GGI lectures on EFT

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Abstract: `Draft Version`. Pls do not distribute. There are many typos, and possibly some errors too. Many references are missing.

Contents

0 Goals, notation, conventions

This is a write-up of my lectures on EFT given at the GGI school in January 2024. My intention was to prepare lecture notes at a very basic, introductory level, without assuming any prior knowledge of EFT techniques. On the other hand, I assume the reader is versed in Quantum Field Theory (QFT) roughly at the Peskin-Schroeder level [1]. The focus is on relativistic EFTs used in high-energy physics, though with some excursions to lower energies where non-relativistic description becomes relevant.

This course consists of five lectures. Section 1 is a general introduction to EFT, illustrated with several prominent examples. In Section 2 I discuss in more detail a particularly important EFT: the so-called SMEFT, which is the effective theory extension of the SM. In Section 3 I discuss the non-relativistic limit of EFTs and some of its applications in nuclear physics. Section 4 deals with a particular application of EFT to physics of neutrino production, oscillation, and detection. Finally, in Section 5 I will briefly introduce some modern amplitude techniques and discuss its EFT applications, in particular for constructing bases and calculating RG running.

I work with the mostly minus Minkowski metric $\eta_{\mu\nu} = (1, -1, -1, -1)$. The metric is used to raise and lower indices, e.g. $A^\mu = \eta^{\mu\nu} A_\nu$. As usual, repeated Lorentz and other indices are implicitly summed over, unless otherwise noted. Since the Lorentz contractions are unambiguous, sometimes I may write contracted Lorentz indices on the same level (e.g. $A^\mu A^\mu$ instead of $A^\mu A_\mu$) if this improves the aesthetics (usually when there are many other indices). The sign convention for the totally anti-symmetric Levi-Civita tensor $\epsilon^{\mu\nu\rho\alpha}$ is $\epsilon^{0123} = 1$, which implies $\epsilon_{0123} = -1$. In the context of general relativity (GR), the Christoffel connection and the Riemann and Ricci tensors built from the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are defined as

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} \left(\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho} \right), \tag{0.1}$$

$$R^\alpha{}_{\mu\nu\beta} = \partial_\nu \Gamma^\alpha_{\mu\beta} - \partial_\beta \Gamma^\alpha_{\mu\nu} + \Gamma^\rho_{\mu\beta} \Gamma^\alpha_{\rho\nu} - \Gamma^\rho_{\mu\nu} \Gamma^\alpha_{\rho\beta}, \tag{0.2}$$

$$R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\rho_{\mu\nu} \Gamma^\alpha_{\rho\alpha} - \Gamma^\rho_{\mu\alpha} \Gamma^\alpha_{\nu\rho}. \tag{0.3}$$

When I refer to a vector I always mean a Lorentz 4-vector. For 3-vectors I use the bold notation rather than an arrow top: $\boldsymbol{x} \equiv \vec{x}$.

I always use the natural units, $\hbar = c = 1$. Energy, momentum, area, distance, time, etc. are expressed in appropriate powers of electronvolts (eV).

I use the 2-component spinor formalism, following the conventions of Ref. [2]. A Dirac fermion is described by a pair anti-commuting fields f_α, $\bar{f}^c_{\dot{\alpha}}$ transforming respectively under the first and the second component of the $SU(2) \otimes SU(2)$ Lorentz algebra. The spinor index can be raised and lowered by the anti-symmetric ϵ tensor, $f^\alpha = \epsilon^{\alpha\beta} f_\beta$, $\epsilon^{12} = -\epsilon^{21} = 1$, and then Lorentz invariant contractions can be easily constructed by marrying the upper and lower undotted and dotted indices. For example, $f^c f \equiv f^{c\,\alpha} f_\alpha$ and $\bar{f}^c \bar{f} \equiv f^c_{\dot{\alpha}} f^{\dot{\alpha}}$ are Lorentz invariant, whereas $f^c_\alpha f_\alpha$, $f^c_{\dot{\alpha}} f_{\dot{\alpha}}$, or $f_\alpha \bar{f}^{\dot{\alpha}}_c$ are *not* Lorentz invariant. The fermion kinetic and mass terms are written as $\mathcal{L} = i \bar{f} \bar{\sigma}^\mu \partial_\mu f + i f^c \sigma^\mu \partial_\mu \bar{f}^c - m f^c f - m \bar{f} \bar{f}^c$, where $\sigma^\mu = (1, \boldsymbol{\sigma})$, $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$, $\bar{f} \equiv f^*$, $\bar{f} \bar{\sigma}^\mu \partial_\mu f \equiv \bar{f}_{\dot{\alpha}} [\bar{\sigma}^\mu]^{\dot{\alpha}\alpha} \partial_\mu f_\alpha$. $f^c \sigma^\mu \partial_\mu \bar{f}^c \equiv f^{c\,\alpha} [\sigma^\mu]_{\alpha\dot{\alpha}} \partial_\mu \bar{f}^{c\dot{\alpha}}$. If you're not familiar with this notation... that's very bad, you should learn this as soon as possible, it's an essential part of modern education of a particle physicist. But if you don't want to learn, you can always quickly translate to the 4-component Dirac fermion using the map

$$F = \begin{pmatrix} f \\ \bar{f}^c \end{pmatrix}, \qquad \bar{F} \equiv F^\dagger \gamma^0 = \begin{pmatrix} f^c & \bar{f} \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{0.4}$$

For example, $\bar{f}\bar{\sigma}^\mu \partial_\mu f = \bar{F}\gamma^\mu \partial_\mu P_L F$, $f^c \sigma^\mu \partial_\mu \bar{f}^c = \bar{F}\gamma^\mu \partial_\mu P_R F$, $f^c f = \bar{F} P_L F$, $\bar{f}\bar{f}^c = \bar{F} P_R F$, where $P_{L,R} = \frac{1 \mp \gamma_5}{2}$ are the Dirac chirality projectors.

The 1σ uncertainty on theoretical or experimental quantities is often expressed either using the bracket notation, e.g. $x = 1.234(56)$ is the same as $x = 1.234 \pm 0.056$. The former notation is especially useful when precision reaches many digits.

1 Illustrated philosophy of EFT

The notion of effective theory is one of the pillars of physics. Not just particle physics, but all of physics. Much like mathematics, it constitutes an integral part of the language, to the extent that it may easily blend into invisibility. The central idea is that things usually appear simpler when viewed from a distance. For countless physical systems complexity is dramatically reduced by focusing on the large-scale behavior and projecting out the short-distance degrees of freedom. It is fair to say that physics would be impossible if not for that fortunate fact. For example, if we had to know the theory of everything to describe the notions of planets in the Solar System, even Newton and Einstein could not work it out. Fortunately for physics and physicists, planets and stars made of gazillions of atoms who are made of electrons and protons who are made of quarks who are probably made of something else too, to a very good approximation obey simple Newtonian laws of motion that can be taught to schoolchildren. Einstein found that these laws are modified at large velocities and in strong fields, but again the modification can be concisely described with a help of a tad more advanced mathematics. This works because distance scales between celestial bodies is typically much larger than other scales in the problem, such as the planetary radii, atomic scales, etc. The simplicity of the laws governing the motion of planets relies on the fact that these small scales have been already integrated out, to use the EFT jargon.

In this lecture I will walk you through a series of examples that illustrate some salient features of effective theories.

1.1 Engineer's tale, or the multipole expansion

Let us start with a very simple example that may be familiar even to non-physicists. Consider a system of many static electric charges confined to a region of space of size R, see Fig. 1. A near observer positioned at a distance $L \sim R$ must trace the position of each charge to accurately determine the electric field in her vicinity. However, for a far observer at $r \gg R$ the details of the charge distribution are not essential. Instead, the electric field at large r can be described using just a handful of effective parameters. These parameters can be conveniently chosen to be the multipole moments of the charge distribution: the

Figure 1. Simple illustration of the EFT idea using the example of multipole expansion of the potential produced by static electric charges.

total charge Q_0, the dipole moment Q_1^i, the quadrupole moment Q_2^{ij}, etc. The potential around the far observer can be approximated by

$$\Phi(\boldsymbol{r}) = \frac{1}{4\pi}\left\{\frac{Q_0}{r} - \boldsymbol{Q_1}\cdot\boldsymbol{\nabla}\frac{1}{r} + Q_2^{ij}\nabla_i\nabla_j\frac{1}{r} + \ldots\right\}. \tag{1.1}$$

The power counting of this effective theory is controlled by R/r. The multipoles have dimensions $[Q_n] = \text{mass}^{-n}$. By dimensional analysis, we must have $Q_n \sim R^n$. Thus, the first (monopole) term in the curly bracket is $\mathcal{O}(1/r)$, the second (dipole) is suppressed by a relative $\mathcal{O}(R/r)$ factor, the third (quadrupole) is suppressed by $\mathcal{O}(R^2/r^2)$ compared to the monopole, etc. The power counting gives us a means to establish the hierarchy between different multipoles for $r \gg R$: the monopole is the leading term, the dipole is subleading (unless $Q_0 = 0$), the quadrupole is subsubleading (unless $Q_0 = Q_1^i = 0$), etc. The error of this effective approximation is controlled by the ratio $(R/r)^n$, where $n-1$ is the number of multipoles taken into account. For large enough r only a first few multipoles need to be included to adequately describe the electric field, and this way the description of a possibly complex system with many degrees of freedom can be described by a small number of parameters. As r approaches R, more and more multipoles need to be taken into account. For $r \sim R$ and infinite number of them would be needed, in which case the effective description breaks down and one should revert to the "UV theory" where the degrees of freedom are the positions of each charge.

1.2 Toy story, or an EFT for a single scalar

For the rest of the chapter I will discuss EFTs which are relativistic QFTs. The simplest example one can imagine is the EFT of a single real scalar ϕ with mass m, whose UV completion contains another real scalar H with mass $M \gg m$. This is an ultimate spherical cow with no practical applications I'm aware of. However, in this simple setting it is relatively easy to introduce some profound concepts underlying the philosophy of EFT.

The Lagrangian of the UV theory is

$$\mathcal{L}_{\rm UV} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m_0^2\phi^2 + (\partial_\mu H)^2 - M^2H^2\right]$$

$$-\frac{\lambda_0}{4!}\phi^4-\frac{\lambda_1}{2}M\phi^2H-\frac{\lambda_2}{4}\phi^2H^2. \tag{1.2}$$

Note that I factored out the heavy mass scale M in the dimensionful coefficient of the trilinear term, which will affect the power counting below. I'm imposing the $\mathbb{Z}_2$ symmetry $\phi\to-\phi$, thus odd powers of ϕ do not appear in the Lagrangian. The H^3 and H^4 interactions are irrelevant for this discussion, and for simplicity I'm assuming they are absent in the Lagrangian (even if this assumption is not stable against radiative corrections, since H^3 and H^4 counterterms will be needed at the loop level).

We want to derive the EFT valid at $E\ll M$ where H is integrated out. The EFT Lagrangian has to be of the form

$$\mathcal{L}_{\rm EFT}=\frac{1}{2}\left[(\partial_\mu\phi)^2-m^2\phi^2\right]-C_4\frac{\phi^4}{4!}-C_6\frac{\phi^6}{6!}+\mathcal{O}(M^{-4}). \tag{1.3}$$

The interaction terms are organized according to canonical dimensions, $\sum_i C_D O_D$. Each operator O_D has the canonical dimension D, and the Wilson coefficient have dimensions $[C_D]=\text{mass}^{4-D}$. Dimensional analysis suggests that $[C_D]\sim M^{4-D}$, thus the expansion is in the heavy mass scale M. In this simple EFT there is only one non-redundant operator at the dimension-4 level, and only one at the dimension-6 level. Operators with an odd number of ϕ's do not appear because of the $\mathbb{Z}_2$ symmetry $\phi\to-\phi$ of the UV Lagrangian in Eq. (1.2), which is inherited by the low-energy theory. I will not trace here the operators with $D>6$. although it is easy to complicate the analysis and truncate the Lagrangian at some higher d.

Note that one could write other possible operators at $\mathcal{O}(M^{-2})$, e.g.

$$\hat{O}_6\equiv(\Box\phi)^2,\quad \tilde{O}_6\equiv\phi^3\Box\phi,\quad \tilde{O}_6'\equiv\phi^2\Box\phi^2,\quad \tilde{O}_6''\equiv\phi^2\partial_\mu\phi\partial_\mu\phi,\quad \ldots \tag{1.4}$$

It turns out that the operators in Eq. (1.4) are *redundant*, that is to say, adding them to the Lagrangian in Eq. (1.3) does not change the physical content of the theory. First, $\tilde{O}_6''$ and $\tilde{O}_6'$ can be traded for $\tilde{O}_6$ via integration by parts: $\phi^2(\partial_\mu\phi)^2=-\frac{1}{3}\phi^3\Box\phi$, $\phi^2\Box\phi^2=\frac{4}{3}\phi^3\Box\phi$. On the other hand, $\hat{O}_6$ and $\tilde{O}_6$ can be eliminated in favor of the interaction term present in Eq. (1.3) by using the classical equations of motion. It was proven in Ref. [3] that shifting the higher-dimensional operators by a term proportional to the classical equations of motion does not change the S-matrix elements, even at the loop level. The point is that trading one interaction term for another using the equations of motion is the same as redefining the fields in the Lagrangian in a non-linear way. It is rather intuitive that the manner in which you define your fields should not affect the physical content of the theory. Independence of the S-matrix on field redefinitions is the consequence of the *equivalence theorem* [4, 5].

In our case, the equation of motion for ϕ reads

$$\Box\phi+m^2\phi+\frac{C_4}{6}\phi^3=\mathcal{O}(M^{-2}). \tag{1.5}$$

For our purpose, we don't need to write down the $\mathcal{O}(M^{-2})$ piece explicitly as it is relevant only for manipulating $\mathcal{O}(M^{-4})$ terms in the Lagrangian. Using the equation of motion we find, for example, the following operator equation:

$$\tilde{C}_6\phi^3\Box\phi=-m^2\tilde{C}_6\phi^4-\frac{C_4\tilde{C}_6}{6}\phi^6+\mathcal{O}(M^{-4}). \tag{1.6}$$

This means that $\Delta\mathcal{L}_{\rm EFT} = \tilde{C}_6 \phi^3 \Box \phi$ has the same effect on on-shell amplitudes as a particular linear combination of the terms already present in Eq. (1.3) with the Wilson coefficients fixed as

$$
\begin{aligned}
C_4 =& m^2 \tilde{C}_6, \\
C_6 =& 5 C_4 \tilde{C}_6.
\end{aligned}
\tag{1.7}
$$

Since the coefficients in Eq. (1.3) are free parameters at this point, $\tilde{O}_6$ can be left out without any loss of generality. Conversely, one can use Eq. (1.5) to trade O_6 for $\tilde{O}_6$, leading to the Lagrangian

$$
\mathcal{L}_{\rm EFT} =\frac{1}{2}\left[(\partial_\mu \phi)^2 - m^2\phi^2\right] - \tilde{C}_4 \frac{\phi^4}{4!} + \frac{\tilde{C}_6}{4!}\phi^3\Box\phi + \mathcal{O}(M^{-4}). \tag{1.8}
$$

In my jargon, the Lagrangian in Eq. (1.3) is written in the *unbox basis*, and the one in Eq. (1.8) is written in the *box basis*. In this toy model the basis of dimension-6 operators is one-dimensional, but that is just because I picked a particularly simple example to introduce the concept; in general the basis of operators may span a multi-dimensional space, as we will see in subsequent examples. At any order in perturbation theory, both Lagrangians give equivalent predictions for all on-shell scattering amplitudes up to $\mathcal{O}(M^{-2})$ terms. The two sets of predictions are related by the map

$$
\begin{aligned}
\tilde{C}_4 =& C_4 - \frac{m^2}{5}\frac{C_6}{C_4}, \\
\tilde{C}_6 =& \frac{C_6}{5 C_4}.
\end{aligned}
\tag{1.9}
$$

Exercise: Express the operator $\hat{O}_6$ by the ones present in the Lagrangian of Eq. (1.3) . Write down the map between the double-box basis, and the unbox basis.

We would like to understand how the EFT parameters in Eq. (1.3) or Eq. (1.8) are related to the “fundamental" UV parameters in Eq. (1.2). The general procedure is to

1. Calculate scattering amplitudes in both the UV theory and the EFT;
2. Expand the former in inverse powers of the large mass scale M;
3. Adjust the EFT parameters such that the EFT amplitudes match the UV ones to a requested order in the $1/M$ expansion.

This procedure is called *matching.*

Calculating tree-level amplitudes is straightforward but at this order the matching can be further simplified by a useful hack. Quite generally, it can be proven that the tree-level EFT Lagrangian can be obtained by solving the equations of motion for the heavy fields in terms of the light fields, and plugging the solution back into the UV Lagrangian. In the case at hand the equation of motion for the heavy field H is solved by

$$H(\phi) = -\frac{\lambda_1 M}{2}\left[M^2 + \Box + \frac{\lambda_2}{2}\phi^2\right]^{-1}\phi^2. \tag{1.10}$$

To obtain the tree-level effective Lagrangian, this solution should be inserted in the UV Lagrangian:

$$\begin{aligned}\mathcal{L}^{(0)}_{\rm EFT}(\phi) =&\mathcal{L}_{\rm UV}(\phi, H(\phi))\\ =&\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_0^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2}M\phi^2 H_c(\phi) - \frac{1}{2}H_c(\phi)\left[\Box + M^2 + \frac{\lambda_2}{2}\phi^2\right]H_c(\phi)\\ =&\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_0^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4 + \frac{\lambda_1^2 M^2}{8}\phi^2\left[M^2 + \Box + \frac{\lambda_2}{2}\phi^2\right]^{-1}\phi^2. \end{aligned}\tag{1.11}$$

Expanding this up to order $1/M^2$:

$$\begin{aligned}\mathcal{L}^{(0)}_{\rm EFT} =&\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_0^2}{2}\phi^2 - \left(\lambda_0 - 3\lambda_1^2\right)\frac{\phi^4}{4!} - \frac{\lambda_1^2\lambda_2}{16M^2}\phi^6 - \frac{\lambda_1^2}{8M^2}\phi^2\Box\phi^2 + \mathcal{O}(M^{-4})\\ =&\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_0^2}{2}\phi^2 - \left(\lambda_0 - 3\lambda_1^2\right)\frac{\phi^4}{4!} - 45\lambda_1^2\lambda_2\frac{\phi^6}{6!M^2} - 4\lambda_1^2\frac{\phi^3\Box\phi}{4!M^2} + \mathcal{O}(M^{-4}). \end{aligned}\tag{1.12}$$

Note that both the ϕ^6 and $\phi^3\Box\phi$ dimension-6 terms appear as a result of this procedure. One can simplify the effective theory by projecting it into one of the bases. Using Eq. (1.7) to get rid of the box term one obtains the tree-level matching in the unbox basis:

$$\begin{aligned} m^2 =&m_0^2,\\ C_4 =&\lambda_0 - 3\lambda_1^2 - 4\lambda_1^2\frac{m_0^2}{M^2},\\ C_6 =&\frac{1}{M^2}\left(45\lambda_1^2\lambda_2 - 20\lambda_0\lambda_1^2 + 60\lambda_1^4\right). \end{aligned}\tag{1.13}$$

The scaling of the Wilson coefficients C_4 and C_6 is consistent with the dimensional analysis, $[C_D] \sim M^{4-D}$, although note that the ratios of the light and heavy mass scales also appear. For m^2 dimensional analysis would suggest $m_0^2 \sim M^2$, but that seems at odds with the requirement that $m \ll M$. This is the first instalment of the hierarchy problem.

The matching between an EFT and its UV completion does not have to be limited to tree-level. Let me just quote without derivation the result of the matching in the dimensionally regularized MSbar scheme up to one loop in the limit $\lambda_1 \to 0$:

$$\begin{aligned} m^2(\mu) =&m_0^2 - \frac{\lambda_2 M^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{M^2}\right) + 1\right],\\ C_4(\mu) =&\lambda_0 - \frac{3\lambda_2^2}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right) - \frac{\lambda_2^2 m_0^2}{48\pi^2 M^2},\\ C_6(\mu) =&\frac{15\lambda_2^3}{32\pi^2 M^2} - \frac{5\lambda_0\lambda_2^2}{48\pi^2 M^2}. \end{aligned}\tag{1.14}$$

Here is μ is an arbitrary dimensionful parameter interpreted as the renormalization scale. We can choose it be whatever, but some choices are better than others. As long as μ is of the order of the high mass scale M, one avoids large logarithms in the matching. These large longs could invalidate perturbation theory even for perturbative couplings λ_2, if $\lambda_2^2 |\log\left(\frac{\mu}{M}\right)| \gg 1$. In particular, choosing $\mu = M$ the matching simplifies to

$$\begin{aligned}
m^2(M) =& m_0^2 - \frac{\lambda_2 M^2}{32\pi^2}, \\
C_4(M) =& \lambda_0 - \frac{\lambda_2^2 m_0^2}{48\pi^2 M^2}, \\
C_6(M) =& \frac{15\lambda_2^3}{32\pi^2 M^2} - \frac{5\lambda_0 \lambda_2^2}{48\pi^2 M^2}.
\end{aligned} \tag{1.15}$$

Notice that logs of the small mass scale m do not show up in the matching equation. The reason is that these logs are of IR origin, thus they are common to both the EFT and the UV theory calculation. The matching of the mass parameter leads to the second instalment of the hierarchy problem: the one loop shift of the EFT mass m is proportional to the heavy mass M. In a way, EFT is craving to recover the dimensional analysis $m \sim M$, or at least $m \sim M/4\pi$. To avoid that and engineer a large hierarchy $m \ll M/4\pi$, it is necessary to fine tune the UV mass parameter m_0.

One last comment is devoted to the RG running. Already from the matching equations we can see that the EFT Wilson coefficients, much as the coupling of the UV theory, should be interpreted as running parameters dependent on the renormalization scale μ. To calculate the EFT couplings at $\mu \ll M$ we need to evolve them using the RG equations. The latter can be obtained by demanding the observables calculated in the EFT do not depend on the renormalization scale. This leads to the equations

$$\begin{aligned}
\frac{dm^2}{d\log\mu} =& \frac{m^2 C_4}{16\pi^2}, \\
\frac{dC_4}{d\log\mu} =& \frac{1}{16\pi^2}\left[3C_4^2 + m^2 C_6\right].
\end{aligned} \tag{1.16}$$

The $\mathcal{O}(M^0)$ terms on the right-hand sides are the standard result in the ϕ^4 theory. There is also an $\mathcal{O}(M^{-2})$ terms in the running equation for C_4 that is proportional to the Wilson coefficient of the dimension-6 operator. In general, the EFT at one loop, Wilson coefficients of higher dimensional operators may affect RG running of lower-dimensional ones (never the other way around) if there are explicit mass parameters in the EFT. To one-loop accuracy the first of the equations in Eq. (1.16) is solved by

$$m^2(\mu) = m^2(M)\left(\frac{\mu}{M}\right)^{\frac{C_4}{16\pi^2}}. \tag{1.17}$$

The good news is that, if we fine tune $m^2(M) \ll M^2$, this relation is radiatively stable within the EFT. Another good news is that large logarithms $\log(M/m)$ in the UV theory can be resummed in the EFT, leading to expression such as Eq. (1.17). Therein lies the power of EFT: large scales are integrated out, leaving only the physical scales of interest, and large logarithms are resummed via RG running. In this sense EFT is not just equivalent to the underlying fundamental theory at low energies, but is also better from the practical point of view.

1.3 Euler-Heisenberg, or let there be light

Another illustrative example of the Euler-Heisenberg[1] EFT [6], which is the effective theory for photons at very low energies (from the particle physics point of view). Let us take QED as the starting point, ignoring all the SM particles heavier than electrons, and ignoring the small couplings to gravity and neutrinos. This is a theory of massless photons minimally coupled to electrons (and their anti-particles), If we are interested only in the propagation of photons (e.g. we study electromagnetic waves), for the photon energies $E \ll \Lambda \approx 2m_e \approx$ 1 MeV we can integrate out the electrons. This leaves us with the Euler-Heisenberg EFT where photons are the only degrees of freedom

The Lagrangian of this EFT takes the general form

$$\mathcal{L}_{\rm EH} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \mathcal{L}_{\rm EH}^{D=8} + \mathcal{L}_{\rm EH}^{D=10} + \ldots \tag{1.18}$$

Each term in this expansion should be Lorentz- and gauge-invariant. Gauge invariance dictates that the photon field A_μ can enter only via the field strength tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ or its dual $\tilde{F}_{\mu\nu} \equiv \epsilon_{\mu\nu\rho\sigma}\partial_\rho A_\sigma$. The first term in Eq. (1.18) is the usual kinetic term of dimension four, which does not contain any interactions. The remaining terms describe higher-dimensional photon self-interactions, which arise after integrating out the electrons. The interactions are organized according to their canonical dimensions, with the Wilson coefficients in each consecutive term being suppressed by more and more powers of the cut-off scale $\Lambda \sim m_e$. At the dimension-6 level there is no possible gauge-invariant interaction due to the identity $F_{\mu\nu} F_{\nu\rho} F_{\rho\mu} = 0$ (more generally, all invariants with an odd number of $F_{\mu\nu}$ vanish). Thus, the leading interactions arise at the level of dimension-8 operators. Ignoring tiny parity-violating effects from weak interactions in the SM, one can assume that all interactions in the Euler-Heisenberg EFT are parity conserving. Then, there are two independent operators at the dimension-8 level. A convenient basis can be chosen as

$$\mathcal{L}_{\rm EH}^{D=8} = \frac{C_1}{16} (F_{\mu\nu} F_{\mu\nu})(F_{\rho\sigma} F_{\rho\sigma}) + \frac{C_2}{16} (F_{\mu\nu} \tilde{F}_{\mu\nu})(F_{\rho\sigma} \tilde{F}_{\rho\sigma}). \tag{1.19}$$

Exercise: Show that the dimension-8 operator $(F_{\mu\rho} F_{\nu\rho})(F_{\mu\sigma} F_{\nu\sigma})$ can be expressed in the basis of Eq. (1.19). What other operators can be added to Eq. (1.19) if we lift the constraint of parity conservation?

The Wilson coefficients C_1 and C_2 can be calculated by matching the Euler-Heisenberg EFT to QED at the scale $E \sim \Lambda \sim 2m_e$. In the UV theory, 2-to-2 photon scattering proceeds through box diagrams with the electron in the loop. For scattering energy $E \ll m_e$

[1] Here, Euler is of course not the famous mathematician, but a PhD student of Heisenberg and Luftwaffe pilot in Nazi Germany. Similarly, Heisenberg is an obscure physicist in 20th century Germany, and not the famous kingpin from Breaking Bad.

we can expand the result in powers of $1/m_e$. In order to match the result to the Euler-Heisenberg EFT we should equate it with the tree-level amplitude obtained using Eq. (1.19). This procedure leads to the following identification:

$$C_1^{\rm QED} = \frac{8\alpha^2}{45 m_e^4}, \qquad C_2^{\rm QED} = \frac{14\alpha^2}{45 m_e^4}. \tag{1.20}$$

Until now we have assumed that QED is the UV completion of the Euler-Heisenberg EFT. But some exotic, so far undetected particles may also contribute to the Wilson coefficients in Eq. (1.19). For example, integrating out a scalar of mass m_s and electric charge Q_s leads to a different pattern of the Wilson coefficients (see e.g. [7]):

$$C_1^{\rm SQED} = \frac{7Q_s^4\alpha^2}{90 m_s^4}, \qquad C_2^{\rm SQED} = \frac{Q_s^4\alpha^2}{90 m_s^4}. \tag{1.21}$$

One could also entertain the possibility of a UV completion being itself an effective theory. Consider the following Lagrangian containing a real scalar a of mass m_a coupled to photons via a dimension-5 interaction:

$$\mathcal{L}_{\rm UV} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial_\mu a)^2 - \frac{m_a^2}{2}a^2 + c_a\frac{\alpha}{4\pi f}aF_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{1.22}$$

This kind of coupling is characteristic for axions, or more generally for Goldstone bosons of a global symmetry spontaneously broken at the scale f where the global symmetry current has a mixed triangle anomaly with the electromagnetic $U(1)$ currents. Integrating out a at tree level leads to the effective dimension-8 interactions in Eq. (1.19) with the Wilson coefficients

$$C_1^{\rm axion} = 0, \qquad C_2^{\rm axion} = \frac{c_a^2\alpha^2}{2\pi^2 f^2 m_a^2}. \tag{1.23}$$

The bottom line is that different UV completions lead to different patterns of Wilson coefficients of dimension-8 operators in the Euler-Heisenberg EFT. Therefore it makes sense to treat C_1 and C_2 as free parameters, to be fixed by experiment. The question we address in the following is what kind of measurements are sensitive to these Wilson coefficients. One way is to search for non-linear effects in electromagnetic waves propagation. Indeed, the interactions in Eq. (1.19) modify Maxwell's equations of electrodynamics, in particular they introduce non-linearities that violate the superposition principle. Following Ref. [8], let us first rewrite the Euler-Heisenberg Lagrangian in engineer's friendly variables $\boldsymbol{E} = -\boldsymbol{\nabla}A^0 - \partial_t\boldsymbol{A}$, $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, where $A^\mu = (A^0, \boldsymbol{A})$. Using $F_{0k} = E^k$, $F_{kl} = -\epsilon^{klm}B^m$, $\tilde{F}_{0k} = B^k$, $\tilde{F}_{kl} = \epsilon^{klm}B^m$, we have

$$\mathcal{L}_{\rm EH} = \frac{1}{2}\Big(\boldsymbol{E}^2 - \boldsymbol{B}^2\Big) + \frac{C_1}{4}\big(\boldsymbol{E}^2 - \boldsymbol{B}^2\big)^2 + C_2\big(\boldsymbol{E}\boldsymbol{B}\big)^2. \tag{1.24}$$

One defines the electric displacement $\boldsymbol{D} \equiv \frac{\partial\mathcal{L}}{\partial\boldsymbol{E}}$, and the magnetic intensity $\boldsymbol{H} \equiv -\frac{\partial\mathcal{L}}{\partial\boldsymbol{B}}$. In QED in vacuum $\boldsymbol{D} = \boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}$, but in the presence of dimension-8 operators in the Euler-Heisenberg Lagrangian these relations are modified as

$$\boldsymbol{D} = \boldsymbol{E} + C_1\big(\boldsymbol{E}^2 - \boldsymbol{B}^2\big)\boldsymbol{E} + 2C_2\big(\boldsymbol{E}\boldsymbol{B}\big)\boldsymbol{B},$$

$$\boldsymbol{H} = \boldsymbol{B} + C_1\big(\boldsymbol{E}^2 - \boldsymbol{B}^2\big)\boldsymbol{B} - 2C_2\big(\boldsymbol{E}\boldsymbol{B}\big)\boldsymbol{E}. \tag{1.25}$$

In electrodynamics one further defines the polarization $\boldsymbol{P} = \boldsymbol{D} - \boldsymbol{E}$ and magnetization $\boldsymbol{M} = \boldsymbol{B} - \boldsymbol{H}$ vectors. For $C_{1,2} \neq 0$ the vacuum behaves as a medium in QED: it has non-zero polarization and magnetization in the presence of external fields. Consider switching on a strong external magnetic field $\boldsymbol{B}_0$. Then we plug in on the right-hand side of Eq. (1.25) $\boldsymbol{E} = \boldsymbol{E}_w$, $\boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_w$, where $\boldsymbol{E}_w$ and $\boldsymbol{B}_w$ are electromagnetic fields of a passing wave. Expanding to the leading order in $\boldsymbol{E}_w$ and $\boldsymbol{B}_w$ we get

$$\begin{aligned} \boldsymbol{D}_w &= \boldsymbol{E}_w - C_1 B_0^2 \boldsymbol{E}_w + 2C_2\big(\boldsymbol{E}_w\boldsymbol{B}_0\big)\boldsymbol{B}_0, \\ \boldsymbol{H}_w &= \boldsymbol{B}_w - C_1 B_0^2 \boldsymbol{B}_w - 2C_1\big(\boldsymbol{B}_w\boldsymbol{B}_0\big)\boldsymbol{B}_0. \end{aligned} \tag{1.26}$$

One consequence of the non-linearities is that in strong magnetic fields is the phenomenon *vacuum birefringence*, where vacuum exhibits different refractive indices for different polarizations of light. Indeed, the electric permittivity and magnetic permeability defined by $\boldsymbol{D} = \epsilon\boldsymbol{H}$ and $\boldsymbol{B} = \mu\boldsymbol{H}$ have a different value for waves with the electric vector polarization parallel or perpendicular to the external magnetic field:

$$\begin{aligned} \epsilon_\perp &= 1 - C_1 B_0^2, \qquad & \epsilon_\parallel &= 1 - C_1 B_0^2 + 2C_2 B_0^2, \\ \mu_\perp &= 1 + 3C_1 B_0^2, \qquad & \mu_\parallel &= 1 + C_1 B_0^2. \end{aligned} \tag{1.27}$$

The index of refraction is $n = \sqrt{\epsilon\mu}$ hence

$$n_\perp = 1 + C_1 B_0^2, \qquad n_\parallel = 1 + C_2 B_0^2. \tag{1.28}$$

As advertised, the refractive index and therefore the speed of propagation $v = 1/n$ is different depending on whether polarization of the electromagnetic wave is parallel or perpendicular to the external magnetic field. Measuring the propagation speed in the two case allows us to access the two independent dimension-8 Wilson coefficients of the Euler-Heisenberg EFT (from the experimental point of view it is usually easier to measure the difference, $\Delta n = n_\perp - n_\parallel$, which probes $C_1 - C_2$). The effect is being searched for in the laboratory settings with high-intensity lasers and strong magnetic fields, most recently in the PVLAS experiment in Italy [8], but for the time being there has been no detection. Another promising environment for observing vacuum birefringence is in the proximity of astrophysical objects such as neutron stars, because these can create immense magnetic fields. There exist some claims of detection in astrophysical settings [9], but to my understanding they are not universally accepted and certainly no measurement of C_i was produced.

This example shows another interesting feature of EFT - it makes the study of collective low-energy phenomena much more tractable. Vacuum birefringence is the consequence of QED, due to soft light quanta interacting with virtual electron-positron pairs. But it would be very difficult to study this effect quantitatively within the QED formalism. On the other hand, after resorting to the Euler-Heisenberg formalism, the calculation of the propagation speed becomes quite trivial.

There is one interesting conceptual consequence of Eq. (1.28). The propagation speed should not exceed the speed of light in the absence of external fields, which translates to $n \geq 1$, otherwise we would be able to send superluminal signal around the CERN tunnel. Therefore in this case causality implies positivity of the Wilson coefficients:

$$C_1 \geq 0, \qquad C_2 \geq 0. \tag{1.29}$$

Unsurprisingly, the Wilson coefficients obtained for specific UV completions, cf. Eqs. (1.20), (1.21) and (1.23), satisfy this inequality. The same conclusion can be reached by considering micro-causality in the sense of Ref. [10], which amounts to exploring analytic properties of scattering amplitudes. In this approach the inequalities can be sharpened to $C_1 > 0$, $C_2 > 0$. This is very interesting. From the naive bottom-up perspective one might suppose that the Wilson coefficients in EFT can in principle take arbitrary values, and any pattern can be reached for suitably designed UV completions. But not anything goes: self-consistency of the theory may impose non-trivial constraints on the EFT parameter space. This is a very active topic of research, with certainly more progress to come.

1.4 GREFT, or quantum gravity for dummies

EFT techniques can be equally well applied to the quantum theory of a massless spin-2 particle, which describes gravitational interactions. This leads to the so-called General Relativity EFT (GREFT), which is an EFT extension of the Einstein theory of general relativity (GR). The quantum field encoding the gravitational degrees of freedom is the spacetime metric $g_{\mu\nu}$, which describes, in the limit of the flat Minkowski background, a massless spin-2 particle called the *graviton*. As in the Euler-Heisenberg EFT, the Lagrangian can be organized into an expansion in canonical dimension. The main difference with the photon EFT is in the choice of symmetry governing the interactions. In order to decouple the unphysical degrees of freedom in the metric the Lagrangian has to be invariant under a different kind of local gauge transformations: the so-called general coordinate transformations. The GREFT Lagrangian takes the form

$$\mathcal{L}_{\rm GREFT} = +\mathcal{L}_{\rm GREFT}^{D=2} + \mathcal{L}_{\rm GREFT}^{D=6} + \mathcal{L}_{\rm GREFT}^{D=8} + \dots. \tag{1.30}$$

Each term in this expansion is separately invariant under general coordinate (GC) transformations. At the leading, dimension-2 order there is only a single GC-invariant term:

$$\mathcal{L}_{\rm GREFT}^{D=2} = \frac{M_{\rm Pl}^2}{2}\sqrt{-g}R, \tag{1.31}$$

where $M_{\rm Pl} \equiv \frac{1}{\sqrt{8\pi G}} \approx 2.44 \times 10^{18}$ GeV This term encapsulates the usual Einsteinian GR. Higher order corrections are constructed from powers of the Riemann tensor $R_{\mu\nu\alpha\beta}$ (more precisely, from its traceless part called the Weyl tensor, $C_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta} - g_{\mu[\alpha}R_{\beta]\nu} + g_{\nu[\alpha}R_{\beta]\mu} + \frac{1}{3}g_{\mu[\alpha}g_{\beta]\nu}R$). The Wilson coefficient in each consecutive term are suppressed by more and more powers of the cutoff scale Λ, which naturally may be comparable to or much smaller than the Planck scale (since GR is not renormalizable, higher-dimensional operators are generated by gravitational loop corrections, therefore the choice $\Lambda \gg M_{\rm Pl}$

would be unnatural, that is unstable under radiative corrections). At dimension four naively it is possible to construct non-trivial GC-invariant operators, e.g. R^2, $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$. However the first two are proportional to the leading order equations of motion $R_{\mu\nu}=0$, and therefore they do not contribute to on shell amplitudes (in other words, they can be eliminated by a suitable field redefinition). The last one can be expressed by the the first two plus a total derivative due to the fact that the so-called Gauss-Bonnet term $R^2 - 4R_{\mu\nu}R_{\mu\nu} + R_{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$ is a total derivative. Thus the leading order GREFT corrections to GR enter at dimension six:

$$\mathcal{L}^{D=6}_{\mathrm{GREFT}} = C_1 C_{\mu\nu\alpha\beta}C_{\alpha\beta\rho\sigma}C^{\rho\sigma\mu\nu} + C_2 C_{\mu\nu\alpha\beta}C_{\alpha\beta\rho\sigma}\tilde{C}^{\rho\sigma\mu\nu}, \tag{1.32}$$

where $\tilde{C}_{\mu\nu\alpha\beta} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}C_{\rho\sigma\alpha\beta}$. If C_i are of order M_{Pl}^{-2}, it is unlikely we will see their effects anytime in our lifetime. However it is conceivable that the cutoff of GREFT is much lower than the Planck scale, and then $C_i \sim \Lambda^{-2}$ may be searched for via a host of GR tests.

How can we calculate amplitudes of quantum processes using Eq. (1.30)? We first need to identify the perturbative degrees of freedom with which we can do QFT. As long as we are away from high curvature regions (a good approximation in most of our local galaxy cluster except near black holes) a good starting point is perturbations around the flat Minkowski space:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_{\mathrm{Pl}}}h_{\mu\nu}. \tag{1.33}$$

The tensor $h_{\mu\nu}$ describes the two polarizations of the massless spin-2 particle - the graviton. The factor $2/M_{\mathrm{Pl}}$ in front ensures that $h_{\mu\nu}$ has canonically normalized kinetic terms. Indeed, expanding Eq. (1.31) to quadratic order in $h_{\mu\nu}$, and integrating by parts while dropping total derivatives to simplify the expression, one finds

$$\mathcal{L}^{D=2}_{\mathrm{GREFT}} = \frac{1}{2}(\partial_\rho h_{\mu\nu})^2 - \frac{1}{2}(\partial_\rho h)^2 - (\partial_\rho h_{\mu\rho})^2 + \partial_\mu h \partial_\rho h_{\mu\rho} + O(h^3), \tag{1.34}$$

where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$. The kinetic terms are invariant under the local gauge transformations $\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. These are needed to decouple the unphysical components of the metric, so that only the two physical polarizations propagate on shell. After gauge fixing, the kinetic terms can be inverted to obtain the propagator, similarly as it is done for massless spin-1 particles.

