potential proportional to e^- number density **$V_{CC} \propto N_e$**



proportional to unity
→ unobservable

observable in
ν_e-related oscill.

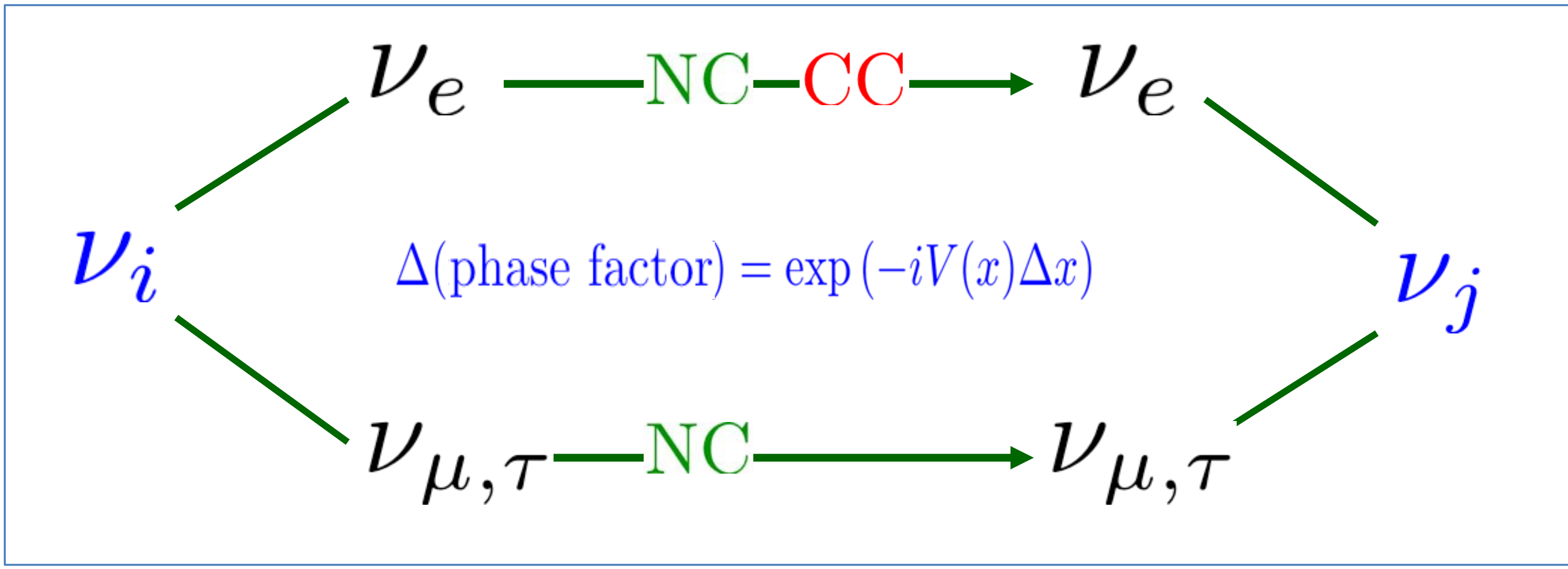
Relevant term is the extra "*ee*" energy (ν potential) **$V_{CC} \propto G_F$**

Potential proportional to e^- number density **$V_{CC} \propto N_e$**

Full calculation (omitted): **$V\ (= V_{CC}) = \sqrt{2}\, G_F\, N_e$**

As anticipated in Lecture I:

Analogy of matter effects with double-slit experiment:
one "arm" (e-flavor) feels a different "refraction index"
through coherent forward scattering (not absorption!)



Governed by a tiny ν "interaction energy" or "potential" V
In general, V=V(x) via $N_e=N_e(x)$ → x-dependent hamiltonian.
Not necessarily periodic effects: oscillations → transitions

3ν MSW hamiltonian in matter:

$$H_f(x) = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}U^\dagger + \begin{pmatrix} A(x) & & \\ & 0 & \\ & & 0 \end{pmatrix}\right]$$

where A=2EV is introduced to make the vacuum and matter terms more similar:

$$A(x) = 2EV = 2\sqrt{2}G_F N_e(x) E$$ [$-A(x)$ for anti-ν]

Huge literature on solutions for various A(x). In general:

Expect **sizeable matter effects** when the two terms in **H** are comparable,

$$\frac{A}{\delta m^2} \sim O(1)$$ or $$\frac{A}{\Delta m^2} \sim O(1)$$

Exercise: Units for matter effects

$$\frac{A}{\Delta m^2_{ij}} = 1.526 \times 10^{-7} \left(\frac{N_e}{\mathrm{mol/cm^3}} \right) \left(\frac{E}{\mathrm{MeV}} \right) \left(\frac{\mathrm{eV^2}}{\Delta m^2_{ij}} \right)$$

