Neutrino Oscillations

Lecture I

Eligio Lisi

(INFN, Bari, Italy)

Outline of lectures I-IV:

Lecture I
Pedagogical introduction + warm-up exercise

Lecture II
3ν osc. in vacuum and matter: notation and basic math

Lecture III
2ν approximations of phenomenological interest

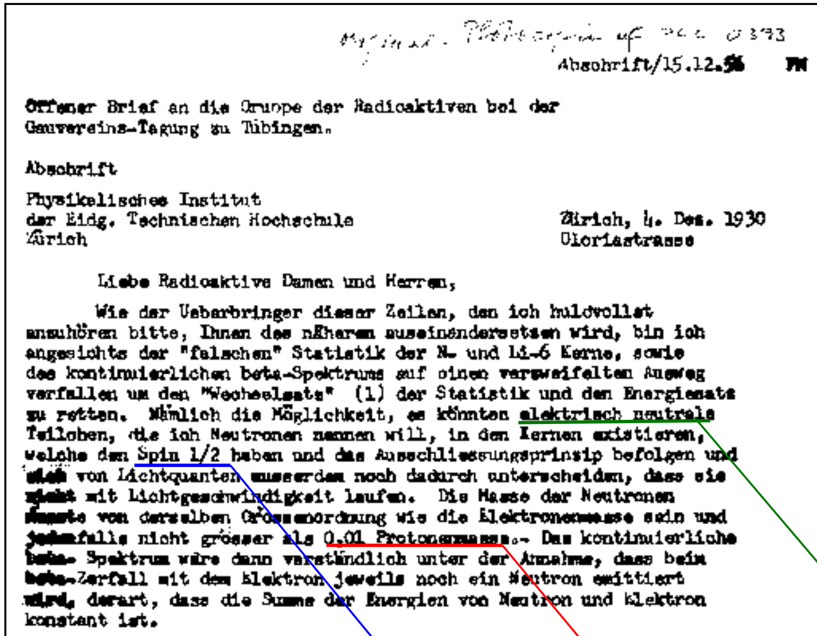
Lecture IV
Back to 3ν oscillations: Status and Perspectives

Feel free to stop me and ask questions at any time!

Pedagogical introduction

1930: ν hypothesis and first kinematical properties

A famous letter by Wolfgang Pauli:



Absohrift/15.12.56 PM

Offener Brief an die Gruppe der Radioaktiven bei der Gauvereins-Tagung zu Tübingen.

Abschrift

Physikalisches Institut
der Eidg. Technischen Hochschule
Zürich

Zürich, 4. Dez. 1930
Gloriastrasse

Liebe Radioaktive Damen und Herren,

Wie der Ueberbringer dieser Zeilen, den ich huldvollst anzuhören bitte, Ihnen des näheren auseinandersetzen wird, bin ich angesichts der "falschen" Statistik der N- und Li-6 Kerne, sowie des kontinuierlichen beta-Spektrums auf einen verzweifelten Ausweg verfallen um den "Wechselsatz" (1) der Statistik und den Energiesatz zu retten. Nämlich die Möglichkeit, es könnten elektrisch neutrale Teilchen, die ich Neutronen nennen will, in den Kernen existieren, welche den Spin 1/2 haben und das Ausschliessungsprinzip befolgen und sich von Lichtquanten ausserdem noch dadurch unterscheiden, dass sie nicht mit Lichtgeschwindigkeit laufen. Die Masse der Neutronen müsste von derselben Grössenordnung wie die Elektronenmasse sein und jedenfalls nicht grösser als 0.01 Protonenmasse.- Das kontinuierliche beta- Spektrum wäre dann verständlich unter der Annahme, dass beim beta-Zerfall mit dem Elektron jeweils noch ein Neutron emittiert wird, derart, dass die Summe der Energien von Neutron und Elektron konstant ist.

spin 1/2, **tiny mass**, **zero electric harge**

1930: $m_\nu < 0.01$ GeV

Today: $m_\nu < 0.1 - 1$ eV

Three years later: ν name and first dynamical properties

A famous paper by Enrico Fermi:

ANNO IV · VOL. II · N. 12 QUINDICINALE 31 DICEMBRE 1933 · XII

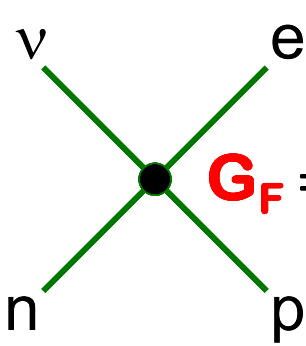
LA RICERCA SCIENTIFICA

ED IL PROGRESSO TECNICO NELL'ECONOMIA NAZIONALE

Tentativo di una teoria dell'emissione dei raggi "beta"

Nota del prof. ENRICO FERMI

Riassunto: Teoria della emissione dei raggi β delle sostanze radioattive, fondata sull'ipotesi che gli elettroni emessi dai nuclei non esistano prima della disintegrazione ma vengano formati, insieme ad un neutrino, in modo analogo alla formazione di un quanto di luce che accompagna un salto quantico di un atomo. Confronto della teoria con l'esperienza.



$G_F = 1.166 \times 10^{-5}\ \mathrm{GeV}^{-2}$

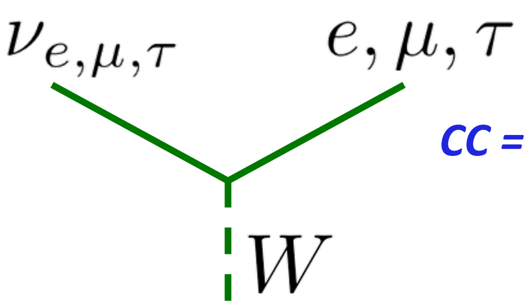
→ Sets the energy scale $\sqrt{(1/G_F)} \sim O(\text{few } 10^2)$ GeV of weak interactions

Many decades of research have revealed other n properties: There are

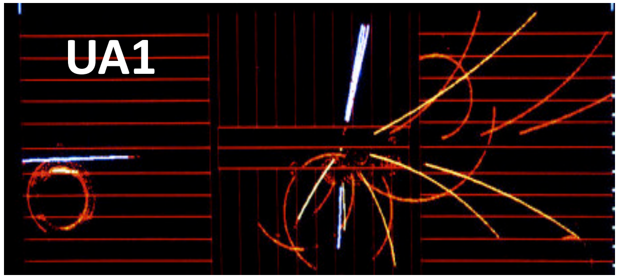
3 different ν "flavors" **e μ τ**

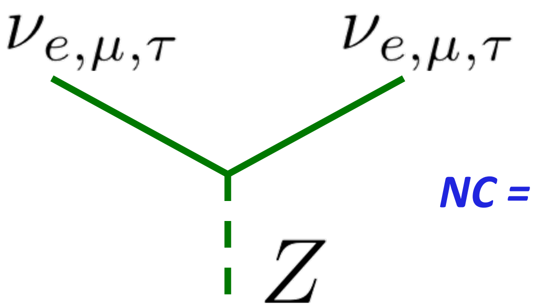
$$\begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \quad \begin{matrix} \leftarrow & q = 0 \\ \leftarrow & q = -1 \end{matrix} \quad (\Delta q = 1)$$

and their Fermi interactions are mediated by a charged vector boson W, with a neutral counterpart, the Z boson



CC = Charged Current (Δq=1)



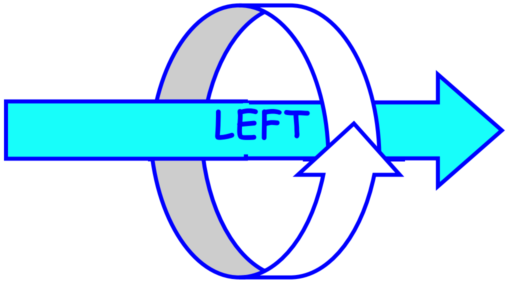


NC = Neutral Current (Δq=0)

Such interactions are chiral (= not mirror-symmetric):

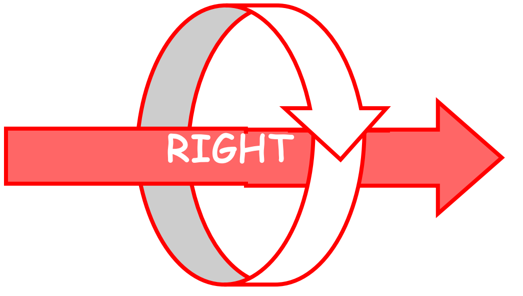
Neutrinos are created in a left-handed (LH) state

ν



Anti-nus are created in a right-handed (RH) state

$\overline{\nu}$



P parity symmetry (space coordinate reversal) is violated

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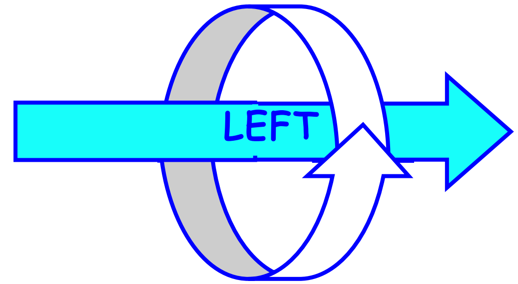