Apart from the kinetic terms, the dimension-2 term in the GREFT Lagrangian contains an infinite series of graviton self-interactions. Just for laughs, let me quote here the cubic self-interactions in the $\partial_\rho h_{\rho\mu}$ gauge copied from Ref. [11] (and appropriately rescaled to account for different normalizations):

$$\begin{aligned}\mathcal{L}^{D=2}_{\mathrm{GREFT}} \supset \frac{1}{M_{\mathrm{Pl}}}\big\{ & 4h_{\mu\nu}\partial_\mu h_{\alpha\beta}\partial_\beta h_{\alpha\nu} - h_{\mu\nu}\partial_\mu h_{\alpha\beta}\partial_\nu h_{\alpha\beta} - 2h_{\mu\nu}\partial_\mu h \partial_\rho h_{\nu\rho} - h\partial_\rho h_{\mu\nu}\partial_\nu h_{\mu\rho} \\ & + \frac{1}{2}h\partial_\rho h_{\mu\nu}\partial_\rho h_{\mu\nu} - 2h_{\mu\nu}\partial_\rho h \partial_\nu h_{\mu\rho} + h_{\mu\nu}\partial_\mu h \partial_\nu h - 2h_{\mu\nu}\partial_\alpha h_{\mu\nu}\partial_\beta h_{\alpha\beta} + h\partial_\mu h \partial_\nu h_{\mu\nu} \\ & + 2h_{\mu\nu}\partial_\rho h_{\mu\nu}\partial_\rho h + \frac{1}{2}h\partial_\mu h \partial_\mu h - 2h_{\mu\nu}\partial_\alpha h_{\mu\beta}\partial_\alpha h_{\nu\beta} + 2h_{\mu\nu}\partial_\alpha h_{\mu\beta}\partial_\beta h_{\nu\alpha}\big\}. \end{aligned} \tag{1.35}$$

It looks scary, but in principle we can derive from it the Feynman rules for the cubic graviton self-interactions. Expanding $\mathcal{L}_{\rm GREFT}^{D=2}$ to higher orders in $h_{\mu\nu}$, we can similarly obtain the quartic and higher graviton vertices. The consecutive terms are suppressed by more and more powers of $M_{\rm Pl}$, which sets the maximum possible validity regime of the theory. With these Feynman rules and the propagator obtained from the quadratic terms we can calculate amplitudes for graviton scattering processes. Similarly $\mathcal{L}_{\rm GREFT}^{D=6}$ and higher-order terms in the GREFT expansion will introduce higher-derivative corrections to the cubic and higher graviton vertices. Schematically,

$$\mathcal{L}_{\rm GREFT}^{D=6} \supset \frac{C_i}{M_{\rm Pl}^3} h^2 \partial^6 h + \mathcal{O}(h^4). \tag{1.36}$$

modifying the cubic and higher graviton vertices by six-derivative terms. Both the GR part and the GREFT corrections can be systematically taken into account in the amplitude calculations. This is extremely tedious using the standard techniques, but in principle there are no showstoppers. At loop level divergences will appear, but these can always be absorbed into counterterms in the GREFT Lagrangian. All in all, GREFT is much like any any other relativistic EFT at the philosophical level, only a tad more difficult at the practical level.

GREFT is an effective theory applicable in the humongous energy regime, possibly $H_0^{-1} \ll E \ll M_{\rm Pl}$. The upper limit is where self-interactions of the graviton become strong, and other fundamental or composite degrees of freedom must emerge to render the graviton scattering amplitudes well-behaved. The lower limit is set roughly by the size of the universe, at which scale QFT in the flat Minkowski background is not applicable. Unlike for the Euler-Heisenberg EFT, we can only speculate about the UV completion of GREFT: it may be some form of string theory, or something completely different. For this reason, we do not know the coupling constants multiplying the higher-derivative interactions terms in the GREFT Lagrangian; they have to be treated as free parameters to be determined one day from experiment. By the way, GREFT is a counterexample to the nonsense you may often hear that *quantum mechanics and general relativity cannot be reconciled*. GREFT obeys the principles of general relativity and is a consistent relativistic quantum theory within its validity range, which is much larger than for other EFTs used in physics.

1.5 Fermi theory, or from the Standard Model to nuclei

Beta decay is the process where a nucleus $\mathcal{N}$ with charge Z decays into another nucleus $\mathcal{N}'$ with charge $Z \pm 1$, while emitting the $\beta^{\mp}$ particle[2] and a (anti-)neutrino. In 1933 Fermi proposed his revolutionary theory of beta decay which, after a couple of tweaks, made it possible to quantitatively describe thousands of such nuclear transitions in nature. At the time, his guide was ingenious bottom-up EFT intuition, as he of course did not know about the SM, weak force, and all that jazz. Here we will derive the Fermi theory top down, using the SM as the starting point.

[2] The name beta particle is one of many living fossils in the nuclear physics literature. The name was coined in the early day after the discovery of radioactivity, and has inexplicably stuck till now, even though the particle was identified with the electron shortly after. In my notation β^- is the electron, β^+ is the positron, and β refers to both.

In the SM, charged current weak interactions responsible for beta decay are mediated by W bosons. After electroweak symmetry breaking, W interacts with the first generation quarks and leptons as

$$\mathcal{L}_{\rm SM} \supset -\frac{g_L}{\sqrt{2}} W_\mu^+ \left[V_{ud}\bar{u}\bar{\sigma}^\mu d + \bar{\nu}\bar{\sigma}^\mu e\right] + \text{h.c.}, \tag{1.37}$$

where g_L is the gauge couplings of the SM $SU(2)_L$, and V_{ud} is an element of the CKM matrix. It is possible to view e.g. β^+ decay as the process where an up quark in the nucleus changes into a down quark while emitting a W^+ boson, which subsequently decay into a positron and a neutrino. This is however very impractical. The typical momentum exchange q in beta decay is of the order of MeV, while the W boson mass is $\mathcal{O}(100)$ GeV. Attempting such a fundamental-level description of the low-energy process introduces unnecessary complications, while on the quantitative level the calculations will be plagues by the large $\log(m_W/q)$ logarithms. In this case it is much advantageous to go down the rabbit hole of EFT and come up with a less fundamental description using the degrees of freedom at the relevant energy scales.

The first step is to get rid of the W boson. At energies $E \lesssim m_W$ we can work with an EFT of the weak force that I call Weak Effective Field Theory (WEFT), where the W and Z bosons (and also the Higgs and the top quark) are integrated out. Consequently, the gauge symmetry of the SM is reduced down to $SU(3)_C \times U(1)_{\rm em}$, as there are no longer gauge bosons corresponding to the full $SU(3)_C \times SU(2)_W \times U(1)_Y$. Also, the weak force is not longer mediated by W and Z, but instead by contact 4-fermion interactions between the SM fermions. Indeed, integrating W at tree-level we get

$$\mathcal{L}_{\rm WEFT} \supset -\frac{g_L^2}{2m_W^2}\left[V_{ud}\bar{u}\bar{\sigma}^\mu d + \bar{\nu}\bar{\sigma}^\mu e\right]\left[V_{ud}\bar{d}\bar{\sigma}_\mu u + \bar{e}\bar{\sigma}_\mu \nu\right]. \tag{1.38}$$

It is convenient to trade $m_W = \frac{g_L v}{2}$ so that g_L cancels out. The scale v controls the strength of weak interactions at low energies, and is directly measurable in experiment. In particular, from muon decay one can infer $v = 246.21965(6)$ GeV.[3] Focusing only on the semi-leptonic interaction relevant for beta decay:

$$\mathcal{L}_{\rm WEFT} \supset -\frac{2V_{ud}}{v^2}(\bar{u}\bar{\sigma}^\mu d)(\bar{e}\bar{\sigma}_\mu \nu) + \text{h.c.} \tag{1.39}$$

We can use this Lagrangian down to energies $E \sim 2$ GeV, when the strong interactions start to set in (the Wilson coefficient depends on the energy scale, but in this case only the electromagnetic interactions induce running at one loop and the effect is therefore small). To press on toward lower energies we will have to get our hands dirty with some non-perturbative physics. In particular, the degrees of freedom change. Quarks are not good degrees of freedom below ~ 2 GeV, and instead nucleons (protons and neutrons) emerge as the particles directly relevant for nuclear processes. We want to match the nucleon-level

[3] In the literature this result is more often presented in terms of the Fermi constant $G_F \equiv \frac{1}{\sqrt{2}v^2} = 1.1663787(6) \times 10^{-5}\ \text{GeV}^{-2}$. However, the Fermi constant leads to proliferation of square roots and I prefer not to use it.

Lagrangian to the UV completion in Eq. (1.39). To this end, we focus on the simplest beta decay process: the neutron decay. We will first calculate the neutron decay amplitude starting from the quark-level Lagrangian, and then write down the nucleon-level Lagrangian which reproduces the same amplitude in the limit where the proton momentum is small in the neutron rest frame. Using Eq. (1.39) we have

$$\mathcal{M}(n \to p e^- \bar{\nu}) = -\frac{2V_{ud}}{v^2} (\bar{x}_e \bar{\sigma}_\mu y_\nu) \langle p(k_p)| \bar{u}\bar{\sigma}^\mu d \, |n(p_n)\rangle . \tag{1.40}$$

The leptonic side is perturbative, and we proceed in the same way as when calculating the amplitude from Feynman diagrams, replacing the fields with the appropriate spinor wave functions. The hadronic side is tricky, because we have to deal with a non-perturbative matrix element of a quark bi-linear between the nucleon external states. We will not able to calculate it from the first principles, and we will parametrize our ignorance instead. We have to respect Lorentz and little group covariance. The matrix element has to consist of the available covariant objects: the spinor wave functions of the neutron and proton, and their momenta. We will make our life simpler by taking the zero recoil limit: $k_p = p_n \equiv p$. This is a good approximation, because the neutron-proton mass difference is small compared to their masses, $m_n - m_p \sim 1$ MeV$\ll m_p \sim 1$ GeV. This means that proton barely moves in the neutron rest frame. In this limit, the most general matrix element takes the form

$$\begin{aligned} \langle p| \bar{u}\bar{\sigma}^\mu d \, |n\rangle =& \alpha_L (\bar{x}_p \bar{\sigma}^\mu x_n) + \alpha_R (y_p \sigma^\mu \bar{y}_n) \\ =& \frac{g_V}{2} (\bar{x}_p \bar{\sigma}^\mu x_p + y_p \sigma^\mu \bar{y}_p) + \frac{g_A}{2} (\bar{x}_p \bar{\sigma}^\mu x_p - y_p \sigma^\mu \bar{y}_p). \end{aligned} \tag{1.41}$$

Here, and x_p, y_p are the spinor wave functions of the neutron and proton (they are the same for both in the zero recoil limit). Other possible structures with the correct Lorentz properties, e.g. $p^\mu (y_p x_p \pm \bar{x}_p \bar{y}_p)$, can reduced to the ones occurring above. The numerical constant $\alpha_{L,R}$ should be $\mathcal{O}(1)$ by dimensional analysis, but are not calculable from first principles. In the second line we traded them for commonly used vector and axial charges of the nucleon, which multiply the combinations of the spinor wave functions with definite parity. The reason is that these variables facilitate folding in some additional information due to symmetry considerations. Namely, the Ademollo-Gatto theorem [12] states that $g_V = 1$ up to second order effects in isospin breaking, that is to say up to corrections of order 10^{-4}. On the other hand, g_A cannot be fixed by symmetry considerations, and instead has to be calculated on the lattice or measured in experiment. The lattice value quoted by the FLAG aggregator is $g_A = 1.246(28)$ [13], so it is indeed order one as expected by dimensional analysis.

> *Exercise:* Show that $g_V = 1$ in the limit of unbroken isospin symmetry ($m_u = m_d$ and electromagnetic interactions switched off).

Putting together Eqs. (1.40) and (1.41)

$$\mathcal{M}(n \to p e^- \bar{\nu}) = -\frac{V_{ud}}{v^2} (\bar{x}_e \bar{\sigma}_\mu y_\nu) \big[g_V (\bar{x}_p \bar{\sigma}^\mu x_p + y_p \sigma^\mu \bar{y}_p) + g_A (\bar{x}_p \bar{\sigma}^\mu x_p - y_p \sigma^\mu \bar{y}_p) \big]. \tag{1.42}$$

This can be derived from the nucleon-level effective Lagrangian

$$\mathcal{L}_{\rm Fermi} = - \left[C_V^+(\bar{p}\bar{\sigma}^\mu n + p^c\sigma^\mu\bar{n}^c) - C_A^+(\bar{p}\bar{\sigma}^\mu n - p^c\sigma^\mu\bar{n}^c)\right](\bar{e}\bar{\sigma}_\mu\nu), \tag{1.43}$$

with the leading order matching

$$C_V^+ = g_V\frac{V_{ud}}{v^2}, \qquad C_A^+ = -g_A\frac{V_{ud}}{v^2}. \tag{1.44}$$

From this Lagrangian one can calculate all observables in neutron decay using the standard QFT techniques. For example, the differential decay spectrum can be found to be

$$\frac{d\Gamma}{dE_e} = \frac{(C_V^+)^2 + 3(C_A^+)^2}{4\pi^3}p_e E_e(E_e^{\rm max} - E_e)^2, \tag{1.45}$$

where E_e is the relativistic energy of the outgoing electron, $p_e = \sqrt{E_e^2 - m_e^2}$ is the magnitude of its 3-momentum, and $E_e^{\rm max} \approx m_n - m_p$ is the endpoint energy. This shape of the spectrum is in fact the prediction of the Fermi theory not only for the neutron decay but for a large class of nuclear β decays (the so-called *allowed* decays in another example of curious fossil nomenclature). Here let me just give you one very simple example how to calculate the amplitude for decays of nuclear states. Consider the situation where both the mother nucleus $\mathcal{N}$ and the daughter nucleus $\mathcal{N}'$ have spin zero and positive parity, which is denoted as 0^+. Then

$$\mathcal{M}(\mathcal{N} \to \mathcal{N}'e^-\bar{\nu}) = -(\bar{x}_e\bar{\sigma}_\mu y_\nu)\left[C_V^+\langle 0^+|\,\bar{p}\bar{\sigma}^\mu n + p^c\sigma^\mu\bar{n}^c\,|0^+\rangle - C_A^+\langle 0^+|\,\bar{p}\bar{\sigma}^\mu n - p^c\sigma^\mu\bar{n}^c\,|0^+\rangle\right]. \tag{1.46}$$

Now we have to deal with matrix elements of nucleon fields between external nuclear states. In general, this is a complicated non-perturbative object, but in the zero recoil limit things simplify considerably because the momentum p^μ of the mother nucleus is the only Lorentz-covariant object available, and we must have $\langle 0^+|\,\bar{p}\bar{\sigma}^\mu n \pm p^c\sigma^\mu\bar{n}^c\,|0^+\rangle \sim p^\mu$. Furthermore, parity of QCD, inherited by nuclear interactions, determines that the axial matrix element is zero, because one cannot construct any object with required parity and Lorentz transformation properties. The amplitude simplifies to to

$$\mathcal{M}(\mathcal{N} \to \mathcal{N}'e^-\bar{\nu}) = -2m_{\mathcal{N}}C_V^+M_F(\bar{x}_e\bar{\sigma}_\mu y_\nu)p^\mu, \tag{1.47}$$

where M_F is the numerical proportionality constant in the vector matrix element. From this point the differential decay width can be calculated using the standard QFT techniques, and one ends up with the result very similar to the one for neutron decay:

$$\frac{d\Gamma}{dE_e} = M_F^2\frac{(C_V^+)^2}{4\pi^3}p_e E_e(E_e^{\rm max} - E_e)^2. \tag{1.48}$$

For decays between members of the isospin multiplet M_F is actually calculable from symmetry, and then at leading order everything the spectrum is fully calculable, in spite of dealing with the brown muck of nuclear physics.

1.6 Chiral perturbation theory, or the tale of broken symmetries

Chiral perturbation theory (χPT) carries perhaps the most uninspiring name in particle physics but its theoretical and practical importance is immense. It is a theory of the lightest pseudo-scalar hadrons bound by the QCD strong force: pions, and sometimes kaons and η mesons, valid in a relatively narrow energy range from 100 MeV (the pion masses) to ~ 700 MeV where the tower of QCD resonances begins. Here, a new kind of complications appears. We know the UV completion perfectly well but nevertheless we cannot derive the effective theory from first principles, because QCD becomes non-perturbative. Instead we will have to rely on approximate symmetries and EFT power counting, with also some input from experiment and lattice QCD.

For the sake of the present discussion imagine that the SM contains only a single generation of fermions, with the up and down quarks charged under the color $SU(3)$ group. Each quark is described by a pair of 2-component spinors: u, $\bar{u}^c$ and d, $\bar{d}^c$ transforming in the fundamental representation of $SU(3)$. The relevant QCD Lagrangian is then

$$\begin{aligned}\mathcal{L}_{\rm QCD} =& i\bar{u}\bar{\sigma}_\mu D_\mu u + iu^c\sigma_\mu D_\mu \bar{u}^c + i\bar{d}\bar{\sigma}_\mu D_\mu d + id^c\sigma_\mu D_\mu \bar{d}^c - m_u\left(u^c u + \bar{u}\bar{u}^c\right) - m_d\left(d^c d + \bar{d}\bar{d}^c\right)\\ =& i\bar{q}\bar{\sigma}_\mu D_\mu q + iq^c\sigma_\mu D_\mu \bar{q}^c - q^c M_q q - \bar{q}M_q^\dagger \bar{q}^c, \end{aligned} \tag{1.49}$$

where the derivatives are covariant with respect to the color SU(3), $D_\mu = \partial_\mu + ig_s G^a_\mu T^a$. In the second line we introduced a more compact notation with $q = (u, d)$, $q^c = (u^c, d^c)$, $M_q = \mathrm{diag}(m_u, m_d)$. In the limit of massless quarks, $M_q \to 0$, the Lagrangian possesses a global $SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$ symmetry. the vector symmetry $U(1)_V$, which acts as $q \to e^{i\alpha} q$, $q^c \to e^{-i\alpha} q^c$ on the quarks, can be identified with the baryon number, and it does not play a role in χPT where all the degrees of freedom carry zero baryon number. It turns out that the axial symmetry $U(1)_A$, which acts as $q \to e^{i\alpha} q$, $q^c \to e^{i\alpha} q^c$, is not a symmetry of the quantum theory due to anomalies, and we will not discuss it anymore.[4] We therefore focus on the two $SU(2)$ factors which act on the quarks as

$$q \to Lq, \qquad q^c \to qR^\dagger, \tag{1.50}$$

where L, R are unitary 2×2 matrices with unit determinant. That is to say, $SU(2)_L$ rotates the left-handed up and down quarks, and $SU(2)_R$ rotates the right-handed quarks. One also defines the vector combination, $SU(2)_V$, corresponding to $R = L$, and the axial combination, $SU(2)_A$, corresponding to $R^\dagger = L$.

We expect the global symmetries of QCD will be somehow reflected in χPT. Indeed, $SU(2)_V$, under the nickname of *isospin*, is well known to hadronc known In χPT the pions form an isospin triplet, (π^+, π^0, π^-), whose all component have approximately the same mass, $m_{\pi^\pm} - m_{\pi^0} \approx 4.6$ MeV$\ll m_\pi$. If χPT is extended to include kaons, then also the doublets (K^+, K^0), $(K^-, \bar{K}^0)$ are approximately degenerate in mass. In the EFT for baryons, isospin is responsible for the tiny proton - neutron mass difference, and if Fermi were a botanist he would certainly invoke scores of other isospin multiplets. All in all, isospin is a good approximate of the hadronic world at the spectroscopic level, and in fact

[4] See the TASI lectures of Anson Hook for a nice discussion [14].

also at the interactions level. On the other hand, $SU(2)_A$ is nowhere to be seen at the spectroscopic level. We interpret this fact as an evidence that $SU(2)_A$ is spontaneously broken by the QCD vacuum. That is to say, the vacuum expectation value of the quark bi-linear is non-zero:

$$\langle 0| q^c q |0\rangle \neq 0. \tag{1.51}$$

Since $q^c q$ is invariant under $SU(2)_V$ but transforms under $SU(2)_A$, this preserves $SU(2)_V$ but spontaneously breaks $SU(2)_A$. This is just as good, because by the Nambu-Goldstone theorem the $SU(2)_L \times SU(2)_R \to SU(2)_V$ global symmetry breaking pattern should result in 3 approximately massless Goldstone bosons filling the adjoint representation of $SU(2)_V$. These can be readily identified with the pion isospin triplet. To summarize, a combination of phenomenological and theoretical arguments allows us to identify χPT as the theory of pseudo-Goldstone pions from the spontaneous $SU(2)_L \times SU(2)_R \to SU(2)_V$ global symmetry breaking of QCD (pseudo- because this symmetry is only approximate - broken by the quark masses).

At this point we need to make one executive decision. We can implement the $SU(2)_L \times SU(2)_R \to SU(2)_V$ breaking in χPT in a linear or in a non-linear way. The first option is akin to the Higgs sector in the SM, and in addition to the Goldstone bosons it results in another scalar "radial" isospin singlet state. But nothing like the narrow and light Higgs boson resonance exists in the hadronic spectrum.[5] Therefore one pick the non-linear realization of the symmetry, which does not introduce additional scalar states. The common formalism is to introduce a unitary matrix U transforming linearly under $SU(2)_L \times SU(2)_R$ but depending in a non-linear way on the pion fields:

$$U \to L U R^\dagger, \qquad U = \exp\left(i \frac{\pi^k \sigma^k}{F_\pi}\right). \tag{1.52}$$

Here π^k is the triplet of pion field, σ^k are the Pauli matrices, and F_π is called the pion decay constant whose phenomenological value is $F_\pi = 93.2$ MeV. The point of this complicated definition is to ensure that pions transform as triplets, $\pi \to L U L^\dagger$ under the isospin symmetry ($R = L$), and by a shift (as characteristic for Goldstone bosons) under $SU(2)_A$, $\delta\pi^k \sim F_\pi \epsilon_A^k + \dots$. This way the variables π^k encode the symmetry breaking structure, which determines the interactions. Other parametrizations encoding the same symmetry pattern will differ by fields and coupling redefinitions, but will lead to the same physics.

Let us pursue the discussion of the limit $M_q \to 0$, in which case $SU(2)_L \times SU(2)_R$ is unbroken. Then the Lagrangian of χPT is organized in an expansion according to a number of derivatives

$$\mathcal{L}_{\chi\mathrm{PT}} = \mathcal{L}_{\chi\mathrm{PT}}^{(2)} + \mathcal{L}_{\chi\mathrm{PT}}^{(4)} + \mathcal{L}_{\chi\mathrm{PT}}^{(6)} + \dots \tag{1.53}$$

Each term is a function of U and its derivatives, and is invariant under global $SU(2)_L \times SU(2)_R$. The upper index marks the number of space-time derivatives involved. There

[5] The $f_0(500)$ scalar meson is much heavier than the pion, with its mass of 500 MeV give or take a cow, and the width comparable to its mass.

is no zero-derivative term because $U^\dagger U = 1$ is independent of the pion field. At the two derivative level there is only a single term we can write:

$$\mathcal{L}^{(2)}_{\chi\mathrm{PT}} = \frac{F_\pi^2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U]. \tag{1.54}$$

Inserting the definition of U in terms of pion fields, this provides canonically normalized kinetic terms, as well as quartic and higher interaction terms. $\mathcal{L}^{(2)}_{\chi\mathrm{PT}}$ is the starting point that can be used to calculate pion scattering amplitudes at leading order. In particular, the 2-to-2 amplitude reads

$$\mathcal{M}^{(2)}[\pi^a\pi^b \to \pi^c\pi^d] = \frac{1}{F_\pi^2}\Big\{s\delta^{ab}\delta^{cd} + t\delta^{ac}\delta^{bd} + u\delta^{ad}\delta^{bc}\Big\}. \tag{1.55}$$

We can already see that this amplitude grows as E^2/F_π^2. Therefore the cutoff of the theory cannot by larger than $4\pi F_\pi \sim 1$ GeV, where perturbative control of the EFT is completely lost. This is fine, because we know that at $E \sim 700$ MeV new vector resonances appear, affecting the pion scattering amplitudes, so we do not intend to use χPT beyond that.

Going to the next order, the 4-derivative terms in the χPT Lagrangian read

$$\mathcal{L}^{(4)}_{\chi\mathrm{PT}} = \frac{l_1}{4}\big(\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U]\big)^2 + \frac{l_2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U]\mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U]. \tag{1.56}$$

The Wilson coefficients $l_{1,2}$ are dimensionless. By dimensional analysis, they contribute to the next-to-leading order amplitude as $\mathcal{M}^{(4)}[\pi^a\pi^b \to \pi^c\pi^d] \sim l_i\frac{E^4}{F_\pi^4}$. Now, the power counting of χPT, commonly referred to as the *chiral counting*, assumes $l_i \sim \frac{1}{(4\pi)^2}$. This way $\mathcal{M}^{(4)}$ hits the strong coupling around $E \sim 4\pi F_\pi$, much as $\mathcal{M}^{(2)}$. This kind of power counting is characteristic for strongly coupled UV completions. The consequence is that the contributions proportional to l_i is of the same order as the one-loop contributions calculated on the basis $\mathcal{L}^{(2)}_{\chi\mathrm{PT}}$. Therefore we have to put both of these contributions on the same footing. The divergences appearing in the one loop calculation can be absorbed into l_i. This way we can systematically calculate the pion scattering amplitudes order by order in the chiral expansion with the result being of the form

$$\mathcal{M}[\pi^a\pi^b \to \pi^c\pi^d] \sim \frac{E^2}{F_\pi^2}\Big\{1 + \frac{E^2\log E}{(4\pi F_\pi)^2} + \dots\Big\}. \tag{1.57}$$

This makes it clear that χPT is an expansion in $E/4\pi F_\pi$ and makes sense as long $E \ll 4\pi F_\pi$. In practice this is a rather narrow range, given the pion masses are already ~ 100 MeV.

Let us now include the quark masses and understand their effect on χPT. A quick look into PDG tells us that $m_u \approx 2.2$ MeV, $m_d \approx 4.7$ MeV, with remarkably small errors not exceeding 20%. This is indeed much smaller than F_π, so our initial starting point of unbroken $SU(2)_L \times SU(2)_R$ is well justified. Whenever we deal with a small order parameter breaking the symmetry, a very useful trick is the spurion analysis. This consists in pretending that the symmetry breaking parameter itself transforms under the symmetry, such that the symmetry is formally restored. In the case at hand, $SU(2)_L \times SU(2)_R$ is formally restored in Eq. (1.49) if we assign the transformation

$$M_q \to R M_q L^\dagger. \tag{1.58}$$

We now have a new object M_q to build invariant from in the χPT Lagrangian. The simplest one is

$$\Delta\mathcal{L}_{\chi\mathrm{PT}} = \tilde{\Lambda} F_\pi^2 \mathrm{Tr}[M_q U] + \mathrm{h.c.} \tag{1.59}$$

We can write $M_q = \frac{m_u+m_d}{2}\mathbf{1} + \frac{m_u-m_d}{2}\sigma^3$, and then $\mathrm{Tr}[M_q U] + \mathrm{h.c.} = -\frac{\mathrm{m_u+m_d}}{\mathrm{F_\pi^2}}\pi^{\mathrm{k}}\pi^{\mathrm{k}} + \mathcal{O}(\pi^3)$. Therefore at this point one predicts universal pion masses[6] given by

$$m_\pi^2 = 2\tilde{\Lambda}(m_u + m_d). \tag{1.60}$$

Given that $m_\pi \approx 140$ MeV, we can fix $\tilde{\Lambda} \approx 1.4$ GeV, which is close to $4\pi F_\pi$. Note that we cannot directly relate quark masses to pion masses without non-perturbative calculations. In fact, this exercise is more fruitful when strange pseudo-scalar mesons are included in χPT, as it allows one to predict the Gell-Mann Okuba relation between the pion, kaon, and eta meson masses, see e.g. [15].

1.7 What have EFTs ever done for us

The examples discussed in this section demonstrate a wide range of applications of EFT techniques in high-energy physics. The EFT may be a convenient framework when the UV completion is known (toy scalar, χPT, Euler-Heisenberg) and when it is not (GREFT). In the former case, the EFT Wilson coefficients may be analytically calculable (toy scalar, Euler-Heisenberg) or not (Fermi theory, χPT). The calculable Wilson coefficient may appear at the tree level (toy scalar) or only at the loop level (Euler-Heisenberg). The degrees of freedom may be fundamental from the point of view the UV completion (Euler-Heisenberg), or not (χPT), or we don't know (GREFT). In spite of the limitations particular to each case, organizing the Lagrangian in a systematic expansion taking into account the symmetries of the low-energy theory always results in considerable simplifications of the qualitative and quantitative nature.

The overview attempted in this lecture is by no means exhaustive. One of the most important EFT frameworks underlying a good part of physics beyond the SM these days, the so-called SMEFT, will be discussed in a bit more detail in the following section. There are countless other important applications of EFTs in particle physics, which may or may not involve a Lorentz-invariant local Lagrangians. For further reading, I wholeheartedly recommend the general EFT reviews of David Kaplan [15] and Ira Rothstein [16]. Parts of my lectures have been inspired, consciously and unconsciously, by these two references, but they also contain a lot of material not discussed here. For reviews focused on more specific applications, it is worth reading e.g. Ref. [14] on the effective theory of axions, Ref. [17] for the effective theory of heavy mesons, Refs. [18, 19] for the EFT approach to gravity, or Ref. [20] for the effective theory of excitations in superconductors.

2 Introduction to SMEFT

The SMEFT philosophy has been employed in high-energy physics since more than 40 years [21], but only recently, around the year 2010, the theory gained large prominence.

[6] The small difference between charged and neutral pion masses is due to electromagnetic loop corrections, which is another isospin breaking effect.

SMEFT is an EFT of the SM degrees of freedom: the photon, the gluon octet, the W and Z bosons, the Higgs boson, and the 3 generations of quarks and leptons. Much as in the SM, the action is exactly invariant under the local (gauge) $SU(3) \times SU(2) \times U(1)$ symmetry. The SMEFT Lagrangian contains the SM one, but also an infinite set of higher-dimensional gauge-invariant interaction terms.[7] The latter interactions, which are non-renornalizable in the old parlance, describe the effects of heavy particles from beyond the SM. Under very broad assumptions, which will be spelled out in Section 2.1, SMEFT is *the* theory of fundamental interactions in the energy range 100 GeV $\lesssim E \lesssim \Lambda$, where $\Lambda \gg m_W$ is the scale at which non-SM particles appear. The Lagrangian is organized in a systematic expansion based on the canonical dimensions of the interaction terms, with the operators of canonical dimension D suppressed by Λ^{D-4}. The operators with $D = 5$ and $D = 6$ are expected to provide the leading deformations of the SM Lagrangian. Most often, the expansion is truncated at $D = 6$, with the $D > 6$ operators deemed as irrelevant at the currently available energies.

The following is basically a shortened version of my lectures collected in [22].

2.1 Assumptions behind SMEFT

In theory, SMEFT is a perfectly consistent EFT of the SM degrees of freedom. However, it is not guaranteed that there is any energy range where SMEFT is the *relevant* EFT to describe physical processes. For this to happen, two broad assumptions have to be satisfied:

#1 **Mass Gap**. The mass scale Λ of the non-SM particles is much larger than the electroweak scale, $\Lambda \gg m_W$.

#2 **Gauge Symmetry**. The Lagrangian describing interactions above the electroweak scale is invariant under the SM gauge symmetry $SU(3)_C \times SU(2)_W \times U(1)_Y$.

Strictly speaking, Assumption #1 is is false. Indeed, the degrees of freedom at the electroweak scale include not only the SM spectrum, but also a massless spin-2 particle called the *graviton*, which mediates the gravitational interactions. Thus, in order to describe all known physics at that scale we should also include the graviton in our EFT, which leads to the construction called GRSMEFT [23]. Nevertheless, gravity is expected to be very weak around the electroweak scale. Consistency of the theory requires the leading order coupling of matter to gravitons to be universal and controlled by the scale $M_{\rm Pl} \simeq 10^{18}$ GeV, leading to the suppression by powers of TeV$/M_{\rm Pl} \sim 10^{-15}$ at the LHC energies. Subleading graviton couplings are controlled by the GRSMEFT expansion scale $\Lambda_{\rm GRSMEFT}$, which is unknown, but the (safe) assumption here is that $\Lambda_{\rm GRSMEFT} \gg m_W$, perhaps even $\Lambda_{\rm GRSMEFT} \sim M_{\rm Pl}$. If that is satisfied, graviton emission is totally irrelevant at the LHC and in other experiments that focus on non-gravitational interactions. For those experiments, SMEFT provides an adequate description.

[7]In the EFT jargon, these higher-dimensional interactions terms are often referred to as *operators*, for no good reason. The coupling constants multiplying these operators are often referred to as the *Wilson coefficients*. I will use this jargon in the following.

Are there other light non-SM degrees of freedom except for the graviton? This is an open question at present. Theorist have hypothesized countless light particles, some of which are even well motivated, and sometimes even hinted at by some experiments. As examples one could mention the sterile neutrinos, the axion, and a light dark matter particle. An affirmative answer to our question will be provided if we are very lucky and such a particle is discovered in some ongoing or future experiment. However a negative answer may never be established, because in many scenarios the coupling of the new particle to the SM matter is a free parameter that can be adjusted to arbitrary small values. From our point of view, a more immediate question is whether the non-SM degrees of freedom are relevant at the LHC energies. Again, this is an open question that may be difficult to settle in the near future. For all we know, a new light particle could for example couple to the Higgs boson, leading to an invisible Higgs branching fraction up to $\mathcal{O}(10)\%$. Such decays cannot be described within the SMEFT framework. All in all, it is reasonable to assume that the graviton is the only non-SM light degree of freedom, however it certainly requires a certain leap of faith. In case the existence of a new light particle is established, and its couplings to the SM matter turn out to be significant, the SMEFT approach may have to be abandoned.

Assumption #2 is more mysterious. In the SM, the action is exactly invariant under the $SU(3)_C \times SU(2)_W \times U(1)_Y$ local symmetry, which in the global limit acts as a linear transformation on the fields in the Lagrangian. This symmetry is not visible at the level of the spectrum because it is spontaneously broken by a vacuum expectation value (VEV) of the Higgs field. With some experimental input about the quantum numbers of SM matter, the gauge principle has led to highly non-trivial and successful predictions. For example, the interactions strength of all left-handed fermions in the flavor basis with the W boson are predicted to be universal (in the tree-level approximation) and controlled by the $SU(2)_L$ gauge coupling g_L, while the interactions with the Z boson are predicted non-universal but controlled only by the fermion's quantum numbers and one universal parameter called the weak mixing angle $\sin\theta_W$. All in all, gauge symmetry has proved to be one of the deepest foundational ideas in QFT, and the SM gauge symmetry has time and again proved to be extremely successful phenomenologically. That's all very impressive, but why should SMEFT respect the same gauge symmetry as the SM? In the end, the goal of SMEFT is to provide a model independent description of heavy new physics *beyond* the SM. The discussion is further complicated by the fact that, in the modern view, gauge symmetry is not a real symmetry of the physical system, but merely a redundancy of ifs description. Why do we insist on imposing that particular redundancy on SMEFT?

Let us recall what is the true purpose of gauge symmetry, or gauge redundancy [24]. The point is that a consistent, unitary QFT that is manifestly Lorentz invariant and contains massless spin-1 particles *must* be equipped with gauge redundancy, one generator for each massless spin-1 particle. Heuristically, this is because a spin-1 particle is described in QFT by a 4-component vector field A_μ, $\mu = 0 \ldots 3$, or equivalently by the associated polarization wave function $\epsilon_\mu(p)$. Since, an on-shell massless spin-1 particle has 2 degrees of freedom, corresponding to the two helicities, two of the four components must be somehow projected from $\epsilon_\mu(p)$. One can be taken care of in a Lorentz invariant way by the transversality condition $p_\mu \epsilon^\mu(p) = 0$. It turns out that the only Lorentz invariant way to project out the other spurious degree of freedom is to identify the states described by the polarization wave functions $\epsilon^\mu(p)$ and $\epsilon^\mu(p) + p^\mu$, that is by imposing gauge redundancy on the theory.

In the SMEFT we have two kinds of massless spin-1 particles: a photon and a gluon octet. Accordingly, we need 9 generators of local symmetry to have a consistent and manifestly Lorentz-invariant theory. An input from phenomenology is needed to identify that $SU(3)_C \times U(1)_{\rm em}$ provides a correct description of these degrees of freedom, because the gluons all self-interact with each other, thus they are described by the non-abelian $SU(3)$ factor, while the photons do not have self-interactions, thus they are described by the abelian $U(1)$ factor. But this raises another question: why do we insist on the larger $SU(3)_C \times SU(2)_W \times U(1)_Y$ local symmetry if the smaller $SU(3)_C \times U(1)_{\rm em}$ is enough to satisfy the consistency principles of QFT?

In fact, an EFT for the SM degrees of freedom, where only the $SU(3)_C \times U(1)_{\rm em}$ gauge symmetry is realized linearly, does exist and is most often referred to as HEFT (as in Higgs EFT). In HEFT, the generators of the larger $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge symmetry that do not belong to $SU(3)_C \times U(1)_{\rm em}$ are realized as a non-linear transformation of the scalar Goldstone bosons eaten by W and Z, akin to the realization of the $SU(2)_L \times SU(2)_R / SU(2)_V$ in χPT. While the formal difference between HEFT and SMEFT is clear, the physical difference between the two EFTs is more subtle and was elucidated only recently [25, 26]. The long story short: HEFT is an effective theory for *non-decoupling* UV physics, that is for theories where the masses of non-SM particles are dominated by contributions from electroweak symmetry breaking. A simple toy model for such a UV completion is a real scalar field S without a mass term but with the quartic interaction with the Higgs field: $\mathcal{L} \supset -\lambda |H|^2 S^2$. After electroweak symmetry breaking S acquires mass $m_S^2 = 2\lambda |H|^2$, which can be large if the quartic coupling λ is $\mathcal{O}(1)$ or larger. Integrating out S will lead to an EFT described by the HEFT framework rather than SMEFT. Another less artificial example is the SM with 4 generations of *chiral* fermions, in which case all fermions are massless in the limit of the Higgs VEV going to zero. Integrating out the 4th generation will again lead to HEFT rather than SMEFT. On the other hand, integrating out the 4th generation of *vector-like* fermions, where the masses of the non-SM fermions are dominated by a vector-like mass term $M \gg v$, will lead to SMEFT rather than HEFT.

In the end, the gauge symmetry Assumption #2 turns out to be closely related to the mass gap Assumption #1. Indeed, in non-decoupling theories masses of non-SM particles are of the form $m_i \sim g_i v$, where g_i is some gauge or Yukawa coupling. Since couplings are restricted by perturbativity to be $|g_i| \lesssim 4\pi$, the masses are $m_i \lesssim 4\pi v$. This means the new particles in non-decoupling theories are within the reach of the LHC or just around the corner. Conversely, if new physics enters at the scale $\Lambda \gtrsim 4\pi v \sim 3$ TeV, then the physics below Λ is necessarily described by SMEFT and not HEFT. By imposing Assumption #2 we make an implicit decision to neglect the possibility of non-decoupling UV completions. Note that large swathes of non-decoupling theories have already been experimentally excluded; for example, the chiral 4th generation was definitely excluded by the Higgs production rate measurements at the LHC. Even though, at present, one cannot formally exclude

Field	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$	Name	Spin	Dimension
G^a_μ	$\mathbf{8}$	$\mathbf{1}$	0	Gluons	1	1
W^k_μ	$\mathbf{1}$	$\mathbf{3}$	0	Weak SU(2) bosons	1	1
B_μ	$\mathbf{1}$	$\mathbf{1}$	0	Hypercharge boson	1	1
Q	$\mathbf{3}$	$\mathbf{2}$	1/6	Quark doublets	1/2	3/2
U^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	-2/3	Up-type anti-quarks	1/2	3/2
D^c	$\bar{\mathbf{3}}$	$\mathbf{1}$	1/3	Down-type anti-quarks	1/2	3/2
L	$\mathbf{1}$	$\mathbf{2}$	-1/2	Lepton doublets	1/2	3/2
E^c	$\mathbf{1}$	$\mathbf{1}$	1	Charged anti-leptons	1/2	3/2
H	$\mathbf{1}$	$\mathbf{2}$	1/2	Higgs field	0	1

Table 1. Transformation properties of the SM fields under the SM gauge group. We also display the spin of the associated particle and the canonical dimension of the field. The matter fields (rows 4-8) are 3-vectors in the generation space: $Q = (q_1, q_2, q_3)$, $U^c = (u^c_1, u^c_2, u^c_3) \equiv (u^c, c^c, t^c)$, $D^c = (d^c_1, d^c_2, d^c_3) \equiv (d^c, s^c, b^c)$, $L = (l_1, l_2, l_3) = \left(\begin{pmatrix} \nu_e \\ e \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \right)$, $E^c = (e^c_1, e^c_2, e^c_3) \equiv (e^c, \mu^c, \tau^c)$.

the existence of non-decoupling new physics, and some wiggle room remains for certain constructions, it is a very unlikely possibility in my opinion. Focusing on decoupling new physics, and thus restricting our scope to SMEFT, seems a very reasonable assumption.