Note: **N_e** is the number of electrons per unit volume. If the chemical composition is known (say, the **average Z/A**), one can connect N_e and the matter density ρ:

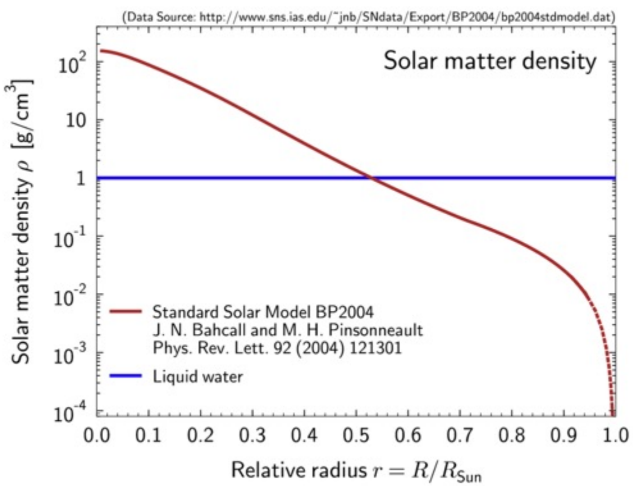
$$\frac{N_e}{\mathrm{mol/cm^3}} \simeq \left\langle \frac{Z}{A} \right\rangle \frac{\rho}{\mathrm{g/cm^3}}$$