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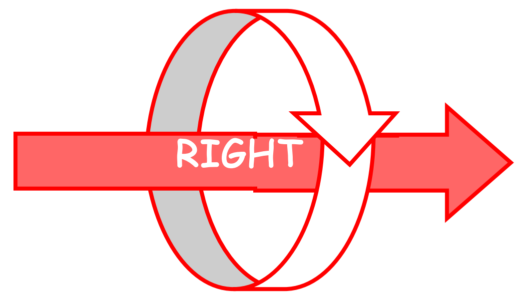


P parity symmetry (space coordinate reversal) is violated

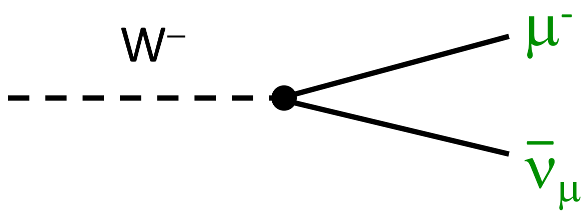
We shall consider also other discrete symmetries:

C charge conjugation (particle-antiparticle exchange)

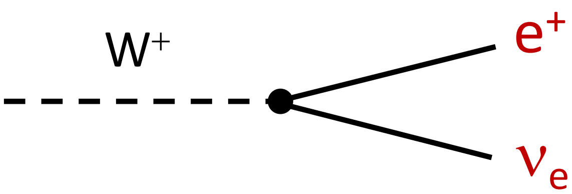
T time reversal (change arrow of time)

Note: combined **CPT** *symmetry always conserved in QFT*

CC processes at **production** provide an operative definition of ν flavor, via the corresponding charged (anti)lepton. E.g., in leptonic decays:

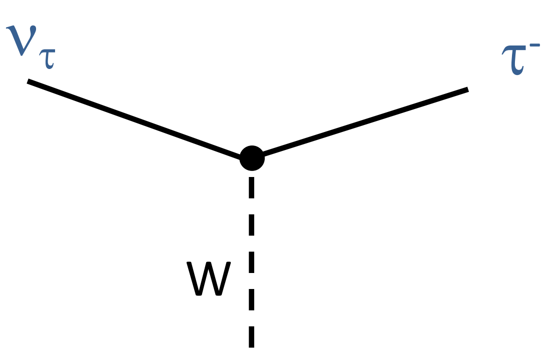


Creation of a muon antineutrino (RH)



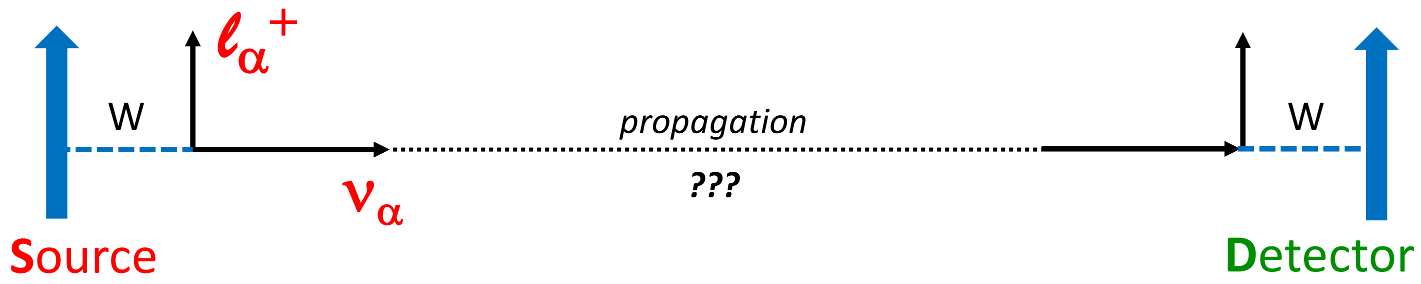
Creation of an electron neutrino (LH)

Similarly at **detection**, e.g.:



Absorption of a tau neutrino (LH)

But… what happens in between?



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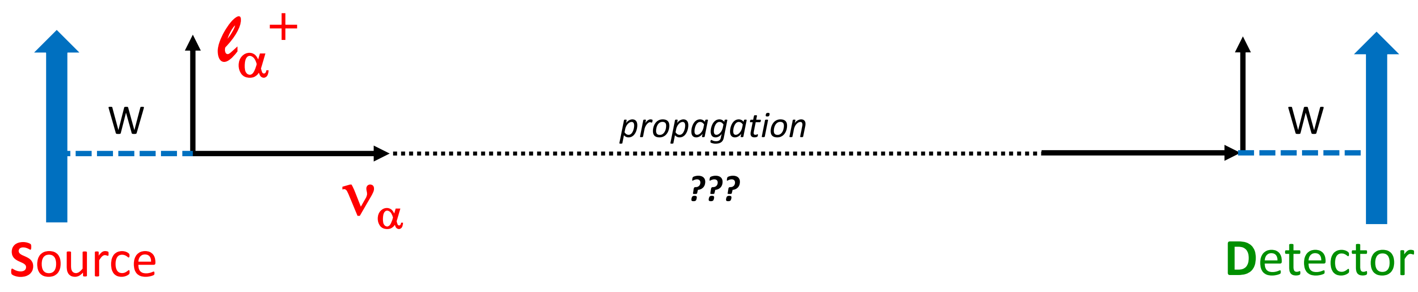
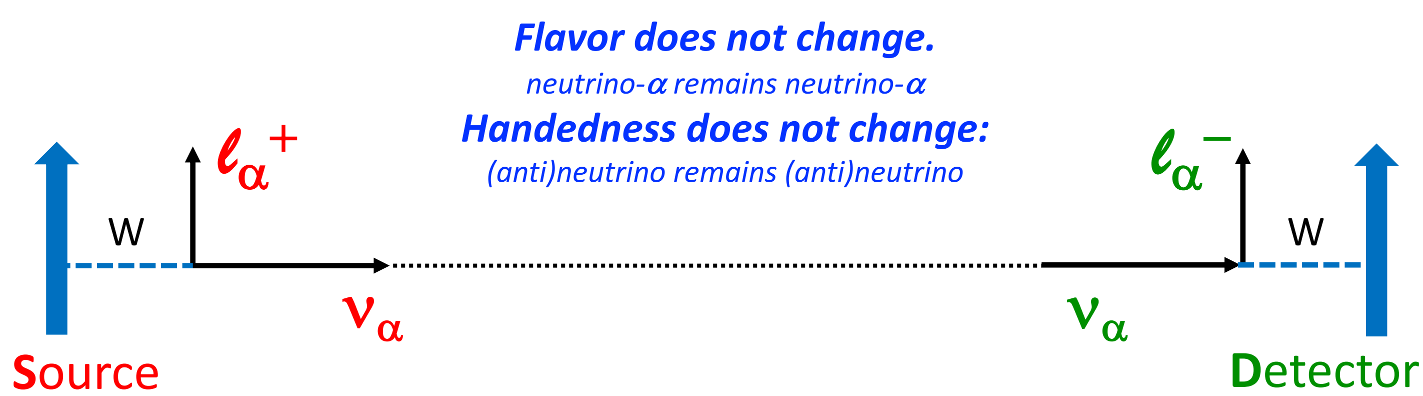


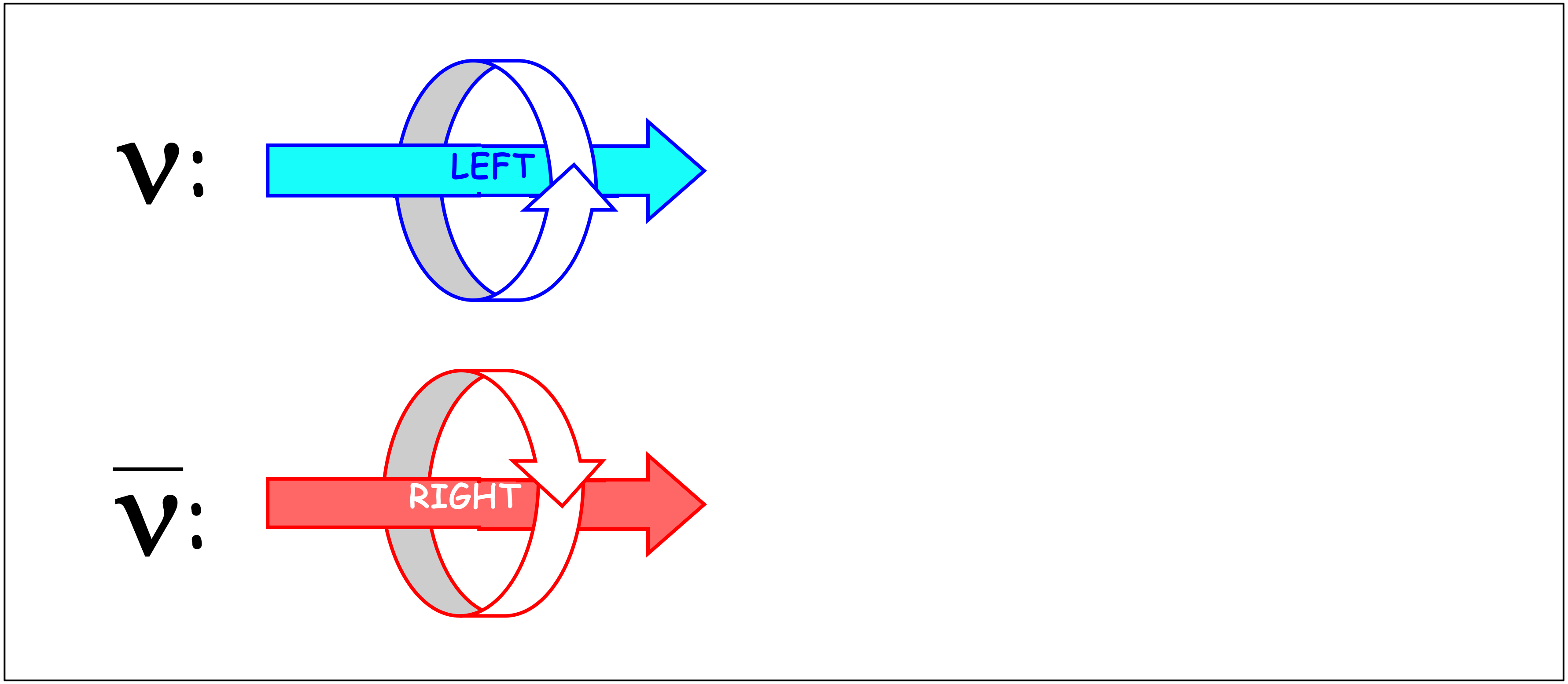
If neutrinos are massless: ***v=c*** **→ "clock" is frozen → no change!**