2.2 Constructing SMEFT

This section reviews a systematic prescription to construct the SMEFT Lagrangian. The fields corresponding to the SM particles and their representations under the gauge symmetry are summarized in Table 1. Using these fields as building blocks, we will write down the most general Lagrangian consistent with the assumptions spelled out in Section 2.1.

Because the SMEFT Lagrangian is non-renornalizable, it contains an infinite number of interaction terms. Even if we wanted to arbitrarily restrict to a finite number of interactions, loop corrections would force us to introduce an infinite number of counterterms to cancel the UV divergences. In order to make the theory usable in practice we need *power counting*, which is the EFT jargon for an organizing principle that allows us to establish a relative importance of different interaction terms. In SMEFT, a natural power counting is based on the canonical dimension of an interaction. We organize the SMEFT Lagrangian as

$$\mathcal{L}_{\mathrm{SMEFT}} = \sum_{D=2}^{\infty} \mathcal{L}_D, \tag{2.1}$$

where each term $\mathcal{L}_D$ in this series contains operators $O_{i,D}$ of canonical dimension D:

$$\mathcal{L}_D = \sum_i C_{i,D} O_{i,D}. \tag{2.2}$$

Above, i indexes all independent gauge-invariant operators constructed out of the SM fields at a given dimension, and $C_{i,D}$ are field-independent coupling constants called the Wilson coefficients. By definition, the dimension of $O_{i,D}$ is D, which we write as $[O_{i,D}] = D$. Since the Lagrangian has dimension four, $[\mathcal{L}] = 4$, it follows that $[C_{i,D}] = 4 - D$. We can write down the Wilson coefficients in the form

$$C_{i,D} = \frac{c_{i,D}}{\Lambda^{D-4}}, \tag{2.3}$$

where $c_{i,D}$ are dimensionless, and Λ is a common mass scale entering all Wilson coefficients. At this point Eq. (2.3) is completely general. The scale Λ can be identified with the mass scale of new particles in the UV completion of SMEFT. Then the dimensionless coefficients $c_{i,D}$ are functions of the couplings and mass ratios in the UV completion of SMEFT, as well as of the SM couplings. Now, the standard SMEFT power counting relies on the assumption that $|c_{i,D}| \sim 1$, that is to say

$$C_{i,D} \sim \frac{1}{\Lambda^{D-4}}, \tag{2.4}$$

which is basically dimensional analysis. In such a case we have a simple estimate of the relative relevance of different Wilson coefficients. Matching the dimensions in tree-level scattering amplitudes (which are dimensionless) one finds that, for the relevant scattering energy E much larger than the particles' mass, a Wilson coefficient at a given D will enter as

$$\mathcal{M} \sim C_{i,D} E^{D-4} \sim \left(\frac{E}{\Lambda}\right)^{D-4}. \tag{2.5}$$

For example, the effects of dimension-4 operators are unsuppressed, the effects of dimension-5 operators are suppressed by E/Λ, the effects of dimension-6 operators are suppressed by $(E/\Lambda)^2$, and so on. The higher the dimension of the operator, the larger is the suppression. Thus, operators with lower dimensions will have a larger impact on phenomenology, assuming $E \ll \Lambda$, that is when SMEFT is used at the energy scale well below the mass scale of the UV completion. We can thus truncate the SMEFT Lagrangian at some particular D, ignoring the contributions of all but a finite number of operators. Conversely, for $E \sim \Lambda$ the suppression of higher-dimensional operators is no more, and one should take into account the whole infinite series of operators in the Lagrangian to correctly evaluate the amplitude. Obviously, in this regime SMEFT in unusable, and thus Λ is the cutoff scale of SMEFT, beyond which it should be replaced by a more fundamental theory.

One important consequence of the standard power counting is that it allows one to define SMEFT at the quantum level. Recall that SMEFT is non-renormalizable, thus in principle an infinite number of unknown counterterms has to be introduced to properly define loop corrections to amplitudes of physical processes. However, working at $E \ll \Lambda$, we can declare that we drop from the amplitudes all the contributions that are $\mathcal{O}(\Lambda^{4-D_{\max}-1})$ or smaller. By dimensional analysis it is easy to see that the counterterms corresponding to operators of dimension $D_{\max}+1$ are moot and we can neglect them in our analysis. This leaves a *finite* number of operators of dimension $D \le D_{\max}$, together with the associated counterterms. Thus, SMEFT with the standard power counting and truncated at a finite $D_{\max}$ is as renormalizable as the renormalizable theories in the standard sense ($D_{\max} = 4$). From the SM it differs only by a larger number of counterterms (if $D_{\max} > 4$), thus a larger number of free parameters that have to be fixed by experiments.

The standard power counting sketched above has the advantage of being simple and self-consistent. One should remember however that it is not the only option, and it may not be the most sound one from the physics point of view. A run-of-the-mill UV completion will not generate all Wilson coefficients universally; typically it will generate a handful of operators at tree level, while others will be suppressed by loop factors, leading to hierarchies not captured by Eq. (2.5). Moreover, certain types of operators can never be generated at tree level, independently of the UV completion. Next, flavor or other symmetries in the UV completion may lead to special patterns in SMEFT, leading to additional suppression of Wilson coefficients. For example, Eq. (2.5) suggests that Wilson coefficients corresponding to analogous operators involving say, up and top quarks scale in the same way, however if the UV completion incorporates something akin to SM flavor hierarchies (which is very likely) one expects the former will be suppressed compared to the latter by a small factor $(m_u/m_t)^n$. Finally, Eq. (2.5) ignores the dependence of the Wilson coefficients on the coupling strength in the UV theory. Consider a UV theory with a single coupling g_*. Very often, Wilson coefficients of dimension-6 and -8 operators will scale $C_{i,6} \sim \frac{g_*^2}{\Lambda^2}$ and $C_{i,8} \sim \frac{g_*^2}{\Lambda^4}$. In the standard power counting, $C_{i,6}^2$ is always of the same order as $C_{i,8}$, which is indeed the case for $g_* \sim 1$. But for $g_* \ll 1$ we have $C_{i,8} \gg C_{i,6}^2$, whereas for $1 \ll g_* \lesssim 4\pi$ we have $C_{i,8} \ll C_{i,6}^2$, in both case the parametric hierarchy being missed in the standard power counting.

Nevertheless, let us brush aside these caveats for the time being and proceed under the assumption that the canonical dimension of an operator is the central determinant of its relevance for the low-energy phenomenology at $E \ll \Lambda$. Consequently, we will build the SMEFT Lagrangian starting from the operators of lowest dimensions, and working up towards higher D.

The sum in Eq. (2.1) starts at $D = 2$ because there is nothing at lower dimensions: $D = 0$ would be a field-independent constant, which has no physical consequences in non-gravitational theories, while there is no gauge invariant $D = 1$ operators because there are no singlet scalars in the spectrum in Table 1. At $D = 2$ there is a single gauge invariant operator, the Higgs mass squared:

$$\mathcal{L}_{D=2} = \mu_H^2 H^\dagger H. \tag{2.6}$$

The Wilson coefficient in this case has mass dimension 2 and is denoted as μ_H^2. According to our power counting in Eq. (2.4), we should have $\mu_H \sim \Lambda \gg v$. In reality we expect $\mu_H \lesssim v$ because the Higgs mass term triggers electroweak symmetry breaking by the Higgs VEV. In the SM, where there are no free unknown parameters anymore, we know precisely the tree level value $\mu_H \simeq 88$ GeV. In SMEFT I cannot give you a number for μ_H because unknown higher dimensional operators also affect the Higgs VEV. Nevertheless, $\mu_H \gg v$ would be unnatural as it would require large cancelations between μ_H and higher-dimensional operators to arrive at the correct value of v. We thus have a puzzle. On one hand, power counting predicts $\mu_H \sim \Lambda \gg v$. On the other hand, phenomenological and naturalness arguments imply $\mu_H \lesssim v$. This clash is nothing else but the *hierarchy problem.*[8] Not so

[8] In fact, the hierarchy problem can be formulated in the most transparent fashion in the EFT language

long ago, the hierarchy problem was considered an almost certain indication that there are new degrees of freedom at the electroweak scale, for example the supersymmetric partners or the Kaluza-Klein modes of the SM particles. If that were the case, SMEFT would not be a useful theory in any energy range. However, the results from the LHC strongly suggest that the SM degrees of freedom are all there is near the electroweak scale, and that SMEFT is the correct description of physics, at least in the energy range from 100 GeV up to a few TeV. That's good for SMEFT and for me personally because I can lecture about it in Florence, however the hierarchy problem remains puzzling. Have we somehow missed the degrees of freedom responsible for stabilizing the electroweak scale? Can the hierarchy problem be addressed with no new degrees of freedom at the electroweak scale? Do we misunderstand something about how QFT works? Is the SM more fundamental than we think? It is fair to say that no one has presented a convincing solution so far.

All in all, the standard EFT power counting fails us at $D = 2$. Nevertheless let us press on and apply the standard power counting to SMEFT operators of dimensions higher than two. At $D = 3$ again there are no gauge invariant operators because there are no singlet fermions in the spectrum in Table 1.[9] At $D = 4$ there are multiple gauge-invariant operators. Here is the complete list:[10]

$$\begin{aligned}\mathcal{L}_{D=4} = & -\frac{1}{4}\sum_{V\in B,W^i,G^a} V_{\mu\nu}V^{\mu\nu} + \sum_{f\in Q,L} i\bar{f}\bar{\sigma}^\mu D_\mu f + \sum_{f\in U,D,E} if^c\sigma^\mu D_\mu \bar{f}^c \\ & -\left(\bar{Q}\tilde{H}Y_u\bar{U}^c + \bar{Q}HY_d\bar{D}^c + \bar{L}HY_e\bar{E}^c + \text{h.c.}\right) + D_\mu H^\dagger D^\mu H - \lambda(H^\dagger H)^2 \\ & +\tilde{\theta}G^a_{\mu\nu}\tilde{G}^a_{\mu\nu},\end{aligned} \tag{2.7}$$

where $V^a_{\mu\nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu - gf^{abc}V^b_\mu V^c_\nu$, $D_\mu X = \partial_\mu X + ig_s G^a_\mu T^a X + ig_L W^i_\mu \frac{\sigma^i}{2}X + ig_Y B_\mu Y X$, $\tilde{H}_a = \epsilon^{ab}H^*_b$, $\tilde{G}^a_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta\ a}$, and Y_f are 3×3 matrices in the generation space. Dimensional analysis dictates that all the couplings in the dimension-4 Lagrangian: the gauge couplings g_X, the Yukawa couplings Y_f, and the quartic coupling λ, are dimensionless. The standard power counting in Eq. (2.4) treats them all as $\mathcal{O}(\Lambda^0)$ couplings. In reality, this is reasonably well borne out for the gauge and quartic couplings, but not for most of the elements of Y_f. Clearly Eq. (2.4) does not know about flavor hierarchies. Some of the $D = 4$ Wilson coefficients are extremely suppressed, e.g. $[Y_e]_{11} \simeq 3\times 10^{-6}$ (in a convenient basis). It is envisageable that contributions of some $D > 4$ operators to certain scattering amplitudes will be larger than the effects proportional to the electron Yukawa coupling, which would represent another break down of the standard power counting. But, overall, the standard power counting is a very successful principle at $D = 4$: all but the last term

as a breakdown of dimensional analysis. In the SM on the other hand, the hierarchy problem cannot be properly formulated. There, it is often explained via the quadratic divergences in the calculation of the Higgs mass, but that is a regularization-dependent statement; for example using dimensional regularization there is no quadratic dependence on a dimensionful regulator.

[9]Dimension-3 operators are present for example in the extension of SMEFT featuring singlet (right-handed) neutrinos.

[10]We only wrote the so-called θ-term $G^a_{\mu\nu}\tilde{G}^a_{\mu\nu}$ for the $SU(3)_C$ gauge bosons because analogous terms for other group factors have no physical effect. The θ-term is moot for $U(1)$ gauge groups, while for $SU(2)_W$ the term $W^k_{\mu\nu}\tilde{W}^k_{\mu\nu}$ can be redefined away via a chiral transformation.

in Eq. (2.7) have been experimentally shown to exist (again assuming that hey are not somehow mimicked by higher-dimensional operators). Of course, $\mathcal{L}_{D=2} + \mathcal{L}_{D=4}$ is nothing else than the SM Lagrangian, so the success of SMEFT with the standard power counting is to reproduce the SM as the leading terms in its EFT expansion. Concerning the last term in Eq. (2.7), the current constraints are $|\tilde{\theta}| \lesssim 10^{-12}$. The lack of experimental evidence for the θ term, which is referred to as the *strong CP problem*, is as much puzzling from the EFT perspective as it is within the SM. Fortunately, unlike for the hierarchy problem, we have some reasonable ideas about the solution. The smallness of $\tilde{\theta}$ could be explained by a new particle called the *QCD axion*, which effectively makes $\tilde{\theta}$ a dynamical quantity settled in a minimum where $\tilde{\theta} \sim 0$. But at this point one cannot completely exclude the possibility that we misunderstand something fundamental about QCD, and in reality the physical effects of $\tilde{\theta}$ are additionally suppressed by an unknown mechanism.[11] Or that the parameter $\tilde{\theta}$ is very small by pure accident.

We move to $D = 5$, that is beyond the SM. At this order in the SMEFT expansion we have the following gauge-invariant interactions:

$$\mathcal{L}_{D=5} = -(\bar{L}H^\dagger)C_5(\bar{L}H^\dagger) + \text{h.c.} \tag{2.8}$$

This is often called the Weinberg's operator, who postulate its existence in Ref. [21]. The Wilson coefficients C_5 form a 3×3 matrix in the generation space. Here and in most of the following, the generation indices are implicitly contracted, so that one should read Eq. (2.8) as $\mathcal{L}_{D=5} = -\sum_{J,K=1}^{3}(\bar{l}_J H^\dagger)[C_5]_{JK}(\bar{l}_K H^\dagger) + \text{h.c.}$. Dimensional analysis dictates that $[C_5] = \text{mass}^{-1}$, and standard power counting treats them as $\mathcal{O}(\Lambda^{-1})$ parameters. The $SU(2)_W$ indices of the lepton and Higgs doublets are contracted via the epsilon tensor: $lH \equiv \epsilon^{ab} l^a H^b$. After electroweak symmetry breaking, Eq. (2.8) gives rise to Majorana neutrino masses:

$$\mathcal{L}_{D=5} \to -\frac{v^2}{2}\bar{\nu}_J[C_5]_{JK}\,\bar{\nu}_K + \text{h.c.} \tag{2.9}$$

Incidentally, neutrinos are known to be massive particles.[12] While we do not know the absolute values of the masses, we know the mass differences (squared) with a good accuracy, see e.g. [29]. Given this, one can estimate $C_5 v^2 \sim 10^{-1}$ eV, that is to say $C_5 \sim \frac{1}{10^{15}\text{GeV}}$.

One cannot emphasize enough what an enormous success of the SMEFT paradigm this is. In SMEFT, the most relevant phenomenological effects at $E \ll \Lambda$ are expected from the

[11] Claims that $\tilde{\theta}$ has no physical effects appear on arXiv in regular intervals. However, these are at odds with the preliminary evidence from lattice calculations [27, 28], which observe nucleon electric dipole moments (EDMs) proportional to $\tilde{\theta}$ at large pion masses.

[12] There is no doubt that at least two neutrinos have masses, but their precise nature is experimentally an open question. There are two different mechanisms to implement the neutrino masses in the Lagrangian. The simplest option is to write down the so-called *Majorana* mass term for a left-handed neutrino ν: $\Delta\mathcal{L} = -\frac{1}{2}m_M\nu\nu + \text{h.c.}$. Another option is to add a new degree of freedom to the SM - the right-handed neutrino ν^c - together with the *Dirac* mass term $\Delta\mathcal{L} = -m_D\nu^c\nu + \text{h.c.}$. The two options lead to different contributions to the neutrinoless double beta decay. The jury is still out whether the SM neutrino masses are of the Majorana, or the Dirac, or the mixed type. In the following I will be assuming without any further comment that the masses are purely Majorana.

$D = 2$ and $D = 4$ operators, which are those of the SM, and which are indeed seen in nature. Furthermore, the standard power counting predicts that the most relevant deviations from the SM should be due to $D = 5$ operators. This prediction was spectacularly confirmed by the discovery of neutrino masses in the Super-Kamiokande detector in 1998 [30], almost 20 years after Weinberg's paper [21].

At the same time, this very success carries a premonition of doom. The neutrino masses turn out to be quite small, leading to the appearance of a very large scale in the denominator of C_5. Since in the standard power counting $C_5 \sim \Lambda^{-1}$, it would be most natural to conclude that the SMEFT expansion parameter $\Lambda = 10^{15}$ GeV. This would not be a problem for SMEFT - on the contrary, it would mean that the expansion is very quickly convergent, and thus the operators up to $D = 5$, maybe plus a handful of operators at $D = 6$ are enough to describe all physics at available energy scales. But this would be a problem for you and for me. It would mean that the gap between the electroweak scale and the new physics scale is enormous, which would make the options for fundamental research very limited. The directions worth pursuing would be neutrino physics, and perhaps proton decay. Otherwise one could switch to astrophysics, cosmology, quantum computing, nuclear fusion, climate science, or banking. Not much point for future colliders, flavor physics, charged lepton flavor violation, which would only serve to confirm ad nauseam the SM predictions.

This may be the future, but it does not have to be. Even within the SMEFT paradigm (no new light degrees of freedom), it is quite possible that the expansion parameter Λ is much smaller than 10^{15} GeV. New physics responsible for the operators in Eq. (2.8) may be much lighter, perhaps even near the TeV scale, but coupled very weakly to the SM fermions. A sharper argument can be formulated by noticing that the operators is Eq. (2.8) are very special, as they violate the lepton number symmetry acting as $L \to e^{i\alpha} L$, $E^c \to e^{-i\alpha} E^c$. This is an *accidental* symmetry at the $D \leq 4$ level, as one simply cannot construct a gauge invariant operators with $D \leq 4$ that violates it, and thus $D = 5$ is the lowest dimension where lepton-number-violating operators can appear. One can modify the standard power counting by assuming that there are *two* scales governing the SMEFT expansion. One, call it Λ_L, corresponds to the mass scale of $B - L$-violating new physics, and it happens to be very high, $\Lambda_L \sim 10^{15}$ GeV. Another, let's keep calling it Λ without a sub-index, corresponds to the mass scale of $B - L$-conserving new physics. It is then perfectly natural to have a huge gap between these two scales, $\Lambda \ll \Lambda_L$. Symmetry consideration forbid new physics at the scale Λ to generate $D = 5$ operators, and the lowest dimension it can show up is $D = 6$. This assumption of the two-scale expansion gives us a rationale for exploring the SMEFT Lagrangian at $D = 6$ and higher, and we will tacitly adopt this point of view in all of the following.

We have arrived at dimension-6 operators, which is the main focus of the SMEFT research these days. At $D = 2$ there is a single operator; the $D = 4$ Lagrangian can fit a t-shirt; at $D = 5$ there is basically a single operator but, taking into account the three fermion generations, it counts as 12 operators.[13] At $D = 6$, all hell breaks loose: we have...

[13] C_5 in Eq. (2.8) is a symmetric matrix in the generation space, thus it has six independent complex

wait for it... 3045 independent operators. They contribute to phenomenology in virtually all areas of particle physics, such as Higgs physics, electroweak precision observables, flavor physics, nuclear physics, electric dipole moments, and much more. Below I will present a quick survey of dimension-6 operators using the set proposed in Ref. [31] and known under the name of the *Warsaw basis.* To organize the presentation, let me divide them into several classes:

$$\mathcal{L}_{D=6} = \mathcal{L}_{D=6}^{\text{bosonic}} + \mathcal{L}_{D=6}^{\text{Yukawa}} + \mathcal{L}_{D=6}^{\text{current}} + \mathcal{L}_{D=6}^{\text{dipole}} + \mathcal{L}_{D=6}^{4-\text{fermion}}. \tag{2.10}$$

The bosonic operators, as the name suggest, are constructed out of the SM gauge and Higgs fields, without involving any fermionic fields. In the Warsaw basis there are 15 bosonic operators:

$$\begin{aligned}\mathcal{L}_{D=6}^{\text{bosonic}} =& C_H (H^\dagger H)^3 + C_{H\Box}(H^\dagger H)\Box(H^\dagger H) + C_{HD}|H^\dagger D_\mu H|^2 + C_{HWB} H^\dagger \sigma^k H\, W^k_{\mu\nu} B_{\mu\nu} \\ &+ C_{HG} H^\dagger H\, G^a_{\mu\nu} G^a_{\mu\nu} + C_{HW} H^\dagger H\, W^k_{\mu\nu} W^k_{\mu\nu} + C_{HB} H^\dagger H\, B_{\mu\nu} B_{\mu\nu} \\ &+ C_W \epsilon^{klm} W^k_{\mu\nu} W^l_{\nu\rho} W^m_{\rho\mu} + C_G f^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} \\ &+ C_{H\widetilde{G}} H^\dagger H\, \widetilde{G}^a_{\mu\nu} G^a_{\mu\nu} + C_{H\widetilde{W}} H^\dagger H\, \widetilde{W}^k_{\mu\nu} W^k_{\mu\nu} + C_{H\widetilde{B}} H^\dagger H\, \widetilde{B}_{\mu\nu} B_{\mu\nu} \\ &+ C_{H\widetilde{W}B} H^\dagger \sigma^k H\, \widetilde{W}^k_{\mu\nu} B_{\mu\nu} + C_{\widetilde{W}} \epsilon^{klm} \widetilde{W}^k_{\mu\nu} W^l_{\nu\rho} W^m_{\rho\mu} + C_{\widetilde{G}} f^{abc} \widetilde{G}^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu},\end{aligned} \tag{2.11}$$

where $\Box \equiv \partial_\mu \partial^\mu$ and σ^k are the three Pauli matrices. Already this relatively small subset of dimension-6 operators contains rich phenomenology. C_H changes the shape of the Higgs potential, in particular it affects the cubic Higgs boson self-coupling - perhaps the last landmark measurement to be delivered by the LHC. $C_{H\Box}$ contributes to the Higgs boson kinetic term and thus, indirectly, affects universally all Higgs boson production and decay rates. The following two operators contribute to electroweak precision observables measured long ago by the LEP collider. C_{HD} contributes to the Z boson mass, while C_{HWB} contributes to the kinetic mixing between the photon and the Z boson. Through these intermediaries, they affect the whole lot of electroweak precision observables. In fact, these two are just the famous oblique S and T parameters of Peskin and Takeuchi [32] in another (more modern) guise. Furthermore, C_{HWB} as well as the Wilson coefficients C_{HG}, C_{HW}, C_{HB} in the second line contribute to the ever important Higgs boson interaction strengths with gluons, W, Z, and photons, which are measured at the LHC. In the third line, C_W and C_G induce 3-derivative anomalous cubic interactions of electroweak gauge bosons and gluons, respectively. The final two lines contain CP violating interactions. They can be searched for in colliders, but more easily discernible effects appear via their loop contributions to electric dipole moments of the electron or the neutron.

The next class of dimension-6 operators we discuss are Yukawa-like interactions:

$$\mathcal{L}_{D=6}^{\text{Yukawa}} = H^\dagger H(\bar{L} H C_{eH} \bar{E}^c) + H^\dagger H(\bar{Q}\tilde{H} C_{uH} \bar{U}^c) + H^\dagger H(\bar{Q} H C_{dH} \bar{D}^c) + \text{h.c.} \tag{2.12}$$

Each C_{fH} is a 3×3 complex matrix in the generation space, thus each comes with 18 free parameters, which makes 54 parameters overall. These operators contribute to the fermion

components. A complex operator, that is to say one that is distinct from its hermitian conjugate, by convention is counted as two operators.

masses, but that is unobservable because it merely renormalizes the unknown Yukawa matrices in Eq. (2.7). The observables effect is the modification of the Higgs boson Yukawa couplings to the fermions. In the SM, the Yukawa coupling is not a free parameter but it is uniquely fixed by the fermion's mass. In the presence of the operator is Eq. (2.12) that relation no longer holds, and the Higgs boson couplings to fermions become free parameters independent of fermion masses. Moreover, a qualitatively new effect of flavor violation in Higgs interactions may appear. That is to say, the Higgs boson can couple to two fermions from different generations, e.g. $\mathcal{L}_{\rm SMEFT} \supset h\bar{e}\bar{\mu}^c$, which does not occur in the SM.

Next, we have what I call the current operators:

$$\begin{aligned}\mathcal{L}_{D=6}^{\rm current} =& iH^\dagger \overleftrightarrow{D}_\mu H(\bar{L}C_{Hl}^{(1)}\bar{\sigma}^\mu L) + iH^\dagger \sigma^k \overleftrightarrow{D}_\mu H(\bar{L}C_{Hl}^{(3)}\bar{\sigma}^\mu \sigma^k L) + iH^\dagger \overleftrightarrow{D}_\mu H(E^c C_{He}\sigma^\mu \bar{E}^c) \\ &+ iH^\dagger \overleftrightarrow{D}_\mu H(\bar{Q}C_{Hq}^{(1)}\bar{\sigma}^\mu Q) + iH^\dagger \sigma^k \overleftrightarrow{D}_\mu H(\bar{Q}C_{Hq}^{(3)}\bar{\sigma}^\mu \sigma^k Q) + iH^\dagger \overleftrightarrow{D}_\mu H(U^c C_{Hu}\sigma^\mu \bar{U}^c) \\ &+ iH^\dagger \overleftrightarrow{D}_\mu H(D^c C_{Hd}\sigma^\mu \bar{D}^c) + \left\{ i\tilde{H}^\dagger D_\mu H(U^c C_{Hud}\sigma^\mu \bar{D}^c) + \text{h.c.} \right\},\end{aligned} \tag{2.13}$$

where $H^\dagger \overleftrightarrow{D}_\mu H \equiv H^\dagger D_\mu H - D_\mu H^\dagger H$. The Wilson coefficient C_{Hf} are matrices in the generation space, but now only C_{Hud} is a general complex matrix, while the remaining ones are Hermitian matrices (thus with 9 free parameters each). This adds up to 81 free parameters in Eq. (2.13). These operators contribute to the W and Z bosons interactions with fermions, which have been precisely measured in the LEP, Tevatron, and LHC colliders. Several qualitatively new effects are introduced by Eq. (2.13). One is the W boson couplings to right-handed quarks, e.g. $\mathcal{L}_{\rm SMEFT} \supset W_\mu(t^c\sigma^\mu \bar{b}^c)$, whereas in the SM W couples only to left-handed quarks. Another is tree-level flavor-changing neutral currents, that is Z boson couplings to quarks or leptons of different generations, e.g. $\mathcal{L}_{\rm SMEFT} \supset Z_\mu(\bar{b}\bar{\sigma}^\mu s)$.

Next, we have the dipole operators

$$\begin{aligned}\mathcal{L}_{D=6}^{\rm dipole} =& (\bar{Q}\sigma^k \tilde{H} C_{uW}\bar{\sigma}^{\mu\nu}\bar{U}^c)W_{\mu\nu}^k + (\bar{Q}\tilde{H}C_{uB}\bar{\sigma}^{\mu\nu}\bar{U}^c)B_{\mu\nu} + (\bar{Q}\tilde{H}C_{uG}T^a\bar{\sigma}^{\mu\nu}\bar{U}^c)G_{\mu\nu}^a \\ &+ (\bar{Q}\sigma^k H C_{dW}\bar{\sigma}^{\mu\nu}\bar{D}^c)W_{\mu\nu}^k + (\bar{Q}HC_{dB}\bar{\sigma}^{\mu\nu}\bar{D}^c)B_{\mu\nu} + (\bar{Q}HC_{dG}T^a\bar{\sigma}^{\mu\nu}\bar{D}^c)G_{\mu\nu}^a \\ &+ (\bar{L}\sigma^k H C_{eW}\bar{\sigma}^{\mu\nu}\bar{E}^c)W_{\mu\nu}^k + (\bar{L}HC_{eB}\bar{\sigma}^{\mu\nu}\bar{E}^c)B_{\mu\nu} + \text{h.c.}\end{aligned} \tag{2.14}$$

Given that C_{fV} are 3×3 complex matrices in the generation space, the above introduces 144 free parameters. An important effect of the operators in Eq. (2.13) is their contribution to the anomalous magnetic dipole moments of fundamental particles. In particular, the Wilson coefficients $[C_{eW}]_{22}$ and $[C_{eB}]_{22}$ contribute to the muon $g-2$ which, at the time of writing, may or may not deviate from the SM prediction. The imaginary parts of these Wilson coefficients contribute to electric dipole moments. Moreover, the operators in Eq. (2.13) can mediate certain processes that are forbidden in the SM, e.g. the $\mu \to e\gamma$ decay.

The dimension-6 operators introduced so far come with $15 + 54 + 81 + 144 = 294$ free parameters. It follows that a large majority of dimension-6 operators are hiding in the last term in Eq. (2.10), which contains 4-fermion operators. For the sake of this discussion let me split them further into four sub-classes:

$$\mathcal{L}_{D=6}^{\rm 4-fermion} = \mathcal{L}_{D=6}^{4L} + \mathcal{L}_{D=6}^{2L2Q} + \mathcal{L}_{D=6}^{4Q} + \mathcal{L}_{D=6}^{3Q1L}, \tag{2.15}$$

defined by the number of lepton and of quark fields. The first sub-class in Eq. (2.15) is the 4-lepton operators:

$$\mathcal{L}^{4L}_{D=6} = \frac{1}{2}(\bar{L}\bar{\sigma}^\mu L)C_{ll}(\bar{L}\bar{\sigma}_\mu L) + \frac{1}{2}(E^c\sigma_\mu \bar{E}^c)C_{ee}(E^c\sigma_\mu \bar{E}^c) + (\bar{L}\bar{\sigma}^\mu L)C_{le}(E^c\sigma_\mu \bar{E}^c). \tag{2.16}$$

This time and for all 4-fermion operators in the following, the Wilson coefficients are 4-index tensors $[C_X]_{JKLM}$ in the generation space. The indices are implicitly contracted with the generation indices of the fermions on the left and on the right; for example, the first term in Eq. (2.16) should be read as $\frac{1}{2}\sum_{J,K,L,M=1}^{3}(\bar{l}_J\bar{\sigma}^\mu l_K)[C_{ll}]_{JKLM}(\bar{l}_L\bar{\sigma}_\mu l_M)$. Hermiticity of the Lagrangian implies that the Wilson coefficients in Eq. (2.16) are Hermitian in the first two and the last two indices: $[C_{XY}]_{JKLM} = [C_{XY}]^*_{KJML}$. For C_{ll} and C_{ee} there is an additional complication stemming from the fact that $(\bar{l}_J\bar{\sigma}^\mu l_K)(\bar{l}_K\bar{\sigma}_\mu l_L)$ and $(\bar{l}_K\bar{\sigma}^\mu l_J)(\bar{l}_J\bar{\sigma}_\mu l_K)$ are the same. Thus, for example, Eq. (2.16) contains

$$\begin{aligned}\mathcal{L}^{4L}_{D=6} \supset& \frac{1}{2}[C_{ll}]_{1221}(\bar{l}_1\bar{\sigma}^\mu l_2)(\bar{l}_2\bar{\sigma}_\mu l_1) + \frac{1}{2}[C_{ll}]_{2112}(\bar{l}_2\bar{\sigma}^\mu l_1)(\bar{l}_1\bar{\sigma}_\mu l_2)\\ =&\frac{1}{2}([C_{ll}]_{1221} + [C_{ll}]_{2112})(\bar{l}_1\bar{\sigma}^\mu l_2)(\bar{l}_2\bar{\sigma}_\mu l_1)\\ =&\mathrm{Re}\,[C_{ll}]_{1221}(\bar{l}_1\bar{\sigma}^\mu l_2)(\bar{l}_2\bar{\sigma}_\mu l_1).\end{aligned} \tag{2.17}$$

Therefore the components $[C_{ll}]_{JKKJ}$ and $[C_{ee}]_{JKKJ}$ can be declared real, as their imaginary parts do not enter the Lagrangian.[14] Four-lepton operators containing electron fields are relevant for physics at LEP-2, where e^+e^- pair were collided with the center-of-mass energy above the Z pole. A subset of interactions in Eq. (2.16) mediate tree-level charge-lepton-flavor violating processed, where the overall lepton number is conserved, but the separate electron, muon, or tau numbers are not. Such processes are forbidden in the SM, while they are mediated at loop level via the $D = 5$ intermediaries but with very suppressed rates due to the smallness of the neutrino masses. For example, $[C_{ee}]_{1112}$ mediates the $\mu^- \to e^-e^-e^+$ decay, which is subject to current experimental searches [34]. Finally, it is worth mentioning that $[C_{ll}]_{1221}$ contributes to the usual muon decay $\mu^- \to e^-\bar{\nu}_e\nu_\mu$, which in the SM is a standard candle to determine the Fermi constant. By disrupting this standard candle, $[C_{ll}]_{1221}$ indirectly affects SM predictions for countless precision measurements.

The next sub-class in Eq. (2.15) are semi-leptonic operators, that is 4-fermion operators containing two quark and two lepton fields:

$$\begin{aligned}\mathcal{L}^{2L2Q}_{D=6} =&(\bar{L}\bar{\sigma}^\mu L)C^{(1)}_{lq}(\bar{Q}\bar{\sigma}_\mu Q) + (\bar{L}\bar{\sigma}^\mu\sigma^k L)C^{(3)}_{lq}(\bar{Q}\bar{\sigma}_\mu\sigma^k Q)\\ &+(E^c\sigma_\mu\bar{E}^c)C_{eu}(U^c\sigma_\mu\bar{U}^c) + (E^c\sigma_\mu\bar{E}^c)C_{ed}(D^c\sigma_\mu\bar{D}^c)\\ &+(\bar{L}\bar{\sigma}^\mu L)C_{lu}(U^c\sigma_\mu\bar{U}^c) + (\bar{L}\bar{\sigma}^\mu L)C_{ld}(D^c\sigma_\mu\bar{D}^c) + (E^c\sigma_\mu\bar{E}^c)C_{eq}(Q\bar{\sigma}_\mu Q)\\ &+\left\{(\bar{L}\bar{E}^c)C_{ledq}(D^cQ) + \epsilon^{kl}(\bar{L}^k\bar{E}^c)C^{(1)}_{lequ}(\bar{Q}^l\bar{U}^c) + \epsilon^{kl}(\bar{L}^k\bar{\sigma}^{\mu\nu}\bar{E}^c)C^{(3)}_{lequ}(\bar{Q}^l\bar{\sigma}^{\mu\nu}\bar{U}^c) + \text{h.c.}\right\}.\end{aligned} \tag{2.18}$$

[14] Another convention existing in the literature, see e.g. [33], is to set $[C_{ll}]_{JKKJ}$ and $[C_{ee}]_{JKKJ}$ to zero for $J > K$. This leads to a factor of two difference in the dependence of observables on these Wilson coefficients with $J = K$, as compared to the convention used in these lectures.

The semi-leptonic operators affect myriads of important precision observables: hadronic cross sections at LEP-2, Drell-Yan production of leptons in hadron colliders, electric dipole moments, beta decays, and so on. They also play a major role in flavor physics, where they contribute to semileptonic flavor transitions. These are often under reasonable theoretical control, such that reliable SM predictions can be established, and thus stringent constraints on the dimension-6 operators can de derived. Moreover, the importance of the operators contributing to flavor-changing neutral currents is amplified by the suppression of these processes in the SM. One of many relevant examples of this kind is the $B_s \to \mu^+\mu^-$ decay, whose branching fraction is currently measured with 10% precision, and the SM prediction is know with a similar accuracy. That decay rate is affected, among others, by the Wilson coefficient $[C_{lq}^{(1)}]_{2232}$.

The third sub-class singled out in Eq. (2.15) are four-quark operators:

$$\begin{aligned}
\mathcal{L}_{D=6}^{4Q} =& \frac{1}{2}(\bar{Q}\bar{\sigma}^\mu Q)C_{qq}^{(1)}(\bar{Q}\bar{\sigma}_\mu Q) + \frac{1}{2}(\bar{Q}\bar{\sigma}^\mu \sigma^k Q)C_{qq}^{(3)}(\bar{Q}\bar{\sigma}_\mu \sigma^k Q) \\
&+\frac{1}{2}(U^c\sigma_\mu \bar{U}^c)C_{uu}(U^c\sigma_\mu \bar{U}^c) + \frac{1}{2}(D^c\sigma_\mu \bar{D}^c)C_{dd}(D^c\sigma_\mu \bar{D}^c) \\
&+(U^c\sigma_\mu \bar{U}^c)C_{ud}^{(1)}(D^c\sigma_\mu \bar{D}^c) + (U^c\sigma_\mu T^a \bar{U}^c)C_{ud}^{(8)}(D^c\sigma_\mu T^a \bar{D}^c) \\
&+(Q^c\sigma_\mu \bar{Q}^c)C_{qu}^{(1)}(U^c\sigma_\mu \bar{U}^c) + (Q^c\sigma_\mu T^a \bar{Q}^c)C_{qu}^{(8)}(U^c\sigma_\mu T^a \bar{U}^c)] \\
&+(Q^c\sigma_\mu \bar{Q}^c)C_{qd}^{(1)}(D^c\sigma_\mu \bar{D}^c) + (Q^c\sigma_\mu T^a \bar{Q}^c)C_{qd}^{(8)}(D^c\sigma_\mu T^a \bar{D}^c) \\
&+\left\{\epsilon^{kl}(\bar{Q}^k\bar{U}^c)C_{quqd}^{(1)}(\bar{Q}^l\bar{D}^c) + \epsilon^{kl}(\bar{Q}^k T^a\bar{U}^c)C_{quqd}^{(1)}(\bar{Q}^l T^a\bar{D}^c) + \text{h.c.}\right\}.
\end{aligned} \tag{2.19}$$

These play arguably a lesser role in phenomenology. The reason is that their effects have to compete with QCD processes, which are typically abundant and poorly controlled theoretically, especially at hadron colliders. Nevertheless, some of the operators in Eq. (2.19) will appear later in our story in the context of precision observables.

The final sub-class in Eq. (2.15) is perhaps the most exciting one, as it consists of operators violating the baryon and lepton numbers:

$$\mathcal{L}_{D=6}^{3Q1L} = (D^cU^c)C_{duq}(\bar{Q}\bar{L}) + (QQ)C_{qqu}(\bar{U}^c\bar{E}^c) + (QQ)C_{qqq}(QL) + (D^cU^c)C_{duu}(U^cE^c) + \text{h.c.} \tag{2.20}$$

Above, the quark color indices are implicitly contracted by epsilon tensors, e.g. $qqq \equiv \epsilon^{abc}q^aq^bq^c$. These operators violate the baryon number B defined as the global symmetry transformation $Q \to e^{i\beta/3}Q$, $U^c \to e^{-i\beta/3}U^c$, $D^c \to e^{-i\beta/3}D^c$. Baryon number is a symmetry for all operators with $D \leq 5$.[15] They also violate the lepton number defined as the global symmetry transformation $L \to e^{i\alpha}L$, $E^c \to e^{-i\alpha}E^c$. The violation of baryon and lepton number implies that these operators can mediate proton decay, in particular the $p \to \pi^0 e^+$ process can be mediated at tree level by the operators involving the first generation fermions. Since experimental bounds on proton decay are extremely stringent,

[15] This kind of symmetry is called *accidental*: the choice of the gauge symmetry and the field content in SMEFT automatically imply that operators of dimension up to five cannot violate baryon number, without any need to impose this symmetry by hand. Similarly, lepton number is an accidental symmetry for $D \leq 4$.