electron fraction $Y_e \sim 1/2$

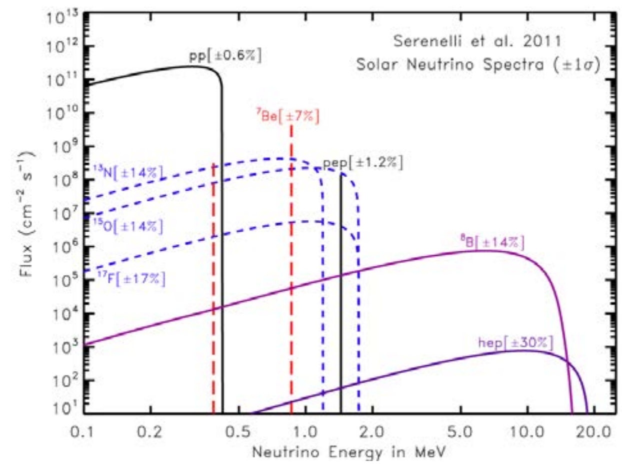
Order-of-magnitude expectations →

Solar neutrinos

Solar density profile



Solar neutrino spectrum

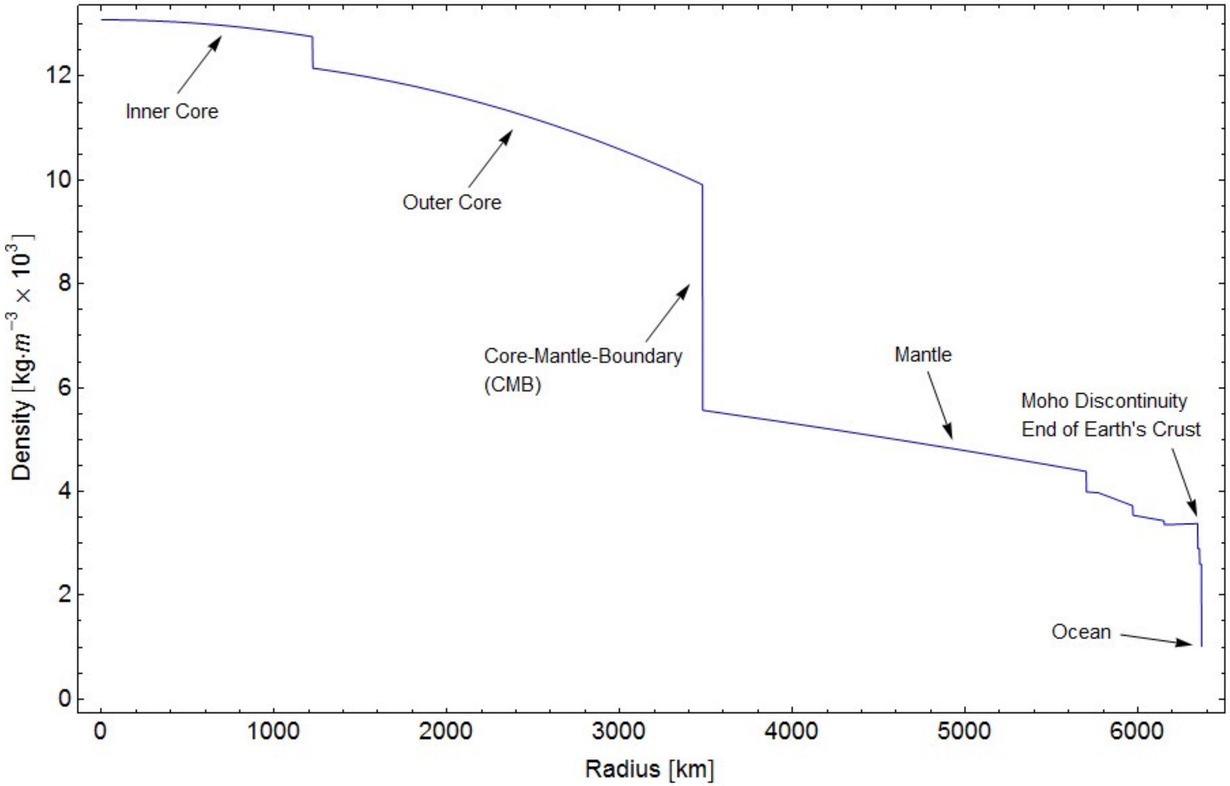


$$\frac{A_{\rm Sun}(0)}{\delta m^2} \sim O(1)$$ just in the middle of the E spectrum…

$$\frac{A_{\rm Earth}}{\delta m^2} \sim O({\rm few} \times 10^{-2})$$ at $E \sim O(10)$ MeV….

Large effects in solar matter; subleading day-night effects in Earth

Earth density profile

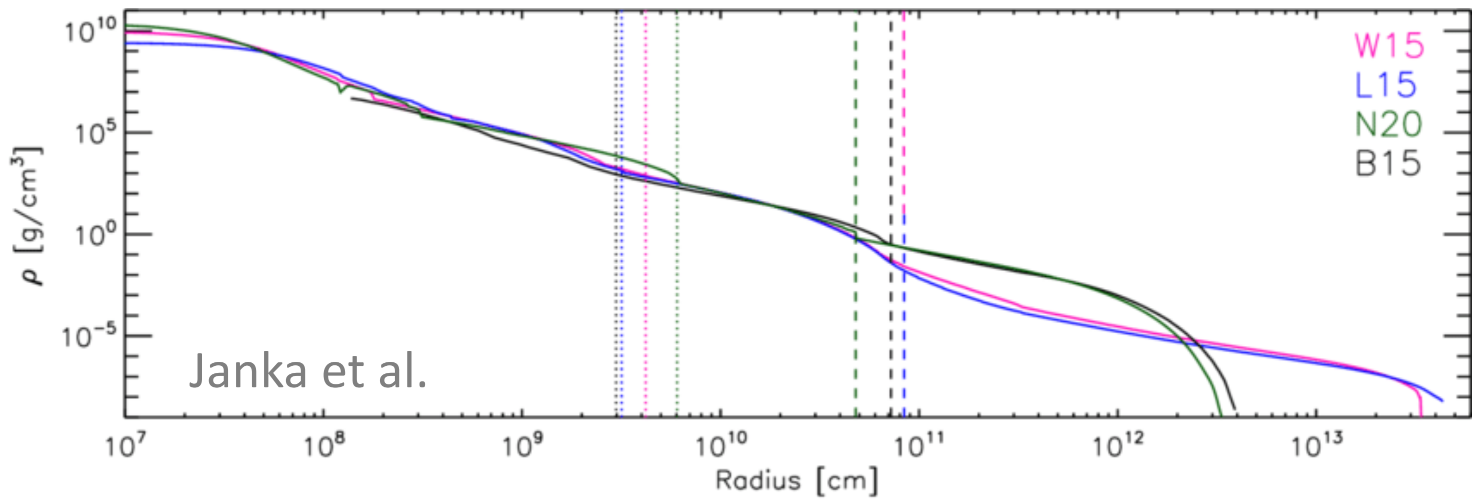


$$\frac{A_{\rm crust}}{\Delta m^2} \sim O(10^{-1})$$ at $E \sim O(1)$ GeV....