However, If ν have mass, interesting things may happen to handedness and flavor…

Handedness: is a constant of motion for massless neutrinos

[You would see handedness reversal if you could travel faster… but you can't (v=c)!]

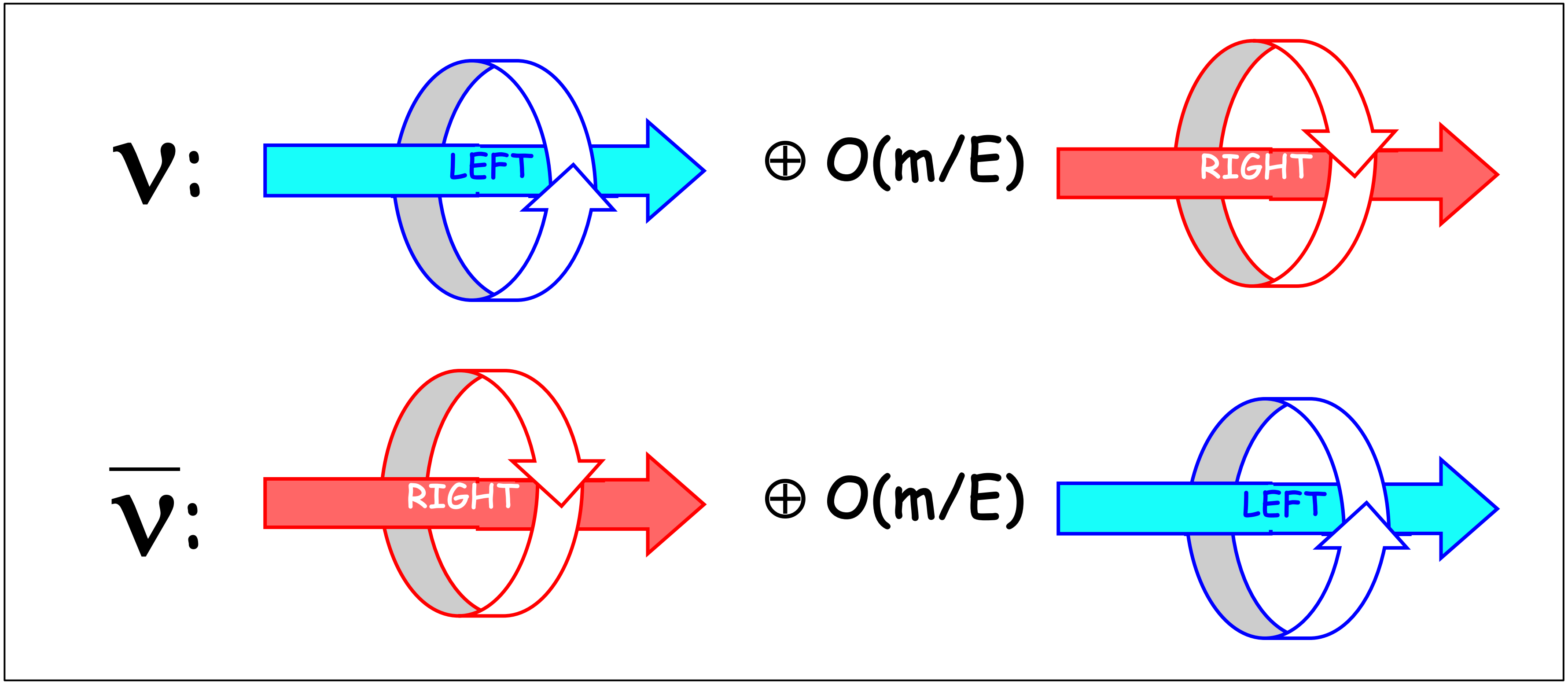


This is a massless "Weyl" two-spinor with 2 independent d.o.f

[And this was also the theoretical prejudice in the construction of the Standard Model]

Massive ν can develop the "wrong" handedness at O(m/E)

E= neutrino energy; the Dirac equation couples RH and LH states for m ≠ 0

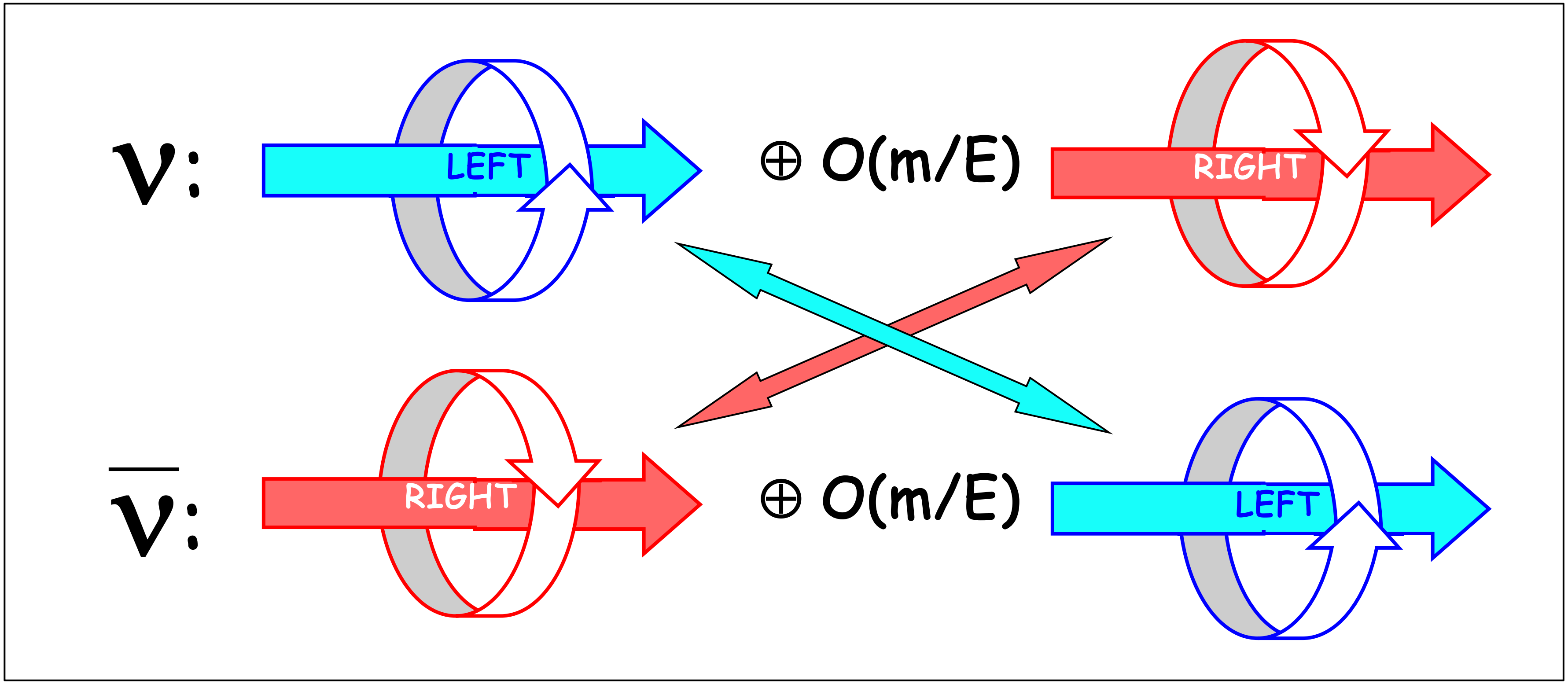


If these 4 d.o.f. are independent: massive "Dirac" four-spinor

Nu and anti-nu are different, as the charged fermions are. Can define a conserved "lepton number"

But, for neutral fermions, two components might be identical !

[Cannot pair components between electron and positron is forbidden: violate electric charge.]