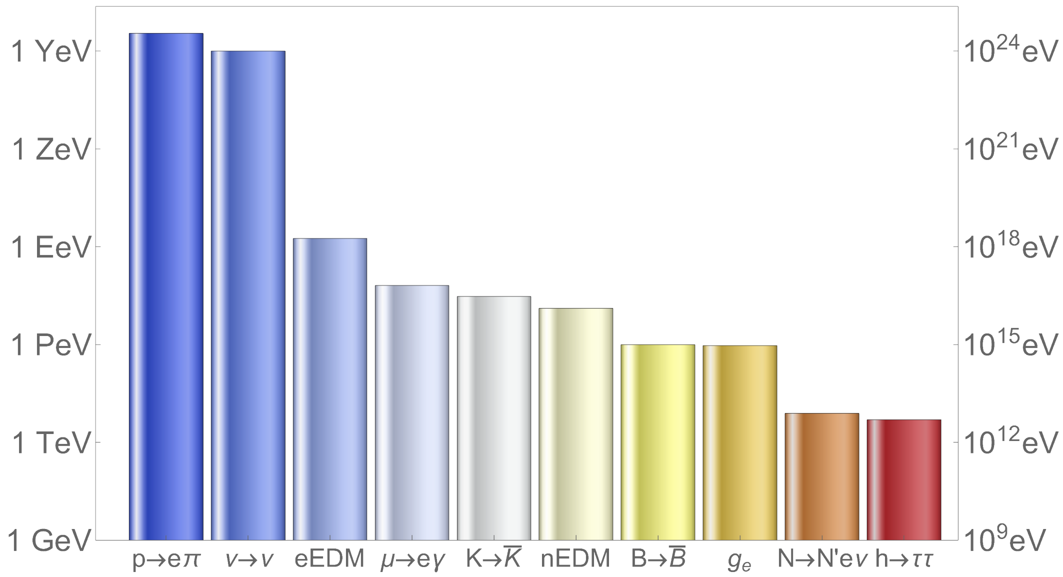

Figure 2. The scale suppressing higher-dimensional SMEFT operators probed by selected observables. From left to right: proton decay, neutrino oscillations, electron EDM, $\mu \to e\gamma$, kaon mixing, neutron EDM, B-meson mixing, electron anomalous magnetic moment, beta decay, Higgs decay to tau leptons.

some of the Wilson coefficients in Eq. (2.20) must be suppressed by a very high scale. To my knowledge, among all processes mediated by higher dimensional operators, proton decay probes the highest scale, not too far from the mythical Planck scale. Comparison of different scales probed by different precision experiments is shown in Fig. 2.

The sum in Eq. (2.1) extends to $D = \infty$, and one could press on, but, more often than not, the discussion in SMEFT stops at dimension six. What lies beyond? At $D = 7$ we have 1542 independent operators, at $D = 8$ the number is 44807, at $D = 9$ it grows to 90456, and at $D = 10$ we have a whopping 2092441 operators (two million!) [35]. There is a good chance, however, that in your research you won't ever deal with this cornucopia of higher-dimension operators. The exponential growth of the number of operators with increasing D is one reason, as it quickly makes any systematic analysis difficult. But that practical difficulty is not the only reason. The philosophy of SMEFT with the standard power counting is that, the higher the dimension of the operator, the more suppressed its effect is. Since at present we do not have any observational evidence of dimension-6 operators, it is hard to believe that $D > 6$ operators might show up in any experiment in a foreseeable future. This general conclusion should hold for other reasonable power counting beyond the standard one. There are a few exceptions, however, that one should be aware of. First of all, if new physics is close to the electroweak scale, the effect of higher-dimensional operators may be non-negligible, especially for observables probing the high-energy tail of differential distributions at the LHC, such as for example the Drell-Yan production of leptons, $pp \to \ell^+\ell^-$. But that is of course also the situation where the SMEFT itself is least useful, since the central assumption of the mass gap is not quite satisfied. A more relevant case is when a qualitatively new phenomenon, which cannot be induced by $D \leq 6$ operators, appears at $D > 6$. $D = 7$ operators such as e.g. $(lHd^c)(lq)$ may provide leading contributions to double beta decay in some situations [36]; tree-level contributions to light-by-light scattering enter at $D = 8$ from operators such as $(B_{\mu\nu}B^{\mu\nu})^2$; neutron-antineutron oscillations arise at $D = 9$ from operators such as $(\bar{u}^c\bar{d}^c)^3$ [37]. When analyzing this kind of observables, one should however pay attention whether new physics generating these higher-dimensional operators does not generate more easily detectable $D \leq 6$ operators.

2.3 From operators to observables

Currently, the SM is the reference point for any physical theory at the electroweak scale. It enjoys huge success, correctly accounting for a host of phenomena measured in colliders and low-energy precision experiments. SMEFT differs from the SM by the presence of interactions originating from operators with dimension $D = 5, 6$, and higher. In this section we discuss, in more precise and quantitative terms, the observable effects of these operators.

There are many ways in which higher-dimensional operators can affect observables. To organize the following discussion, it is convenient to divide them into three broad classes:

1. **New vertices:** interaction vertices in the SMEFT Lagrangian that do not occur in the SM Lagrangian, due to symmetries or accidental reasons.

2. **New Lorentz structures:** interaction vertices that do occur in the SM Lagrangian, but which appear in the SMEFT with a different number of derivatives, different contractions of Lorentz or spinor indices, etc.

3. **Modified couplings:** corrections to the coupling strengths of the interaction terms present in the SM Lagrangian.

In the following I will discuss each of these classes in turn.

New vertices. The most spectacular effects of SMEFT occur when higher-dimensional operators *violate an exact global symmetry* of the SM. One important example of this kind is baryon and lepton number violation[16] by the dimension-6 operators in Eq. (2.20). Let us take one of these operators at random, say

$$\mathcal{L}_{\text{SMEFT}} \supset C_{duu}(d^c u^c)(u^c e^c) + C^*_{duu}(\bar{u}^c\bar{d}^c)(\bar{e}^c\bar{u}^c), \tag{2.21}$$

where I abbreviated $C_{duu} \equiv [C_{duu}]_{1111}$ and explicitly displayed both the operator and its Hermitian conjugate. The latter mediates the quark-level process $uu \to \bar{d}e^+$. In a bit hand-waving but intuitive way, one can think of this process as transforming two up quarks from the proton (uud) into a down antiquark, leading to a $d\bar{d}$ meson state. Consequently, the operator can mediate proton decay into meson states. One important example is $p \to e^+\pi^0$

[16]Strictly speaking, baryon or lepton number are an exact symmetry only at the perturbative level in the SM, but they are both violated by non-perturbative effects. Only one linear combination of the two, $B - L$, is conserved at the non-perturbative level. This subtlety is however irrelevant for the discussion in this section.

- the decay to a positron and a neutral pion (who is a combination of $d\bar{d}$ and $u\bar{u}$ quark states, the latter pair you can think of as being pulled from the vacuum sea during the decay). To calculate the rate for this process, one needs to take the on-shell matrix elements of the operator in Eq. (2.21) between the initial and final states:

$$\mathcal{M}(p \to e^+\pi^0) = C^*_{duu} \langle e^+(k_3)\pi^0(k_2)| (\bar{u}^c\bar{d}^c)(\bar{e}^c\bar{u}^c) |p(p_1)\rangle \,, \tag{2.22}$$

where p_1 is the incoming momentum of the proton, k_2, k_3 are the outgoing momenta of the pion and positron, and we also define $q = p_1 - k_2$. The electron field acts on the annihilation operator of the positron final state, leaving the spinor wave function $\bar{x}_3$ corresponding to the momentum k_3. We get

$$\mathcal{M}(p \to e^+\pi^0) = C^*_{duu}\bar{x}_3 \langle \pi^0(k_2)| (\bar{u}^c\bar{d}^c)\bar{u}^c |p(p_1)\rangle \,. \tag{2.23}$$

The remaining matrix element between the proton and pion states is non-perturbative, and we cannot calculate it using the familiar textbook methods. Nevertheless, the Poincare and little group covariance of the S-matrix tell us that it has to be proportional to a linear combination of the spinor wave functions of the incoming proton (pion is a scalar particle): We can thus parametrize

$$\langle \pi^0(k_2)| (\bar{u}^c\bar{d}^c)\bar{u}^c |p(p_1)\rangle = \frac{1}{\sqrt{2}}\big(W_0\bar{y}_1 + W_1\frac{q^\mu}{m_p}\bar{\sigma}_\mu x_1\big), \tag{2.24}$$

where W_0 and W_1 encode the information about the non-perturbative brown muck. To know its value you have to ask your lattice friends, and they may reply $W_0 \approx 0.15$ GeV^2, $W_1 \approx -0.13$ GeV^2, with roughly a 20% error [44]. Thus

$$\mathcal{M}(p \to e^+\pi^0) = \frac{C^*_{duu}}{\sqrt{2}}\Big[W_0(\bar{x}_3\bar{y}_1) + W_1 q^\mu(\bar{x}_3\bar{\sigma}_\mu x_1)\Big]. \tag{2.25}$$

The rest is standard QFT manipulations. Taking the square of the amplitude, summing/averaging over the positron/proton spins, and plugging the result into the formula for the decay width one gets

$$\Gamma(p \to e^+\pi^0) = \frac{|C_{duu}|^2 m_p W_0^2}{32\pi}\left(1 - \frac{m^2_{\pi_0}}{m^2_p}\right)^2. \tag{2.26}$$

Above I approximated the electron mass by zero, which is perfectly legitimate given the uncertainty on W_0 (actually, even approximating the much larger m_{π_0} as zero would be fine). In this limit the contribution of the W_1 form factor drops out. I presented this calculation here so that you can get familiar with hadronic matrix element, but as long as you are not interested in precision calculation you could easily obtain the order of magnitude of the result via dimensional analysis: $\Gamma(p \to e^+\pi^0) \sim m_p^5|C_{duu}|^2/16\pi$.

Now, a sneak peak into the Particle Data Group booklet [45] reveals that the limit on this proton decay channel is $\Gamma(p \to e^+\pi^0) \le 1.3 \times 10^{-66}$ GeV at 90% confidence level (CL), which translates into the limit on the Wilson coefficient

$$|C_{duu}| \le \left(\frac{1}{3.5 \times 10^{15}\ \text{GeV}}\right)^2. \tag{2.27}$$

This limit is valid assuming only a single baryon-number violating operator is present in the Lagrangian; otherwise proton decay constrains a linear combination of various baryon-number violating Wilson coefficients.

It is mind-blowing that low-energy experiments searching for proton decay allow us to probe new physics up to scales not so far from the Planck scale! To my knowledge, this is the highest scale we can indirectly access via low-energy experiments within a sane theoretical framework (unitary, causal, local, Lorentz-symmetric). The reason for this extreme sensitivity is that it is feasible to amass astronomical number of protons for a long period of time in a controlled setting, for example in a tank filled with water and surrounded by photo-detectors [46]. Moreover, the detection capabilities are impressive and would allow us to see the signal even if a handful of the protons in the tank decayed. Thanks to this combination of favorable circumstances, the limit on the proton lifetime can be orders of magnitude larger than the age of the universe! The final fact making the limit so strong is that we search for an effect that is predicted to be zero in the SM, so we do not have to face the uncertainty due to imprecise theory predictions.

Baryon and lepton number violation is certainly the most spectacular prediction of SMEFT. Nevertheless, higher-dimensional operators can also break other exact or approximate global symmetries of the SM, with quite interesting consequences. A nice example is the decay $\mu \to e\gamma$. In the SM, not only the overall lepton number L is conserved, but also the individual lepton numbers L_α for each generation, $\alpha = e, \mu, \tau$. The process $\mu \to e\gamma$ preserves L, but breaks L_μ and L_e, that is to say, these quantum numbers are different for the initial and final states. Therefore in the SM $\mu \to e\gamma$ is forbidden, and the predicted branching ratio is exactly zero. In the SMEFT, dimension-5 operators break L and consequently each L_α, but the smallness of the neutrino masses (translating to the large scale suppressing the dimension-5 Wilson coefficients) suppresses their contribution to $\mu \to e\gamma$ to an unobservable level. On the other hand, there are many dimension-6 operators that break L_e and L_μ, and their contributions may be more significant. In particular, one of the dipole operators in Eq. (2.14) reads

$$\mathcal{L}_{D=6} \supset [C_{eB}]_{12} (\bar{l}_1 H \bar{\sigma}^{\alpha\beta} \bar{\mu}^c) B_{\alpha\beta} + \text{h.c.}, \tag{2.28}$$

where $l_1 = (\nu_e, e)$ is the doublet of left-handed first generation leptons. After electroweak symmetry breaking, this operator leads to the interaction term

$$\mathcal{L}_{\text{SMEFT}} \supset \frac{\cos\theta_W}{\sqrt{2}} [C_{eB}]_{12} v (\bar{e} \bar{\sigma}^{\alpha\beta} \bar{\mu}^c) F_{\alpha\beta} + \text{h.c.} \tag{2.29}$$

mediating $\mu \to e\gamma$ at tree level, with $\cos\theta_W \approx 0.89$ being the cosine of the Weinberg angle. In the presence of this interaction, the branching ratio for the $\mu \to e\gamma$ can be calculated to be

$$\text{Br}(\mu \to e\gamma) = \frac{\cos^2\theta_W \left|[C_{eB}]_{12}\right|^2 v^2 m_\mu^3}{8\pi\Gamma_\mu}, \tag{2.30}$$

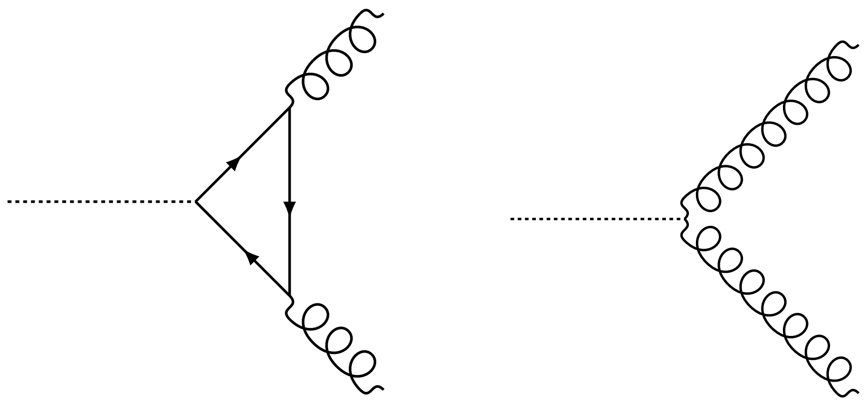

Figure 3. Feynman diagrams for Higgs production via gluon fusion. Left: via a fermion loop, as in the SM. Right: via a contact interaction due to the dimension-6 operator in Eq. (2.32).

where $\Gamma_\mu \approx 3 \times 10^{-19}$ GeV is the total muon decay width.[17] We again peek into Particle Data Group, and find the 90% CL experimental constraint $\mathrm{Br}(\mu \to e\gamma) \leq 4.2 \times 10^{-13}$ coming from the MEG experiment [47]. This translates into the constraint on the Wilson coefficient

$$\left|[C_{eB}]_{12}\right| \leq \frac{1}{\left(6.5 \times 10^7 \text{ GeV}\right)^2}. \tag{2.31}$$

The scale probed by $\mu \to e\gamma$ is less impressive than that probed by proton decay, but it is nevertheless several orders of magnitude above the direct reach of the LHC. Again, it helps that we consider a process forbidden in the SM, so we avoid dealing with theoretical errors on the SM prediction. Moreover, muons are relatively long-lived (Γ_μ in the denominator of Eq. (2.30) is small), and that we can easily produce and handle large amounts of them.

There is one caveat concerning the scale probed by $\mu \to e\gamma$. In typical BSM models, the mass of the new particles that can be excluded by this constraint will be much smaller. One reason is that in perturbative models the operator in Eq. (2.28) cannot be generated at tree level, thus it will appear with at least one loop suppression factor. Furthermore, the BSM model is likely to have some form of the chiral symmetry, with some small parameters suppressing the transitions between left- and right-handed fermions. If that symmetry is akin to the one in the SM (for example, if chirality is violated only by the SM Yukawa interactions), the operator in Eq. (2.28) will be generated with the $y_\mu \sim 10^{-3}$ suppression factor. But even if $[C_{eB}]_{12} = \frac{y_\mu e}{16\pi^2 \Lambda^2}$, with Λ identified as the scale of BSM particles, $\mu \to e\gamma$ still probes $\Lambda \sim 10^5$ GeV, comfortably above the LHC reach.

New SMEFT vertices violating established SM symmetries may be our best path to new physics at high energy scales. Nevertheless, not all new SMEFT vertices are of this

[17]In principle, new physics contributions to $\mu \to e\gamma$ affect Γ_μ, but experimental constraints ensure this is a tiny effect that can be safely ignored.

type. There are many examples of SMEFT interactions that preserve all SM symmetries but do not appear in the SM Lagrangian, usually due to renormalizability of the latter. As an example, consider this dimension-6 operator:

$$\mathcal{L}_{D=6} \supset C_{HG} H^\dagger H\, G^a_{\mu\nu} G^a_{\mu\nu}. \tag{2.32}$$

Its effect is to induce the Higgs boson couplings to gluons:

$$\mathcal{L}_{\text{SMEFT}} \supset v^2 C_{HG} \frac{h}{v} G^a_{\mu\nu} G^a_{\mu\nu}, \tag{2.33}$$

which permits the Higgs boson to mutate into two gluons or vice-versa. Such a contact interaction term between the Higgs and gluons is absent in the SM. However, the process where two gluons collide to produce a Higgs boson does appear in the SM at the one-loop level, see Fig. 3. This is in fact the most common way the Higgs is created at the LHC. The SM loops and the contact interaction in Eq. (2.33) are in principle distinguishable experimentally, in particular they lead to a different p_T distribution of the Higgs production in hadron colliders. In practice, we can best distinguish them indirectly via global fits to Higgs data, as the SM process and the Eq. (2.33) are differently correlated with the associated tth production rate. The resulting bounds are in the ballpark of $|C_{HG}| \lesssim \frac{1}{(10\text{ TeV})^2}$, see e.g. Ref. [48]. This is visibly less spectacular than the bounds discussed previously in this subsection. The sensitivity to $\mathcal{O}(10)$ TeV scale is rather typical for new SMEFT vertices not violating any exact or approximate symmetries of the SM.

New Lorentz structures. We turn to another class of effects of higher-dimensional operators, which are related to interaction terms with different Lorentz structures compared to those in the SM. Perhaps the most iconic example in this class are the interactions contributing to the anomalous magnetic and electric moments of elementary particles. The dimension-6 SMEFT Lagrangian contains

$$\mathcal{L}_{D=6} \supset [C_{eB}]_{11} (\bar{l}_1 H \bar{\sigma}^{\mu\nu} \bar{e}^c) B_{\mu\nu} + \text{h.c.}, \tag{2.34}$$

where $[C_{eB}]_{11}$ is assumed to be real. In the presence of this operator, the Lagrangian after electroweak symmetry breaking contains

$$\mathcal{L}_{\text{SMEFT}} \supset -\frac{\Delta\mu_e}{4} F_{\mu\nu} (e^c \sigma^{\mu\nu} e) + \text{h.c.} \tag{2.35}$$

where

$$\Delta\mu_e = -2\sqrt{2} v \cos\theta_W \text{Re}\,[C_{eB}]_{11}. \tag{2.36}$$

The effect of the operator in Eq. (2.34) is to introduce another vertex with two electrons and one photon, but with a different Lorentz structure, in particular with the photon entering via the field strength $F_{\mu\nu}$. This effect can be identified as the contribution to the anomalous magnetic moment of the electron, which can be described by the electron g-factor

$$\frac{g_e - 2}{2} = \frac{g_{\text{loops}} - 2}{2} + \Delta\mu_e \frac{m_e}{q_e e}, \tag{2.37}$$

where $q_e = -1$. See Section 3.2 for more details and derivations.

We want to determine the constraint on $[C_{eB}]_{11}$ from the measurement of the anomalous magnetic moment of the electron. These days, g_e is predicted and measured with the incredible 10^{-13} accuracy, which is often hailed as a triumph of both experimental and theoretical prowess and a powerful demonstration of the robustness of the QFT framework. The most recent experimental result comes from Ref. [49]: $\frac{g_e-2}{2} = 0.00115965218059(13)$. To constrain new physics we also need the SM prediction $g_e^{\rm SM}$. This can be calculated perturbatively, mainly in function of the fine structure constant α. Unfortunately, at this point in time there is some confusion about α. Measurements using rubidium atoms in Ref. [50] lead to $1/\alpha(0) = 137.035999206(11)$, while those using cesium atoms in Ref. [51] find $1/\alpha(0) = 137.035999046(27)$, the two disagreeing at more than 5 sigma. Clearly, one or both experiments underestimated their systematic errors. To deal with this kind of situations, Particle Data Group developed a completely ad-hoc but nevertheless very useful procedure. The idea is to punish both groups indiscriminately by inflating the error bars to the point where the two measurements become consistent with each other at 1 sigma. Using this procedure, I find the combined value $1/\alpha(0) = 137.035999183(56)$, where the errors are inflated by $S = 5.5$. This is a loss of precision by a factor of 5(2) compared to the more (less) precise input, but, undeniably, this combination better reflects our current knowledge of α than the two individual results with the smaller errors. With this value of α one can obtain the prediction $g_e^{\rm SM}/2 = 1.00115965218045(48)$. Note that the theoretical error is now almost 4 times larger than the experimental one.[18]

At this point we have all the ingredients to constrain the Wilson coefficient $[C_{eB}]_{11}$. Using Eq. (3.18) and replacing $\Delta\mu_e$ using Eq. (2.36) we get $[C_{eB}]_{11} = 1.4(5.0)\times 10^{-13}\,{\rm GeV}^{-2}$ or

$$\big|[C_{eB}]_{11}\big| \lesssim \frac{1}{(940\ {\rm TeV})^2} \tag{2.38}$$

at 95% CL. We can see that the anomalous magnetic moment of the electron probes very high scales, although not as high as, say, $\mu \to e\gamma$. Moreover, similar caveat as the one discussed below Eq. (2.31) applies: in natural BSM models the chiral symmetry will typically be implemented, leading to $[C_{eB}]_{11} \sim \frac{y_e}{16\pi^2\Lambda^2}$. The same scaling is true when $[C_{eB}]_{11}$ is induced by other dimension-6 operators via renornalization group running. If that is the case, measurements of g_e currently probe the very unimpressive new physics scale $\Lambda \sim 100$ GeV, such that the validity range of SMEFT would be null in this scenario. The most accurate precision experiment in physics may not be accurate enough to reach new physics above a TeV !

Modified couplings. We turn to yet another important class of effects of higher-dimensional operators. The SM has merely 18 free parameters (not counting the theta term), and in terms of those it predicts countless interaction strengths between particles in the SM Lagrangian. For example, *all* interactions of the Higgs boson are uniquely predicted in terms

[18] Because of that, it would make sense to actually fix $\alpha(0)$ using the g_e measurement, while the traditional α measurements using atom spectroscopy would then be used to constrain new physics. This would be completely equivalent for the sake of constraining the C_{eB} Wilson coefficient, and just a tad more tricky at the level of the theoretical formalism, see the discussion of input parameters in the following subsection.

of the Higgs VEV v and the SM particles masses:

$$\mathcal{L}_{\rm SM} \supset \frac{h}{v}\Big\{2m_W^2 W_\mu^+ W_\mu^- + m_Z^2 Z_\mu Z^\mu - \sum_{f=d,u,s,c,b,t} m_f\big[f^c f + \bar{f}\bar{f}^c\big]\Big\} - \frac{m_h^2}{2v}h^3 - \frac{m_h^2}{8v^2}h^4. \tag{2.39}$$

All of the parameters above are well known. The experimental precision varies (e.g. m_Z is measured with a relative 10^{-4} error, while for the up and down quark the accuracy is closer to 10%) but is invariably better than what is needed to adequately predict the LHC rates of Higgs production and decay. Staying within the SM paradigm, the LHC measurements of Higgs cross sections and branching ratios teach us next to nothing about fundamental interactions. Things are completely different in SMEFT, where literally every interaction strength in Eq. (2.39) can be altered by higher-dimensional operators. As an example, consider the dimension-6 operator in Eq. (2.11) that modifies the Higgs boson coupling to tau leptons:

$$\mathcal{L}_{D=6} \supset [C_{eH}]_{33} H^\dagger H (\bar{l}_3 H \bar{\tau}^c) + \text{h.c.} \tag{2.40}$$

After electroweak symmetry breaking this becomes

$$\begin{aligned}\Delta\mathcal{L}_{\rm SM} =& [C_{eH}]_{33}\frac{(v+h)^3}{2\sqrt{2}}(\bar{\tau}\bar{\tau}^c) + \text{h.c.}\\ =& [C_{eH}]_{33}\frac{v^3 + 3v^2 h + 3vh^2 + h^3}{2\sqrt{2}}(\bar{\tau}\bar{\tau}^c) + \text{h.c.}\end{aligned} \tag{2.41}$$

The h^2 and h^3 are new vertices in the nomenclature of this section. They have currently very limited phenomenological relevance, so let us leave them aside. The first two terms shift the τ mass term and the Higgs Yukawa coupling to τ:

$$\mathcal{L}_{\rm SMEFT} \supset -\frac{v}{\sqrt{2}}\bigg([Y_e]_{33} - \frac{v^2}{2}[C_{eH}]_{33}\bigg)\big[\tau^c\tau + \bar{\tau}\bar{\tau}^c\big] - \frac{h}{\sqrt{2}}\bigg([Y_e]_{33} - \frac{3v^2}{2}[C_{eH}]_{33}\bigg)\big[\tau^c\tau + \bar{\tau}\bar{\tau}^c\big], \tag{2.42}$$

where, to simplify this discussion, I assume that $[C_{eH}]_{33}$ does not have an imaginary part. By convention, I always work in a basis where $Y_e + (v^2/2)C_{eH}$ is diagonal and real. Therefore we can identify the τ mass as $m_\tau = \frac{v}{\sqrt{2}}\big([Y_e]_{33} - \frac{v^2}{2}[C_{eH}]_{33}\big)$, and rewrite the Yukawa

$$\begin{aligned}\mathcal{L}_{\rm SMEFT} \supset& -\frac{h}{\sqrt{2}}\bigg([Y_e]_{33} - \frac{v^2}{2}[C_{eH}]_{33}\bigg)\bigg(1 - \frac{v^2}{[Y_e]_{33}}[C_{eH}]_{33}\bigg)\big[\tau^c\tau + \bar{\tau}\bar{\tau}^c\big]\\ =& m_\tau\big(1 + \delta y_\tau\big)h\big[\tau^c\tau + \bar{\tau}\bar{\tau}^c\big], \qquad \delta y_\tau = -\frac{v^3}{\sqrt{2}m_\tau}[C_{eH}]_{33},\end{aligned} \tag{2.43}$$

where I'm neglecting $\mathcal{O}([C_{eH}]_{33}^2) \sim \mathcal{O}(\Lambda^{-4})$ effects. Thus, $[C_{eH}]_{33}$ destroys the correlation between the Higgs boson Yukawa coupling to the τ lepton and the τ lepton mass. In other words, that coupling is modified, deviating from the SM prediction. This is in fact the best way to constrain $[C_{eH}]_{33}$. Particle Data Group performs an average of the ATLAS and CMS bounds on the $h \to \tau\tau$ signal strength, finding $\Gamma(h \to \tau\tau)/\Gamma(h \to \tau\tau)_{\rm SM} = 1.15 \pm 0.15$ [45]. This translates to $\delta y_\tau = 0.08 \pm 0.08$, or

$$\left|[C_{eH}]_{33}\right| \lesssim \frac{1}{(5\ \text{TeV})^2} \tag{2.44}$$

at 95% CL. The reach up to a few TeV is representative to what one can currently squeeze out of Higgs physics.

There are two main lessons from this simple example. One is the importance of precision measurements. C_{eH} is just one of many dimension-6 operators that shift interaction strengths away from the SM value. Searching for such effects relies not only on improving experimental accuracy, but also on a good control of the theoretical predictions. This is often challenging, but the payoff is important: increasing precision of the measurements directly translates into increased scale of higher-dimensional operators (thus, increased scale of new physics) that we can probe. The second lesson concerns the importance of properly identifying the input parameters in SMEFT. In the above example, one had to take into account that $[C_{eH}]_{33}$ contributes not only to the $h\tau\tau$ Yukawa, but also to the τ mass term. Had we forgotten about it, and just naively looked at the Yukawa term in Eq. (2.42), we would have obtained a wrong answer for δy_τ. In this case the error committed would be of order one, but it can be much more dramatic. Consider another example, where we switch on the four-fermion operator

$$\mathcal{L}_{D=6} \supset [C_{ll}]_{1221}(\bar{l}_1\bar{\sigma}^\rho l_2)(\bar{l}_2\bar{\sigma}_\rho l_1), \tag{2.45}$$

where $[C_{ll}]_{1221}$ is real in our conventions (see the comment below Eq. (2.16)). This operator contains the interaction terms $\mathcal{L}_{\rm SMEFT} \supset [C_{ll}]_{1221}(\bar{e}\bar{\sigma}^\rho\mu)(\bar{\nu}_\mu\bar{\sigma}_\rho\nu_e) + \text{h.c.}$, which contributes to muon decay, $\Gamma(\mu^- \to e^-\bar{\nu}_e\nu_\mu)/\Gamma(\mu^- \to e^-\bar{\nu}_e\nu_\mu)_{\rm SM} = 1 - v^2[C_{ll}]_{1221}$. Now, muon decay is measured with impressive precision, with the relative error of order 10^{-6}. One might naively jump to the conclusion that $[C_{ll}]_{1221}$ is stringently constrained, $v^2|[C_{ll}]_{1221}| \lesssim 10^{-6}$, that is to say $|[C_{ll}]_{1221}| \lesssim \frac{1}{(300\ \text{TeV})^2}$. This would be terribly wrong. The reason is that, in the usual approach to SM precision tests, muon decay is the standard candle that determines one of the unknown parameters of the SM - the Higgs VEV v. Indeed, the tree-level formula[19] $\Gamma^{\rm SM}_{\mu\to e\nu\nu} = m_\mu^5/(384\pi^3 v^4)$ allows one to precisely fix v, given that m_μ is known with an even better accuracy. On the other hand, in SMEFT in the presence of $[C_{ll}]_{1221}$ one has $\Gamma^{\rm SMEFT}_{\mu\to e\nu\nu} = m_\mu^5(1 - v^2[C_{ll}]_{1221})/(384\pi^3 v^4)$. Since both v and $[C_{ll}]_{1221}$ are a-priori unknown parameters, muon decay does not fix either, but just one combination of the two. Let us repeat it loudly and clearly: muon decay alone leads to *no constraint at all* on C_{ll} !

Nevertheless, the effects of $[C_{ll}]_{1221}$ do not jus disappear, when regarded from a more global perspective. To understand how $[C_{ll}]_{1221}$ re-emerges we need to do a small detour first, and discuss the input parameters for electroweak precision tests. The latter can be defined as a set of observables that, in the SM at tree level, depend on the parameters g_L, g_Y and v in the electroweak sector. The numerical values of these parameters are traditionally fixed by three precisely known input observables:

[19] Of course, at this level of precision, one should also take into account the radiative and $\mathcal{O}(m_e^2/m_\mu^3)$ corrections when relating v to the observable decay width. This does not interfere with the following discussion.

1. **The Fermi constant** G_F, extracted from the measured muon lifetime using the formula $\Gamma^{\rm SM}_{\mu\to e\nu\nu} = G_F^2 m_\mu^5/(192\pi^3)$ plus radiative corrections [52].

2. **The electromagnetic structure constant** α, currently best extracted from the spectroscopy of rubidium [50] and cesium [51] atoms.

3. **The Z boson mass** m_Z, extracted from the position of the corresponding resonance in e^+e^- scattering in the LEP-1 collider [53].

In SMEFT, in the presence of $[C_{ll}]_{1221}$, working at $\mathcal{O}(\Lambda^{-2})$ and at tree[20] level, these input observables are connected to the SMEFT parameters as

$$G_F = \frac{1}{\sqrt{2}v_0^2}\left(1 - \frac{v^2}{2}[C_{ll}]_{1221}\right), \qquad \alpha = \frac{g_{L0}^2 g_{Y0}^2}{4\pi(g_{L0}^2+g_{Y0}^2)}, \qquad m_Z = \frac{\sqrt{g_{L0}^2+g_{Y0}^2}\,v_0}{2}, \tag{2.46}$$

where I re-labeled the parameters from the SM Lagrangian using the subscript zero to distinguish them from the g_L, g_Y and v parameters in the following, which will differ by $\mathcal{O}(\Lambda^{-2})$, and which will be assigned definite numerical values. Note at, at $\mathcal{O}(\Lambda^{-2})$, it does not matter whether I write $v^2 C_{ll}$ or $v_0^2 C_{ll}$. If $[C_{ll}]_{1221} = 0$ then, as in the SM, Eq. (2.46) relates 3 parameters to 3 observables and we can readily solve for v_0, g_{L0}, and g_{Y0}. However when $[C_{ll}]_{1221} \neq 0$ we have 3 equations for 4 parameters. In this case it is convenient to use a trick: we can get rid of $[C_{ll}]_{1221}$ from Eq. (2.46) by absorbing it into the other parameters. These can be achieved by defining

$$v_0 = v(1+\delta v), \qquad g_{L,0} = g_L(1+\delta g_L), \qquad g_{Y,0} = g_Y(1+\delta g_Y), \tag{2.47}$$

where

$$\delta v = -\frac{v^2}{4}[C_{ll}]_{1221}, \qquad \delta g_L = \frac{g_L^2 v^2}{4(g_L^2-g_Y^2)}[C_{ll}]_{1221}, \qquad \delta g_Y = -\frac{g_Y^2 v^2}{4(g_L^2-g_Y^2)}[C_{ll}]_{1221}. \tag{2.48}$$

The shift δv removes the $[C_{ll}]_{1221}$ pollution from G_F at $\mathcal{O}(\Lambda^{-2})$. The other two shifts are then needed to prevent $[C_{ll}]_{1221}$ from popping up in m_Z and α. After the shift Eq. (2.46) becomes

$$G_F = \frac{1}{\sqrt{2}v^2}, \qquad \alpha = \frac{g_L^2 g_Y^2}{4\pi(g_L^2+g_Y^2)}, \qquad m_Z = \frac{\sqrt{g_L^2+g_Y^2}\,v}{2}, \tag{2.49}$$

which means that g_L, g_Y, and v are related to observables in exactly the same way as the corresponding SM parameters, and therefore they can be assigned exactly the same numerical values. Plugging in the numbers into Eq. (2.49), $G_F = 1.1663787(6)\times 10^{-5}$ GeV^{-2}, $\alpha(m_Z) = 7.81549(55)\times 10^{-3}$, $m_Z = 91.1876(21)$ GeV [45], one finds[21]

$$v = 246.219651(63)\ \text{GeV}, \qquad g_L = 0.648457(10), \qquad g_Y = 0.357968(18). \tag{2.50}$$

[20] Once again, radiative corrections from $D \le 4$ operators must be taken into account in the matching of the input parameters to observables in order to meet the required precision level. The procedure is in fact very similar to our treatment of higher-dimensional effects.

[21] Note that I use $\alpha(m_Z)$ rather than $\alpha(0)$ to extract the numerical values of the electroweak couplings in SMEFT, even though the former has a much larger error due to non-perturbative contributions to the running from the low-energy up to the electroweak scale. This choice is more convenient in practice, and the incurred error is negligible for most applications.

OK, we managed to assign numerical values to electroweak couplings in SMEFT, but where is $[C_{ll}]_{1221}$ now? The point is that, due to the shift in Eq. (2.47), that Wilson coefficient will pop in practically every other electroweak precision observable. Let us focus on just one of them - the W boson mass. Starting from the tree-level formula $m_W = g_{L0} v_0/2$ and applying the shift in Eq. (2.47) one finds that the correction to the W boson mass in the presence of $[C_{ll}]_{1221}$ is given by

$$\frac{\Delta m_W}{m_W} = \frac{g_Y^2 v^2}{4(g_L^2 - g_Y^2)} [C_{ll}]_{1221}. \tag{2.51}$$

Now we are ready to constrain $[C_{ll}]_{1221}$. Using the average of experimental measurements from Particle Data Group, $m_W = 80.377(12)$ [45], as well as their SM prediction $m_W^{\rm SM} = 80.361(6)$ [45], one obtains $v^2 [C_{ll}]_{1221} = 1.8(1.5) \times 10^{-3}$. This translates to

$$\big|[C_{ll}]_{1221}\big| \lesssim \frac{1}{(3.5 \text{ TeV})^2} \tag{2.52}$$

at 95% CL. $[C_{ll}]_{1221}$ contributes to many other electroweak precision observables via the shift in Eq. (2.47), therefore the true bound is somewhat stronger than what one obtains based on the W mass alone. Using the global likelihood from Ref. [54] I obtain $\big|[C_{ll}]_{1221}\big| \lesssim \frac{1}{(5.7 \text{ TeV})^2}$ at 95% CL.

One more thing. In this lecture, for the sake of simplicity, the discussion of phenomenological effects of higher-dimensional operators is divided into “new vertices", “new Lorentz structures", and "modified couplings" parts. It is however important to mention that sometimes there is no invariant way to make this distinction. Consider the following example of self-interactions of the Higgs boson h:

$$\mathcal{L}_{\rm SMEFT} \supset \frac{1}{2}(\partial_\mu h)^2 - \frac{m_h^2}{2} h^2 - \frac{m_h^2}{2v}\left(1 + \delta_1\right) h^3 - \frac{\delta_2}{v} h \partial_\mu h \partial_\mu h + \ldots \tag{2.53}$$

where the dots denote terms with 4 and more Higgs bosons. Two possible effects of higher-dimensional operators appear above. The one proportional to δ_1 changes the magnitude of the triple Higgs self-coupling, which is already present in the SM Lagrangian but with the magnitude strictly fixed by the Higgs boson mass. This is a modified coupling in our nomenclature. The other effect proportional to δ_2 is a two-derivative Higgs self-interaction term which does not appear in the SM Lagrangian in its canonical form. This is a new Lorentz structure in our nomenclature. Both δ_i can be generated by dimension-6 operators, therefore we will treat δ_i as $\mathcal{O}(\Lambda^{-2})$. For example, switching on the Wilson coefficients C_H and $C_{H\Box}$ in Eq. (2.11), one gets $\delta_1 = 3v^2 C_{H\Box} - 5\frac{v^4}{m_h^2} C_H$, $\delta_2 = 2v^2 C_{H\Box}$. Both δ_i contribute in a non-trivial way to the Higgs scattering amplitudes, for example to $hh \to hh$, or to double Higgs production at the LHC once interactions of h with the rest of the SM are taken into account. Nevertheless, we can equivalently work with an effective Lagrangian where the 2-derivative $h(\partial_\mu h)^2$ interaction is completely eliminated via field redefinitions. To this end we redefine the Higgs boson field as

$$h \to h + \frac{\delta_2}{2v} h^2. \tag{2.54}$$

After this redefinition the effective Lagrangian of Eq. (2.53) takes the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu h)^2 - \frac{m_h^2}{2}h^2 - \frac{m_h^2}{2v}(1+\delta_1+\delta_2)\,h^3 + \ldots, \tag{2.55}$$

where I ignored $\delta_i^2 \sim \mathcal{O}(\Lambda^{-4})$ terms resulting from the redefinition. Seemingly, the Lagrangians in Eqs. (2.53) and (2.55) are different, as they contain different interaction terms. However, the equivalence theorem ensures that field redefinitions cannot change the physical content of the theory, as was discussed in **??** in the context of SMEFT bases. Therefore, the two Lagrangians give exactly the same predictions for physical observables, at any order in the perturbation theory, which can be verified by explicit calculations. This demonstrates that we can get rid of some of the new Lorentz structures generated by higher-dimensional operators and absorb them into modified couplings. Conversely, a shift $h \to h - \frac{\delta_1}{2v}h^2$ applied to Eq. (2.53) would erase the modified coupling in favor of the new Lorentz structure. Once the fermions are included in the discussion, the above shift of the Higgs boson field produces Yukawa-like interactions between two powers of the Higgs boson and the SM fermions. Such interactions are absent in the SM Lagrangian, thus they are "new vertices" in our nomenclature. Therefore the boundaries between new vertices and modified couplings or new Lorentz structures are also blurred by field redefinitions. The sharp boundary persists only for the new vertices violating the SM symmetries, as those can never be redefined away into modified couplings or new Lorentz structures.