Subleading matter effects in long-baseline accelerator neutrinos

Depend on sign of numerator vs denominator: handle on NO/IO

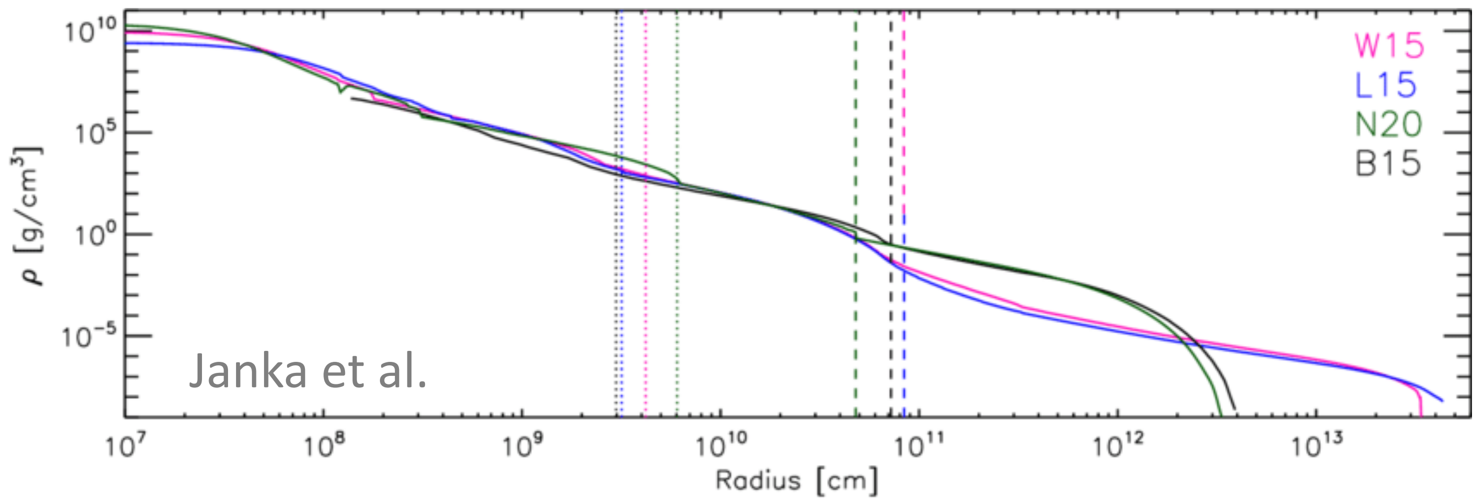
Supernova neutrino density profile(s)



The most dynamical (radius- and time-dependent) matter background:

$N_e = N_e(x,t)$

Supernova neutrino density profile(s)



The most dynamical (radius- and time-dependent) matter background:

$N_e = N_e(x,t)$

Moreover, for a few seconds, neutrinos are a background to themselves!

$N_\nu \sim O(N_e)$

→ "Self-interaction" effects, "collective" highly-nonlinear flavor evolution

$H=H(\nu)$ → density matrix formalism

Physics & math to be (better) understood: a research topic in its own!

3ν MSW hamiltonian in matter:

$$H_f(x) = \frac{1}{2E}\left[U\begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}U^\dagger + \begin{pmatrix} A(x) & & \\ & 0 & \\ & & 0 \end{pmatrix}\right]$$

Huge related literature on numerical and (semi)analytical solutions at given A(x)

Oscillating behavior makes numerical solutions prone to error accumulation:
brute-force application of Runge-Kutta codes may fail [published examples…]
(Semi)Analytical solutions/approximations useful whenever A(x) is "simple"

We shall consider the two simplest cases of phenomenological interest:

A(x) = constant	**[dA/dx = 0]**	**←**	**constant density case**
A(x) = slowly varying	**[dA/dx = "small"]**	**←**	**adiabatic case**
[A(x) = rapidly varying	**[dA/dx = "large"]**	**←**	**non-adiabatic case]**

Effective parameters. In general, $H_f(x)$ can be diagonalized at each point x:

$$H_f(x) = \frac{1}{2E}\left[U \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix} U^\dagger + \begin{pmatrix} A(x) & & \\ & 0 & \\ & & 0 \end{pmatrix}\right]$$

$$= \frac{1}{2E}\tilde{U}(x) \begin{pmatrix} \tilde{m}_1^2(x) & & \\ & \tilde{m}_2^2(x) & \\ & & \tilde{m}_3^2(x) \end{pmatrix} \tilde{U}^\dagger(x)$$

where

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \tilde{U}(x) \begin{bmatrix} \tilde{\nu}_1(x) \\ \tilde{\nu}_2(x) \\ \tilde{\nu}_3(x) \end{bmatrix}$$

"effective..." ...mixing matrix ...mass states ...squared masses... ...in matter

A = constant: diagonalize only once (at any energy E) and exponentiate

$$\tilde{S}_f = e^{-i\tilde{H}_f x}$$

Get the same probability functions as in vacuum, but with "effective" energy-dependent mass-mixing oscillation parameters in matter.
Relevant application: matter effects in LBL accelerator experiments