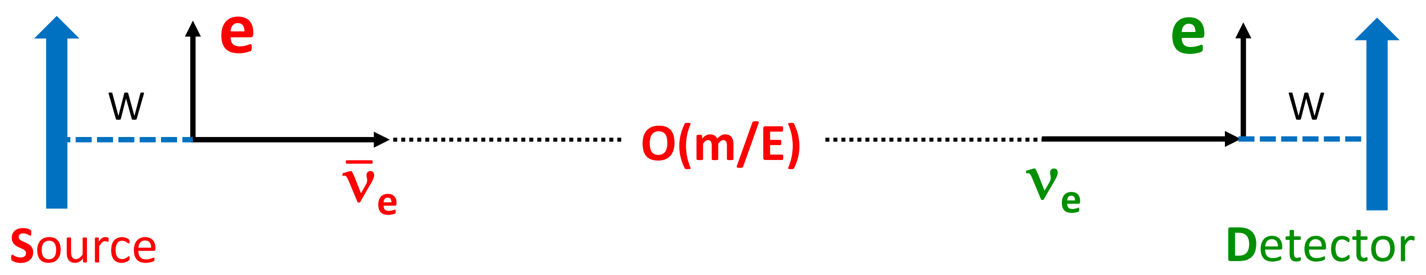


Massive “Majorana” four-spinor with only 2 independent d.o.f.

No fundamental distinction between nu / antinu, up to a possible “Majorana phase”:

A *very* neutral particle with no electric charge, no leptonic number …

E.g., if neutrinos are Majorana, expect $\bar{\nu} \rightarrow \nu$ transition:



But, we haven't seen anything like that so far …
E.g., reactions induced by neutrinos haven't been observed with anti-neutrinos …

Paradox? No!

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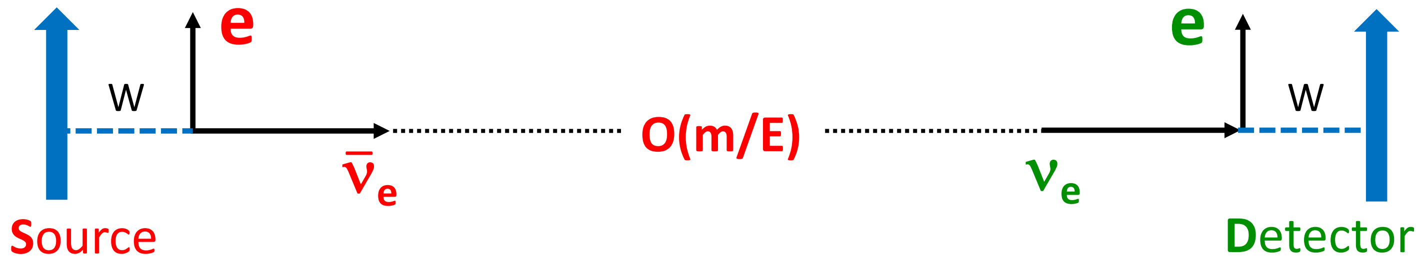

E.g., in the highest-statistics ν experiments, using reactor sources of anti-ν_e with E ~ few MeV, we have seen $O(10^7)$ events from inverse beta-decay (IBD) reaction:

$$\bar{\nu}_e + p \rightarrow e^+ + n \quad \checkmark \text{ (IBD)}$$

If neutrinos are Dirac, the same reaction with initial ν_e is *strictly forbidden*:

$$\nu_e + p \rightarrow e^+ + n \quad \text{✗ (not seen)}$$

If neutrinos are Majorana, it *is allowed* in principle, but in practice is suppressed by $O(m/E) < 10^{-7}$ and becomes ~unobservable: <O(1) reactor event in 10^7 (if any)

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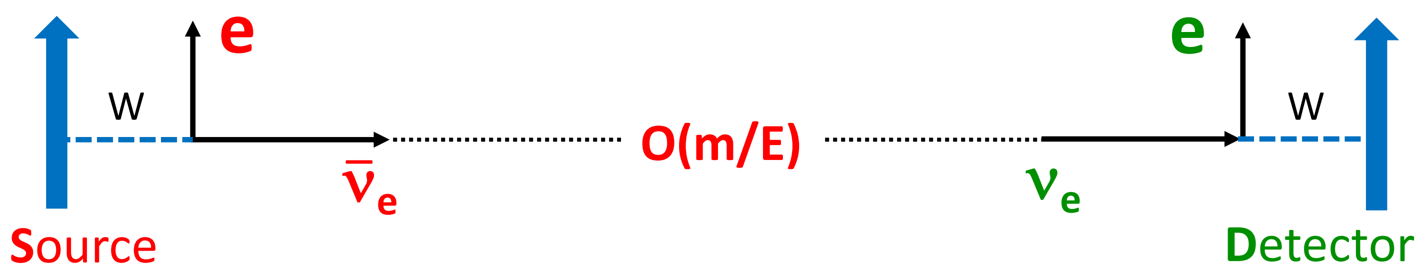

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To observe nu→antinu transitions, better compare O(1) event to ~0, than to 10^N !

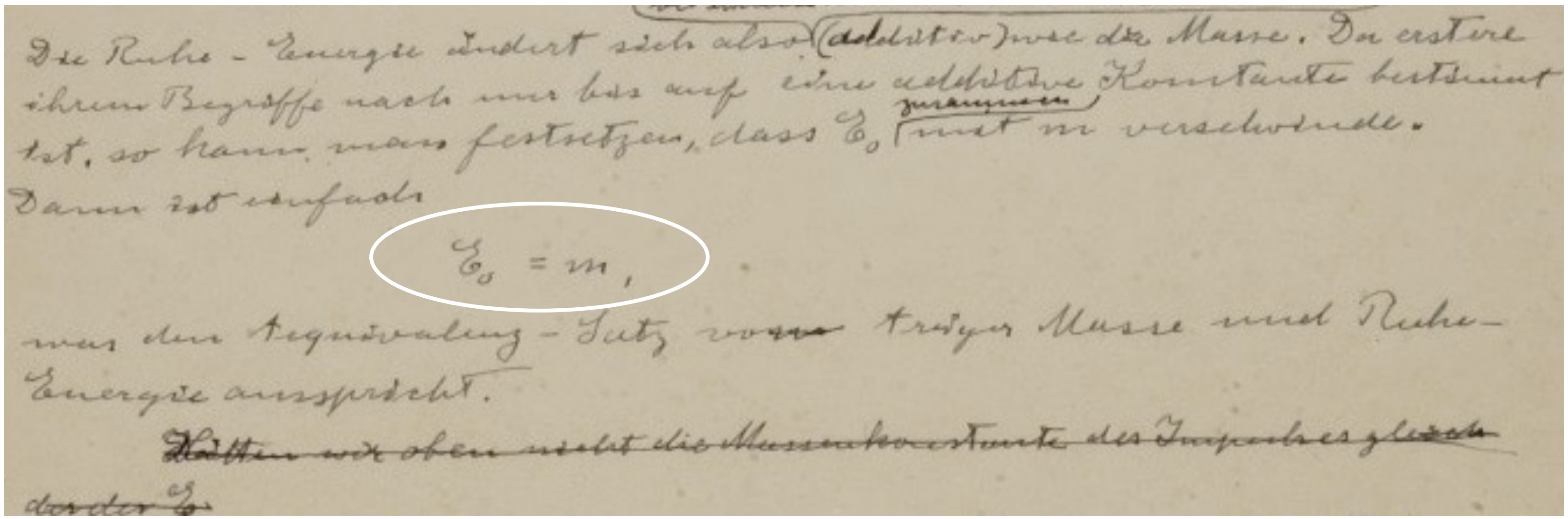
→ Search for **neutrinoless ββ decay**: occurs if and only if neutrinos are Majorana

(microscopic process with ν production and absorption in the same nucleus)

Proof that ν are massive has been provided (1998+) by an alternative process, involving macroscopic distances x=L and observable when $O(m^2L/E) \sim O(1)$:

Neutrino flavor oscillations (or transitions)

Let's start from a celebrated equation, already handwritten in natural units:



Die Ruhe-Energie ändert sich also (additiv) wie die Masse. Da erstere ihrem Begriffe nach nur bis auf eine additive Konstante bestimmt ist, so kann man festsetzen, dass E_0 zusammen mit m verschwinde. Dann ist einfach

$$E_0 = m,$$

was den Äquivalenz-Satz von träger Masse und Ruhe-Energie ausspricht.