3 Non-relativistic EFT with applications

3.1 From Dirac to Schrödinger

Consider a massive spin 1/2 particle, e.g. an electron or a nucleon. In a relativistic QFT it can be described by a pair of 2-component spinor fields f and $\bar{f}^c$ with the free Dirac Lagrangian:

$$\mathcal{L}_{\rm Dirac} = i\bar{f}\bar{\sigma}^\mu\partial_\mu f + if^c\sigma^\mu\partial_\mu\bar{f}^c - mf^cf - m\bar{f}\bar{f}^c. \tag{3.1}$$

We know this actually describes 4 degrees of freedom: two polarizations of the particle, and two polarizations of the anti-particle. Both ψ and ψ^c contain the ladder operators for both particle and anti-particle states. On the other hand, for 3-momenta $|\boldsymbol{p}| \ll m$, creation of particle-anti-particle pairs is kinematically impossible. In this regime it may be worthwhile to use a different set of variables where particles and anti-particles are separated.

The original fields are expanded in terms of the ladder operators as

$$\begin{aligned} f &= \int d\Phi_k\big[a_k x_k e^{-ikx} + b_k^\dagger y_k e^{ikx}\big], \\ \bar{f}^c &= \int d\Phi_k\big[a_k \bar{y}_k e^{-ikx} + b_k^\dagger \bar{x}_k e^{ikx}\big],, \end{aligned} \tag{3.2}$$

where $d\Phi_k \equiv \frac{d^3k}{(2\pi)^3 2E_k} = \frac{d^4k\delta(k^2-m^2)\theta(k^0)}{(2\pi)^3}$ and $E_k \equiv \sqrt{m^2+\boldsymbol{k}^2}$. Consider the combination

$$\left(1 + \frac{i}{2m}\boldsymbol{\sigma}\boldsymbol{\nabla}\right)f_\alpha + \left(1 - \frac{i}{2m}\boldsymbol{\sigma}\boldsymbol{\nabla}\right)\bar{f}^{c\,\alpha}$$

$$
\begin{aligned}
&= \int d\Phi_k \Big\{ a_k e^{-ikx} \Big[\Big(1 - \frac{\boldsymbol{k\sigma}}{2m}\Big) x_{k\alpha} + \Big(1 + \frac{\boldsymbol{k\sigma}}{2m}\Big) \bar{y}_k^{\alpha} \Big] \\
&\quad + b_k^{\dagger} e^{ikx} \Big[\Big(1 + \frac{\boldsymbol{k\sigma}}{2m}\Big) y_{k\alpha} + \Big(1 - \frac{\boldsymbol{k\sigma}}{2m}\Big) \bar{x}_k^{\alpha} \Big] \Big\} \\
&= \frac{1}{2} \int d\Phi_k \Big\{ a_k e^{-ikx} \Big(1 + \frac{E_k}{m}\Big) \big[x_{k\alpha} + \bar{y}_k^{\alpha}\big] + b_k^{\dagger} e^{ikx} \Big(1 - \frac{E_k}{m}\Big) \big[y_{k\alpha} + \bar{x}_k^{\alpha}\big] \Big\} \\
&= \int d\Phi_k a_k e^{-ikx} \big[x_{k\alpha} + \bar{y}_k^{\alpha}\big] + \mathcal{O}(\boldsymbol{\nabla}^2).
\end{aligned} \tag{3.3}
$$

We used the Dirac equations $k\bar{\sigma}x_k = (E_k\sigma^0 + \boldsymbol{k\sigma})x_k = m\bar{y}_k$, $k\bar{\sigma}y_k = (E_k\sigma^0 + \boldsymbol{k\sigma})y_k = -m\bar{x}_k$, $k\sigma\bar{y}_k = (E_k\sigma^0 - \boldsymbol{k\sigma})\bar{y}_k = mx_k$, $k\sigma\bar{x}_k = (E_k\sigma^0 - \boldsymbol{k\sigma})\bar{x}_k = -my_k$. We see that the $b^\dagger$ part, which creates antiparticles, cancels out, up to $\boldsymbol{k^3}/m^3$ corrections, while the a part, which annihilates particles, survives. It is possible to write down a combination of f, $\bar{f}^c$ where the anti-particle part cancels out to all orders, but that won't be necessary for our purpose. There is one more thing about this combination that is suboptimal from the low-energy point of view: it contains the rapidly oscillating factor $e^{-iE_k t} \approx e^{-imt}$, which knows about the rest energy of the particle. In non-relativistic physics we would rather work with the kinetic energy instead, $E_{\text{kin}} \equiv k^0 - m \approx \boldsymbol{k}^2/2m$. This motivates the definition of the non-relativistic particle field

$$
\psi = \frac{e^{imt}}{\sqrt{2}} \Big\{ \Big(1 + \frac{i}{2m}\boldsymbol{\sigma\nabla}\Big) f_\alpha + \Big(1 - \frac{i}{2m}\boldsymbol{\sigma\nabla}\Big) \bar{f}^{c\,\alpha} \Big\} + \mathcal{O}(\boldsymbol{\nabla}^2). \tag{3.4}
$$

In the same way the non-relativistic anti-particle field defined as

$$
\psi^{c\dagger} = \frac{e^{-imt}}{\sqrt{2}} \Big\{ \Big(1 - \frac{i}{2m}\boldsymbol{\sigma\nabla}\Big) f_\alpha - \Big(1 + \frac{i}{2m}\boldsymbol{\sigma\nabla}\Big) \bar{f}^{c\,\alpha} \Big\} + \mathcal{O}(\boldsymbol{\nabla}^2) \tag{3.5}
$$

contains only $b^\dagger$ and not a and oscillates with $e^{+iE_{\text{kin}}t}$. One can invert the relation between the relativistic and non-relativistic fields:

$$
\begin{aligned}
f_\alpha &= \frac{1}{\sqrt{2}} \Big\{ e^{-imt} \Big(1 + \frac{i}{2m}\boldsymbol{\sigma\nabla} - \frac{\boldsymbol{\nabla}^2}{8m^2}\Big) \psi_\alpha - e^{imt} \Big(1 - \frac{i}{2m}\boldsymbol{\sigma\nabla} - \frac{\boldsymbol{\nabla}^2}{8m^2}\Big) \psi^{c\dagger\,\alpha} \Big\} + \mathcal{O}(\boldsymbol{\nabla}^3), \\
\bar{f}^{c\,\alpha} &= \frac{1}{\sqrt{2}} \Big\{ e^{-imt} \Big(1 - \frac{i}{2m}\boldsymbol{\sigma\nabla} - \frac{\boldsymbol{\nabla}^2}{8m^2}\Big) \psi_\alpha + e^{imt} \Big(1 + \frac{i}{2m}\boldsymbol{\sigma\nabla} - \frac{\boldsymbol{\nabla}^2}{8m^2}\Big) \psi^{c\dagger\,\alpha} \Big\} + \mathcal{O}(\boldsymbol{\nabla}^3).
\end{aligned} \tag{3.6}
$$

Apart from inverting, I generalized the transformation to the quadratic order in spatial derivatives, because it will be needed for the following calculation.

> *Exercise:* Generalize the change of variables in Eq. (3.6) such that particles and anti-particles are separated (that is to say, ψ depends only on the particle annihilation operator a, and ψ^c depends only on the anti-particle annihilation operators b) to all orders in the $\boldsymbol{\nabla}/m$ expansion.

We can treat Eq. (3.6) as any other change of variables in QFT. What happens is we insert this change of variable into the free Lagrangian Eq. (3.1) ? Tracing only the particle field we have

$$
\begin{aligned}
i\bar{f}\bar{\sigma}^\mu\partial_\mu f = & \frac{ie^{imt}}{2}\left[\psi^\dagger - \frac{i}{2m}\boldsymbol{\nabla}\psi^\dagger\boldsymbol{\sigma} - \frac{\boldsymbol{\nabla}^2\psi^\dagger}{8m^2}\right]\bar{\sigma}^\mu\partial_\mu\left[e^{-imt}\left(\psi + \frac{i}{2m}\boldsymbol{\sigma}\boldsymbol{\nabla}\psi - \frac{\boldsymbol{\nabla}^2\psi}{8m^2}\right)\right] + \mathcal{O}(\boldsymbol{\nabla}^3) \\
= & \frac{i}{2}\left[\psi^\dagger - \frac{i}{2m}\boldsymbol{\nabla}\psi^\dagger\boldsymbol{\sigma} - \frac{\boldsymbol{\nabla}^2\psi^\dagger}{8m^2}\right]\left[-im\left(\psi + \frac{i}{2m}\boldsymbol{\sigma}\boldsymbol{\nabla}\psi - \frac{\boldsymbol{\nabla}^2\psi}{8m^2}\right) + \partial_t\psi - \boldsymbol{\sigma}\boldsymbol{\nabla}\left(\psi + \frac{i}{2m}\boldsymbol{\sigma}\boldsymbol{\nabla}\psi\right)\right] + \mathcal{O}(\boldsymbol{\nabla}^3) \\
= & \frac{1}{2}\left[\psi^\dagger - \frac{i}{2m}\boldsymbol{\nabla}\psi^\dagger\boldsymbol{\sigma} - \frac{\boldsymbol{\nabla}^2\psi^\dagger}{8m^2}\right]\left[m\psi - \frac{i}{2}\boldsymbol{\sigma}\boldsymbol{\nabla}\psi + i\partial_t\psi + \frac{3}{8m}\boldsymbol{\nabla}^2\psi\right] + \mathcal{O}(\boldsymbol{\nabla}^3) \\
= & \frac{1}{2}\left\{m\psi^\dagger\psi - \frac{i}{2}\boldsymbol{\nabla}[\psi^\dagger\boldsymbol{\sigma}\psi] + i\psi^\dagger\partial_t\psi + \frac{\psi^\dagger\boldsymbol{\nabla}^2\psi}{2m}\right\} + \mathcal{O}(\boldsymbol{\nabla}^3).
\end{aligned}
\tag{3.7}
$$

$$
if^c\sigma^\mu\partial_\mu\bar{f}^c = \frac{1}{2}\left\{m\psi^\dagger\psi + \frac{i}{2}\boldsymbol{\nabla}[\psi^\dagger\boldsymbol{\sigma}\psi] + i\psi^\dagger\partial_t\psi + \frac{\psi^\dagger\boldsymbol{\nabla}^2\psi}{2m}\right\} + \mathcal{O}(\boldsymbol{\nabla}^3).
\tag{3.8}
$$

$$
\begin{aligned}
mf^cf + \mathrm{h.c.} = & \frac{\mathrm{m}}{2}\left(\psi^\dagger + \frac{\mathrm{i}}{2\mathrm{m}}\boldsymbol{\nabla}\psi^\dagger\boldsymbol{\sigma} - \frac{\boldsymbol{\nabla}^2\psi^\dagger}{8\mathrm{m}^2}\right)\left(\psi + \frac{\mathrm{i}}{2\mathrm{m}}\boldsymbol{\sigma}\boldsymbol{\nabla}\psi - \frac{\boldsymbol{\nabla}^2\psi}{8\mathrm{m}^2}\right) + \mathrm{h.c.} \\
= & \frac{1}{2}\left\{m\psi^\dagger\psi + \frac{i}{2}\boldsymbol{\nabla}[\psi^\dagger\boldsymbol{\sigma}\psi] + \mathcal{O}(\boldsymbol{\nabla}^3)\right\} + \mathrm{h.c.} = \mathrm{m}\psi^\dagger\psi + \mathcal{O}(\boldsymbol{\nabla}^3).
\end{aligned}
\tag{3.9}
$$

In this derivation I used integration by parts. Thus the free Dirac Lagrangian in the non-relativistic variables becomes

$$
\mathcal{L}_{\mathrm{Dirac}} = i\psi^\dagger\partial_t\psi + \frac{\psi^\dagger\boldsymbol{\nabla}^2\psi}{2m} + \mathcal{O}(\boldsymbol{\nabla}^3).
\tag{3.10}
$$

The equation of motion following from the Lagrangian is $i\psi^\dagger\partial_t\psi = -\frac{\boldsymbol{\nabla}^2\psi}{2m}$, which is nothing but the Schrödinger equation. If we worked to all orders in $\boldsymbol{\nabla}/m$ we would find the relativistic generalization of the Schrödinger equation: $i\psi^\dagger\partial_t\psi = m\left(\sqrt{1 - \frac{\boldsymbol{\nabla}^2}{m^2}} - 1\right)\psi$.

Exercise: Show that the change of variables Eq. (3.6) applied to the free Dirac Lagrangian Eq. (3.1) achieves separation of the particle ψ and anti-particle ψ^c fields at the quadratic level.

3.2 Electromagnetic interactions at low energies

Consider a Dirac particle f coupled to the electromagnetic field. The most general Lagrangian is

$$
\mathcal{L}_{\mathrm{em}} = -qeA_\mu\left[\bar{f}\bar{\sigma}^\mu f + f^c\sigma^\mu\bar{f}^c\right] - \left\{\frac{\Delta\mu - id}{4}F_{\mu\nu}(f^c\sigma^{\mu\nu}f) + \mathrm{h.c.}\right\}.
\tag{3.11}
$$

The first term is the minimal coupling for a particle of charge q. The second term describes the non-minimal, non-renormalizable coupling. The parameters $\Delta\mu$ and d have dimensions mass^{-1}; their physical interpretation will be clarified shortly.

Let us now apply the change of variable in Eq. (3.6). Focusing on the particle field ψ, we have

$$\begin{aligned}\bar f\bar\sigma^\mu f =&\frac{1}{2}\big[\psi^\dagger - \frac{i}{2m}\boldsymbol{\nabla}\psi^\dagger\boldsymbol{\sigma}\big]\bar\sigma^\mu\big[\psi + \frac{i}{2m}\boldsymbol{\sigma}\boldsymbol{\nabla}\psi\big] + \mathcal{O}(\boldsymbol{\nabla}^2)\\ =&\frac{1}{2}\Big\{\psi^\dagger\bar\sigma^\mu\psi + \frac{i}{2m}\Big[\psi^\dagger\bar\sigma^\mu\boldsymbol{\sigma}\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}\psi^\dagger\boldsymbol{\sigma}\bar\sigma^\mu\psi\Big]\Big\} + \mathcal{O}(\boldsymbol{\nabla}^2)\\ f^c\sigma^\mu\bar f^c =&\frac{1}{2}\Big\{\psi^\dagger\sigma^\mu\psi - \frac{i}{2m}\Big[\psi^\dagger\sigma^\mu\boldsymbol{\sigma}\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}\psi^\dagger\boldsymbol{\sigma}\sigma^\mu\psi\Big]\Big\} + \mathcal{O}(\boldsymbol{\nabla}^2).\end{aligned} \tag{3.12}$$

Thus

$$\begin{aligned}\bar f\bar\sigma^0 f + f^c\sigma^0\bar f^c =&\psi^\dagger\psi + \mathcal{O}(\boldsymbol{\nabla}^2)\\ \bar f\bar\sigma^k f + f^c\sigma^k\bar f^c =&-\frac{i}{2m}\Big[\psi^\dagger\sigma^k\boldsymbol{\sigma}\boldsymbol{\nabla}\psi - \boldsymbol{\nabla}\psi^\dagger\boldsymbol{\sigma}\sigma^k\psi\Big] + \mathcal{O}(\boldsymbol{\nabla}^2)\\ =&-\frac{i}{2m}\big[\psi^\dagger\nabla_k\psi - \nabla_k\psi^\dagger\psi\big] + \frac{\epsilon^{klm}}{2m}\nabla_l(\psi^\dagger\sigma^m\psi) + \mathcal{O}(\boldsymbol{\nabla}^2).\end{aligned} \tag{3.13}$$

We also need to work on the tensor, but there we only need the zero-derivative terms:

$$\begin{aligned}f^c\sigma^{0k}f =&\frac{1}{2}\psi^\dagger\sigma^{0k}\psi + \mathcal{O}(\boldsymbol{\nabla}) = -\frac{i}{2}\psi^\dagger\sigma^k\psi + \mathcal{O}(\boldsymbol{\nabla}),\\ f^c\sigma^{kl}f =&\frac{1}{2}\psi^\dagger\sigma^{kl}\psi + \mathcal{O}(\boldsymbol{\nabla}) = \frac{\epsilon^{klm}}{2}\psi^\dagger\sigma^m\psi + \mathcal{O}(\boldsymbol{\nabla}).\end{aligned} \tag{3.14}$$

Plugging that all into Eq. (3.15),

$$\begin{aligned}\mathcal{L}_{\rm em} =&-qeA^0\psi^\dagger\psi + qeA^k\Big(-\frac{i}{2m}\big[\psi^\dagger\nabla_k\psi - \nabla_k\psi^\dagger\psi\big] + \frac{\epsilon^{klm}}{2m}\nabla_l(\psi^\dagger\sigma^m\psi)\Big)\\ &+\Big\{i\frac{\Delta\mu - id}{4}F_{0k}\psi^\dagger\sigma^k\psi + \text{h.c.}\Big\} - \frac{\epsilon^{\rm klm}}{2}\Big\{\frac{\Delta\mu - \rm id}{4}\rm F_{kl} + \text{h.c.}\Big\}.\end{aligned} \tag{3.15}$$

I introduce the Coulomb potential, $V \equiv A^0$, the electric field $\boldsymbol{E} \equiv -\boldsymbol{\nabla}V - \partial_t\boldsymbol{A}$, the magnetic field $\boldsymbol{B} \equiv \boldsymbol{\nabla}\times\boldsymbol{A}$, and the shorthand $\psi^\dagger\overleftrightarrow{\nabla}_k\psi \equiv \psi^\dagger\nabla_k\psi - \nabla_k\psi^\dagger\psi$. Using $F_{0k} = E^k$, $\epsilon^{klm}F_{lk} = -2B^m$, the electromagnetic interactions in the non-relativistic variables become

$$\mathcal{L}_{\rm em} = -qeV\psi^\dagger\psi - \frac{iqe}{2m}A^k\psi^\dagger\overleftrightarrow{\nabla}_k\psi + \Big(\frac{qe}{m} + \Delta\mu\Big)B^k(\psi^\dagger\frac{\sigma^k}{2}\psi) + dE^k(\psi^\dagger\frac{\sigma^k}{2}\psi), \tag{3.16}$$

where The first two terms describe the interaction of fermion's charge with the Coulomb and vector potentials. The remaining two terms describes the interaction of its dipole moments with the external electric and magnetic fields. We can compare them with the textbook expressions for the dipole Hamiltonian: $H_{\rm int} \supset -\boldsymbol{B}\cdot\boldsymbol{\mu} - \boldsymbol{E}\cdot\boldsymbol{d}$. We can thus identify[22] the magnetic and electric moments of the electron at tree level:

$$\boldsymbol{\mu}_f = \Big(\frac{qe}{m} + \Delta\mu\Big)\boldsymbol{s}, \qquad \boldsymbol{d}_f = d\boldsymbol{s}, \tag{3.17}$$

[22]Recall that the sign of the potential terms is flipped between the Hamiltonian, $H = T + V$, and the Lagrangian, $L = T - V$.

where the spin vector is defined as $s^k = \psi^\dagger \frac{\sigma^k}{2}\psi$. Furthermore, defining the g-factor for a charged particle via the relation $\boldsymbol{\mu} = g\frac{qe}{2m}\boldsymbol{s}$, we have

$$\frac{g-2}{2} = \frac{g_{\rm loops}-2}{2} + \Delta\mu \frac{m}{qe}, \tag{3.18}$$

where $g_{\rm loops} - 2$ encodes the electromagnetic loop effects. In summary, the non-relativistic variables make it transparent that $g = 2$ for a minimally coupled Dirac particle at tree-level, and quantify how the coefficients of the non-minimal interactions are related to the anomalous magnetic and electric dipole moments.

3.3 Fermi theory revisited

Another elegant application of the non-relativistic formalism is in relation to β decay. We studied the Fermi theory in Section 1.5. There we worked with the quark-level effective Lagrangian Eq. (1.39) obtained by integrating out the W boson in the SM. Here I will take a more general approach, allowing for arbitrary interactions below the electroweak scale, including those not predicted by the SM. On the other hand, I assume the absence of non-SM degrees of freedom below the electroweak scale. This interactions can be organized in the expansion in the canonical dimension, much as in SMEFT. At the leading order, which is dimension-6 in this EFT expansion, the charged current 4-fermion interactions between the first generation quarks and leptons take the most general form

$$\begin{aligned}\mathcal{L}_{\rm WEFT} \supset -\frac{2V_{ud}}{v^2}\Big\{ &(1+\epsilon_L)\,(\bar e\bar\sigma_\mu\nu)(\bar u\bar\sigma^\mu d) + \epsilon_R(\bar e\bar\sigma_\mu\nu)(u^c\sigma^\mu \bar d^c) \\ &+\frac{\epsilon_S+\epsilon_P}{2}(e^c\nu)(u^c d) + \frac{\epsilon_S-\epsilon_P}{2}(e^c\nu)(\bar u\bar d^c) + \frac{\epsilon_T}{4}(e^c\sigma_{\mu\nu}\nu)(u^c\sigma^{\mu\nu}d)\Big\} + \text{h.c.}\end{aligned} \tag{3.19}$$

Deviations from the SM are parametrized by ϵ_X, and in the limit $\epsilon_X \to 0$ one recovers Eq. (1.39). Apart from the $V-A$ interactions (whose strength may deviate from the SM for $\epsilon_L \neq 0$) this effective Lagrangian also contains the right-handed currents (ϵ_R), as well as scalar (ϵ_S), pseudo-scalar (ϵ_P), and tensor (ϵ_T) interactions. At energies below ~ 2 GeV we can match this Lagrangian to the nucleon-level one. The logic is the same as in the derivation of Eq. (1.39), and the result is

$$\begin{aligned}\mathcal{L}_{\rm Lee-Yang} \supset &- C_V^+(\bar e\bar\sigma_\mu\nu)\left[\bar p\bar\sigma^\mu n + p^c\sigma^\mu\bar n^c\right] + C_A^+(\bar e\bar\sigma_\mu\nu)\left[\bar p\bar\sigma^\mu n - p^c\sigma^\mu\bar n^c\right] \\ &- C_S^+(e^c\nu)\left[p^c n + \bar p\bar n^c\right] - C_P^+(e^c\nu)\left[p^c n - \bar p\bar n^c\right] - \frac{1}{2}C_T^+(e^c\sigma_{\mu\nu}\nu)(p^c\sigma^{\mu\nu}n) + \text{h.c.}\end{aligned} \tag{3.20}$$

This kind of effective Lagrangian was first introduced by Lee and Yang in the seminar 1956 paper [55].[23] The matching between the nucleon-level Wilson coefficients in Eq. (3.20) and

[23] Lee and Yang also allow for right-handed neutrinos, therefore their Lagrangian has ten free parameters rather than five. denoted by C_X (parity-conserving) and C'_X (parity-violating), $X = V, A, S, P, T$. My notation is related to theirs by $C_X^+ = C_X + C'_X$, which explains the plus index. This index may seem uncalled for in our EFT where only left-handed neutrinos are present, but I prefer to keep it to avoid confusion with the Lee-Yang notation, which is still very widely used by the nuclear community.

the quark-level ones in Eq. (3.19) reads

$$
\begin{aligned}
C_V^+ =& g_V \frac{V_{ud}}{v^2}(1+\epsilon_L+\epsilon_R),\\
C_A^+ =& -g_A \frac{V_{ud}}{v^2}(1+\epsilon_L-\epsilon_R),\\
C_S^+ =& g_S \frac{V_{ud}}{v^2}\epsilon_S,\\
C_P^+ =& g_P \frac{V_{ud}}{v^2}\epsilon_P,\\
C_T^+ =& g_T \frac{V_{ud}}{v^2}\epsilon_T.
\end{aligned} \tag{3.21}
$$

The nucleon charges g_X appear due to the non-perturbative nature of the matching. In Eq. (1.39) we already discussed that $g_V \approx 1$ due to isospin symmetry, and $g_A = 1.246(28)$ [13]. For the other nucleon charges, $g_S = 1.02(10)$, $g_T = 0.989(34)$ from a lattice calculation [56], while g_P can be related to g_A by equations of motion, yielding $g_P = 349(9)$ [57] (this large value is due to m_N/m_q enhancement, which can also be understood as due to integrating out the pion).

It seems that the EFT described by Eq. (3.20) is much more complex than the SM limit, with several new Lorentz structures forcing us to calculate many more nuclear matrix elements. However, we will see that the use of the non-relativistic variables simplify the problem considerably, so that the level of complications is not much different than in the SM limit, and no new nuclear matrix elements are needed. The goal will be to re-organize Eq. (3.20) as the non-relativistic expansion in $\boldsymbol{\nabla}/m_N$:

$$
\mathcal{L}_{\text{Lee-Yang}} = \mathcal{L}_{\text{NR}}^{(0)} + \mathcal{L}_{\text{NR}}^{(1)} + \mathcal{L}_{\text{NR}}^{(2)} + \dots \tag{3.22}
$$

The upper indices on the right-hand side label the order in $\boldsymbol{\nabla}/m_N$ (for simplicity, I assume the common nucleon mass for the neutron and the proton, $m_N \equiv \frac{m_n+m_p}{2}$). Since $\boldsymbol{\nabla}$ will be traded for the momentum exchange $\boldsymbol{q}$ in the amplitude, and $\boldsymbol{q}/m_N \sim 10^{-2}, 10^{-3}$ in β decay, this expansion is quickly converging. For our purpose the leading order term in this expansion will be sufficient, and even more advanced calculation today rarely go beyond the next-to-leading order.

In analogy to Eq. (3.6) I define the non-relativistic nucleon fields ψ_N by

$$
\begin{aligned}
N_\alpha =& \frac{e^{-im_N t}}{\sqrt{2}}\left(1+\frac{i}{2m_N}\boldsymbol{\sigma}\boldsymbol{\nabla}\right)\psi_{N\,\alpha} + \mathcal{O}(\psi_N^c) + \mathcal{O}(\boldsymbol{\nabla}^2),\\
\bar{N}^{c\,\alpha} =& \frac{e^{-im_N t}}{\sqrt{2}}\left(1-\frac{i}{2m_N}\boldsymbol{\sigma}\boldsymbol{\nabla}\right)\psi_{N\,\alpha} + \mathcal{O}(\psi_N^c) + \mathcal{O}(\boldsymbol{\nabla}^2),
\end{aligned} \tag{3.23}
$$

for $N = n, p$. Currently there is not that much nuclear physics done with antimatter, so we will not trace the interactions of the anti-nucleon fields ψ_N^c. Let us first translate the scalar bi-linear contractions of the relativistic nucleon fields into the non-relativistic language:

$$
p^c n = \frac{1}{2}\left[\psi_p^\dagger + \frac{i}{2m_N}\boldsymbol{\nabla}\psi_p^\dagger\boldsymbol{\sigma}\right]\left[\psi_n + \frac{i}{2m_N}\boldsymbol{\sigma}\boldsymbol{\nabla}\psi_n\right] + \mathcal{O}(\boldsymbol{\nabla}^2) = \frac{1}{2}\psi_p^\dagger\psi_n + \frac{i}{4m_N}\boldsymbol{\nabla}(\psi_p^\dagger\boldsymbol{\sigma}\psi_n) + \mathcal{O}(\boldsymbol{\nabla}^2),
$$

$$\bar{p}\bar{n}^c = \frac{1}{2}\big[\psi_p^\dagger - \frac{i}{2m_N}\boldsymbol{\nabla}\psi_p^\dagger\boldsymbol{\sigma}\big]\big[\psi_n - \frac{i}{2m_N}\boldsymbol{\sigma}\boldsymbol{\nabla}\psi_n\big] + \mathcal{O}(\boldsymbol{\nabla}^2) = \frac{1}{2}\psi_p^\dagger\psi_n - \frac{i}{4m_N}\boldsymbol{\nabla}(\psi_p^\dagger\boldsymbol{\sigma}\psi_n) + \mathcal{O}(\boldsymbol{\nabla}^2). \tag{3.24}$$

For the vector and tensor contractions we can borrow the results from the previous subsection All in all, the relevant bi-linear contractions occurring in Eq. (3.20) are expressed in terms of the non-relativistic nucleon fields as

$$\begin{aligned}
p^c n + \bar{p}\bar{n}^c =& \psi_p^\dagger\psi_n + \mathcal{O}(\boldsymbol{\nabla}^2), \\
p^c n - \bar{p}\bar{n}^c =& \mathcal{O}(\boldsymbol{\nabla}), \\
\bar{p}\bar{\sigma}^0 n + p^c\sigma^0\bar{n}^c =& \psi_p^\dagger\psi_n + \mathcal{O}(\boldsymbol{\nabla}^2), \\
\bar{p}\bar{\sigma}^0 n - p^c\sigma^0\bar{n}^c =& \mathcal{O}(\boldsymbol{\nabla}), \\
\bar{p}\bar{\sigma}^k n + p^c\sigma^k\bar{n}^c =& \mathcal{O}(\boldsymbol{\nabla}), \\
\bar{p}\bar{\sigma}^k n - p^c\sigma^k\bar{n}^c =& -\psi_p^\dagger\sigma^k\psi_n + \mathcal{O}(\boldsymbol{\nabla}), \\
p^c\sigma^{0k} n =& -\frac{i}{2}\psi_p^\dagger\sigma^k\psi_n + \mathcal{O}(\boldsymbol{\nabla}), \\
p^c\sigma^{kl} n =& \frac{\epsilon^{klm}}{2}\psi_p^\dagger\sigma^m\psi_n + \mathcal{O}(\boldsymbol{\nabla}).
\end{aligned} \tag{3.25}$$

Plugging this back into Eq. (3.20) we obtain

$$\mathcal{L}_{\rm NR}^{(0)} = -(\psi_p^\dagger\psi_n)\big[C_V^+(\bar{e}\bar{\sigma}^0\nu) + C_S^+(e^c\nu)\big] + (\psi_p^\dagger\sigma^k\psi_n)\big[C_A^+(\bar{e}\bar{\sigma}^k\nu) + C_T^+(e^c\sigma^0\bar{\sigma}^k\nu)\big] + \mathcal{O}(\boldsymbol{\nabla}) + \text{h.c.} \tag{3.26}$$

It is notable that in the non-relativistic limit five hadronic structure in Eq. (3.20) reduce to just two, which mediate the so-called Fermi and Gamow-Teller transitions, respectively. In the SM limit $C_S^+ = C_T^+ = 0$. As promised, the more general case is only slightly more complicated than the SM limit. The pseudo-scalar interaction drop out at the leading order in the non-relativistic limit (enhancement due to large $g_P \sim 10^2$ loses against the $\boldsymbol{\nabla}/m_N \sim 10^{-3}$ suppression), and no new hadronic structures arise due to the scalar and tensor interactions. The difference between the vector and scalar interactions lies only in the chirality of the beta particle involved; idem for axial vs tensor. This elucidates the reasons for the confusion in the early days after the Lee-Yang paper: for some time the dominant lore was that beta decay is mediated by scalar and tensor interactions (rather than by the vector and axial ones, as predicted by the SM).

The Lagrangian in Eq. (3.26) is enough to capture the salient feature of the large class of nuclear β transitions called the *allowed* decays. These are basically defined by the matrix element of $\psi_p^\dagger\psi_n$ or $\psi_p^\dagger\sigma^k\psi_n$ being $\mathcal{O}(1)$ in the units of the mass $m_\mathcal{N}$ of the involved nuclei. For this class, the amplitude for β^- decay is given by

$$\begin{aligned}
\mathcal{M}(\mathcal{N}\to\mathcal{N}'e^-\bar{\nu}) =& -\langle\mathcal{N}'|\,\psi_p^\dagger\psi_n\,|\mathcal{N}\rangle\,\big[C_V^+(\bar{x}_e\bar{\sigma}^0 y_\nu) + C_S^+(y_e y_\nu)\big] \\
&+ \langle\mathcal{N}'|\,\psi_p^\dagger\sigma^k\psi_n\,|\mathcal{N}\rangle\,\big[C_A^+(\bar{x}_e\bar{\sigma}^k y_\nu) + C_T^+(y_e\sigma^0\bar{\sigma}^k y_\nu)\big].
\end{aligned} \tag{3.27}$$

The matrix elements appearing here are non-perturbative. They should be understood as matrices in the space of spin J and polarization $s \in (-J, J)$ of the mother and daughter nuclei. The first one is called the Fermi matrix element. At the order we are working we need to know it only in the zero-recoil limit $\boldsymbol{q} \equiv \boldsymbol{p}_{\mathcal{N}} - \boldsymbol{k}_{\mathcal{N}'} \to 0$. We also assume that $\mathcal{N}$ and $\mathcal{N}'$ have the same parity. Then, in the rest frame of the mother nucleus, rotational invariance allows for only a single structure - the identity matrix - to appear on the right-hand side:

$$\langle \mathcal{N}', J', s' | \, \psi_p^\dagger \psi_n \, | \mathcal{N}, J, s \rangle = 2 m_{\mathcal{N}} M_F \delta_{JJ'} \delta_{ss'}. \tag{3.28}$$

The nuclear mass was factored out for dimensional reasons, and the proportionality constant M_F can be calculated in the limit of unbroken isospin as $M_F = \sqrt{j(j+1) - j_z(j_z+1)}$, where j is the isospin quantum number of $\mathcal{N}$ and $\mathcal{N}'$, and j_3 is the projection of the isospin on the z axis for the mother nucleus. For example, for neutron decay the isospin multiple is (p, n), thus $j = 1/2$, $j_z = -1/2$. One can see that the Fermi matrix element is operative only when the spins of the mother and daughter nuclei are equal.

For the other matrix element in Eq. (3.27), rotational covariance implies the right-hand side in the zero-recoil limit must transform as a 3-vector. In the rest frame of the mother nucleus, the only available object is the generator of rotations in the polarization space

$$\langle \mathcal{N}', J', s' | \, \psi_p^\dagger \sigma^k \psi_n \, | \mathcal{N}, J, s \rangle = 2 m_{\mathcal{N}} M_{\mathrm{GT}} [\mathcal{T}^k_{J',J}]_{ss'}. \tag{3.29}$$

The generators can be expressed in terms of the Clebsch-Gordan coefficients

$$[\mathcal{T}^1_{J',J}]_{ss'} \equiv -\sqrt{\frac{1}{2}} \Big[C(J, s; 1, 1; J', s') - C(J, s; 1, -1; J', s') \Big], \tag{3.30}$$

$$[\mathcal{T}^2_{J',J}]_{ss'} \equiv \frac{i}{2} \Big[C(J, s; 1, 1; J', s') + C(J, s; 1, -1; J', s') \Big], \tag{3.31}$$

$$[\mathcal{T}^3_{J',J}]_{ss'} \equiv C(J, s; 1, 0; J', s'). \tag{3.32}$$

The precise form of the right-hand sides is less important for our purpose. What I wanted to demonstrate by displaying this formula is that, due to the triangle rule for the Clebsch-Gordan coefficients, $\mathcal{T}^k_{(J',J)} = 0$ for $|J - J'| > 1$. This explains the phenomenological selection rules for allowed beta decays: they are possible only if the mother and daughter nuclei have the same parity and this spins differing by zero or one. If these conditions are not satisfied, β decay must proceed through higher-order operators in the $\boldsymbol{\nabla}/m_N$ expansion of the Lagrangian and/or $\boldsymbol{q}/m_{\mathcal{N}}$ corrections to the Fermi or Gamow-Teller matrix element. This leads to a large suppression of the matrix element and therefore typically much longer lifetime than for allowed decays.[24] In a curious misnomer, such transitions are called *forbidden* in the nuclear literature.

The proportionality constant M_{GT} is not calculable, except for neutron decay where $M_{\mathrm{GT}} = \sqrt{3}$ via a perturbative calculation. This inhibits our ability to predict the rate of

[24]Typical lifetimes for allowed beta decay are in the seconds to minutes regime, while for forbidden decays they range from days to longer than the age of the universe. Note however that the lifetime depends not only on the matrix element but also, very strongly, on the volume of the available phase space, which leads to a large spread of the lifetimes for transitions belonging to the same classes. For example, tritium undergoes allowed beta decay with the life-time of approximately 18 years due to $m_{^3H} - m_{^3He} \approx 0.53$ MeV, barely above the electron mass of $m_e \approx 0.51$ MeV.

many nuclear transitions, with the exception of pure Fermi ones defined by the vanishing Gamow-Teller matrix element. This is however not a fatal obstruction, as there remain many differential distribution that can be predicted in terms of the EFT parameters C_X^+, once $M_{\rm GT}$ is extracted phenomenologically (for example by measuring the lifetime). Let me give you one example demonstrating how C_X^+ can be constrained by experiment. Consider a β^+ decay process where the mother and daughter nuclei have spin zero and the same parity (there are dozens of such transitions in nature, e.g. $^{14}O \to {}^{14}Ne^+\nu$). One has $\mathcal{T}^k_{0,0} = 0$, therefore the The amplitude takes the form

$$\mathcal{M} = -2m_{\mathcal{N}} M_F \big[\bar{C}_V^+ (\bar{x}_\nu \bar{\sigma}^0 y_e) + \bar{C}_S^+ (\bar{x}_\nu \bar{x}_e)\big], \tag{3.33}$$

where, as discussed previously M_F can be calculate given the isospin quantum numbers of the involved nuclei. From this point the differential decay width can be calculated using the standard QFT techniques. Taking the square and summing over the lepton spins we have

$$\begin{aligned}
\sum |\mathcal{M}|^2 =& 4m_{\mathcal{N}}^2 M_F^2 \big[\bar{C}_V^+ (\bar{x}_\nu \bar{\sigma}^0 y_e) + \bar{C}_S^+ (\bar{x}_\nu \bar{x}_e)\big]\big[C_V^+ (\bar{y}_e \bar{\sigma}^0 x_\nu) + C_S^+ (x_e x_\nu)\big] \\
=& 4m_{\mathcal{N}}^2 M_F^2 \mathrm{Tr}\Big\{ |C_V^+|^2 [k_\nu \sigma \bar{\sigma}^0 k_e \bar{\sigma}^0] + |C_S^+|^2 [k_\nu \sigma k_e \bar{\sigma}] - 2m_e \mathrm{Re}\,[C_V^+ \bar{C}_S^+][k_\nu \sigma \bar{\sigma}^0] \Big\} \\
=& 8m_{\mathcal{N}}^2 M_F^2 \Big\{ |C_V^+|^2 [E_\nu E_e + \boldsymbol{k_e k_\nu}] + |C_S^+|^2 [E_\nu E_e - \boldsymbol{k_e k_\nu}] - 2\mathrm{Re}\,[C_V^+ \bar{C}_S^+] m_e E_\nu \Big\}.
\end{aligned} \tag{3.34}$$

The differential decay width is given by

$$\begin{aligned}
\frac{d\Gamma}{dE_e d\Omega_e d\Omega_\nu} =& \frac{p_e E_\nu}{512 m_{\mathcal{N}}^2 \pi^5} \sum |\mathcal{M}|^2 = \frac{p_e E_e (E_e^{\max} - E_e)^2 M_F^2}{64\pi^5} \\
& \times \Big\{ |C_V^+|^2 \Big[1 + \frac{\boldsymbol{k_e k_\nu}}{E_e (E_e^{\max} - E_e)}\Big] + |C_S^+|^2 \Big[1 - \frac{\boldsymbol{k_e k_\nu}}{E_e (E_e^{\max} - E_e)}\Big] - 2\mathrm{Re}\,[C_V^+ \bar{C}_S^+] \frac{m_e}{E_e} \Big\}.
\end{aligned} \tag{3.35}$$

I approximated $E_\nu \approx E_e^{\max} - E_e$ in the zero-recoil limit, where $E_e^{\max}$ is the end-point energy, usually known from spectroscopic measurement independent of β transitions. The total width measurement constrains one combination of C_V^+ and C_S^+, namely $|C_V^+|^2 + |C_S^+|^2 - 2\langle m_e/E_e \rangle \mathrm{Re}\,[C_V^+ \bar{C}_S^+]$ We can constrain another combination either by measuring the beta energy spectrum $\frac{d\Gamma}{dE_e}$, which allows us to isolate $\mathrm{Re}\,[C_V^+ \bar{C}_S^+]$ of the so-called Fierz term, or by measuring the angular correlation between the electron and neutrino directions[25] which gives us access to the difference $|C_V^+|^2 - |C_S^+|^2$. All in all, using the information from $0^+ \to 0^+$ transitions allows us to independently constrain C_V^+ and C_S^+. Moreover, neutron lifetime and angular distributions constrain also C_A^+ and C_T^+. The bottom line is that all four parameters in the leading order Lagrangian in Eq. (3.26) can be precisely measured from experiment. The current precision is $\mathcal{O}(10^{-4})$ for C_V^+ and C_A^+, and $\mathcal{O}(10^{-3})$ for C_S^+ and C_T^+, see [58].

[25]Neutrino is not observed in precision measurements of β transitions to date, but its momentum can be inferred by measuring the momenta of the beta particle and the nuclei, and using momentum conservation.