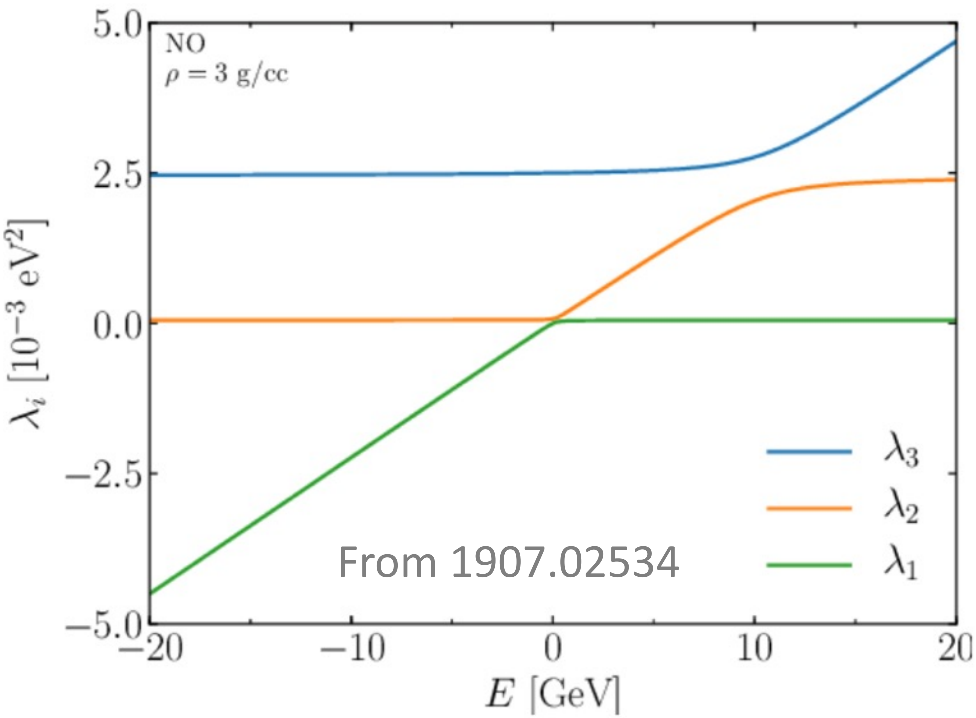
A = constant: diagonalize only once (at any energy E) and exponentiate

$$\tilde{S}_f = e^{-i\tilde{H}_f x}$$

Get the same probability functions as in vacuum, but with "effective" energy-dependent mass-mixing oscillation parameters in matter.
Relevant application: matter effects in LBL accelerator experiments

Examples of effective $\tilde{m}^2_i$ for Earth crust density as functions of energy E

[Here, formally E<0 for anti-nu, since A → –A]

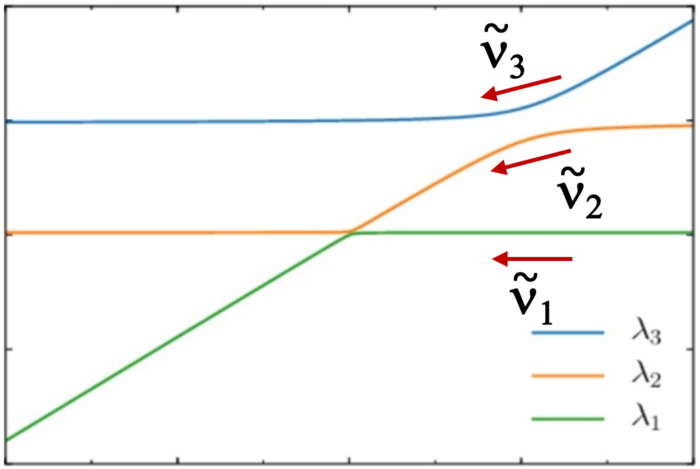


A(x) slowly varying: diagonalize step by step, and patch the solutions

Exercise: $$d\tilde{U}/dx \simeq 0 \;\Rightarrow\; \tilde{\nu}_i(0) \to \tilde{\nu}_i(x)$$

i.e., the effective mass states in matter evolve independently.
The adiabaticity condition can be formulated precisely (omitted).
Applicable, e.g., in stars with smoothly decreasing density…

← **decreasing A**



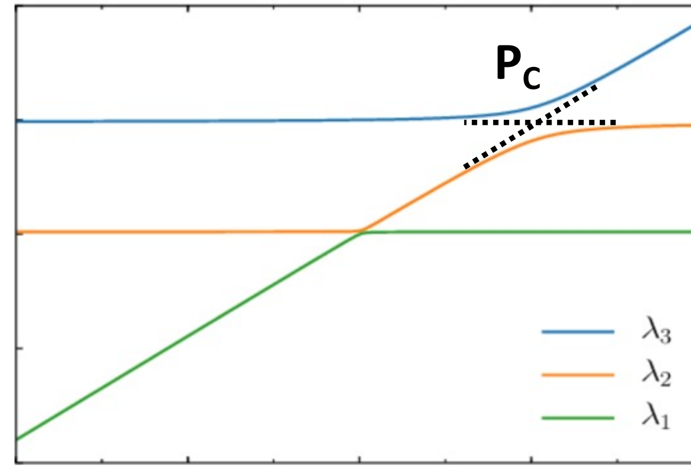
…. for the oscillation parameters chosen by Nature!
Relevant applications: (1) matter effects for solar neutrinos
(2) for SN neutrinos, up to shock-wave and collective effects