… namely, for p≠0:

$$E = \sqrt{m^2 + p^2}$$

Expand at small *p/m* or *m/p* →

Our ordinary experience takes place in the limit: $p \ll m$

$$E \simeq m + \frac{p^2}{2m}$$

mass

kinetic energy

Our ordinary experience takes place in the limit: $p \ll m$

$$E \simeq m + \frac{p^2}{2m}$$

... while neutrinos experience the opposite limit: $p \gg m$

$$E \simeq p + \frac{m^2}{2p}$$

Energy difference between two neutrinos ν_i e ν_j with mass m_i e m_j in the same beam $(p_i = p_j \simeq E)$:

$$\Delta E \simeq \frac{\Delta m_{ij}^2}{2E}$$

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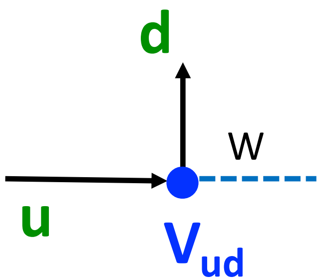
Tiny ΔE → Probed at large (macroscopic) L~Δt from uncertainty relation:

$$1 \sim \Delta E \Delta t \simeq \frac{m_i^2 - m_j^2}{2E} L$$

Besides (different) neutrino masses, a second important ingredient of neutrino oscillations is **mixing.** In the Standard Model, mixing matrices arise, after SSB, in CC interaction vertices involving massive fermions:

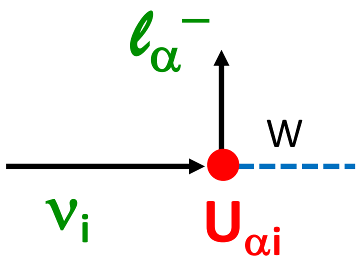
Quarks:



● = CC strength
∝ V element
with $VV^{\dagger}=1$

CKM = Cabibbo-Kobayashi-Maskawa

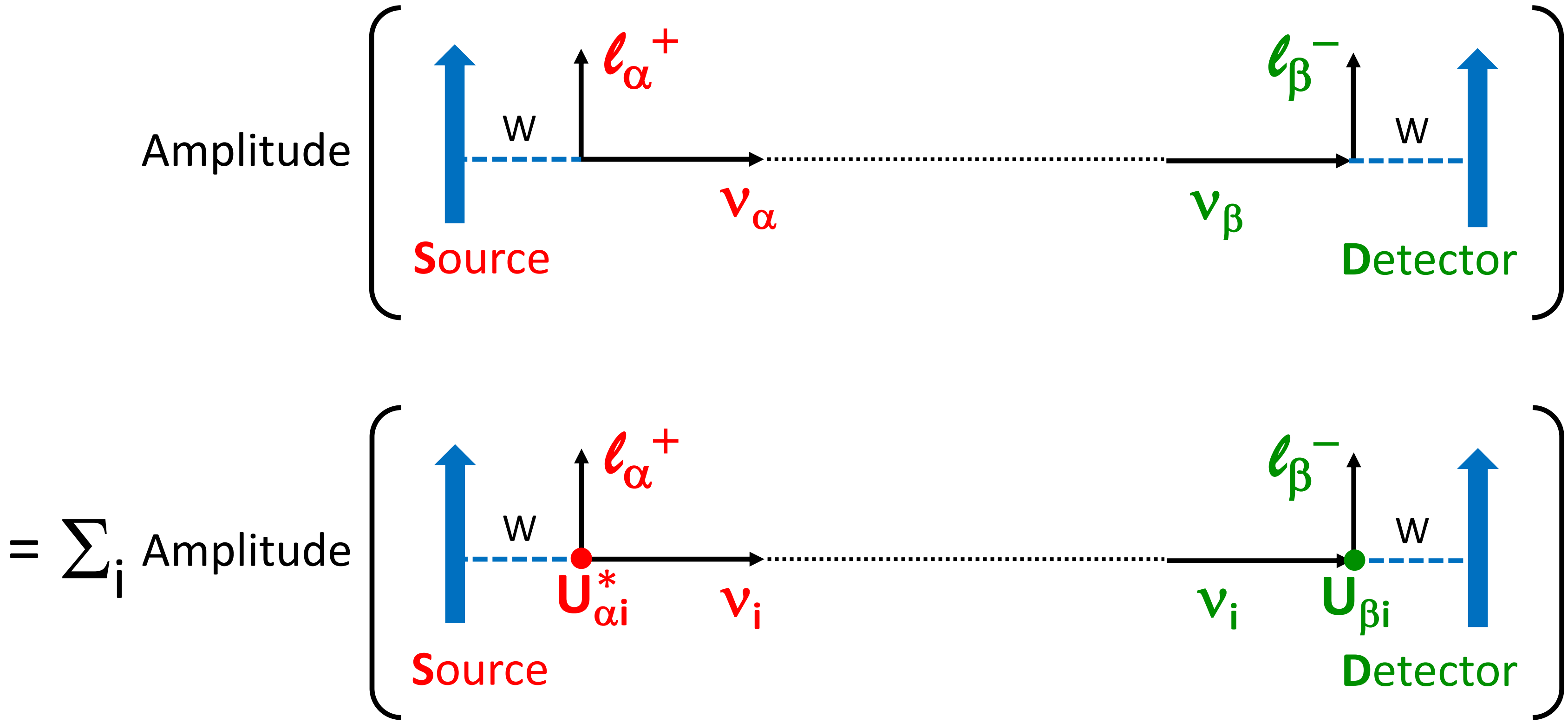
Leptons:



● = CC strength
∝ U element
with $UU^{\dagger}=1$

PMNS = Pontecorvo-Maki-Nakagawa-Sakata

With both ingredients… **flavor may change from α (production) to β (detection)!**



$$\nu_\beta = U_{\beta i}\nu_i$$

$$\nu_i = U^*_{\alpha i}\nu_\alpha$$

Oscill. probability = |Amplitude|^2

Note: for $\bar{\nu}$

$$\ell_\alpha^\pm \to \ell_\alpha^\mp$$

$$U \to U^*$$

Warm-up exercise:
The simplest oscillation probability

Hereafter, some excellent approximations for neutrino oscillations with m≪E:

(1) Take $x \simeq t$ and $\partial x \simeq \partial t$

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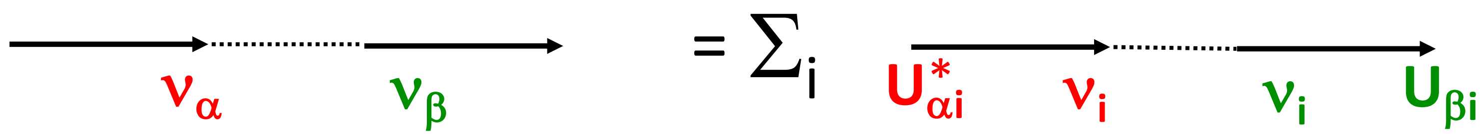
(2) Forget about initial and final interactions, isolate ν propagation only:



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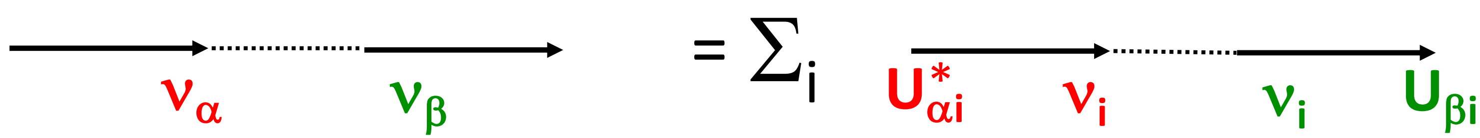

(3) Forget about spin, Majorana/Dirac… treat ν as "scalar" wavefunctions

$$\nu_\alpha = \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} \qquad \nu_i = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}$$

flavor basis

mass basis

with $\begin{cases} |\nu_e|^2 + |\nu_\mu|^2 + |\nu_\tau|^2 = 1 \\ |\nu_1|^2 + |\nu_2|^2 + |\nu_3|^2 = 1 \end{cases}$

e.g., a pure ν_e state in flavor basis is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Next steps: find the hamiltonian of ν propagation → Schroedinger equation:

$$H_f \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = i\frac{d}{dx} \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix}$$

Solve, square amplitudes, get oscillation probabilities, discuss phenomenology!

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H_f = hamiltonian in flavor basis (3x3 matrix). Relation with ***H_m*** in mass basis:

$$H_f = U\, H_m\, U^\dagger$$

U ↓

$$\begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix} = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix}$$

$U^\dagger$ ↓

$$\begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{bmatrix} = \begin{bmatrix} U^*_{e1} & U^*_{\mu 1} & U^*_{\tau 1} \\ U^*_{e2} & U^*_{\mu 2} & U^*_{\tau 2} \\ U^*_{e3} & U^*_{\mu 3} & U^*_{\tau 3} \end{bmatrix} \begin{bmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{bmatrix}$$

[$\nu \to \bar{\nu}$: $U \to U^*$]

The simplest case: two neutrinos evolving in vacuum

(flavors α and β, masses m_i and m_j)

U is real:

(tomorrow: back to complex vs real U)

$$\begin{bmatrix} \nu_\alpha \\ \nu_\beta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \nu_i \\ \nu_j \end{bmatrix}$$

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H_m is easy:

(diagonal energies)

$$H_m = \begin{bmatrix} E_i & \\ & E_j \end{bmatrix} \simeq p \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \frac{1}{2E} \begin{bmatrix} m_i^2 & \\ & m_j^2 \end{bmatrix}$$

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Evolution op.:

(overall phases ∝ **1** are unobservable)

$$S_m = e^{-iH_m x} = \begin{bmatrix} e^{-i\frac{m_i^2 x}{2E}} & \\ & e^{-i\frac{m_j^2 x}{2E}} \end{bmatrix}$$

In flavor basis:

(nondiagonal → flavor change)

$$S_f = U S_m U^\dagger$$

$$P(\nu_\alpha \to \nu_\beta) = P_{\alpha\beta} = |S_{\beta\alpha}|^2$$

Swap of indices reflects opposite writing (from right to left) of algebraic operations:

$$S_{\beta\alpha} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} e^{-i\frac{m_i^2 x}{2E}} & 0 \\ 0 & e^{-i\frac{m_j^2 x}{2E}} \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

ν_β	U	S_m	$U^\dagger$	ν_α
Final flavor	Back to flavor basis	Evolve in mass basis	Go to mass basis	Initial flavor

[Take care of correct indices; e.g., for three neutrinos, in general it is $\mathbf{P_{\alpha\beta} \neq P_{\beta\alpha}}$]