3.4 From Klein-Gordon to Schrödinger

So far in this section I have discussed only the non-relativistic limit for the massive spin-1/2 Dirac fermions. But, obviously, the non-relativistic limit makes sense for particles of any spin. I have focused on spin 1/2 because it's more relevant for practical applications (all of the known stable massive matter particles turn out to be spin-1/2), but also because I find it be the simplest case conceptually. The reason is that passing to the non-relativistic limit consists in a specific field redefinitions in Eq. (3.6) such that the new variables satisfy Schrödinger equation. This cannot be exactly the same for bosons because in this case the equations of motion are second order (as opposed to first-order equations of motion for fermions). There cannot be any field redefinition that transforms second-order equations of motion into the Schrödinger equation, which is first order in the time derivative. Instead, the procedure for boson Φ is to define the change of variables $(\Phi, \pi) \to (\psi, \psi^c)$, where π is the canonical momentum associated with Φ.

Let us illustrate the procedure for a complex scalar field ϕ. The free Lagrangian is

$$\mathcal{L}_{\rm scalar} = \partial_\mu \phi^\dagger \partial_\mu \phi - m^2 \phi^\dagger \phi. \tag{3.36}$$

The scalar satisfies the Klein-Gordon equation $(\Box + m^2)\phi = 0$. The canonical momentum is $\pi = \frac{\partial \mathcal{L}_{\rm scalar}}{\partial(\partial_t \phi)} = \partial_t \phi^\dagger$. We can rewrite Eq. (3.36) as

$$\mathcal{L}_{\rm scalar} = \pi \partial_t \phi + \pi^\dagger \partial_t \phi^\dagger - \pi^\dagger \pi + \phi^\dagger (\boldsymbol{\nabla}^2 - m^2)\phi. \tag{3.37}$$

Treating π and ϕ as independent, the equations of motion are $\pi = \partial_t \phi^\dagger$, $-\partial_t \pi^\dagger + (\boldsymbol{\nabla}^2 - m^2)\phi = 0$, which are coupled first-order differential equations equivalent to the second-order Klein-Gordon equation. The field ϕ and the canonical momentum π can be expressed in terms of the ladder operators as

$$\begin{aligned}\phi =& \int d\Phi_k \big(a_k e^{-ikx} + b_k^\dagger e^{ikx}\big),\\ \pi =& i \int d\Phi_k E_k \big(a_k^\dagger e^{ikx} - b_k e^{-ikx}\big).\end{aligned} \tag{3.38}$$

Given $[a_{k'}, a_k^\dagger] = [b_{k'}, b_k^\dagger] = 2E_k (2\pi)^3 \delta^3(\boldsymbol{k} - \boldsymbol{k'})$, this leads to the usual equal-time commutation relations

$$\begin{aligned}[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] =& i \int d\Phi_k d\Phi_{k'} E_{k'} \big(e^{-i(k_0 - k_0')t + i\boldsymbol{kx} - i\boldsymbol{k'y}} + \text{h.c.}\big) 2\mathrm{E_k}(2\pi)^3 \delta^3(\mathrm{k} - \mathrm{k'})\\ =& i \int d\Phi_k E_k \big(e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{y})} + \text{h.c.}\big) = \mathrm{i}\delta^3(\mathrm{x} - \mathrm{y}).\end{aligned} \tag{3.39}$$

We can now define the change of variables:

$$\begin{aligned}\psi =& \sqrt{\frac{m}{2}} e^{imt} \left[\left(1 - \frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/4} \phi + \frac{i}{m} \left(1 - \frac{\boldsymbol{\nabla}^2}{m^2}\right)^{-1/4} \pi^\dagger \right],\\ \psi^c =& \sqrt{\frac{m}{2}} e^{imt} \left[\left(1 - \frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/4} \phi^\dagger + \frac{i}{m} \left(1 - \frac{\boldsymbol{\nabla}^2}{m^2}\right)^{-1/4} \pi \right].\end{aligned} \tag{3.40}$$

Once again, the motivation for this particular mixture of ϕ and π is to isolate the particles from anti-particles. Indeed

$$
\begin{aligned}
&\left(1-\frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/4}\phi+\frac{i}{m}\left(1-\frac{\boldsymbol{\nabla}^2}{m^2}\right)^{-1/4}\pi^\dagger\\
&=\int d\Phi_k\left\{a_k e^{-ikx}\left[\sqrt{\frac{E_k}{m}}+\frac{E_k}{m}\sqrt{\frac{m}{E_k}}\right]+b_k^\dagger e^{ikx}\left[\sqrt{\frac{E_k}{m}}-\frac{E_k}{m}\sqrt{\frac{m}{E_k}}\right]\right\}=2\int d\Phi_k\sqrt{\frac{E_k}{m}}a_k e^{-ikx}\\
&\left(1-\frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/4}\phi^\dagger+\frac{i}{m}\left(1-\frac{\boldsymbol{\nabla}^2}{m^2}\right)^{-1/4}\pi\\
&=\int d\Phi_k\left\{a_k^\dagger e^{ikx}\left[\sqrt{\frac{E_k}{m}}-\frac{E_k}{m}\sqrt{\frac{m}{E_k}}\right]+b_k e^{-ikx}\left[\sqrt{\frac{E_k}{m}}+\frac{E_k}{m}\sqrt{\frac{m}{E_k}}\right]\right\}=2\int d\Phi_k\sqrt{\frac{E_k}{m}}b_k e^{-ikx}.
\end{aligned}
\tag{3.41}
$$

Thus, the non-relativistic fields are expressed in terms of the ladder operators as

$$
\begin{aligned}
\psi&=\int\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}a_k e^{i(m-\sqrt{m^2+\boldsymbol{k}^2})t}e^{i\boldsymbol{kx}},\\
\psi^c&=\int\frac{d^3k}{(2\pi)^3\sqrt{2E_k}}b_k e^{i(m-\sqrt{m^2+\boldsymbol{k}^2})t}e^{-i\boldsymbol{kx}}.
\end{aligned}
\tag{3.42}
$$

The normalizations in Eq. (3.40) are chosen to ensure $[\psi(t,\boldsymbol{x}),\psi^\dagger(t,\boldsymbol{y})]=[\psi^c(t,\boldsymbol{x}),\psi^{c\,\dagger}(t,\boldsymbol{y})]\delta^3(\boldsymbol{x}-\boldsymbol{y})$. In other words, $i\psi^\dagger$ behaves like the canonical momentum, which means that the kinetic terms constain $\mathcal{L}\supset i\psi^\dagger\partial_t\psi$ as in Eq. (3.10).

We can easily invert Eq. (3.40):

$$
\begin{aligned}
\phi=&\frac{1}{\sqrt{2m}}\left(1-\frac{\boldsymbol{\nabla}^2}{m^2}\right)^{-1/4}\left[e^{-imt}\psi+e^{imt}\psi^{c\dagger}\right],\\
\pi=&i\sqrt{\frac{m}{2}}\left(1-\frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/4}\left[e^{imt}\psi^\dagger-e^{-imt}\psi^c\right].
\end{aligned}
\tag{3.43}
$$

Now

$$
\begin{aligned}
\pi^\dagger\pi=&\frac{m}{2}\left[e^{imt}\psi^\dagger-e^{-imt}\psi^c\right]\left(1-\frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/2}\left[e^{-imt}\psi-e^{imt}\psi^{c\dagger}\right]\\
\pi\partial_t\phi=&\frac{1}{2}\left[e^{imt}\psi^\dagger-e^{-imt}\psi^c\right]\left[e^{-imt}(i\partial_t\psi+m\psi)+e^{imt}(i\partial_t\psi^{c\dagger}-m\psi^{c\dagger})\right],\\
\phi^\dagger(\boldsymbol{\nabla}^2-m^2)\phi=&-\frac{m}{2}\left[e^{-imt}\psi+e^{imt}\psi^{c\dagger}\right]\left(1-\frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/2}\left[e^{-imt}\psi+e^{imt}\psi^{c\dagger}\right],.
\end{aligned}
\tag{3.44}
$$

Plugging that back into Eq. (3.37),

$$
\begin{aligned}
\mathcal{L}_{\text{scalar}}=&\frac{i}{2}\left\{\psi^\dagger\partial_t\psi-\psi\partial_t\psi^\dagger+\psi^{c\dagger}\partial_t\psi^c-\psi^c\partial_t\psi^{c\dagger}+e^{2imt}\partial_t\left(\psi^\dagger\psi^{c\dagger}\right)-e^{-2imt}\partial_t\left(\psi^c\psi\right)\right\}\\
&+m\left\{\psi^\dagger\psi+\psi^{c\dagger}\psi^c-e^{2imt}\psi^\dagger\psi^{c\dagger}-e^{-2imt}\psi\psi^c\right\}-m\left\{\psi^\dagger\left(1-\frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/2}\psi+\psi^c\left(1-\frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/2}\psi^{c\dagger}\right\}.
\end{aligned}
\tag{3.45}
$$

Finally, integrating by parts

$$\mathcal{L}_{\rm scalar} = i\psi^\dagger \partial_t \psi + \psi^\dagger \left[m - \left(1 - \frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/2} \right] \psi + i\psi^{c\dagger} \partial_t \psi^c + \psi^{c\dagger} \left[m - \left(1 - \frac{\boldsymbol{\nabla}^2}{m^2}\right)^{1/2} \right] \psi^c. \tag{3.46}$$

We have achieved two things. One is that the particle and anti-particle degrees of freedom separate to all orders in $\boldsymbol{\nabla}/m$. The other is that both the particle and anti-particle satisfy the relativistic generalization of the Schrödinger equation, $i\partial_t \psi = [(1 - \frac{\boldsymbol{\nabla}^2}{m^2})^{1/2} - m]\psi$. Expanding to the second order in $\boldsymbol{\nabla}/m$ one obtains the same Lagrangian as in Eq. (3.10), leading to the bona fide Schrödinger equation.

Exercise: Derive the transformation to non-relativistic variables for a massive spin-1 vector.

4 EFT approach to neutrino oscillations

Neutrino oscillations are a specific class of precision experiments. What is being observed is the rate with which neutrinos emitted by a source at a macroscopic distance interact with particles in the detector. Quite often, the energy spectrum and flavor composition of the incoming neutrinos are also reconstructed. The distinguishing feature is the oscillatory dependence of the neutrino detection rate as a function of the neutrino energy E_ν and the distance L between the neutrino source and detector.

Collider physics and cosmology have established that there are at least three distinct neutrino states. Neutrino oscillations have in addition demonstrated that these states have at least three distinct masses.[26] The large body of experimental data so far is consistent with the predictions of the SM supplemented with dimension-5 interactions leading to Majorana masses for the SM neutrinos. In other words, neutrino masses and oscillations are a generic prediction of the SMEFT framework. This should not be underappreciated, especially given that successful predictions have been rare after the theoretical completion of the SM.

In this lecture I will assume that the SMEFT paradigm indeed holds. This implies the existence of *exactly* three neutrinos and their anti-particles, much as in the SM. Sterile neutrino states will be absent from this discussion, though they can be easily included and analyzed within the same formalism. Dimension-5 SMEFT operators lead to Majorana masses, which result in mixing between different flavors, similarly as in the quark sector but with some subtle differences. Furthermore, dimension-6 and higher operators may also affect the phenomenology of neutrino production, scattering, and propagation. This last, less explored point will be the main focus of this lecture.

4.1 Neutrino rates in QFT

More often than not, discussion of neutrinos oscillations in the literature and talks takes you back to school, with quantum mechanics, hamiltonians, etc. I find it somewhat confusing.

[26] If you read this a decade from now, the overall scale of neutrino masses have likely been determined from cosmological observations.

Neutrinos are part of the SMEFT, which is a QFT, and neutrino oscillation can be discussed in a similar language as any other particle process. This is what we will do.

Consider neutrinos produced in the process $S \to X_\alpha \nu$, and detected via the charge-current process $\nu T \to Y_\beta$. Here, S and T are both one-body particle states localized away from each other in the coordinate space, describing the neutrino source (e.g. a pion or a neutron) and target (e.g. a proton or a nucleus in a detector). X_α and Y_β are some n_x- and n_y-body states, with $n_{x,y} \geq 1$. The indices α and β indicate that these states contain charged lepton ℓ_α and ℓ_β respectively, but otherwise their precise identity is irrelevant for this discussion. The charged leptons are here to provide a handle on the neutrino flavor: if $\alpha \neq \beta$ we can say that the neutrino has oscillated.

The goal is to calculate the rate of neutrino detection at the target. Thing become more transparent if we treat the production and detection as a single process:

$$ST \to X_\alpha Y_\beta. \tag{4.1}$$

In this approach, neutrino is merely an intermediate particle and never shows up in asymptotic states. The initial states S and T are represented by wave packets:

$$|ST\rangle = \int d\Phi(p_S) d\Phi(p_T) f_S(p_S) f_T(p_T) e^{ibp_T} |p_S p_T\rangle_{\rm in}. \tag{4.2}$$

Here, $|p_j\rangle_{\rm in}$ are the usual plane wave momentum eigenstates you are accustomed to. They are normalized as ${}_{\rm in}\langle q_j | p_j \rangle_{\rm in} = (2\pi)^3 2E_j \delta^3(\mathbf{p}_j - \mathbf{q}_j)$. The functions $f_j(p_j)$ are the wave packets describing ensuring localization of the source and target particles in a specific region of space. The spatial separation between the source and the target is assured by the e^{ibp_T} factor, with b^μ space-like, $b^2 < 0$. The momentum spread of the wave packets is denoted as $1/\sigma$, which by Heisenberg translates to a $\sim \sigma$ spread in the coordinate space. We assume $\sigma \gg 1/E_{S,T}$. The phase space element is as usual defined as $d\Phi(p_j) \equiv \frac{d^4 p_j}{(2\pi)^3} \delta^+(p_j^2 - m_j^2) = \frac{d^3 p_j}{(2\pi)^3 2E_j}$ for $j = S, T$. We normalize $\int d\Phi(p) |f_j(p)|^2 = 1$, which implies $\langle ST | ST \rangle = 1$. For the outgoing states we do need any wave packet gymnastics: they can be taken as the usual momentum eigenstates, ${}_{\rm out}\langle k_1 \dots k_n|$, where n is the total number of particles in the X_α and Y_β states. The number of events with the final state particles in the infinitesimal volume of the final state phase space is

$$dN_{\alpha\beta} = \left| {}_{\rm out}\langle k_1 \dots k_n| \, |ST\rangle \right|^2 \Pi_{i=1}^n d\Phi(k_i). \tag{4.3}$$

A bit more work is needed to rewrite this expression in terms of familiar scattering amplitudes, factorize the amplitude into the production and detection part, take the limit of relativistic neutrinos and large separation between the source and target, get rid of the wave packets, etc. These steps are actually similar in spirit to those applied in the the textbook derivation of the cross section formula. The interested reader is referred to Appendix A of Ref. [59] for a detailed derivation. When the smoke clears, one obtains the following formula for the differential rate $R_{\alpha\beta} \equiv \frac{dN_{\alpha\beta}}{dt}$ per source and per target particle

$$dR_{\alpha\beta} = \frac{1}{32\pi L^2 m_S m_T E_\nu} e^{-i\frac{L\Delta m_{kl}^2}{2E_\nu}} \mathcal{M}_{\alpha k}^P \bar{\mathcal{M}}_{\alpha l}^P \mathcal{M}_{\beta k}^D \bar{\mathcal{M}}_{\beta l}^D d\Pi_P d\Pi_D. \tag{4.4}$$

Here, L is the distance between the source and the target, m_S and m_T are the masses of the source and target particles, and E_ν is the neutrino energy. The amplitudes $\mathcal{M}^P_{\alpha k} \equiv \mathcal{M}(S \to X_\alpha \nu_k)$ and $\mathcal{M}^D_{\beta k} \equiv \mathcal{M}(\nu_k T \to Y_\beta)$ describe the neutrino production and detection processes, respectively. The indices $k, l = 1 \ldots 3$ label the three neutrino mass eigenstates, and they are implicitly summed over (but α, β are not!). These are related to the neutrino fields in the flavor basis ν_α via the unitary rotation

$$\nu_\alpha = \sum_{k=1}^{3} U_{\alpha k} \nu_k, \tag{4.5}$$

where U is called the $PMNS$ matrix. The mass squared differences of the eigenstates are defined as $\Delta m^2_{kl} \equiv m^2_k - m^2_l$. The phase space elements $d\Pi_P$ and $d\Pi_D$ for the production and detection processes are defined in the standard way: $d\Pi \equiv \frac{d^3 k_1}{(2\pi)^3 2E_1} \cdots \frac{d^3 k_n}{(2\pi)^3 2E_n} (2\pi)^4 \delta^4(\mathcal{P} - \sum k_i)$, where $\mathcal{P}$ is the total momentum of the initial state and k_i are the 4-momenta of the final states. The production $d\Pi_P$ includes the neutrino phase space $\frac{d^3 k_\nu}{(2\pi)^3 2E_\nu}$. Eq. (4.4) assumes the source and target are at rest, and that the source is unpolarized, so that neutrinos are emitted isotropically. More general expressions can easily be derived but we will not need them here.

Eq. (4.4) is directly related to physical observables in oscillation experiments, and contains all relevant physics. However in the neutrino community one prefers to work with the oscillation probability $P_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta)$. Let us make a connection with that language. We can define the oscillation probability by normalizing the differential rate in the presence of oscillation by neutrino flux at the source times cross section at the target. Thus

$$P_{\alpha\beta} = \frac{\int \frac{dR_{\alpha\beta}}{dE_\nu}}{\frac{d\Phi_\alpha}{dE_\nu} \sigma_\beta}, \tag{4.6}$$

where the neutrino differential flux at the source is $\frac{d\Phi_\alpha}{dE_\nu} \equiv \frac{1}{4\pi L^2} \sum_k \frac{d\Gamma(S \to X_\alpha \nu_k)}{dE_\nu}$, and $\sigma_\beta \equiv \sum_l \sigma(\nu_l T \to Y_\beta)$ is the detection cross section at the target. These can be calculated by the standard QFT techniques. All in all we get

$$P_{\alpha\beta} = \frac{\sum_{kl} e^{-i \frac{L \Delta m^2_{kl}}{2E_\nu}} \int d\Pi_{P'} \mathcal{M}^P_{\alpha k} \bar{\mathcal{M}}^P_{\alpha l} \int d\Pi_D \mathcal{M}^D_{\beta k} \bar{\mathcal{M}}^D_{\beta l}}{\int d\Pi_{P'} \sum_k |\mathcal{M}^P_{\alpha k}|^2 \int d\Pi_D \sum_l |\mathcal{M}^D_{\beta l}|^2}, \tag{4.7}$$

where $d\Pi_P \equiv d\Pi_{P'} dE_\nu$.

Exercise: Show that the oscillation probability in Eq. (4.7) is real and satisfies $0 \le P_{\alpha\beta} \le 1$.

Before proceeding, let us check that Eq. (4.7) reproduces the familiar formulas for oscillation probability in the case of the SM-like theory with massive neutrinos. Here, SM-like means that one can choose a basis where neutrino couple to matter via flavor diagonal and flavor universal V-A interactions. Rotating from this basis to the mass eigenstates as in Eq. (4.5), the interactions of the latter with matter are proportional to $U_{\alpha k}$. As a consequence, the production and detection amplitudes depends on the neutrino eigenstate index only via the PMNS matrix elements: $\mathcal{M}^P_{\alpha k} = U^*_{\alpha k} A^P_\alpha$, $\mathcal{M}^D_{\alpha k} = U_{\alpha k} A^D_\alpha$, where the reduced amplitudes are independent of k up to completely negligible corrections due to neutrino masses. Then

$$
\begin{aligned}
P^{\rm SM}_{\alpha\beta} =& \frac{\sum_{kl} e^{-i\frac{L\Delta m^2_{kl}}{2E_\nu}} U^*_{\alpha k} U_{\alpha l} \int d\Pi_{P'} |A^P_\alpha|^2 U_{\beta k} U^*_{\beta l} \int d\Pi_D |A^D_\beta|^2}{\sum_k U^*_{\alpha k} U_{\alpha k} \int d\Pi_{P'} |A^P_\alpha|^2 \sum_l U_{\beta l} U^*_{\beta l} \int d\Pi_D |A^D_\beta|^2} = \sum_{k,l} e^{-i\frac{L\Delta m^2_{kl}}{2E_\nu}} U^*_{\alpha k} U_{\alpha l} U_{\beta k} U^*_{\beta l} \\
=& \sum_{k,l} \cos\left(\frac{L\Delta m^2_{kl}}{2E_\nu}\right) U^*_{\alpha k} U_{\alpha l} U_{\beta k} U^*_{\beta l} - i \sum_{k,l} \sin\left(\frac{L\Delta m^2_{kl}}{2E_\nu}\right) U^*_{\alpha k} U_{\alpha l} U_{\beta k} U^*_{\beta l} \\
=& \delta_{\alpha\beta} - 2\sum_{k,l} \sin^2\left(\frac{L\Delta m^2_{kl}}{4E_\nu}\right) U^*_{\alpha k} U_{\alpha l} U_{\beta k} U^*_{\beta l} - i \sum_{k,l} \sin\left(\frac{L\Delta m^2_{kl}}{2E_\nu}\right) U^*_{\alpha k} U_{\alpha l} U_{\beta k} U^*_{\beta l}.
\end{aligned}
\tag{4.8}
$$

In the first step, the phase space integrals have canceled out, and the PMNS factor in the denominator are equal to 1 thanks to $U^\dagger U = 1$. In the last step we again used the unitarity of the PMNS matrix. The result indeed reduces to the well-known formula for the oscillation probability, see e.g. the review in Ref. [60]. For antineutrinos the discussion is the same up to interchanging $U \leftrightarrow U^*$.

The first two pieces are CP-conserving, that is they yield $P^{\rm SM}_{\alpha\beta} = P^{\rm SM}_{\beta\alpha}$. The last piece is CP-violating:

$$
\begin{aligned}
P^{\rm SM}_{\alpha\beta} - P^{\rm SM}_{\beta\alpha} =& - i \sum_{kl} \sin\left(\frac{L\Delta m^2_{kl}}{2E_\nu}\right) \left[U^*_{\alpha k} U_{\alpha l} U_{\beta k} U^*_{\beta l} - U^*_{\beta k} U_{\beta l} U_{\alpha k} U^*_{\alpha l}\right] \\
=& 4 \sum_{k>l} \sin\left(\frac{L\Delta m^2_{kl}}{2E_\nu}\right) \mathrm{Im}\left[U^*_{\alpha k} U_{\alpha l} U_{\beta k} U^*_{\beta l}\right],
\end{aligned}
\tag{4.9}
$$

and CP violation is proportional to the Jarlskog invariant of the PMNS matrix.

4.2 Neutrino rates in EFT

At the dimension-4 level in SMEFT (that is to say, in the SM) neutrinos are massless and there are no oscillations. Dimension-5 SMEFT operators induce neutrino masses. These also lead to neutrino mixing controlled by the PMNS matrix. The resulting oscillation probability is that in Eq. (4.8). In the following I will discuss how dimension-6 SMEFT operators modify that picture.

I will illustrate it in a concrete setting where (anti)neutrinos are produced in pion decay and detected via inverse beta decay:

$$
\pi^- p \to \mu^- \bar\nu p \to \mu^- e^+ n.
\tag{4.10}
$$

In our notation, $S = \pi^-$, $X_\alpha = \mu^-$, $T = p$, and $Y_\beta = e^+ n$. In other words $\alpha = \mu$, $\beta = e$, so in the usual parlance we are looking at muon antineutrinos oscillating into electron ones. The source pions are assumed to decay at rest, which results in a monochromatic neutrinos with energy $E_0 \approx 29.8$ MeV. I'm neglecting pion decays to electrons, which are a small correction due to chiral suppression. For kinematics reasons only electrons can be created at the detection. I am not aware of this particular experimental set-up existing in real life, however its building blocks are quite generic. In particular, stopped pions as a neutrino source (albeit π^+) are used in the celabrated COHERENT experiment [61], while inverse beta decay has historically been the most prominent neutrino detection mechanism, starting with the Poltergeist project of Cowan and Reines [62]. I've chosen this setting because the production and detection amplitudes are simple enough to calculate on blackboard, and they will allow me to illuminate some interesting points about non-SM corrections to oscillations.

To calculate the oscillation rate in Eq. (4.4) we need two ingredients: the production and the detection amplitude. To simplify the discussion, in this subsection I will be assuming that dimension-6 SMEFT operators affect production only; generalization to new physics in detection is straightforward and will be briefly discussed in the following subsection.

Let us begin with the production amplitude. It describes pion decay into a muon and a mass eigenstate of an anti-neutrino. The suitable EFT is the quark level one below the electroweak scale - the one I call WEFT. The relevant part of the Lagrangian is

$$\mathcal{L}_{\rm WEFT} \subset -\frac{2\,V_{ud}}{v^2}\Big\{(\delta^{\alpha\gamma}+\epsilon_L^{\alpha\gamma})\big(\bar u\bar\sigma^\mu d\big)\big(\bar\ell_\alpha\bar\sigma_\mu\nu_\gamma\big)+\epsilon_R^{\alpha\gamma}\big(u^c\sigma^\mu\bar d^c\big)\big(\bar\ell_\alpha\bar\sigma_\mu\nu_\gamma\big) \\ +\frac{1}{2}\epsilon_P^{\alpha\gamma}\big(u^c d-\bar u\bar d^c\big)\big(\ell^c_\alpha\nu_\gamma\big)\Big\}+\text{h.c.} \tag{4.11}$$

Leading non-SM corrections are parametrized by $\epsilon_X^{\alpha\gamma}$, which are 3×3 matrices in the lepton flavor space with a completely arbitrary flavor structure. The scalar and tensor interactions do not contribute to pion decay so they are not displayed here. After rotating to neutrino mass eigenstates as in Eq. (4.5) one gets

$$\mathcal{L}_{\rm WEFT} \subset -\frac{V_{ud}}{v^2}\Big\{(\delta^{\alpha\gamma}+\epsilon_V^{\alpha\gamma})\big(\bar u\bar\sigma^\mu d+u^c\sigma^\mu\bar d^c\big)\big(\bar\ell_\alpha\bar\sigma_\mu\nu_k\big)+(\delta^{\alpha\gamma}+\epsilon_A^{\alpha\gamma})\big(\bar u\bar\sigma^\mu d-u^c\sigma^\mu\bar d^c\big)\big(\bar\ell_\alpha\bar\sigma_\mu\nu_k\big) \\ +\epsilon_P^{\alpha\gamma}\big(u^c d-\bar u\bar d^c\big)\big(\ell^c_\alpha\nu_k\big)\Big\}U_{\gamma k}+\text{h.c.}, \tag{4.12}$$

where $\epsilon_V^{\alpha\gamma}\equiv\epsilon_L^{\alpha\gamma}+\epsilon_R^{\alpha\gamma}$, $\epsilon_A^{\alpha\gamma}\equiv\epsilon_L^{\alpha\gamma}-\epsilon_R^{\alpha\gamma}$. Given this Lagrangian, the production amplitude is[27]

$$\mathcal{M}^P_{\mu k}=\frac{V_{ud}}{v^2}\Big\{(\delta^{\mu\gamma}+\epsilon^{\mu\gamma})\big(\bar x_\mu\bar\sigma_\rho y_\nu\big)\langle 0|\,u^c\sigma^\rho\bar d^c-\bar u\bar\sigma^\rho d\,|\pi^-\rangle+\epsilon_P^{\mu\gamma}\big(y_\mu y_\nu\big)\langle 0|\,\bar u\bar d^c-u^c d\,|\pi^-\rangle\Big\}U_{\gamma k}, \tag{4.13}$$

where y_ν, x_μ, y_μ are the spinor wave functions corresponding to the outgoing anti-neutrino and charged lepton. Because we calculate pion decays from a quark-level Lagrangian, the amplitude necessarily includes a non-perturbative element: the QCD matrix element of the quark operators sandwiched between the one-pion state and the vacuum. I already used the fact that $\langle 0|\,u^c\sigma^\mu\bar d^c+\bar u\bar\sigma^\mu d\,|\pi^-\rangle=0$ due to parity invariance of QCD and the negative parity quantum number of the pion. We cannot calculate the matrix element from first

[27]Recall our conventions for the matrix elements: $\langle k|\,O\,|p\rangle_x\equiv\langle k|\,O(x)\,|p\rangle=Ae^{i(k-p)x}$ if x is displayed explicitly, and $\langle k|\,O\,|p\rangle=A$ if x is not displayed.

principles, but we can always paramtrize them so as swipe our ignorance and under the carpet. In the case at hand, for the decaying pion momentum p_π we can parametrize

$$\langle 0|\, u^c \sigma^\mu \bar{d}^c - \bar{u}\bar{\sigma}^\mu d\, |\pi^-\rangle_x = i p_\pi^\mu f_\pi e^{-i p_\pi x}, \tag{4.14}$$

where f_π is the non-perturbative parameter call the pion decay constant. The dependence on x follows from translational invariance, and k_π^μ appears because it is the only Lorentz vector available to make the Lorentz transformations the same on both sides of the equation. The other matrix element is not independent, but can be derived from the above by using the equations of motion:

$$\begin{aligned} &i\partial_\mu \langle 0|\, u^c \sigma^\mu \bar{d}^c - \bar{u}\bar{\sigma}^\mu d\, |\pi^-\rangle_x = i \langle 0|\, \partial_\mu (u^c \sigma^\mu \bar{d}^c) - \partial_\mu (\bar{u}\bar{\sigma}^\mu d)\, |\pi^-\rangle_x \\ &= \langle 0|\, m_d (u_c d) - m_u (\bar{u}\bar{d}^c) - m_d (\bar{u}\bar{d}^c) + m_u (u^c d)\, |\pi^-\rangle_x = (m_d + m_u) \langle 0|\, u_c d(x) - \bar{u}\bar{d}^c(x)\, |\pi^-\rangle_x . \end{aligned} \tag{4.15}$$

On the other hand

$$i\partial_\mu \langle 0|\, u^c \sigma^\mu \bar{d}^c - \bar{u}\bar{\sigma}^\mu d\, |\pi^-\rangle_x = i p_\pi^2 f_\pi e^{-i p_\pi x} = i m_\pi^2 f_\pi e^{-i p_\pi x}. \tag{4.16}$$

Putting this together

$$\langle 0|\, \bar{u}\bar{d}^c - u_c d\, |\pi^-\rangle_x = -\, i \frac{m_\pi^2}{m_u + m_d} f_\pi e^{-i p_\pi x}, \tag{4.17}$$

and the production amplitude simplifies to

$$\begin{aligned} \mathcal{M}^P_{\mu k} =& i\frac{V_{ud} f_\pi}{v^2} \Big\{ (\delta^{\mu\gamma} + \epsilon_A^{\mu\gamma}) (\bar{x}_\mu \bar{\sigma}_\rho y_\nu) p_\pi^\rho + \epsilon_P^{\mu\gamma} (y_\mu y_\nu) \frac{m_\pi^2}{m_u + m_d} \Big\} U_{\gamma k} \\ =& i\frac{V_{ud} f_\pi m_\mu}{v^2} (y_\mu y_\nu) \big[U_{\mu k} + \epsilon_{\mu k} \big]. \end{aligned} \tag{4.18}$$

In the last step I traded $p_\pi = k_e + k_\nu$ and used the equations of motions for the spinor wave functions x and y. I also introduce the short-hand notation $\epsilon_{\mu k} \equiv \epsilon_A^{\mu\gamma} U_{\gamma k} - \frac{m_\pi^2}{m_\mu (m_u + m_d)} \epsilon_P^{\mu\gamma} U_{\gamma k}$. The combination ϵ_k encodes all non-SM corrections to oscillations entering through the production process.

The production amplitude that enters the rate in Eq. (4.4) as $\int d\Pi_{P'} \mathcal{M}^P_{\mu k} \bar{\mathcal{M}}^P_{\mu l}$. The phase space in the case at hand is simple two-body:

$$\begin{aligned} d\Pi_P =& \frac{d^4 k_\nu}{(2\pi)^3} \delta^+(k_\nu^2) \frac{d^4 k_\mu}{(2\pi)^3} \delta^+(k_\mu^2 - m_\mu^2) (2\pi)^4 \delta^4(k_\nu + k_\mu - p_\pi) \\ =& \frac{d^4 k_\nu}{4\pi^2} \delta^+(k_\nu^2) \delta^+((p_\pi - k_\nu)^2 - m_\mu^2) = \frac{dE_\nu p_\nu^2 dp_\nu d\Omega_\nu}{4\pi^2} \delta^+(E_\nu^2 - p_\nu^2) \delta^+(m_\pi^2 - m_\mu^2 - 2 p_\pi k_\nu) \\ =& \frac{E_\nu dE_\nu d\Omega_\nu}{8\pi^2} \delta^+(m_\pi^2 - m_\mu^2 - 2 p_\pi k_\nu). \end{aligned} \tag{4.19}$$

For pions at rest $p_\pi = (m_\pi, \mathbf{0})$, thus

$$d\Pi_{P'} = \frac{E_\nu}{16\pi^2 m_\pi} \delta(E_\nu - E_0) d\Omega_\nu, \qquad E_0 \equiv \frac{m_\pi}{2} \left(1 - \frac{m_\mu^2}{m_\pi^2} \right). \tag{4.20}$$

It follows that

$$\int d\Pi_{P'} \mathcal{M}^P_{\mu k} \bar{\mathcal{M}}^P_{\mu l} = N_P \big[U_{\mu k} + \epsilon_{\mu k}\big] \big[U^*_{\mu l} + \epsilon^*_{\mu l}\big], \tag{4.21}$$

where the overall normalization is

$$\begin{aligned} N_P =& \frac{V_{ud}^2 f_\pi^2 m_\mu^2 E_\nu}{16\pi^2 m_\pi v^4} \delta(E_\nu - E_0) \int d\Omega_\mu \sum_{\text{spin}} \big|y_\mu y_\nu\big|^2 = \frac{V_{ud}^2 f_\pi^2 m_\mu^2 E_\nu}{8\pi^2 m_\pi v^4} \delta(E_\nu - E_0) \int d\Omega_\nu k_\mu k_\nu \\ =& \frac{V_{ud}^2 f_\pi^2 m_\mu^2 E_\nu}{8\pi^2 m_\pi v^4} \delta(E_\nu - E_0) \int d\Omega_\nu \big[(m_\pi - E_\nu) E_\nu - \boldsymbol{k}_\mu \boldsymbol{k}_\nu\big] \\ =& \frac{V_{ud}^2 f_\pi^2 m_\mu^2 E_\nu^2 (m_\pi - E_\nu)}{4\pi m_\pi v^4} \delta(E_\nu - E_0). \end{aligned} \tag{4.22}$$

Note that N_P is proportional to the muon mass squared; for decays to electrons the calculation is analogous with $\mu \leftrightarrow e$, such that the rate will be suppressed by the relative $m_e^2/m_\mu^2 \sim 10^{-4}$. Furthermore, note that $N_P \sim \delta(E_\nu - E_0)$, reflecting the monochromaticity of neutrinos from pion decay.

The other ingredient to calculate the oscillation rate is the detection amplitude for the $2 \to 2$ scattering process $\nu p \to e^+ n$. Recall that our assumption for the time being is no new physics in detection. Therefore the amplitude will be calculated starting from a nucleon-level EFT approximation of the SM. Since neutrinos from pion decay at rest have low enough energy, we can use the non-relativistic Lagrangian at zero recoil. After rotating the neutrino to the mass basis we have

$$\mathcal{L}_{\text{Fermi}} \supset \frac{V_{ud}}{v^2} \bigg\{ - (\psi_n^\dagger \psi_p)(\bar{\nu}_k \bar{\sigma}^0 e) U^*_{ek} + g_A (\psi_n^\dagger \sigma^k \psi_p)(\bar{\nu}_k \bar{\sigma}^j e) U^*_{ek} \bigg\} + \text{h.c.} \tag{4.23}$$

The detection amplitude follows

$$\mathcal{M}^D_{ek} = \frac{V_{ud} m_N}{v^2} \bigg\{ - (\zeta^\dagger_{s'} \zeta_s)(\bar{y}_\nu \bar{\sigma}^0 y_e) + g_A (\zeta^\dagger_{s'} \sigma^k \zeta_s)(\bar{y}_\nu \bar{\sigma}^j y_e) \bigg\} U^*_{ek}. \tag{4.24}$$

For simplicity I'm assuming the common nucleon mass, $m_N = m_p = m_n$. The spins of the two nucleons are encoded by the spinor wave functions $\zeta_s = (1,0)$ for $s = +$ and $\zeta_s = (0,1)$ for $s' = -$. The combination appearing in the numerator of the oscillation probability is

$$\int d\Pi_D \mathcal{M}^D_{ek} \mathcal{M}^D_{el} = \frac{V_{ud}^2 m_N^2}{v^4} \int d\Pi_D \sum_{\text{spin}} \bigg\{ (\bar{y}_\nu \bar{\sigma}^0 y_e)(\bar{y}_e \bar{\sigma}^0 y_\nu) + g_A^2 (\bar{y}_\nu \bar{\sigma}^j y_e)(\bar{y}_e \bar{\sigma}^j y_\nu) \bigg\} U^*_{ek} U_{el}. \tag{4.25}$$

I already summed/averaged over the nucleon spins. Summing, over the lepton spins, we use $\sum_{\text{spin}} (\bar{y}_\nu \bar{\sigma}^0 y_e)(\bar{y}_e \bar{\sigma}^0 y_\nu) = 2(E_e E_\nu + \boldsymbol{k}_e \boldsymbol{k}_\nu)$, $\sum_{\text{spin}} (\bar{y}_\nu \bar{\sigma}^j y_e)(\bar{y}_e \bar{\sigma}^j y_\nu) = 2(3 E_e E_\nu - \boldsymbol{k}_e \boldsymbol{k}_\nu)$. The detection phase space is

$$\begin{aligned} d\Pi_D =& (2\pi)^4 \delta^4(k_\nu + p_p - k_n - k_e) \frac{d^4 k_n}{(2\pi)^3} \delta^+(k_n^2 - m_n^2) \frac{d^4 k_e}{(2\pi)^3} \delta^+(k_e^2 - m_e^2) \\ =& \frac{p_e E_e}{4\pi^2} dE_e d\Omega_e \delta^+(k_e^2 - m_e^2) \delta^+(m_e^2 + m_p^2 - m_n^2 - 2 k_e k_\nu + 2 p_p k_\nu - 2 p_p k_e) \end{aligned}$$

$$
\begin{aligned}
=&\frac{E_e}{8\pi^2}dE_e d\Omega_e \delta(m_e^2+m_p^2-m_n^2+2\sqrt{E_e^2-m_e^2}E_\nu\cos\theta_e+2m_p(E_\nu-E_e)-2E_eE_\nu)\\
=&\frac{E_e}{16\pi^2\sqrt{E_e^2-m_e^2}E_\nu}dE_e d\phi_e,
\end{aligned}\tag{4.26}
$$

where $\phi_e \in [0,2\pi]$, and the positron production angle in the target rest frame is fixed as $\cos\theta_e = \frac{m_p(E_e-E_\nu)+E_eE_\nu+(m_n^2-m_p^2-m_e^2)/2}{\sqrt{E_e^2-m_e^2}E_\nu}$. For $m_e \ll E_\nu \ll m_N$ we can neglect the electron mass and expand in $1/m_N$, in which case the phase space reduces to

$$
d\Pi_D \approx \frac{1}{16\pi^2 E_\nu}dE_e d\phi_e, \tag{4.27}
$$

with the angle fixed as $\cos\theta_e = \frac{m_N(E_e+\Delta-E_\nu)+E_eE_\nu}{E_eE_\nu}$, implying E_e constrained to the range $E_e \in [E_\nu-\Delta-\frac{2E_\nu^2}{m_N}, E_\nu-\Delta]$, where $\Delta \equiv m_n - m_p \approx 1.3$ MeV. Integrating the amplitudes over the phase space one finds

$$
\int d\Pi_D \mathcal{M}_{ek}^D \mathcal{M}_{el}^D = N_D U_{ek}^* U_{el}, \tag{4.28}
$$

where the overall normalization is $N_D \equiv \frac{V_{ud}^2 m_N E_\nu^3(1+3g_A^2)}{2\pi v^4}$.