A(x) rapidly varying: "crossing" probability between effective states

[In QM: "tunnelling" between E eigenstates subject to rapid externals field variations]

$$P_c = P(\tilde{\nu}_i \to \tilde{\nu}_j)$$

For a 2-level QM system:
Solved independently by
Majorana, Landau, Zener, Stueckelberg in 1932
(at leading order)



This would have happened to solar neutrinos at very small mixing!
[so-called Small Mixing Angle MSW solution, a prejudice for many years…]

Relevant applications: (1) steps in Earth density profile
(2) Shock-wave front in SN neutrinos (discussion omitted)

Similar effects (re)analyzed independently in many subfields of physics but with different "jargon" – e.g. recently in qbit manipulations across QM levels

Recap

3ν oscillations in vacuum (may be CP violating):

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J_{\alpha\beta}^{ij} \sin^2\left(\frac{\Delta m_{ij}^2 x}{4E}\right) - 2\sum_{i<j} \mathrm{Im}\, J_{\alpha\beta}^{ij} \sin\left(\frac{\Delta m_{ij}^2 x}{2E}\right)$$

Vacuum oscillation effects expected to be significant when:

$$\frac{\delta m^2 L}{4E} \sim O(1)$$

or

$$\frac{\Delta m^2 L}{4E} \sim O(1)$$

Recap

3ν oscillations in vacuum (may be CP violating):

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4\sum_{i<j} \mathrm{Re}\, J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} x}{4E}\right) - 2\sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij} x}{2E}\right)$$

Vacuum oscillation effects expected to be significant when:

$$\frac{\delta m^2 L}{4E} \sim O(1)$$ or $$\frac{\Delta m^2 L}{4E} \sim O(1)$$

$P_{\alpha\beta}$ in matter (constant or slowly changing): vacuum param. → effective param.

Matter effects expected to be significant when:

$$\frac{A}{\delta m^2} \sim O(1)$$ or $$\frac{A}{\Delta m^2} \sim O(1)$$

End of Lecture II

Solutions to exercises: extra slides →

Exercise: General 3ν oscillations in vacuum

- Flavor evolution operator: $S_f = U S_m U^+$

  where: $S_m = \text{diag}\left(e^{-i\frac{m_1^2 x}{2E}},\ e^{-i\frac{m_2^2 x}{2E}},\ e^{-i\frac{m_3^2 x}{2E}}\right)$

- Inserting indices: $S_{\beta\alpha} = \sum_i U_{\alpha i}^* U_{\beta i}\, e^{-i\frac{m_i^2 x}{2E}}$ for any $\alpha, \beta \in (e, \mu, \tau)$

- Flavor oscillation probability is obtained as $P = |S|^2$ by reorganizing terms:

$$
\begin{aligned}
P(\nu_\alpha \to \nu_\beta) &= |S_{\beta\alpha}|^2 \\
&= \left| \sum_i U_{\alpha i}^* U_{\beta i}\, e^{-i\frac{m_i^2}{2E}x} \right|^2 \\
&= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*\, e^{-i\frac{m_i^2}{2E}x}\, e^{+i\frac{m_j^2}{2E}x} \\
&= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left(e^{i\frac{m_j^2 - m_i^2}{2E}x} - 1 + 1\right) \\
&= \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \left(e^{i\frac{m_j^2 - m_i^2}{2E}x} - 1\right) + \sum_{ij} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*
\end{aligned}
$$

$\longrightarrow$ cont'd

$$= \left(\sum_{i<j} + \sum_{i>j} \right) U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \left(e^{i \frac{m^2_j - m^2_i}{2E} x} - 1 \right) + \sum_i U^*_{\alpha i} U_{\beta i} \sum_j U_{\alpha j} U^*_{\beta j}$$

$$= \sum_{i>j} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \left(e^{i \frac{m^2_j - m^2_i}{2E} x} - 1 \right) + \sum_{i>j} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} \left(e^{-i \frac{m^2_j - m^2_i}{2E} x} - 1 \right) + \delta_{\alpha\beta} \cdot \delta_{\alpha\beta}$$