Expect to be sensitive to phase differences, thus to $\mathbf{\Delta m^2 = m_i^2 - m_j^2}$

Exercise: Pontecorvo's formula

$$P_{\alpha\beta} = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$

Exercise: Change of units

$$\frac{\Delta m^2 x}{4E} = 1.267\left(\frac{\Delta m^2}{\mathrm{eV}^2}\right)\left(\frac{x}{\mathrm{m}}\right)\left(\frac{\mathrm{MeV}}{E}\right)$$

In many textbooks: 1.267 $\simeq$ 1.27, no longer adequate in subpercent precision expts!

$$P(\nu_\alpha \to \nu_\beta) = 4\sin^2\theta\cos^2\theta\sin^2\left(\frac{\Delta m_{ij}^2 L}{4E}\right)$$

"Amplitude"
(vanishes for $\theta=0$ or $\pi/2$, is maximal for $\theta=\pi/4$)

"Phase"
(vanishes for degenerate masses or small L/E)

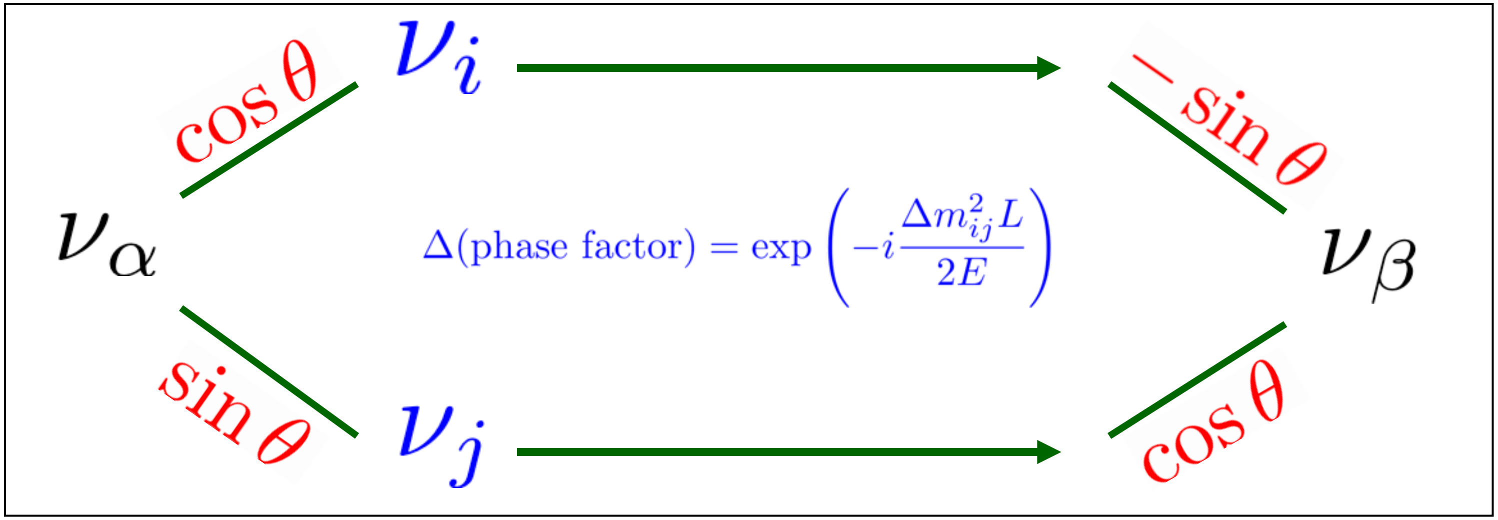
This is the flavor "appearance" probability $P_{\alpha\beta}$. The "survival" probability $P_{\alpha\alpha}$ for the flavor α is the complement to unity: $P_{\alpha\alpha} = 1 - P_{\alpha\beta}$

The oscillation effect depends on the *difference* of (squared) masses, not on the *absolute masses themselves*.

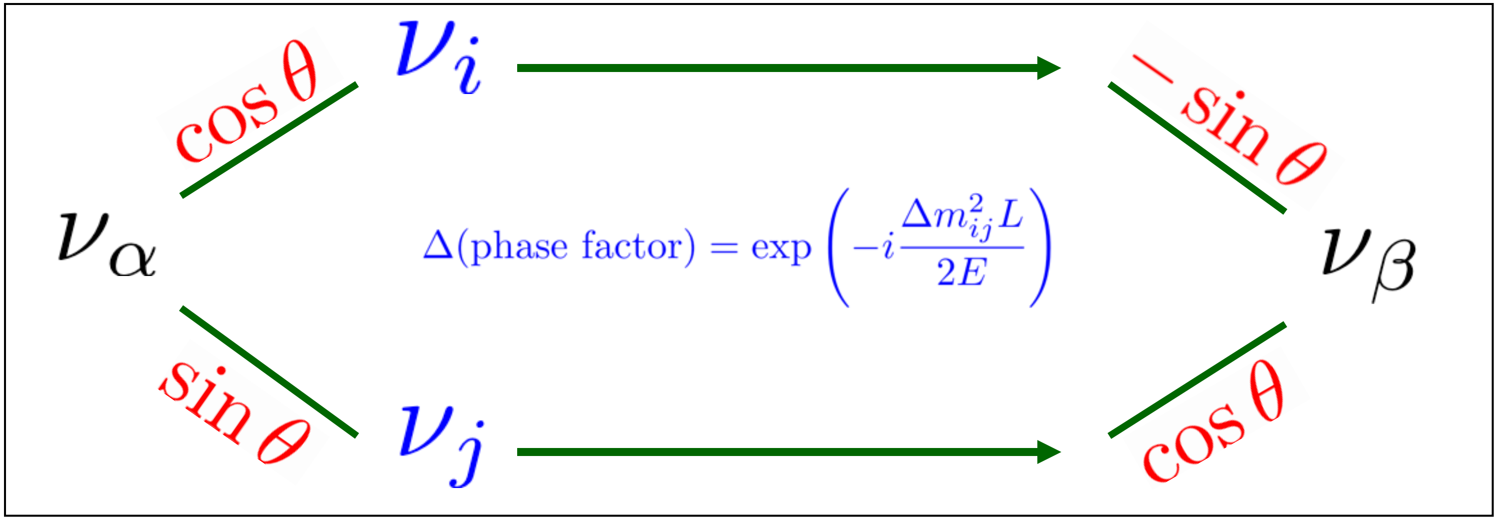
The oscillating term is squared, repeats at $n\pi$. Oscillation length: $L_{osc} = 4\pi E/\Delta m^2$

The above probability is octant-symmetric: amplitude does not change when $\theta \to \pi/2-\theta$, namely, $c_\theta \to s_\theta$

Analogy with a double-slit interference experiment in vacuum:



Analogy with a double-slit interference experiment in vacuum:

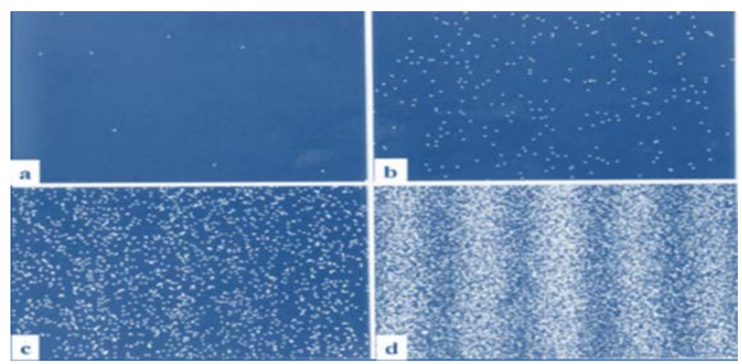


Fringes best observed when

$$1 \sim \Delta E \Delta t \simeq \frac{m_i^2 - m_j^2}{2E} L$$

Vanishing fringes when $\ll 1$

Unresolved fringes (gray screen) when $\gg 1$



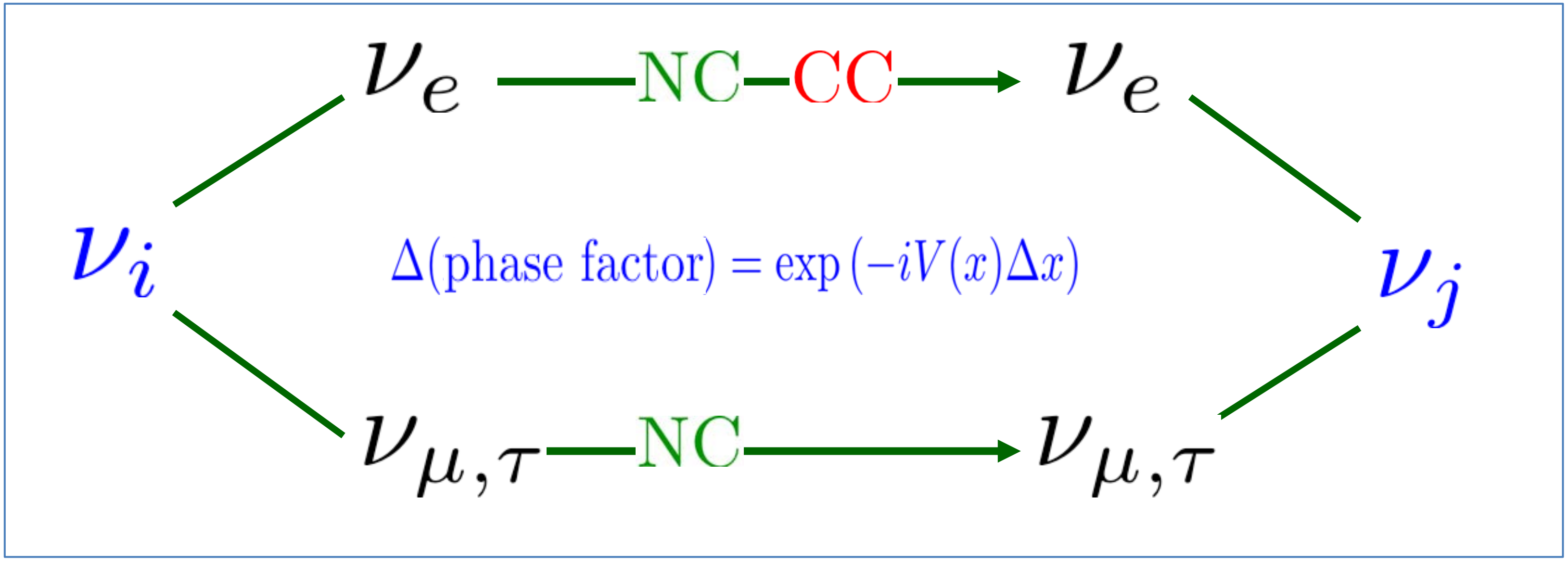
Orders of magnitude of L, E for some past/current oscillation experiments