We have all the ingredients to write down the oscillation rate. Plugging in Eqs. (4.21) and (4.28) into Eq. (4.4) we obtain

$$
\frac{dR_{\mu e}}{dE_\nu} = \kappa e^{-i\frac{L\Delta m_{kl}^2}{2E_\nu}}\left[U_{\mu k}+\epsilon_{\mu k}\right]\left[U_{\mu l}^*+\epsilon_{\mu l}^*\right]U_{ek}^*U_{el}, \tag{4.29}
$$

where we abbreviated the overall normalization of the rate

$$
\kappa \equiv \frac{N_P N_D}{32\pi L^2 m_\pi m_N E_\nu} = \frac{V_{ud}^4(1+3g_A^2)f_\pi^2 m_\mu^2 E_\nu^4(m_\pi-E_\nu)}{256\pi^3 L^2 m_\pi^2 v^8}\delta(E_\nu-E_0). \tag{4.30}
$$

Let us expand the rate in powers of new physics contributions: $R_{\mu e} = R_{\mu e}^{(0)} + R_{\mu e}^{(1)} + R_{\mu e}^{(2)}$. The SM-like piece is

$$
\frac{dR_{\mu e}^{(0)}}{dE_\nu} = \kappa e^{-i\frac{L\Delta m_{kl}^2}{2E_\nu}} U_{\mu k}U_{\mu l}^* U_{ek}^* U_{el}. \tag{4.31}
$$

It may be worth visualizing this expression in the 2-flavor approximation. This is often a good approximation in neutrino physics, in particular in the case when $L\Delta m_{21}^2/E_\nu \ll 1$, but $L\Delta m_{31}^2/E_\nu \gtrsim 1$. In this approximation the PMNS matrix is orthogonal, $U = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$ (the phase can be redefined away for two flavors), and then

$$
\begin{aligned}
\frac{dR_{\mu e}^{(0)}}{dE_\nu} =& \kappa\left[U_{\mu 1}U_{\mu 1}U_{e1}U_{e1}+U_{\mu 2}U_{\mu 2}U_{e2}U_{e2}+e^{i\frac{L\Delta m^2}{2E_\nu}}U_{\mu 1}U_{\mu 2}U_{e1}U_{e2}+e^{-i\frac{L\Delta m^2}{2E_\nu}}U_{\mu 2}U_{\mu 1}U_{e2}U_{e1}\right]\\
=&2\kappa\sin^2\theta\cos^2\theta\left[1-\cos\left(\frac{L\Delta m^2}{2E_\nu}\right)\right] = \kappa\sin^2(2\theta)\sin^2(L/L_{\rm osc}),
\end{aligned}\tag{4.32}
$$

where the oscillation length is $L_{\rm osc} \equiv \frac{4E_\nu}{\Delta m^2}$. The rate is manifestly positive, vanishes in the absence of mixing ($\theta = 0$), and vanishes at short-distances, $L \ll L_0$.

Let us move to corrections linear in EFT Wilson coefficients. The former is defined as

$$\frac{dR^{(1)}_{\mu e}}{dE_\nu} = \kappa e^{-i\frac{L\Delta m^2_{kl}}{2E_\nu}} \big[U_{\mu k}\epsilon^*_{\mu l} + U^*_{\mu l}\epsilon_{\mu k}\big] U^*_{ek} U_{el}. \tag{4.33}$$

In the two-flavor approximation

$$\begin{aligned}\frac{dR^{(1)}_{\mu e}}{dE_\nu} =& \kappa \bigg[2\mathrm{Re}\,\epsilon_{\mu 1} U_{\mu 1} U_{e1} U_{e1} + 2\mathrm{Re}\,\epsilon_{\mu 2} U_{\mu 2} U_{e2} U_{e2} \\ & + e^{i\frac{L\Delta m^2}{2E_\nu}} (U_{\mu 1}\epsilon^*_{\mu 2} + U_{\mu 2}\epsilon_{\mu 1}) U_{e1} U_{e2} + e^{-i\frac{L\Delta m^2}{2E_\nu}} (U_{\mu 2}\epsilon^*_{\mu 1} + U_{\mu 1}\epsilon_{\mu 2}) U_{e2} U_{e1} \bigg] \\ =& \frac{\kappa}{2}\sin(2\theta) \bigg[2\cos\theta \mathrm{Re}\,\epsilon_{\mu 1} + 2\sin\theta \mathrm{Re}\,\epsilon_{\mu 2} - \Big(e^{i\frac{L\Delta m^2}{2E_\nu}} (\sin\theta\epsilon^*_{\mu 2} + \cos\theta\epsilon_{\mu 1}) + \text{h.c.} \Big) \bigg] \\ =& \kappa \sin(2\theta) \bigg[2\sin^2(L/L_{\rm osc}) \mathrm{Re}\,\big[\cos\theta\epsilon_{\mu 1} + \sin\theta\epsilon_{\mu 2}\big] + \sin(2L/L_{\rm osc}) \mathrm{Im}\,\big[\cos\theta\epsilon_{\mu 1} - \sin\theta\epsilon_{\mu 2}\big] \bigg].\end{aligned} \tag{4.34}$$

The first term has the same form as the SM-like oscillations in Eq. (4.32). In fact, the two are indistinguishable because the non-SM effect can be absorbed into a redefinition of the θ angle: $\theta \to \theta_\epsilon \equiv \theta + \frac{\mathrm{Re}\,[\cos\theta\epsilon_{\mu 1} + \sin\theta\epsilon_{\mu 2}]}{2\cos(2\theta)}$. After this redefinition, the rate contains $\frac{dR_{\mu e}}{dE_\nu} \supset [1 + \mathcal{O}(\epsilon^2)]\kappa \sin^2(L/L_{\rm osc}) \sin^2(2\theta_\epsilon)$ Note that experimentalists trying to measure θ in our setup would be fooled, as they really have access only to the polluted mixing angle θ_ϵ. The best they can do is to perform several measurements of this kind in different experimental setups, which would allow one to measure different θ_{ϵ_i} with a different dependence on EFT parameters. If these different measurements come out compatible, it constrains non-SM effects proportional to ϵ (though still does not fix θ in a model-independent way, as there could be a conspiracy between different $\mathcal{O}(\epsilon)$ contributions). Conversely, if the measurements results in different θ_{ϵ_i} (what in the future may be called Pontecorvo anomaly), it would constitute a hint for dimension-6 contributions to oscillations.

On the other hand, the second term in the big square bracket in $R^{(1)}_{\mu e}$ is genuinely observable. It is a CP-violating effect in oscillations. Experimentalist could pinpoint it by placing the detector on rails and moving it to different distances from the source, so as to determine that the oscillatory dependence on L is distinct than for the standard CP-conserving oscillations. A signal of this kind would constitute a discovery of a new source of CP violation, in addition to the phase of the CKM matrix and independent of the phase of the PMNS matrix. To be fair, one should mention however that such a signal is not very likely, as CP violation involving the first generation is very strongly constrained by EDM constraints.

An important thing to notice about Eq. (4.34): lepton-flavor-off-diagonal new physics appears in the observable at linear level, that is to say, it interferes with the SM amplitude. This is a peculiar feature of oscillation observables. In the absence of oscillations, lepton-flavor-off-diagonal new physics would appear only at the quadratic level due to lack of interference with the lepton-flavor conserving SM amplitude.

Let us press on to look at quadratic effects in EFT Wilson coefficients. In the following we purely production quadratic effects given by

$$\frac{dR^{(2)}_{\mu e}}{dE_\nu} = \kappa e^{-i\frac{L\Delta m^2_{kl}}{2E_\nu}} \epsilon_{\mu k}\epsilon^*_{\mu l} U^*_{ek} U_{el}. \tag{4.35}$$

In the two flavor approximation

$$\begin{aligned}
\frac{dR^{(2)}_{\mu e}}{dE_\nu} =& \kappa \Big\{ \epsilon_{\mu 1}\epsilon^*_{\mu 1} U_{e1} U_{e1} + \epsilon_{\mu 2}\epsilon^*_{\mu 2} U_{e2} U_{e2} + \big(e^{i\frac{L\Delta m^2}{2E_\nu}} \epsilon_{\mu 1}\epsilon^*_{\mu 2} U_{e1} U_{e2} + \text{h.c.} \big) \Big\} \\
=& \kappa \Big\{ |\epsilon_{\mu 1}|^2 \cos^2\theta + |\epsilon_{\mu 2}|^2 \sin^2\theta - 2\sin\theta\cos\theta \big(\cos(L/L_{\rm osc}) \mathrm{Re}\,[\epsilon_{\mu 1}\epsilon^*_{\mu 2}] - \sin(L/L_{\rm osc}) \mathrm{Im}\,[\epsilon_{\mu 1}\epsilon^*_{\mu 2}] \Big\} \\
=& \kappa \Big\{ \big| \cos\theta \epsilon_{\mu 1} - \sin\theta \epsilon_{\mu 2} \big|^2 + 2\sin(2\theta)\sin^2(L/L_{\rm osc}) \mathrm{Re}\,[\epsilon_{\mu 1}\epsilon^*_{\mu 2}] - \sin(L/L_{\rm osc}) \mathrm{Im}\,[\epsilon_{\mu 1}\epsilon^*_{\mu 2}] \Big\}.
\end{aligned} \tag{4.36}$$

The last two terms are an old acquaintance at this point. The second is an $\mathcal{O}(\epsilon^2)$ correction to the standard oscillations, which again can be redefined away into θ_ϵ. The third is an $\mathcal{O}(\epsilon^2)$ correction to the CP-violating oscillations; the novel element here is that it is operative even in the absence of PMNS mixing. The first term, on the other hand, brings in a new qualitative element: it does not depend on L at all, in particular it does not vanish in the limit $L \to 0$ unlike all the other terms. In the neutrino literature such contributions sometimes go under the pythonesque name of "zero-length oscillations". From the EFT point of view its appearance is by no means mysterious. Indeed, unwrapping the combination of $\epsilon_{\mu k}$ surviving at $L \to 0$ into the original Wilson coefficients in Eq. (4.11) one finds

$$\cos\theta \epsilon_{\mu 1} - \sin\theta \epsilon_{\mu 2} = \epsilon_A^{\mu e} - \frac{m_\pi^2}{m_\mu(m_u + m_d)} \epsilon_P^{\mu e}. \tag{4.37}$$

That is to say, the zero-length piece describes the source pion decaying into a muon and an *electron neutrino* due to flavor-non-diagonal interactions in the Lagrangians. The corresponding amplitude does not interfere with the SM one due to the distinct final states, therefore the effect appears only at the quadratic order in EFT Wilson coefficients.

4.3 Generalizations

Moving source

Very often neutrino experiments deal with sources moving in the rest frame of the detector: Let us modify our previous example to allow for pions moving toward the target. The amplitude is Lorentz invariant, so the only thing that changes is the production phase space. We follow the same derivation as in Eq. (4.19), except that $p_\pi = (E_\pi, 0, 0, \sqrt{E_\pi^2 - m_\pi^2})$:

$$\begin{aligned}
d\Pi_P =& \frac{E_\nu dE_\nu d\Omega_\nu}{8\pi^2} \delta^+(m_\pi^2 - m_\mu^2 - 2p_\pi k_\nu) \\
=& \frac{E_\nu dE_\nu d\Omega_\nu}{8\pi^2} \delta^+(m_\pi^2 - m_\mu^2 - 2E_\pi E_\nu + 2\sqrt{E_\pi^2 - m_\pi^2} E_\nu \cos\theta),
\end{aligned} \tag{4.38}$$

where θ is the azimuthal angle from the source-target axis. The Dirac delta has solution with $-1 \leq \cos\theta \leq 1$ for $E_\nu^\pi \leq E_\nu \leq E_\nu^0$, where $E_\nu(\theta) \equiv \frac{m_\pi^2 - m_\mu^2}{2(E_\pi - \cos\theta\sqrt{E_\pi^2 - m_\pi^2})}$. This collapses to $E_\nu = E_0$ for pions at rest, $E_\pi = m_\pi$, but as soon as $E_\pi > m_\pi$ the neutrinos emitted in pion decay display a continuous spectrum between the two endpoints. Different neutrino energies correspond to a different emission angle in the lab frame. From the experimental point of view, what is relevant is not the neutrino spectrum in the space surrounding the source, but rather the spectrum of neutrinos reaching the detector. The latter is $E_\nu^{\theta_0} < E_\nu \leq E_\nu^0$ where the lower cutoff corresponds to neutrinos just barely missing the detector, and θ_0 depends on the size of the detector D and the distance L as $\theta_0 \approx D/2L$.

In real life, more often that not you will have pion sources moving with different energy, according to some spectrum $\phi(E_S)$. This can be taken into accoun by modifying Eq. (4.4) as

$$\frac{dR_{\alpha\beta}}{dE_\nu} = \frac{1}{32\pi L^2 m_T E_\nu} \int dE_S \frac{\phi(E_S)\beta(E_S)}{E_S} e^{-i\frac{L\Delta m_{kl}^2}{2E_\nu}} \int d\Pi_{P'} \mathcal{M}_{\alpha k}^P \bar{\mathcal{M}}_{\alpha l}^P \int d\Pi_D \mathcal{M}_{\beta k}^D \bar{\mathcal{M}}_{\beta l}^D, \tag{4.39}$$

where $\beta(E_S)$ is the acceptance factor correcting for the non-isotropric emission of neutrinos. This formula is relevant for example for the FASER experiment at CERN, where neutrinos originate from highly boosted pions (and many other hadrons) produced in the forward limit of high-energy proton-proton collisions.

New physics in detection

In the previous subsection we assumed for simplicity that new physics only appears in the neutrino production amplitude while the detection amplitude is the one predicted by the SM. As we will see, qualitatively little changes when this assumption is lifted. I will now calculate the detection amplitude using a more general nucleon level EFT - basically the Lee-Yang Lagrangian but with a more generalized lepton flavor structure. After rotating the neutrino to the mass basis it reads

$$\mathcal{L}_{\rm LY} \supset -(\psi_n^\dagger \psi_p)\Big[\bar{C}_V^\gamma \bar{\nu}_k \bar{\sigma}^0 e + \bar{C}_S^\gamma \bar{\nu}_k \bar{e}^c\Big] U_{\gamma k}^* + (\psi_n^\dagger \sigma^j \psi_p)\Big[\bar{C}_A^\gamma \bar{\nu}_k \bar{\sigma}^j e + \bar{C}_T^\gamma \bar{\nu}_k \bar{\sigma}^j \sigma^0 \bar{e}^c\Big] U_{\gamma k}^* + \text{h.c.} \tag{4.40}$$

The detection amplitude follows

$$\mathcal{M}_{ek}^D = m_N \Big\{ -(\zeta_{s'}^\dagger \zeta_s)\Big[\bar{C}_V^\gamma \bar{y}_\nu \bar{\sigma}^0 y_e + \bar{C}_S^\gamma \bar{y}_\nu \bar{x}_e\Big] + (\zeta_{s'}^\dagger \sigma^j \zeta_s)\Big[\bar{C}_A^\gamma \bar{y}_\nu \bar{\sigma}^j y_e + \bar{C}_T^\gamma \bar{y}_\nu \bar{\sigma}^j \sigma^0 \bar{x}_e\Big]\Big\} U_{\gamma k}^*. \tag{4.41}$$

For simplicity I'm assuming the common nucleon mass, $m_N = m_p = m_n$. The spins of the two nucleons are encoded by the spinor wave functions $\zeta_s = (1, 0)$ for $s = +$ and $\zeta_s = (0, 1)$ for $s' = -$. The combination appearing in the numerator of the oscillation probability is

$$\int d\Pi_D \mathcal{M}_{ek}^D \mathcal{M}_{el}^D = m_N^2 \int d\Pi_D \sum_{\rm spin} \Big\{ [\bar{C}_V^\gamma \bar{y}_\nu \bar{\sigma}^0 y_e + \bar{C}_S^\gamma \bar{y}_\nu \bar{x}_e] [C_V^{\gamma'} \bar{y}_e \bar{\sigma}^0 y_\nu + C_S^{\gamma'} x_e y_\nu]$$

$$+\left[\bar{C}_A^{\gamma}\bar{y}_\nu\bar{\sigma}^j y_e+\bar{C}_T^{\gamma}\bar{y}_\nu\bar{\sigma}^j\sigma^0\bar{x}_e\right]\left[C_A^{\gamma'}\bar{y}_e\bar{\sigma}^j y_\nu+C_T^{\gamma'}x_e\sigma^0\bar{\sigma}^j y_\nu\right]\Big\}U_{\gamma k}^* U_{\gamma' l}. \tag{4.42}$$

I already summed/averaged over the nucleon spins. For the lepton spin sums we use $\sum_{\rm spin}(\bar{y}_\nu\bar{\sigma}^0 y_e)(\bar{y}_e\bar{\sigma}^0 y_\nu)=2(E_eE_\nu+\boldsymbol{k}_e\boldsymbol{k}_\nu)$, $\sum_{\rm spin}(\bar{y}_\nu\bar{\sigma}^j y_e)(\bar{y}_e\bar{\sigma}^j y_\nu)=2(3E_eE_\nu-\boldsymbol{k}_e\boldsymbol{k}_\nu)$, $\sum_{\rm spin}(\bar{y}_\nu\bar{\sigma}^0 y_e)(x_e y_\nu)=-2m_eE_\nu$, $\sum_{\rm spin}(\bar{y}_\nu\bar{\sigma}^j y_e)(x_e\sigma^0\bar{\sigma}^j y_\nu)=-6m_eE_\nu$, $\sum_{\rm spin}\bar{y}_\nu\bar{x}_e x_e y_\nu=2(E_eE_\nu-\boldsymbol{k}_e\boldsymbol{k}_\nu)$, $\sum_{\rm spin}\bar{y}_\nu\bar{\sigma}^j\sigma^0\bar{x}_e x_e\sigma^0\bar{\sigma}^j y_\nu=2(3E_eE_\nu+\boldsymbol{k}_e\boldsymbol{k}_\nu)$. Then

$$\begin{aligned}\int d\Pi_D\mathcal{M}_{ek}^D\mathcal{M}_{el}^D =&2E_\nu m_N^2U_{\gamma k}^*U_{\gamma' l}\int d\Pi_D E_e\Big\{\\ &\bar{C}_V^{\gamma}C_V^{\gamma'}\left(1+\frac{\boldsymbol{k}_e\boldsymbol{k}_\nu}{E_eE_\nu}\right)+\bar{C}_S^{\gamma}C_S^{\gamma'}\left(1-\frac{\boldsymbol{k}_e\boldsymbol{k}_\nu}{E_eE_\nu}\right)-\frac{m_e}{E_e}\left(\bar{C}_V^{\gamma}C_S^{\gamma'}+\bar{C}_S^{\gamma}C_V^{\gamma'}\right)\\ &\bar{C}_A^{\gamma}C_A^{\gamma'}\left(3-\frac{\boldsymbol{k}_e\boldsymbol{k}_\nu}{E_eE_\nu}\right)+\bar{C}_T^{\gamma}C_T^{\gamma'}\left(3-\frac{\boldsymbol{k}_e\boldsymbol{k}_\nu}{E_eE_\nu}\right)-\frac{3m_e}{E_e}\left(\bar{C}_A^{\gamma}C_T^{\gamma'}+\bar{C}_T^{\gamma}C_A^{\gamma'}\right)\Big\}.\end{aligned} \tag{4.43}$$

For $m_e \ll E_\nu \ll m_N$, integrating over the phase space one finds

$$\begin{aligned}\int d\Pi_D\mathcal{M}_{ek}^D\mathcal{M}_{el}^D\approx\frac{m_NE_\nu^3}{2\pi}U_{\gamma k}^*U_{\gamma' l}\Big\{&\bar{C}_V^{\gamma}C_V^{\gamma'}+\bar{C}_S^{\gamma}C_S^{\gamma'}-\frac{m_e}{E_\nu}(\bar{C}_V^{\gamma}C_S^{\gamma'}+\bar{C}_S^{\gamma}C_V^{\gamma'})\\ &+3\bar{C}_A^{\gamma}C_A^{\gamma'}+3\bar{C}_T^{\gamma}C_T^{\gamma'}-\frac{3m_e}{E_\nu}(\bar{C}_A^{\gamma}C_T^{\gamma'}+\bar{C}_T^{\gamma}C_A^{\gamma'})\Big\},\end{aligned} \tag{4.44}$$

This expression is valid up to corrections of order E_ν/m_N, m_e/E_ν, and Δ/E_ν. Finally, to facilitate power counting of new physics corrections, let us rewrite $C_V^\gamma=\frac{V_{ud}}{v^2}(\delta^{\gamma e}+\delta C_V^\gamma)$, $C_A^\gamma=-\frac{g_AV_{ud}}{v^2}(\delta^{\gamma e}+\delta C_A^\gamma)$, $C_S^\gamma=\frac{g_SV_{ud}}{v^2}\delta C_S^\gamma$, $C_T^\gamma=\frac{g_TV_{ud}}{v^2}\delta C_T^\gamma$. Then

$$\begin{aligned}\int d\Pi_D\mathcal{M}_{ek}^D\mathcal{M}_{el}^D\approx\frac{N_D}{1+3g_A^2}U_{\gamma k}^*U_{\gamma' l}\Big\{&(\delta^{\gamma e}+\delta\bar{C}_V^{\gamma})(\delta^{\gamma' e}+\delta C_V^{\gamma'})+3g_A^2(\delta^{\gamma e}+\delta\bar{C}_A^{\gamma})(\delta^{\gamma' e}+\delta C_A^{\gamma'})\\ &-g_S\frac{m_e}{E_\nu}\left[(\delta^{\gamma e}+\delta\bar{C}_V^{\gamma})\delta C_S^{\gamma'}+(\delta^{\gamma' e}+\delta C_V^{\gamma'})\delta C_S^{\gamma}\right]\\ &+3g_Ag_T\frac{m_e}{E_\nu}\left[(\delta^{\gamma e}+\delta\bar{C}_A^{\gamma})\delta C_T^{\gamma'}+(\delta^{\gamma' e}+\delta C_A^{\gamma'})\delta C_T^{\gamma}\right]\\ &+g_S^2\delta\bar{C}_S^{\gamma}\delta C_S^{\gamma'}+3g_T^2\delta\bar{C}_T^{\gamma}\delta C_T^{\gamma'}\Big\},\end{aligned} \tag{4.45}$$

where the overall normalization is $N_D\equiv\frac{V_{ud}^2(1+3g_A^2)m_NE_\nu^3}{2\pi v^4}$. Putting everything together, the oscillation rate up to linear order in new physics in production and detection is given by

$$\begin{aligned}\frac{dR_{\mu e}}{dE_\nu}=&\kappa e^{-i\frac{L\Delta m_{kl}^2}{2E_\nu}}\left[U_{\mu k}U_{\mu l}^*+\epsilon_{\mu k}U_{\mu l}^*+U_{\mu k}\epsilon_{\mu l}^*\right]U_{\gamma k}^*U_{\gamma' l}\\ &\times\Big\{\delta^{\gamma e}\delta^{\gamma' e}+\frac{\delta^{\gamma e}\delta C_V^{\gamma'}+\delta\bar{C}_V^{\gamma}\delta^{\gamma' e}}{1+3g_A^2}+\frac{3g_A^2\left(\delta^{\gamma e}\delta C_A^{\gamma'}+\delta^{\gamma' e}\delta\bar{C}_A^{\gamma}\right)}{1+3g_A^2}\\ &+\frac{m_e}{E_\nu(1+3g_A^2)}\left[-g_S\delta^{\gamma e}\delta C_S^{\gamma'}-g_S\delta^{\gamma' e}\delta\bar{C}_S^{\gamma}+3g_Ag_T\delta^{\gamma e}\delta C_T^{\gamma'}+3g_Ag_T\delta^{\gamma' e}\delta\bar{C}_T^{\gamma}\right]\Big\}\end{aligned} \tag{4.46}$$

or

$$\frac{dR_{\mu e}}{dE_\nu} = \kappa e^{-i\frac{L\Delta m_{kl}^2}{2E_\nu}} \Big\{ U_{\mu k}U_{\mu l}^* U_{ek}^* U_{el} + \left[\epsilon_{\mu k}U_{\mu l}^* + U_{\mu k}\epsilon_{\mu l}^*\right] U_{ek}^* U_{el} + U_{\mu k}U_{\mu l}^* \left[U_{ek}^*\delta C_{el} + U_{el}\delta C_{ek}^*\right] \Big\} \tag{4.47}$$

where the combination controlling the new physics effects in detection is

$$\delta C_{ek} = \left\{ \frac{1}{1+3g_A^2}\delta C_V^\gamma + \frac{3g_A^2}{1+3g_A^2}\delta C_A^\gamma - \frac{m_e}{E_\nu}\frac{g_S}{1+3g_A^2}\delta C_S^\gamma + \frac{m_e}{E_\nu}\frac{3g_T g_A}{1+3g_A^2}\delta C_T^\gamma \right\} U_{\gamma k}^*. \tag{4.48}$$

New physics with different Lorentz structures enters with different weight factors, leading to relative enhancement or suppression of different contributions. This was already the case for new physics in production via pion decay, where the pseudoscalar interactions lead to chiral enhancement. The qualitatively new thing to notice here is that these weight factors may depend on neutrino energy. For the monochromatic beam in our example, this just results in an additional m_e/E_0 suppression of the scalar and tensor contributions. However for the continuous neutrino spectrum, this energy dependence may offer a new handle to discriminate between different scenarios for new physics.

Neutral current detection

Charged current neutrino detection is very common. However some relevant neutrino experiments use another set up, where neutrinos may hit particles in the detector without producing a charged lepton. Instead the neutrino gives a momentum kick to the target and flies off undetected (with or without changing to a different mass eigenstate). In such a situation our discussion has to be modified accordingly. The process we are interested in is

$$ST \to X_\alpha Y \nu_n. \tag{4.49}$$

Compared to Eq. (4.1), we do not have a flavor tag at the detection, and instead in the observables we will sum over the undetected neutrino states labeled by n. Eq. (4.4) is then modified as

$$dR_\alpha = \frac{1}{32\pi L^2 m_S m_T E_\nu} \sum_{kln} e^{-i\frac{L\Delta m_{kl}^2}{2E_\nu}} \mathcal{M}_{\alpha k}^P \bar{\mathcal{M}}_{\alpha l}^P \mathcal{M}_{kn}^D \bar{\mathcal{M}}_{ln}^D d\Pi_P d\Pi_D, \tag{4.50}$$

where $\mathcal{M}_{kn}^D \equiv \mathcal{M}(\nu_k T \to \nu_n Y)$

Neutral current detection is in particular employed in the COHERENT experiment (although in this case the baseline is short enough for oscillations to play a negligible role assuming only three neutrinos exist). One motivation for it is to take advantage of the coherent enhancement of the detection cross section. It turns out that for low enough energy of the neutrinos the scattering on different nuclei adds up coherently at the level of the detection amplitude. The formalism of Eq. (4.50) can be used in this case, and the coherent amplitude including new physics contributions can be calculated starting from the nucleon level effective Lagrangian. See [54] for details. Another advantage is that both neutrinos and anti-neutrinos are detected, even for low-energy neutrinos (whereas only anti-neutrinos are detected via inverse beta decay on protons).

5 On-shell methods for EFT

This section is devoted to on-shell techniques and their EFT applications. This is a hot topic that has seen a lot of interesting progress in recent years. On-shell techniques fit very well with the bottom-up EFT philosophy. They allow one to perform practical calculations, arguably in a simpler and more transparent fashion than the traditional techniques. The most fruitful applications so far have been in the domain of constructing bases and of RG equations for the Wilson coefficients, but this probably not the end of the story.

The plan of this lecture is the following. I will start in Section 5.1 with a very brief introduction to on-shell methods of calculating scattering amplitudes, including a summary and key formulas for helicity spinors techniques. My review is self-contained but very dense. If you have never seen helicity spinors before, it may be a good idea to first read a more extensive review before attacking the following subsections. Then in Section 5.3 I will apply these to study a relatively simple but already non-trivial EFT: the Yang-Mills theory with higher-dimensional operators. I will show how to define this theory on shell, how to organize the EFT expansion in this language, and how does the operators - amplitudes correspondence emerges. Within this EFT, first in Section 5.4 I will show to practically calculate tree level amplitudes. Then in Section 5.5 I will move to the one-loop level. I will give a prescription for calculating one loop amplitudes using unitarity. Finally I will give a sample calculation of RG running of EFT Wilson coefficients using on-shell techniques.

5.1 Lightning introduction to helicity spinors and on-shell methods

Consider a massless particle. Its momentum p^μ satisfies the on-shell condition $p^2 = 0$. From p we can construct the 2×2 matrix $p\sigma \equiv p^\mu \sigma_\mu = \begin{pmatrix} p^0 - p^3 & -p_1 + ip_2 \\ -p_1 - ip^2 & p^0 + p^3 \end{pmatrix}$, where $\sigma^\mu = (\mathbb{1}, \boldsymbol{\sigma})$ and σ^k are the usual Pauli matrices. The massless condition translates into $\det p\sigma = 0$. This means $p\sigma$ can be represented as a product of two two-component vectors:

$$[p\sigma]_{\alpha\dot\beta} = \lambda_{p\,\alpha}\tilde\lambda_{p\,\dot\beta}, \qquad \alpha, \dot\beta \in 1, 2 \tag{5.1}$$

The two vectors λ_p and $\tilde\lambda_p$ are the *helicity spinors* associated with the momentum p. For real p, the hermiticity of σ implies a relation between the two: $(\lambda_{p\,\alpha})^* = \tilde\lambda_{p\,\dot\alpha}$. The reason why I used a twiddle rather than a bar or a star is that on-shell techniques often makes excursions to complex momenta, in which case $\lambda_p = \tilde\lambda_p$ become independent. From Eq. (5.1) the dimension of helicity spinors is $[\lambda_p] = [\tilde\lambda_p] = \text{mass}^{1/2}$.

Using sigma matrix identities one can show that $p\bar\sigma$ is expressed by helicity spinors with raised indices: $[p\bar\sigma]^{\dot\alpha\beta} = \tilde\lambda_p^{\dot\alpha}\lambda_p^{\beta}$, where as usual $\lambda_p^\alpha = \epsilon^{\alpha\beta}\lambda_{p\,\beta}$. This allows us to express momentum contractions in the spinor language:

$$2p_1p_2 = \mathrm{Tr}[p_1\sigma p_2\bar\sigma] = \lambda_{1\,\alpha}\tilde\lambda_{1\,\dot\beta}\tilde\lambda_2^{\dot\beta}\lambda_2^{\alpha} = (\lambda_2\lambda_1)(\tilde\lambda_1\tilde\lambda_2) = (\lambda_1\lambda_2)(\tilde\lambda_2\tilde\lambda_1). \tag{5.2}$$

Here $\lambda_i \equiv \lambda_{p_i}$ are helicity spinors associated with p_i^μ. I use the shorthand notation $(\lambda_1\lambda_2) \equiv \lambda_1^\alpha\lambda_{2\,\alpha} = \epsilon^{\alpha\beta}\lambda_{1\,\beta}\lambda_{2\,\alpha}$, and $(\tilde\lambda_1\tilde\lambda_2) \equiv \tilde\lambda_{1\,\dot\alpha}\tilde\lambda_2^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\tilde\lambda_{1\,\dot\alpha}\tilde\lambda_{2\,\dot\beta}$. Note that the convention for placing the spinor indices is different for the twiddled and untwiddled spinors, which also may differ from the conventions in other references. More generally, contractions of undotted indices are always descending from left to right, while contractions of dotted indices are always ascending, e.g. $(\lambda_1\sigma^\mu\tilde\lambda_2) = \lambda_1^\alpha[\sigma^\mu]_{\alpha\dot\beta}\tilde\lambda_2^{\dot\beta}$ (dotted and undotted indices cannot be contracted together in Lorentz-covariant expressions, e.g. $\sum_{\alpha=1,2}\lambda^\alpha\tilde\lambda_{\dot\alpha}$ would break Lorentz symmetry). From these definitions the antisymmetry of spinor contractions immediately follows: $(\lambda_1\lambda_2) = -(\lambda_2\lambda_1)$, $(\tilde\lambda_1\tilde\lambda_2) = -(\tilde\lambda_2\tilde\lambda_1)$, in particular $(\lambda\lambda) = (\tilde\lambda\tilde\lambda) = 0$. With a bit more work one can prove $(\lambda_1\sigma^\mu\tilde\lambda_2) = (\tilde\lambda_2\bar\sigma^\mu\lambda_1)$, $(\lambda_1\sigma^\mu\bar\sigma^\nu\lambda_2) = -(\lambda_2\sigma^\nu\bar\sigma^\mu\lambda_1)$, $(\tilde\lambda_1\bar\sigma^\mu\sigma^\nu\tilde\lambda_2) = -(\tilde\lambda_2\bar\sigma^\nu\sigma^\mu\tilde\lambda_1)$, etc. Let me also mention that using Fierz identities we can rewrite Eq. (5.1) as $p^\mu = \frac{1}{2}\lambda\sigma^\mu\tilde\lambda$.

You may think that you see the helicity spinors for the first time, but that's not the case. Indeed, the helicity spinors are just the usual 2-component spinor wave functions $x(p,h)$ and $y(p,h)$ in another guise. One can identify $\lambda_p = x(p,-)$ and $\tilde\lambda_p = \bar y(p,+)$ (for massless particles $x(p,+) = y(p,-) = 0$). Then the defining relation in Eq. (5.1) is just the spinor helicity sum rule: $\sum_h x_\alpha(p,h)\bar x_{\dot\beta}(p,h) = [p\sigma]_{\alpha\dot\beta}$. Since the spinor wave functions are usually defined as solutions of the Dirac equation, it follows that both λ_p and $\tilde\lambda_p$ must satisfy the massless Weyl equation. Indeed

$$p\bar\sigma\lambda_p = \tilde\lambda_p(\lambda_p\lambda_p) = 0, \qquad p\sigma\tilde\lambda_p = \lambda_p(\tilde\lambda_p\tilde\lambda_p) = 0, \tag{5.3}$$

It will also be useful to relate spinors corresponding to flipped momenta. In general $\lambda_{-p} = -\omega\lambda_p$, $\tilde\lambda_{-p} = \omega^{-1}\tilde\lambda_p$ with an arbitrary ω (a pure phase in the real case) to ensure $\lambda_{-p}\tilde\lambda_{-p} = -p\sigma$. The choice of ω is a convention. In these lecture $\omega = 1$, thus

$$\lambda_{-p} = -\lambda_p, \qquad \tilde\lambda_{-p} = \tilde\lambda_p, \tag{5.4}$$

but mind that $\omega = -1$ or $\omega = \pm i$ are also encountered in the literature. One warning: do not confuse helicity spinors with quantum fields describing half-integer spin particles, in these lectures usually denoted as ψ to mark the difference. The former commuting objects, in particular $\lambda_\alpha\tilde\lambda_{\dot\beta} = \tilde\lambda_{\dot\beta}\lambda_\alpha$, whereas the latter anti-commute.

The importance of helicity spinors lies in the fact that they encode little group transformations. Indeed, if a pair of spinors satisfies $\lambda_i\tilde\lambda_i = p_i\sigma$, so does any pair obtained by the transformation

$$\lambda_i \to z_i^{-1}\lambda_i, \qquad \tilde\lambda_i \to z\tilde\lambda_i, \tag{5.5}$$

where z is a pure phase for real momenta. Recall that the little group is defined as Lorentz transformations that do not change the momentum. Thus, helicity spinors transform under the $U(1)$ little group pertaining to massless momenta.[28] Now, particle states and consequently scattering amplitudes transform under the little group associated with each involved particles. In this business it is convenient to work with amplitudes where all particles are incoming, to make the little group transformations more transparent (amplitudes with outgoing particles can be obtained from these using crossing symmetry). Let

[28]AA: Comment about continuous spin

us denote the amplitude for scattering of n particles with momenta p_i^μ and helicities h_i as $\mathcal{M}(1^{h_1}2^{h_2}\dots n^{h_n})$. Then, under Eq. (5.5) it transforms us

$$\mathcal{M}(1^{h_1}2^{h_2}\dots n^{h_n}) \to z_1^{2h_1} z_2^{2h_2} \dots z_n^{2h_n} \mathcal{M}(1^{h_1}2^{h_2}\dots n^{h_n}). \tag{5.6}$$

This severely restricts how the amplitude can depend on the helicity spinors. For example, 4-point amplitudes for scattering of spin-1/2 fermions must be of the form

$$\begin{aligned}
\mathcal{M}(1^-2^-3^-4^-) =&(\lambda_1\lambda_2)(\lambda_3\lambda_4)f_-(s,t),\\
\mathcal{M}(1^-2^-3^-4^+) =&0,\\
\mathcal{M}(1^-2^-3^+4^+) =&(\lambda_1\lambda_2)(\tilde\lambda_3\tilde\lambda_4)f_0(s,t),\\
\mathcal{M}(1^-2^+3^+4^+) =&0,\\
\mathcal{M}(1^+2^+3^+4^+) =&(\tilde\lambda_1\tilde\lambda_2)(\tilde\lambda_3\tilde\lambda_4)f_+(s,t),
\end{aligned} \tag{5.7}$$

where $f_i(s,t)$ are some functions of the kinematic variables with zero little group weights (meaning that all spinors can be eliminated in favor of momenta). As we will see soon, in certain cases the condition in Eq. (5.6) *completely* fixes the amplitudes up to a constant.

Helicity spinors is just one ingredient to render calculations of scattering amplitudes more efficient. Another ingredient, much more radical, is referred to as on-shell techniques. These have the ambition to formulate the quantum theory using solely on-shell ingredients, without referring to off-shell concepts such as fields, Lagrangians, or gauge symmetry. In practice, this relies on defining certain lower-point tree-level amplitudes as an input, and then bootstrapping higher-point and loop amplitudes using (generalized) unitarity.

The most primitive object in this approach is the 3-point amplitude $\mathcal{M}_3 \equiv \mathcal{M}(1^{h_1}2^{h_2}3^{h_3})$. [29] Here there is already a conceptual issue that needs to be addressed before we can proceed. At first sight, on-shell 3-point amplitude do not exist for all massless particles. Indeed, all kinematic invariant vanish in this case:

$$2p_ip_j = (p_i+p_j)^2 = p_k^2 = 0, \tag{5.8}$$

where $i,j,k \in 1\dots 3$ are any three different particle labels. I used the on-shell condition $p_i^2=0$ and momentum conservation $p_i+p_j+p_k=0$ (recall that all momenta are treated as incoming, unless otherwise noted). Therefore the only available option for real kinematics is $\mathcal{M}_3 = \text{const}$, which works only for scalars. In order to define on-shell 3-point amplitudes for spinning particles one needs to deform the momenta to the complex domain. Here is where the helicity spinors come in handy. Writing $2p_ip_j = (\lambda_i\lambda_j)(\tilde\lambda_j\tilde\lambda_i)$ it is clear that not all Lorentz invariants have to vanish to satisfy $2p_ip_j=0$; it is enough that either $(\tilde\lambda_i\tilde\lambda_j)=0$, or $(\lambda_i\lambda_j)=0$. These two discrete possibilities are referred to as the *holomorphic* (**H**) and *anti-holomorphic* (**AH**) kinematics, respectively. For the **H** kinematics, one observes that if $(\tilde\lambda_i\tilde\lambda_j)=0$ for one pair of i,j, then the same holds for any pair. This follows from momentum conservation: $0=\lambda_i\tilde\lambda_i+\lambda_j\tilde\lambda_j+\lambda_k\tilde\lambda_k$. Assuming, $(\tilde\lambda_i\tilde\lambda_j)=0$, multiplying the momentum conservation equation from the right by $\tilde\lambda_i$ one derives $(\tilde\lambda_k\tilde\lambda_i)=0$, while multiplying it by

[29] AA: comment on lower-point and form factors without momentum conservation.

$\tilde{\lambda}_j$ one derives $(\tilde{\lambda}_k\tilde{\lambda}_j) = 0$. The vanishing of all anti-holomorphic invariants imply that $\tilde{\lambda}_i \sim \tilde{\lambda}_j \sim \tilde{\lambda}_k$. More precisely, the proportionality constraint can be determined by multiplying the momentum conservation equation from the left e.g. by λ_k: $0 = (\lambda_k\lambda_i)\tilde{\lambda}_i + (\lambda_k\lambda_j)\tilde{\lambda}_j$ which is solved by $\tilde{\lambda}_i = -\frac{(\lambda_k\lambda_j)}{(\lambda_k\lambda_i)}\tilde{\lambda}_j$. The analogous formulas with $\lambda \leftrightarrow \tilde{\lambda}$ can be derived for the **AH** kinematics. In summary

$$\begin{aligned}
\mathbf{H}: \quad & (\tilde{\lambda}_i\tilde{\lambda}_j) = (\tilde{\lambda}_j\tilde{\lambda}_k) = (\tilde{\lambda}_k\tilde{\lambda}_i) = 0, \\
& \tilde{\lambda}_i = -\frac{(\lambda_k\lambda_j)}{(\lambda_k\lambda_i)}\tilde{\lambda}_j, \\
& (\lambda_i\lambda_j), (\lambda_j\lambda_k), (\lambda_k\lambda_i) \neq 0,
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
\mathbf{AH}: \quad & (\lambda_i\lambda_j) = (\lambda_j\lambda_k) = (\lambda_k\lambda_i) = 0, \\
& \lambda_i = -\frac{(\tilde{\lambda}_k\tilde{\lambda}_j)}{(\tilde{\lambda}_k\tilde{\lambda}_i)}\lambda_j, \\
& (\tilde{\lambda}_i\tilde{\lambda}_j), (\tilde{\lambda}_j\tilde{\lambda}_k), (\tilde{\lambda}_k\tilde{\lambda}_i) \neq 0,
\end{aligned} \tag{5.10}$$

for $i \neq j \neq k \in 1\dots 3$.