$$= \sum_{i>j} \left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} + U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} \right) \left[\cos\left(\frac{m^2_j - m^2_i}{2E} x \right) - 1 \right]$$

$$+ \sum_{i>j} \left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} - U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} \right) \left[i \sin\left(\frac{m^2_j - m^2_i}{2E} x \right) \right] + \delta_{\alpha\beta}$$

$$= \delta_{\alpha\beta} - \sum_{i>j} 2 \,\mathrm{Re}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \right) \left[\cos\left(\frac{m^2_j - m^2_i}{2E} x \right) - 1 \right] - \sum_{i>j} 2 \,\mathrm{Im}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \right) \sin\left(\frac{m^2_j - m^2_i}{2E} x \right)$$

$$= \delta_{\alpha\beta} - 4 \sum_{i>j} \mathrm{Re}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \right) \sin^2\left(\frac{m^2_j - m^2_i}{4E} x \right) - 2 \sum_{i>j} \mathrm{Im}\left(U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} \right) \sin\left(\frac{m^2_j - m^2_i}{2E} x \right)$$

$$\overset{i \leftrightarrow j}{=} \delta_{\alpha\beta} - 4 \sum_{i<j} \mathrm{Re}\, J^{ij}_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij}}{4E} x \right) - 2 \sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij}}{2E} x \right)$$

where
$$\begin{cases} \Delta m^2_{ij} = m^2_i - m^2_j \\ J^{ij}_{\alpha\beta} = U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} \end{cases}$$

Exercise: J = Jarlskog invariant

- Define $J \equiv \mathrm{Im}(J^{12}_{e\mu})$
- For any $\alpha \neq \beta$, it turns out that $\mathrm{Im}(J^{ij}_{\alpha\beta}) = \pm J$ where $\begin{cases} +J & (\alpha,\beta) \text{ cyclic over } (e,\mu,\tau) \\ +J & (i,j) \text{ '' '' } (1,2,3) \\ -J & \text{otherwise} \end{cases}$
- This can be checked by inspection of cases. For instance:
- Let's work out $J^{12}_{e\tau}$ as:

$$
\begin{aligned}
\mathrm{Im}(J^{12}_{e\tau}) &= \mathrm{Im}(U_{e1}U^*_{\tau 1}U^*_{e2}U_{\tau 2}) \\
&= \mathrm{Im}(U_{e1}U^*_{e2}(-U^*_{e1}U_{e2} - U^*_{\mu 1}U_{\mu 2})) \qquad \leftarrow \text{since } UU^\dagger = 1 \\
&= \mathrm{Im}(-U_{e1}U^*_{e2}U^*_{\mu 1}U_{\mu 2}) = -\mathrm{Im}(J^{12}_{e\mu}) = -J \,. \qquad (\text{similarly for other } (\alpha,\beta))
\end{aligned}
$$

- Note that $\mathrm{Im}(J^{21}_{e\tau}) = +J = (-1)(-1)J$ since both $(e\tau)$ and (21) are anticyclic.
- One can also write: $\mathrm{Im}(J^{ij}_{\alpha\beta}) = J \cdot \sum_\gamma \epsilon_{\alpha\beta\gamma} \sum_k \epsilon_{ijk}$

where ϵ is the totally antisymmetric tensor of rank 3.

Exercise : P^{CPV} in product form

- To transform a sum into a product, we use the following identity:

  If $x+y+z=0$, then $\sin 2x + \sin 2y + \sin 2z = -4 \sin x \sin y \sin z$ (from trigonometry)

- The identity is applied to $\Delta m^2_{12} + \Delta m^2_{23} + \Delta m^2_{31} = 0$.

- $$P^{CPV}(\nu_\alpha \to \nu_\beta) = -2 \sum_{i<j} \mathrm{Im}\, J^{ij}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{ij}}{2E} x\right)$$

$$= -2\left[\mathrm{Im}\, J^{12}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{12} x}{2E}\right) + \mathrm{Im}\, J^{23}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{23}}{2E} x\right) + \mathrm{Im}\, J^{13}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{13} x}{2E}\right)\right]$$

$$= -2\, \mathrm{Im}\, J^{12}_{\alpha\beta}\left[\sin\left(\frac{\Delta m^2_{12} x}{2E}\right) + \sin\left(\frac{\Delta m^2_{23}}{2E} x\right) - \sin\left(\frac{\Delta m^2_{13} x}{2E}\right)\right]$$

where $-\sin\left(\frac{\Delta m^2_{13} x}{2E}\right) = +\sin\left(\frac{\Delta m^2_{31} x}{2E}\right)$

$$= +8\, \mathrm{Im}\, J^{12}_{\alpha\beta} \sin\left(\frac{\Delta m^2_{12} x}{4E}\right) \sin\left(\frac{\Delta m^2_{23} x}{4E}\right) \sin\left(\frac{\Delta m^2_{31} x}{4E}\right)$$

$$= +8\, \mathrm{Im}\, J^{12}_{\alpha\beta} \prod_{(ij)}^{\text{cyclic}} \sin\left(\frac{\Delta m^2_{ij} x}{4E}\right)$$