	Initial flavors	Typical E	Typical L
Short-baseline (SBL) reactor neutrinos: CHOOZ, Double Chooz, RENO, Daya Bay…	$\bar{\nu}_e$	few MeV	O(1) km
Long-baseline reactor neutrinos: KamLAND	$\bar{\nu}_e$	few MeV	$O(10^2)$ km
Long-baseline (LBL) accelerator neutrinos: K2K, OPERA, T2K, NOvA…	$\overset{(-)}{\nu}_\mu$ mostly	O(1) GeV	$O(10^{2\text{-}3})$ km
Atmospheric neutrinos: MACRO, MINOS, (Super)-Kamiokande, IceCube…)	$\overset{(-)}{\nu}_\mu$ $\overset{(-)}{\nu}_e$	> O(0.1) GeV	$O(10^{1\text{-}4})$ km
Solar neutrinos: Chlorine, Gallium, Super-K, SNO, Borexino…	ν_e	O(1-10) MeV	1 a.u.

In the latter case, L=1 a.u. actually plays a marginal role in oscillations, dominated by matter effects

More in Lecture II and III:

Analogy of matter effects with double-slit experiment:
one "arm" (e-flavor) feels a different "refraction index"
through coherent forward scattering (not absorption!)

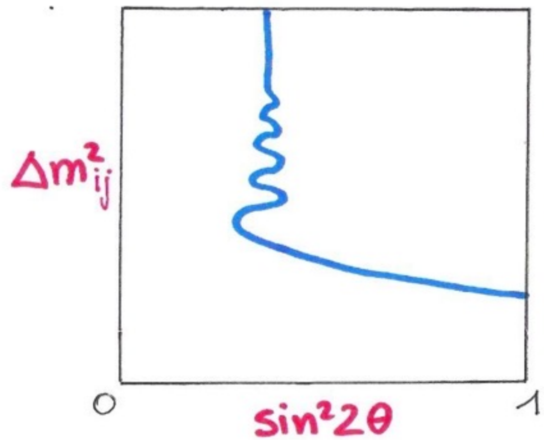


Governed by a tiny ν "interaction energy" or "potential" V
Not necessarily periodic effects: oscillations → transitions

Generic experimental constraints in 2ν approxim.

- Experiments measure some "averaged" $P_{\alpha\beta} \simeq \sin^2 2\theta \left\langle \sin^2\left(\frac{\Delta m^2_{ij}\, x}{4E}\right)\right\rangle$

- Curve of iso-$P_{\alpha\beta}$:

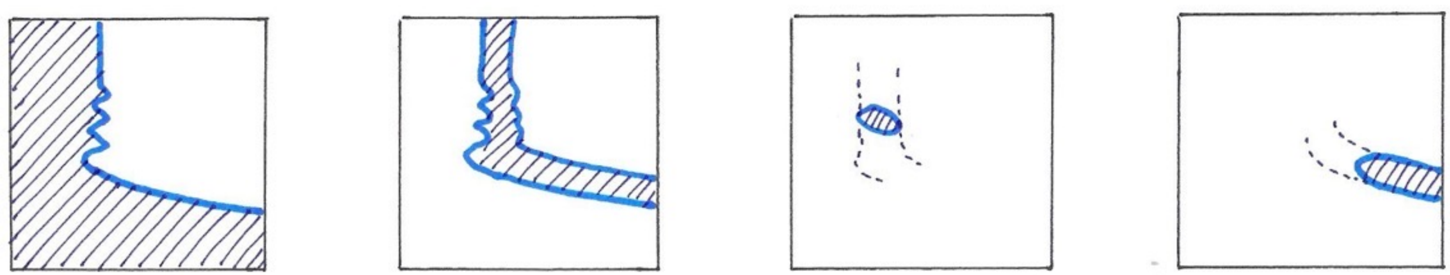


← $\frac{\Delta m^2_{ij}\, x}{4E} \gg 1$, $\langle \cdots \rangle \sim \frac{1}{2}$, fast oscillations

← $\frac{\Delta m^2_{ij}\, x}{4E} \sim \mathcal{O}(1)$

← $\frac{\Delta m^2_{ij}\, x}{4E} \ll 1$, vanishing oscillations

- Possible expt. constraints:



No signal

Signal

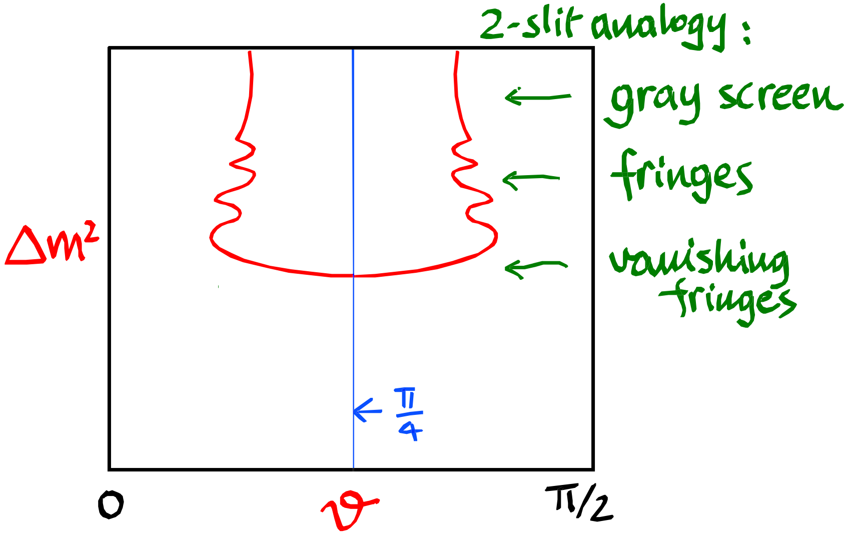
Precise signal at small mixing

Precise signal at large mixing

(need ≥ 2 expts or spectral data in 1 expt)

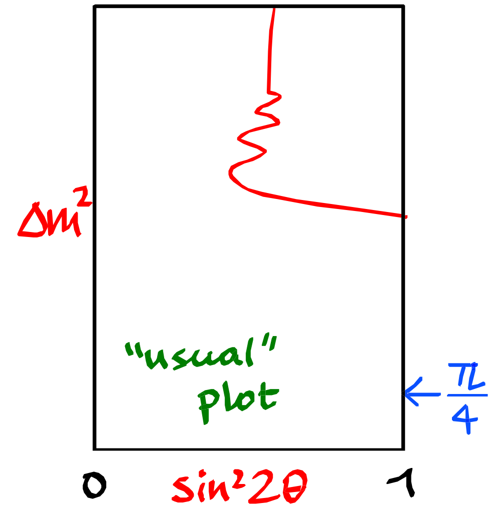
Octant (a)symmetry:

Typical iso-$\langle P_{\alpha\beta} \rangle$ contours



Octant symmetry: $\theta \to \frac{\pi}{2} - \theta$ in $P_{e\mu}$

If 2nd octant folded onto the 1st one:



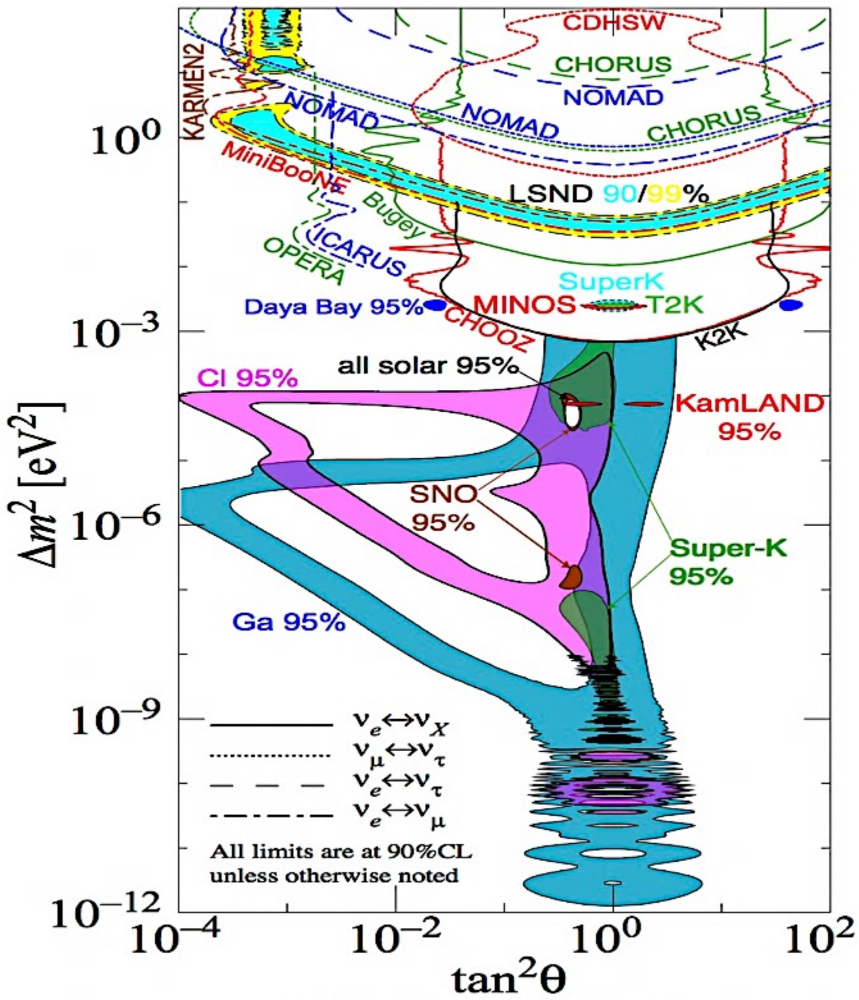
Basically obsolete

In general, better to use $\log \tan^2\theta$ or $\sin^2\theta$

(preserve octant-symmetry)

Note: 2ν Octant symmetry broken by 3ν and/or matter effects

Octant (a)symmetric 2ν contours from Particle Data Group review:



But... patching 2ν approximations in different oscillation channels, in order to get a full 3ν picture, is no longer a useful approach:

Better to go the other way around, from the full 3ν case to 2ν limits

RECAP

If neutrinos are massless: "clock" is frozen, no change in propagation

Flavor does not change.
Handedness does not change.

ν_α $\longrightarrow \cdots\cdots \longrightarrow$ ν_α

If neutrinos are massive, the other handedness develops at $O(m/E) \ll 1$

Iff neutrinos are Majorana:
(other lectures in this School)

ν $\longrightarrow \cdots\cdots \longrightarrow$ $\bar{\nu}$

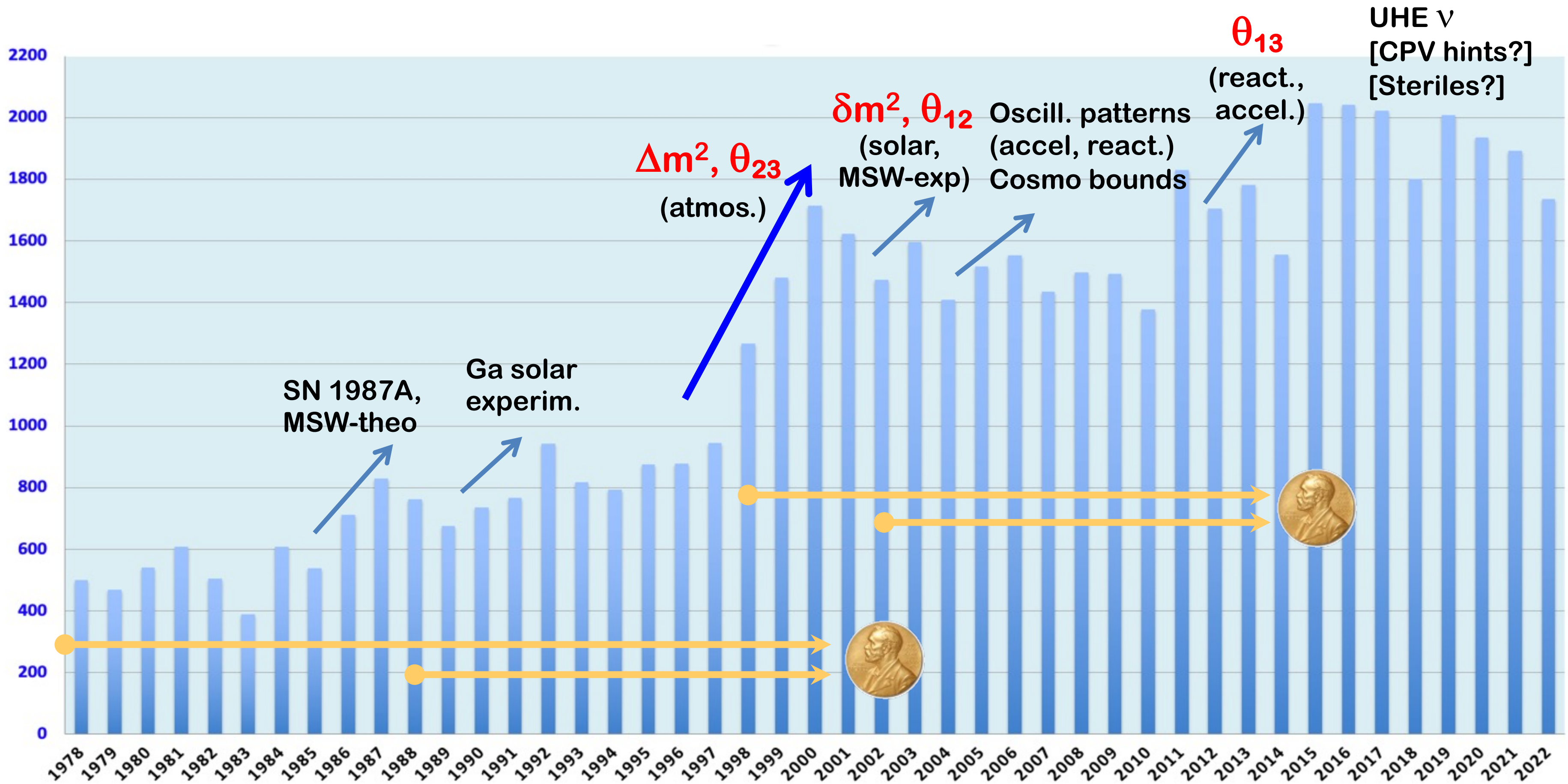
If ν are massive and mixed, other flavors develop at $O(\Delta m^2\ L/E)$

Flavor oscillations / transitions:
(these lectures)

ν_α $\longrightarrow \cdots\cdots \longrightarrow$ ν_β

An active research field…

Papers with *neutrino* in the title, yearly trend from INSPIRE HEP



future → … ?

End of Lecture I

Solutions to exercises: extra slides →

Exercise: Pontecorvo's formula

$$S_{\beta\alpha} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} e^{-i\frac{m_i^2 x}{2E}} & \\ & e^{-i\frac{m_j^2 x}{2E}} \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= s_\theta c_\theta \left(-e^{-i\frac{m_i^2 x}{2E}} + e^{-i\frac{m_j^2 x}{2E}} \right)$$

To get $|S_{\beta\alpha}|^2$, use the identities :
$$\begin{cases} |a+b|^2 = |a|^2 + |b|^2 + 2\,\mathrm{Re}\,(a^* b) \\ 2 s_\varphi^2 = 1 - \cos(2\varphi) \end{cases}$$

$$\rightarrow P_{\alpha\beta} = |S_{\beta\alpha}|^2$$

$$= 2 s_\theta^2 c_\theta^2 \left(1 - \cos\left(\frac{\Delta m^2 x}{2E} \right) \right)$$

$$= 4 s_\theta^2 c_\theta^2 \sin^2\left(\frac{\Delta m^2 x}{4E} \right)$$

$$= \sin^2 2\theta \sin^2\left(\frac{\Delta m^2 x}{4E} \right)$$

Exercise: Change of units (from natural to eV, m)

- Remember that: $1 = \hbar c = 197.327 \text{ MeV}\cdot\text{fm}$ ← $1\text{ fm} = 10^{-15}\text{ m}$

  $(1 \simeq 0.2 \text{ GeV}\cdot\text{fm})$ ← "Rule of thumb"

  Thus: $1 \text{ MeV}\cdot 1 \text{ m} = 5.0677 \times 10^{+12}$

- Rewrite the oscillation phase as:

$$\left(\frac{\Delta m^2 x}{4E}\right) = \frac{1}{4}\left(\frac{\Delta m^2}{eV^2}\cdot eV^2\right)\left(\frac{x}{m}\cdot m\right)\left(\frac{MeV}{E}\cdot\frac{1}{MeV}\right)$$

$$= \frac{1}{4}\left(\frac{1\,eV^2\cdot 1\,m}{1\,MeV}\right)\left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{x}{m}\right)\left(\frac{MeV}{E}\right)$$

$$= \frac{10^{-12}}{4}(MeV\cdot m)\left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{x}{m}\right)\left(\frac{MeV}{E}\right)$$

$$= 1.267\left(\frac{\Delta m^2}{eV^2}\right)\left(\frac{x}{m}\right)\left(\frac{MeV}{E}\right)$$