Exercise : Useful units for matter effects

- Remember that:
$$\begin{cases} 1 \text{ mol} = N_A \text{ particles} = 6.022 \times 10^{23} \text{ particles} \\ 1 \text{ MeV} \cdot 1 \text{ m} = 5.0677 \times 10^{+12} \\ G_F = 1.1664 \times 10^{-5} \text{ GeV}^{-2} = 1.1664 \times 10^{-11} \text{ MeV}^{-2} \end{cases}$$

- $$1 \frac{\text{mol}}{\text{cm}^3} = \frac{6.022 \times 10^{23}}{10^{-6} \text{m}^3} \left(\frac{\text{MeV}^3}{\text{MeV}^3} \right) = 6.022 \times 10^{29} \frac{\text{MeV}^3}{(\text{m} \cdot \text{MeV})^3} = \frac{6.022 \times 10^{29}}{(5.0677 \times 10^{12})^3} \text{MeV}^3 = 4.627 \times 10^{-9} \text{MeV}^3$$

- $$\frac{A}{\Delta m^2_{ij}} = \frac{2\sqrt{2} G_F N_e E}{\Delta m^2_{ij}} = 2\sqrt{2} \left(1.1664 \times 10^{-11} \text{MeV}^{-2} \right) \left(\frac{N_e}{\text{mol/cm}^3} \text{mol/cm}^3 \right) \left(\frac{E}{\text{MeV}} \cdot \text{MeV} \right) \left(\frac{\text{eV}^2}{\Delta m^2_{ij}} \frac{1}{\text{eV}^2} \right)$$

$$= 3.299 \times 10^{-11} \frac{\text{MeV}^{-2} \text{MeV}}{\text{eV}^2} \frac{\text{mol}}{\text{cm}^3} \left(\frac{N_e}{\text{mol/cm}^3} \right) \left(\frac{E}{\text{MeV}} \right) \left(\frac{\text{eV}^2}{\Delta m^2_{ij}} \right)$$

$$= 3.299 \times 10^{-11} \frac{10^{12}}{\text{MeV}^3} \times 4.627 \times 10^{-9} \text{MeV}^3 \left(\frac{N_e}{\text{mol/cm}^3} \right) \left(\frac{E}{\text{MeV}} \right) \left(\frac{\text{eV}^2}{\Delta m^2_{ij}} \right)$$

$$= 1.526 \times 10^{-7} \left(\frac{N_e}{\text{mol/cm}^3} \right) \left(\frac{E}{\text{MeV}} \right) \left(\frac{\text{eV}^2}{\Delta m^2_{ij}} \right)$$

Exercise: Adiabatic evolution

- In flavor basis one can write always:

$$i\frac{d}{dx}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = H_f\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} \quad \text{where} \quad H_f = \frac{1}{2E}\tilde{U}\begin{pmatrix}\tilde{m}_1^2 & & \\ & \tilde{m}_2^2 & \\ & & \tilde{m}_3^2\end{pmatrix}U \quad \text{and} \quad \begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \tilde{U}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix}$$

- Therefore:

$$i\frac{d}{dx}\tilde{U}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} = \left[\frac{1}{2E}\tilde{U}\begin{pmatrix}\tilde{m}_1^2 & & \\ & \tilde{m}_2^2 & \\ & & \tilde{m}_3^2\end{pmatrix}\tilde{U}^+\right]U\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} = \frac{1}{2E}\tilde{U}\begin{pmatrix}\tilde{m}_1^2 & & \\ & \tilde{m}_2^2 & \\ & & \tilde{m}_3^2\end{pmatrix}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix}$$

- If $d\tilde{U}/dx \simeq 0$:

$$i\frac{d}{dx}\tilde{U}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} \simeq \tilde{U}\, i\frac{d}{dx}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix}$$

- Multiply by $\tilde{U}^+$ and get:

$$i\frac{d}{dx}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} = \frac{1}{2E}\begin{pmatrix}\tilde{m}_1^2 & & \\ & \tilde{m}_2^2 & \\ & & \tilde{m}_3^2\end{pmatrix}\begin{pmatrix}\tilde{\nu}_1\\ \tilde{\nu}_2\\ \tilde{\nu}_3\end{pmatrix} \longrightarrow$$

no off-diagonal terms,
three decoupled equations,
the $\tilde{\nu}_i$ evolve independently.