With this bit of formalism at hand, we are ready to write down amplitudes. Once the helicities of the three particles are fixed, for either kinematics there is a unique expression (up to an overall constant) for the on-shell 3-point amplitude satisfying the little group constraints:

$$\begin{aligned}
\mathbf{H}: \quad & \mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) = g(\lambda_1\lambda_2)^{h_3-h_1-h_2}(\lambda_2\lambda_3)^{h_1-h_2-h_3}(\lambda_3\lambda_1)^{h_2-h_3-h_1}, \\
\mathbf{AH}: \quad & \mathcal{M}(1^{h_1}2^{h_2}3^{h_3}) = \tilde{g}(\tilde{\lambda}_1\tilde{\lambda}_2)^{h_1+h_2-h_3}(\tilde{\lambda}_2\tilde{\lambda}_3)^{h_2+h_3-h_1}(\tilde{\lambda}_3\tilde{\lambda}_1)^{h_3+h_1-h_2}.
\end{aligned} \tag{5.11}$$

These building blocks can be bootstrapped to construct higher-point amplitudes. How this works in practice will be discussed later in this section in the context of specific EFT examples. Let me now just comment that not all amplitudes Eq. (5.11) lead to consistent quantum theories. Many of the amplitudes, while consistent with the little group scaling in Eq. (5.6) ultimately clash with some fundamental principles, such as unitarity, locality, or the spin-statistics theorem. This is in particular the fate of all amplitudes containing massless particles with the helicity $h > 2$. This leaves only a small finite subset of the possible amplitude in relativistic quantum field theory. QFT turns out to be an incredibly rigid structure, and on-shell methods offer probably the most transparent way to understand why. For more on this story see e.g. the lectures of Cliff Cheung [63] or some of the original publications [64, 65].

In theories with scalars and spin-1/2 fermions only, helicity spinors do not give us much extra mileage. But the usefulness of these variables sharply increases for larger spins of the scattered particles. Already for gauge theories (spin 1) the standard methods may lead to lengthy expressions for the amplitudes, which then reveal their hidden simplicity after switching to helicity spinors. This is even more true for gravitational theories (spin 2), where standard Feynman calculations are hopelessly messy even at tree level. Conversely, gravity is not more difficult than gauge theories when attacked with helicity spinors and on-shell methods.

5.2 Warmup: GREFT revisited

Let us apply the general discussion in the previous subsection to a theory of the graviton, who is a massless spin-2 particle. The graviton has two helicities: $+2 \equiv +$ and $-2 \equiv -$. The on-shell 3-point amplitude has 4 independent helicity configurations: $\mathcal{M}(1^-2^-3^-)$, $\mathcal{M}(1^-2^-3^+)$, $\mathcal{M}(1^-2^+3^+)$, $\mathcal{M}(1^+2^+3^+)$; the permutations of the middle two can be derived using Bose symmetry. The general formula in Eq. (5.11) allows for the following possibilities consistent with little group scaling:

$$
\begin{array}{lll}
 & \boldsymbol{H} & \boldsymbol{AH} \\
\mathcal{M}(1^-2^-3^-) = & C_{---}(\lambda_1\lambda_2)^2(\lambda_2\lambda_3)^2(\lambda_1\lambda_3)^2, & \tilde{C}_{---}\dfrac{1}{(\tilde{\lambda}_1\tilde{\lambda}_2)^2(\tilde{\lambda}_2\tilde{\lambda}_3)^2(\tilde{\lambda}_1\tilde{\lambda}_3)^2}, \\
\mathcal{M}(1^-2^-3^+) = & C_{--+}\dfrac{(\lambda_1\lambda_2)^6}{(\lambda_2\lambda_3)^2(\lambda_1\lambda_3)^2}, & \tilde{C}_{--+}\dfrac{(\tilde{\lambda}_1\tilde{\lambda}_3)^2(\tilde{\lambda}_2\tilde{\lambda}_3)^2}{(\tilde{\lambda}_1\tilde{\lambda}_2)^6}, \\
\mathcal{M}(1^-2^+3^+) = & C_{-++}\dfrac{(\lambda_1\lambda_2)^2(\lambda_1\lambda_3)^2}{(\lambda_2\lambda_3)^6}, & \tilde{C}_{-++}\dfrac{(\tilde{\lambda}_2\tilde{\lambda}_3)^6}{(\tilde{\lambda}_1\tilde{\lambda}_3)^2(\tilde{\lambda}_2\tilde{\lambda}_3)^2}, \\
\mathcal{M}(1^+2^+3^+) = & C_{+++}\dfrac{1}{(\lambda_1\lambda_2)^2(\lambda_2\lambda_3)^2(\lambda_1\lambda_3)^2}, & \tilde{C}_{+++}\dfrac{1}{(\tilde{\lambda}_1\tilde{\lambda}_2)^2(\tilde{\lambda}_2\tilde{\lambda}_3)^2(\tilde{\lambda}_1\tilde{\lambda}_3)^2}.
\end{array}
\tag{5.12}
$$

The dimension of the 3-point amplitude is $[\mathcal{M}_3] = \text{mass}^1$, which determines the couplings dimensions: $[C^{---}] = [\tilde{C}^{+++}] = \text{mass}^{-5}$, $[C_{--+}] = [\tilde{C}_{-++}] = \text{mass}^{-1}$, $[C_{-++}] = [\tilde{C}_{--+}] = \text{mass}^3$, $[C_{+++}] = [\tilde{C}_{---}] = \text{mass}^7$. Not all of these amplitudes can appear in a consistent theory. One can show that the cases where the dimension of the coupling is positive clash with unitarity and locality. Therefore these amplitudes must vanish in the respective kinematics. The general rule is that 3-point amplitudes where the number of spinors in the denominator is larger or equal than the number of spinors in the numerator are inconsistent (in this counting, a momentum should be treated as a pair of spinors). In the Lagrangian language, these would correspond to non-local interactions with some derivatives in the denominator. The unique exception from this rule is the all-scalar amplitude $\mathcal{M}(1^02^03^0) = g$, which must be consistent as it corresponds to the healthy ϕ^3 interaction in a Lagrangian. Furthermore, one can also show that crossing symmetry requires $C_{--+} = \tilde{C}^*_{-++}$ and one can choose the phase conventions such that $C_{--+} = \tilde{C}_{-++} \equiv 1/M$. All in all, Eq. (5.12) simplifies to

$$
\begin{array}{lll}
 & \boldsymbol{H} & \boldsymbol{AH} \\
\mathcal{M}(1^-2^-3^-) = & C(\lambda_1\lambda_2)^2(\lambda_2\lambda_3)^2(\lambda_1\lambda_3)^2, & 0, \\
\mathcal{M}(1^-2^-3^+) = & \dfrac{1}{M}\dfrac{(\lambda_1\lambda_2)^6}{(\lambda_2\lambda_3)^2(\lambda_1\lambda_3)^2}, & 0, \\
\mathcal{M}(1^-2^+3^+) = & 0, & \dfrac{1}{M}\dfrac{(\tilde{\lambda}_2\tilde{\lambda}_3)^6}{(\tilde{\lambda}_1\tilde{\lambda}_3)^2(\tilde{\lambda}_2\tilde{\lambda}_3)^2}, \\
\mathcal{M}(1^+2^+3^+) = & 0, & \tilde{C}\dfrac{1}{(\tilde{\lambda}_1\tilde{\lambda}_2)^2(\tilde{\lambda}_2\tilde{\lambda}_3)^2(\tilde{\lambda}_1\tilde{\lambda}_3)^2},
\end{array}
\tag{5.13}
$$

where $C_{---} \equiv C$, and $\tilde{C}_{+++} \equiv \tilde{C}$. What is the meaning of this ? Let us relate the 3-point amplitude to the GREFT Lagrangian in Eq. (1.30). The leading $D = 2$ term contains the mass scale $M_{\rm Pl}$ and, as discussed around Eq. (1.35), it contains 2-derivatives cubic graviton interactions suppressed by $1/M_{\rm Pl}$. Thus we can identify $M \sim M_{\rm Pl}$ and the corresponding amplitudes with the Einsteinian cubic GR interactions. One should appreciate the stunning simplicity of Eq. (5.13) compared to Eq. (1.35), even though the two contain the same physics!

On the other hand, C and $\tilde{C}$ have to be related to the Wilson coefficients C_1 and C_2. By dimensional analysis, $C \sim \frac{C_i}{M_{\rm Pl}^3}$, consistently with Eq. (1.36). Furthermore, parity relates amplitudes with opposite helicity, thus $C = \tilde{C}$ in the parity invariant theory. Therefore we expect the relation of the form $C = \frac{C_1 + iC_2}{M_{\rm Pl}^3}$, $\tilde{C} = \frac{C_1 - iC_2}{M_{\rm Pl}^3}$. All in all, all of the physics of GREFT up to dimension-6 is captured by simple 3-point on-shell amplitudes in Eq. (5.13). The even higher-dimensional GREFT operators correspond to contact terms in the 4-point and higher on-shell amplitudes. We will discuss this kind of correspondence in a different framework in the next subsections.

5.3 Yang-Mills EFT on-shell

We mover to a theory of colored spin-1 fields. That is to say, the amplitudes are labeled by helicities of the incoming particles, which can take values $h_i = \pm 1$, and by the color label $a, b, c \dots \in [1, N]$. Now the couplings in the 3-point amplitudes will come with color indices $C^{abc}_{h_1 h_2 h_3}$. The usual on-shell route would be to start with the analog of Eq. (5.13) for the Yang-Mills theory, bootstrap that into 4-point amplitudes, and demonstrate that consistency requires $C^{abc}_{h_1 h_2 h_3} \sim f^{abc}$, where f^{abc} are the Yang-Mills structure constants. This is a beautiful story but widely covered elsewhere [63, 65] and a bit laborious so in these lectures I will take a short cut. I will start with the Lagrangian of the Yang-Mills EFT and from that I will derive the 3-point amplitudes. This is in line with a more general philosophy where one does not try to define relativistic quantum theories entirely on shell, but rather one treats the on-shell techniques as a tool that exists in parallel to the standard techniques.

Consider the EFT Lagrangian of the form

$$\mathcal{L}_{\rm YM-EFT} = -\frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} + C_G f^{abc} G^a_{\mu\nu} G^b_{\nu\rho} G^c_{\rho\mu} + C_{\widetilde{G}} f^{abc} G^a_{\mu\nu} G^b_{\nu\rho} \widetilde{G}^c_{\rho\mu} + \dots \tag{5.14}$$

Here $G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g f^{abc} G^b_\mu G^c_\nu$, and G^a_μ are the vector fields referred to as gluons, f^{abc} is the group structure tensor, which antisymmetric in all 3 indices, and g is the dimensionless coupling constant. The first term contains the kinetic and interaction terms of the ordinary Yang-Mills theory. The other two terms are dimension-6, that is the corresponding couplings carry the mass dimensions $[C_G] = [C_{\widetilde{G}}] = {\rm mass}^{-2}$. They are non-renormalizable in the somewhat outdated parlance, and their presence signals that the Lagrangian describes an EFT with a limited validity range. The dots stand for interactions of dimension-8 and higher. The ordinary Yang-Mills Lagrangian augmented with higher-dimensional terms constructed out of the gluon field strength is referred to as the Yang-Mills EFT.

3-point amplitudes originate from cubic interactions terms in the Lagrangian. In the case at hand these are

$$\mathcal{L}_{\rm YM-EFT} \subset g f^{abc}\partial_\mu G^a_\nu G^b_\mu G^c_\nu + 8C_G f^{abc}\partial_{[\mu}G^a_{\nu]}\partial_{[\nu}G^b_{\rho]}\partial_{[\rho}G^c_{\mu]} + 4C_{\widetilde{G}} f^{abc}\epsilon^{\rho\mu\alpha\beta}\partial_{[\mu}G^a_{\nu]}\partial_{[\nu}G^b_{\rho]}\partial_{[\alpha}G^c_{\beta]}, \tag{5.15}$$

where the square brackets indicate anti-symmetrization with the weight 1/2.

Let us focus on the first (dimension-4) term proportional to g. It contributes to the 3-point amplitude as

$$\mathcal{M}(1_a 2_b 3_c) = i g f^{abc}\epsilon^1_\mu \epsilon^2_\nu \epsilon^3_\rho \Big\{ \eta_{\mu\nu}(p_1 - p_2)_\rho + \eta_{\nu\rho}(p_2 - p_3)_\mu + \eta_{\rho\mu}(p_3 - p_1)_\nu \Big\}, \tag{5.16}$$

where $\epsilon^i_\mu \equiv \epsilon_\mu(p_i, h_i)$ is the polarization vector corresponding to the i-th particle. To proceed, we need to express these polarization vectors by spinors. The correct answer is

$$\epsilon^\mu(p,-) = \frac{(\lambda_p \sigma^\mu \tilde{\zeta})}{\sqrt{2}(\tilde{\lambda}_p \tilde{\zeta})}, \qquad \epsilon^\mu(p,+) = \frac{(\zeta \sigma^\mu \tilde{\lambda}_p)}{\sqrt{2}(\lambda_p \zeta)}, \tag{5.17}$$

where ζ and $\tilde{\zeta}$ are reference spinors that are not proportional to λ_p and $\tilde{\lambda}_p$. The freedom of choosing ζ corresponds to the gauge redundancy of shifting the polarization vectors by a piece proportional to the momentum, $\epsilon^\mu(p,h) \to \epsilon^\mu(p,h) + c p^\mu$. As a sanity check, this choice gives correct little group scaling and satisfies $p^\mu \epsilon_\mu(p,h) = 0$, which is a simple consequence of the Weyl equation satisfied by λ_p. The normalization is chosen so that $\epsilon^\mu(p,-)\epsilon_\mu(p,+) = 1$.

Consider Eq. (5.16) evaluated at the $(--+)$ helicity configuration. Since this is the first non-trivial calculation of this kind using several tricks from the on-shell repertoire, we will be visceral. Inserting the appropriate polarization vectors one gets

$$\begin{aligned}
\mathcal{M}(1^-_a 2^-_b 3^+_c) \supset & i\frac{g f^{abc}}{2\sqrt{2}} \frac{(\lambda_1\sigma^\mu\tilde{\zeta})}{(\tilde{\lambda}_1\tilde{\zeta})} \frac{(\lambda_2\sigma^\nu\tilde{\zeta})}{(\tilde{\lambda}_2\tilde{\zeta})} \frac{(\zeta\sigma^\rho\tilde{\lambda}_3)}{(\lambda_3\zeta)} \Big\{ \eta_{\mu\nu}(p_1 - p_2)_\rho + \eta_{\nu\rho}(p_2 - p_3)_\mu + \eta_{\rho\mu}(p_3 - p_1)_\nu \Big\} \\
= & i\frac{g f^{abc}}{\sqrt{2}(\tilde{\lambda}_1\tilde{\zeta})(\tilde{\lambda}_2\tilde{\zeta})(\lambda_3\zeta)} \Big[-(\lambda_1 p_3 \sigma \tilde{\zeta})(\lambda_2\sigma^\nu\tilde{\zeta})(\zeta\sigma_\nu\tilde{\lambda}_3) + (\lambda_1\sigma^\mu\tilde{\zeta})(\lambda_2 p_3\sigma\tilde{\zeta})(\zeta\sigma_\mu\tilde{\lambda}_3) \Big] \\
= & i\frac{\sqrt{2} g f^{abc}}{(\tilde{\lambda}_1\tilde{\zeta})(\tilde{\lambda}_2\tilde{\zeta})(\lambda_3\zeta)} \Big[(\lambda_1 p_3\sigma\tilde{\zeta})(\lambda_2\zeta)(\tilde{\zeta}\tilde{\lambda}_3) - (\lambda_2 p_3\sigma\tilde{\zeta})(\lambda_1\zeta)(\tilde{\zeta}\tilde{\lambda}_3) \Big] \\
= & i\frac{\sqrt{2} g f^{abc}(\tilde{\lambda}_3\tilde{\zeta})^2}{(\tilde{\lambda}_1\tilde{\zeta})(\tilde{\lambda}_2\tilde{\zeta})(\lambda_3\zeta)} \Big[(\lambda_1\zeta)(\lambda_2\lambda_3) - (\lambda_2\zeta)(\lambda_1\lambda_3) \Big] \\
= & -ig\sqrt{2} f^{abc}(\lambda_1\lambda_2)\frac{(\tilde{\lambda}_3\tilde{\zeta})^2}{(\tilde{\lambda}_1\tilde{\zeta})(\tilde{\lambda}_2\tilde{\zeta})}.
\end{aligned} \tag{5.18}$$

For simplicity, I picked the same reference spinor $\tilde{\zeta}$ for the first and second particle. This annihilates the first term in the curly bracket, as $(\lambda_1\sigma^\mu\tilde{\zeta})(\lambda_2\sigma_\mu\tilde{\zeta}) = 0$ after using Fierz identify $(\lambda_1\sigma^\mu\tilde{\lambda}_2)(\lambda_3\sigma_\mu\tilde{\lambda}_4) = -2(\lambda_1\lambda_3)(\tilde{\lambda}_2\tilde{\lambda}_4)$ and $(\tilde{\zeta}\tilde{\zeta}) = 0$. I also used momentum conservation $p_1 + p_2 + p_3$. In the third line I again used that Fierz identity. In the fourth line I replaced $p_3\sigma = \lambda_3\tilde{\lambda}_3$ and rearranged. In the fifth line I used the Schouten identity $(\lambda_1\lambda_2)(\lambda_3\lambda_4) = (\lambda_1\lambda_3)(\lambda_2\lambda_4) - (\lambda_1\lambda_4)(\lambda_2\lambda_3)$.

At this point the 3-point amplitude still depends on the gauge parameter $\tilde{\zeta}$, as one is accustomed to in Yang-Mills theories. However, on-shell, gauge symmetry has no place, which means we should be able to get rid of it. Note first that the amplitude in Eq. (5.18) is proportional to $(\lambda_1\lambda_2)$, therefore it vanishes for the **AH** kinematics where $(\lambda_i\lambda_j) = 0$ for all untwiddled contractions. For the **H** kinematics, starting from momentum conservation $\sum_{i=1}^3 \lambda_i\tilde{\lambda}_i = 0$ multiplied by $\tilde{\zeta}$ on the right and by λ_1 or λ_2 on the right one can derive

$$\frac{(\tilde{\lambda}_3\tilde{\zeta})}{(\tilde{\lambda}_1\tilde{\zeta})} = \frac{(\lambda_1\lambda_2)}{(\lambda_2\lambda_3)}, \qquad \frac{(\tilde{\lambda}_3\tilde{\zeta})}{(\tilde{\lambda}_2\tilde{\zeta})} = -\frac{(\lambda_1\lambda_2)}{(\lambda_1\lambda_3)}. \tag{5.19}$$

Hence, for the **H** kinematics, the on-shell 3-point amplitude in Eq. (5.18) can simplifies to

$$\mathcal{M}(1_a^- 2_b^- 3_c^+) \supset i\sqrt{2} g f^{abc} \frac{(\lambda_1\lambda_2)^3}{(\lambda_1\lambda_3)(\lambda_2\lambda_3)}. \tag{5.20}$$

As advertised, it is independent of the gauge parameter ζ. The calculation of $\mathcal{M}(1_a^+ 2_b^+ 3_c^-)$ is completely analogous, except the results is zero for the **H** kinematics, while for the **AH** kinematics it is obtained from Eq. (5.20) by $\lambda_i \to \tilde{\lambda}_i$.

Moving to the same incoming helicity amplitudes, the calculation is greatly simplified by choosing the same reference spinor for all the three polarization vectors. Then

$$\begin{aligned}\mathcal{M}(1_a^- 2_b^- 3_c^-) \supset & i\frac{g f^{abc}}{2\sqrt{2}} \frac{(\lambda_1\sigma^\mu\tilde{\zeta})}{(\tilde{\lambda}_1\tilde{\zeta})} \frac{(\lambda_2\sigma^\nu\tilde{\zeta})}{(\tilde{\lambda}_2\tilde{\zeta})} \frac{(\lambda_3\sigma^\rho\tilde{\zeta})}{(\tilde{\lambda}_3\tilde{\zeta})} \Big\{ \eta_{\mu\nu}(p_1 - p_2)_\rho + \eta_{\nu\rho}(p_2 - p_3)_\mu + \eta_{\rho\mu}(p_3 - p_1)_\nu \Big\} \\ = & 0, \end{aligned} \tag{5.21}$$

for either kinematics. This follows from the Fierz identity quoted below Eq. (5.18). Of course, this is the only possible result, since g_{abc}^{---} in **??** is dimensionful, whereas the dimension-4 Yang Mills terms does not provide any dimensionful coupling. Similarly, the dimension-4 term cubic interaction in the Yang-Mills Lagrangian does not contribute to the on-shell $\mathcal{M}(1_a^+ 2_b^+ 3_c^+)$. This is an example of vanishing of maximally helicity violating (MHV) amplitudes in the ordinary Yang Mills theory.

We move our focus to the second term proportional to C_G in Eq. (5.15). It contributes to the 3-point amplitude as

$$\mathcal{M}(1_a 2_b 3_c) \supset 6 i C_G f^{abc} \epsilon_\mu^1 \epsilon_\nu^2 \epsilon_\rho^3 \Big\{ p_1^\rho p_2^\mu p_3^\nu - p_1^\nu p_2^\rho p_3^\mu + \dots \Big\}, \tag{5.22}$$

the dots stand for 6 other terms that contain momentum contractions $(p_i p_j)$ and therefore vanish on shell. Dimensional analysis tells us that this can only contribute to the same helicity amplitudes. Evaluating it on the $(- - -)$ helicity configuration,

$$\begin{aligned}\mathcal{M}(1_a^- 2_b^- 3_c^-) = & \frac{3 i C_G f^{abc}}{\sqrt{2}(\tilde{\lambda}_1\tilde{\zeta})(\tilde{\lambda}_2\tilde{\zeta})(\tilde{\lambda}_3\tilde{\zeta})} \Big\{ (\lambda_1 p_2 \sigma\tilde{\zeta})(\lambda_2 p_3\sigma\tilde{\zeta})(\lambda_3 p_1\sigma\tilde{\zeta}) - (\lambda_1 p_3\sigma\tilde{\zeta})(\lambda_2 p_1\sigma\tilde{\zeta})(\lambda_3 p_2\sigma\tilde{\zeta}) \Big\} \\ = & 3i\sqrt{2} C_G f^{abc} (\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1). \end{aligned} \tag{5.23}$$

In the second line I replaced $p_i\sigma$ with $\lambda_i\tilde{\lambda}_i$, after which both terms turn out to be the same, and the reference spinors cancel. This vanishes for the **AH** kinematics, but is non-zero for the **H** kinematics. The calculation for the $(- - -)$ helicity configuration is analogous with $\lambda \to \tilde{\lambda}$.

Exercise: Calculate the contributions proportional to $C_{\widetilde{G}}$ to the 3-point amplitude.

To recapitulate, the 3-point amplitudes in Yang-Mills EFT for the two kinematics are

$$\begin{aligned}
\mathbf{H}: \qquad & \mathcal{M}(1_a^- 2_b^- 3_c^+) = i\sqrt{2} g f^{abc} \frac{(\lambda_1\lambda_2)^3}{(\lambda_1\lambda_3)(\lambda_2\lambda_3)}, \qquad \mathcal{M}(1_a^+ 2_b^+ 3_c^-) = 0, \\
& \mathcal{M}(1_a^- 2_b^- 3_c^-) = 3i\sqrt{2} f^{abc} \mathcal{C}_G (\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1), \qquad \mathcal{M}(1_a^+ 2_b^+ 3_c^+) = 0. \\
\mathbf{AH}: \qquad & \mathcal{M}(1_a^+ 2_b^+ 3_c^-) = i\sqrt{2} g f^{abc} \frac{(\tilde\lambda_1\tilde\lambda_2)^3}{(\tilde\lambda_1\tilde\lambda_3)(\tilde\lambda_2\tilde\lambda_3)}, \qquad \mathcal{M}(1_a^- 2_b^- 3_c^+) = 0, \\
& \mathcal{M}(1_a^+ 2_b^+ 3_c^+) = 3i\sqrt{2} f^{abc} \mathcal{C}_G^* (\tilde\lambda_1\tilde\lambda_2)(\tilde\lambda_2\tilde\lambda_3)(\tilde\lambda_3\tilde\lambda_1), \qquad \mathcal{M}(1_a^- 2_b^- 3_c^-) = 0,
\end{aligned} \tag{5.24}$$

where $\mathcal{C}_G \equiv C_G + iC_{\widetilde{G}}$. In this example, both dimension-4 and dimension-6 interactions contribute to the 3-point amplitudes, although it's not a rule. One should appreciate the simplicity and compactness of these formulas, compared to the clutter of indices in the usual Feynman rules. The gain in simplicity and transparency will only increase once one moves to higher-point amplitudes and beyond tree level.

5.4 Bootstrapping EFT amplitudes at tree level

In quantum theories there is a connection between higher- and lower-point amplitudes due to unitarity. The master equation underlying this is

$$\mathrm{Disc}\mathcal{M}(\alpha \to \beta) = i \sum_X \int d\Pi_X \mathcal{M}(\alpha \to X)\mathcal{M}(X \to \beta), \tag{5.25}$$

where the discontinuity is in the kinematic variable p_α^2, and p_α is the sum of momenta of all particles in α. The sum goes over all possible intermediate states X, with $d\Pi_X$ being the phase space element appropriate to the multiplicity of X. At tree level X are one-particle states, and the master equation fixed the residue of the pole in $\mathcal{M}(\alpha \to \beta)$ corresponding to the exchange of X:

$$\lim_{p_\alpha^2 \to m_X^2} \mathcal{M}(\alpha \to \beta) = -\frac{1}{p_\alpha^2 - m_X^2 + i\epsilon} \mathcal{M}(\alpha \to X)\mathcal{M}(X \to \beta). \tag{5.26}$$

In the following we will use Eq. (5.26) to calculate 4-point amplitudes from the 3-point amplitudes in Eq. (5.24). 4-point amplitudes are functions of the Mandelstam invariants $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_1 + p_4)^2$. In our case everyone is massless, so the poles can occur when one of these invariants approaches zero. We need to calculate the residues like

$$\begin{aligned}
R_s^{h_1h_2h_3h_4} \equiv & \mathrm{Res}_{s\to 0}\mathcal{M}(1_a^{h_1} 2_b^{h_2} 3_c^{h_3} 4_d^{h_4}) = -\sum_h \mathcal{M}(1_a^{h_1} 2_b^{h_2} \to s_e^h)\mathcal{M}(s_e^h 3_c^{h_3} 4_d^{h_4}) \\
= & -\sum_h \mathcal{M}(1_a^{h_1} 2_b^{h_2} (-s)_e^{-h})\mathcal{M}(s_e^h 3_c^{h_3} 4_d^{h_4}),
\end{aligned} \tag{5.27}$$

where $p_s = p_1 + p_2 = -p_3 - p_4$. In the last step we used crossing symmetry. Similarly

$$R_t^{h_1h_2h_3h_4} \equiv \mathrm{Res}_{t\to 0}\mathcal{M}(1_a^{h_1}2_b^{h_2}3_c^{h_3}4_d^{h_4}) = -\sum_h \mathcal{M}(1_a^{h_1}3_c^{h_3}(-t)_e^{-h})\mathcal{M}(t_e^h 2_b^{h_2}4_d^{h_4})$$
$$R_u^{h_1h_2h_3h_4} \equiv \mathrm{Res}_{u\to 0}\mathcal{M}(1_a^{h_1}2_b^{h_2}3_c^{h_3}4_d^{h_4}) = -\sum_h \mathcal{M}(1_a^{h_1}4_d^{h_4}(-u)_e^{-h})\mathcal{M}(u_e^h 2_b^{h_2}3_c^{h_3}), \quad (5.28)$$

where $p_t = p_1 + p_3 = -p_2 - p_4$, $p_u = p_1 + p_4 = -p_2 - p_3$.

Let us start with the all-minus helicity configuration. Then $R_s^{----} = -\sum_h \mathcal{M}(1_a^- 2_b^- (-s)_e^{-h})\mathcal{M}(s_e^h 3_c^- 4_d^-)$. At this point we have to make a choice of kinematics for the two 3-point amplitudes. Let us choose the **H** kinematics for the 12 vertex, which implies $\tilde\lambda_1 \sim \tilde\lambda_2 \sim \tilde\lambda_s$. Then we are forced to choose the **AH** kinematics for the 34 vertex (otherwise we would conclude that $\tilde\lambda_3 \sim \tilde\lambda_4 \sim \tilde\lambda_s \sim \tilde\lambda_1 \sim \tilde\lambda_2$, thus $(\tilde\lambda_i\tilde\lambda_j) = 0$ for $i,j \in 1\dots 4$ leading to vanishing of all Mandelstam invariants). However both $\mathcal{M}(s_e^- 3_c^- 4_d^-)$ and $\mathcal{M}(s_e^+ 3_c^- 4_d^-)$ vanish for the **AH** kinematics, as can be seen by consulting Eq. (5.24). We conclude that $R_s^{----} = 0$. Of course, the same conclusion is obtained if we picked the **AH** kinematics for the 12 vertex, and the **H** kinematics for the 34 vertex, in which case $\mathcal{M}(1_a^- 2_b^- (-s)_e^{-h}) = 0$ for either h. The same argument goes in other channels. We thus have

$$R_s^{----} = R_t^{----} = R_u^{----} = 0. \quad (5.29)$$

In the ordinary Yang-Mills theory this would lead to the conclusion that the 4-point amplitude itself vanishes for the all-minus helicity configuration. This is not necessarily so in the Yang-Mills EFT. All we can say at this point is that $\mathcal{M}(1_a^- 2_b^- 3_c^- 4_d^-)$ does not have poles. Still it can contain expressions without kinematic poles, to which the unitarity-based formula in Eq. (5.26) is blind. These are the so-called *contact terms*. They are subject to several constraints: little group scaling, gauge group covariance, Bose symmetry. Furthermore, there cannot be any spinors in the denominator, to avoid poles or other singularities. For example, a legal contact term is $\mathcal{M}(1_a^- 2_b^- 3_c^- 4_d^-) \supset f^{abe}f^{cde}F[s,t.u](\lambda_1\lambda_3)^2(\lambda_2\lambda_4)^2 + \mathrm{sym}$, where *sym* stands for symmetrization in all four indices to ensure Bose symmetry and $F[s,t.u]$ is any polynomial of Mandelstam variables. Here I will restrict to minimal contact terms where $F[s,t.u] = \mathrm{const}$, as they correspond to the leading effect in the EFT expansion. It is a bit tricky to determine a complete set of all independent contact terms, even the minimal ones. This task is the analogue of finding a basis of EFT operators in the Lagrangian language. The answer depends on the gauge group of the Yang Mills EFT. For example, for the $U(1)$ group there is only a single "color" index and a single minimal contact term for this helicity configuration:

$$\boldsymbol{U(1)}: \quad \mathcal{M}(1^-2^-3^-4^-) = C_-\Big[(\lambda_1\lambda_2)^2(\lambda_3\lambda_4)^2 + (\lambda_1\lambda_3)^2(\lambda_2\lambda_4)^2 + (\lambda_1\lambda_4)^2(\lambda_2\lambda_3)^2\Big]. \quad (5.30)$$

The Wilson coefficient C_- has dimension $[C_-] = \mathrm{mass}^{-4}$, therefore this contact term corresponds to a dimension-8 operator in the Yang-Mills EFT Lagrangian. Non-minimal contact terms, that is the ones where spinor are multiplied by a non-trivial polynomial of the Mandelstam invariants, would correspond to operators of dimension 10 and higher. Note that in the $U(1)$ case $f^{abc}=0$, thus all 3-point amplitudes vanish (no amplitude simultaneously consistent with little group scaling and Bose symmetry is possible), and thus interactions begin at the 4-point and dimension-8 level. This is nothing but the Euler-Heisenberg EFT discussed earlier in Section 1. It should be clear that on-shell methods greatly simplify dealing with this EFT, avoiding dealing with the lengthy Feynman rules and the clutter of gauge indices.

> *Exercise:* Find out which (combination of) dimension-8 operators in the Euler-Heisenberg EFT does the Wilson coefficient C_- in Eq. (5.30) correspond to.

For non-abelian Yang-Mills, the contact terms get more complicated because we have to also include group invariants. For example, for $SU(2)$ there are two independent dimension-8 contact terms:

$$\begin{aligned}\boldsymbol{SU(2)}:\quad \mathcal{M}(1_a^- 2_b^- 3_c^- 4_d^-) =&C_-^{(1)}\left\{\delta^{ab}\delta^{cd}(\lambda_1\lambda_2)^2(\lambda_3\lambda_4)^2+\text{sym}\right\}\\ &+C_-^{(2)}\left\{\delta^{ab}\delta^{cd}\left[(\lambda_1\lambda_3)(\lambda_2\lambda_4)+(\lambda_1\lambda_4)(\lambda_2\lambda_3)\right]^2+\text{sym}\right\},\end{aligned}\tag{5.31}$$

where sym stands for two more terms obtained by the replacement $2\leftrightarrow 3, b\leftrightarrow c$ and $2\leftrightarrow 4, b\leftrightarrow d$.

> *Exercise:* Show that Eq. (5.31) contains the complete basis of dimension-8 contact terms. In particular, use the Schouten identity to show that $(\lambda_1\lambda_3)^2(\lambda_2\lambda_4)^2+(\lambda_1\lambda_4)^2(\lambda_2\lambda_3)^2$ and $(\lambda_1\lambda_3)(\lambda_2\lambda_4)(\lambda_1\lambda_4)(\lambda_2\lambda_3)$ can be reduced to the structures already present in Eq. (5.31).

For $SU(3)$ there are 3 independent terms, and for $SU(N)$ with $N>3$ there are four independent terms at the dimension-8 level [66].

Let us turn our attention to the mostly minus helicity configuration, $\mathcal{M}(1_a^- 2_b^-(-s)_e^{-h})\mathcal{M}(s_e^h 3_c^- 4_d^+)$. The s-channel residue is

$$\begin{aligned}R_s^{---+} =& -\sum_h \mathcal{M}(1_a^- 2_b^-(-s)_e^{-h})\mathcal{M}(s_e^h 3_c^- 4_d^+) = -\mathcal{M}(1_a^- 2_b^-(-s)_e^-)\mathcal{M}(s_e^+ 3_c^- 4_d^+)\\ =&6gf^{abe}f^{edc}\mathcal{C}_G\frac{(\lambda_1\lambda_2)(\lambda_2\lambda_s)(\lambda_s\lambda_1)(\tilde\lambda_s\tilde\lambda_4)^3}{(\tilde\lambda_s\tilde\lambda_3)(\tilde\lambda_4\tilde\lambda_3)} = -6gf^{abe}f^{cde}\mathcal{C}_G\frac{(\lambda_1\lambda_2)(\lambda_2 p_s\sigma\tilde\lambda_4)(\lambda_1 p_s\sigma\tilde\lambda_4)^2}{(\lambda_1 p_s\sigma\tilde\lambda_3)(\tilde\lambda_3\tilde\lambda_4)}\\ =&6gf^{abe}f^{cde}\mathcal{C}_G\frac{(\lambda_1\lambda_2)(\lambda_1\lambda_3)^2(\tilde\lambda_1\tilde\lambda_4)(\tilde\lambda_3\tilde\lambda_4)}{(\tilde\lambda_2\tilde\lambda_3)} = -6gf^{abe}f^{cde}\mathcal{C}_G\frac{(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde\lambda_4 p_1\bar\sigma p_3\sigma\tilde\lambda_4)}{u}\\ =&6gf^{abe}f^{cde}\mathcal{C}_G\frac{(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde\lambda_4 p_1\bar\sigma p_2\sigma\tilde\lambda_4)}{u}\end{aligned}\tag{5.32}$$

In the 2nd step we committed to the **AH** kinematics for the 34 vertex, which eliminates the $h=-$ contribution. In the 3rd step I multiplied the numerator and denominator by $(\lambda_1\lambda_s)$, and used $\lambda_s\tilde\lambda_s=p_s\sigma$. In the 4th step I used $p_s=p_1+p_2=-p_3-p_4$ and canceled some terms between the numerator and denominator. In the 5th step I multiplied the numerator and denominator by $(\lambda_2\lambda_3)$ and used $u=-(\lambda_2\lambda_3)(\tilde\lambda_2\tilde\lambda_3)$. I also rearranged the remaining terms in a way that makes it easy to recognize the correct little group transformation of the result. In the last step I used momentum conservation $p_3 = -p_1 - p_2 - p_4$ together with $p_1 \bar{\sigma} p_1 \sigma = p_1^2 = 0$ and the $p_4 \sigma \tilde{\lambda}_4 = 0$. The residue in the t-(u-)channel can be immediately obtained from the above by $2 \leftrightarrow 3, b \leftrightarrow c$ $(1 \leftrightarrow 3, a \leftrightarrow c)$. All in all, the full set of tree-level residues for this helicity configuration reads

$$\begin{aligned}
R_s^{---+} =& 6g\mathcal{C}_G(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde{\lambda}_4 p_1 \bar{\sigma} p_2 \sigma \tilde{\lambda}_4)\frac{f^{abe}f^{cde}}{u}, \\
R_t^{---+} =& 6g\mathcal{C}_G(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde{\lambda}_4 p_1 \bar{\sigma} p_2 \sigma \tilde{\lambda}_4)\frac{f^{ace}f^{bde}}{u}, \\
R_u^{---+} =& -6g\mathcal{C}_G(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde{\lambda}_4 p_1 \bar{\sigma} p_2 \sigma \tilde{\lambda}_4)\frac{f^{ade}f^{bce}}{s}.
\end{aligned} \tag{5.33}$$

The complication, which is a feature of Yang Mills and more generally of theories with long-range interactions, is that the residue in one channel contains a pole in another channel. Therefore we *cannot* just write the answer as $R_s/s + R_t/t + R_u/u$, as that would not correctly reproduce the residues above. Let us for a second be less ambitious and write an amplitude reproducing just the s- and t- channel residues in Eq. (5.33):

$$\mathcal{M}(1_a^- 2_b^- 3_c^- 4_d^+)_{\text{try}} = 6g\mathcal{C}_G(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde{\lambda}_4 p_1 \bar{\sigma} p_2 \sigma \tilde{\lambda}_4)\left\{\frac{f^{abe}f^{cde}}{su} + \frac{f^{ace}f^{bde}}{tu}\right\}. \tag{5.34}$$

Now if we calculate the u-channel residue of that we get

$$\begin{aligned}
\text{res}_{u\to 0}\mathcal{M}(1_a^- 2_b^- 3_c^- 4_d^-)_{\text{try}} =& 6g\mathcal{C}_G(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde{\lambda}_4 p_1 \bar{\sigma} p_2 \sigma \tilde{\lambda}_4)\left\{\frac{f^{abe}f^{cde}}{s} + \frac{f^{ace}f^{bde}}{t}\right\} \\
=& 6g\mathcal{C}_G(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde{\lambda}_4 p_1 \bar{\sigma} p_2 \sigma \tilde{\lambda}_4)\frac{f^{abe}f^{cde} - f^{ace}f^{bde}}{s} \\
=& 6g\mathcal{C}_G(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde{\lambda}_4 p_1 \bar{\sigma} p_2 \sigma \tilde{\lambda}_4)\frac{-f^{ade}f^{bce}}{s}.
\end{aligned} \tag{5.35}$$

In the 2nd line I used that on this residue $s = -t$ as the consequence of $s + t + u = 0$ and $u = 0$. In the 3rd line I used the Jacobi identity $f^{abe}f^{cde} - f^{ace}f^{bde} + f^{ade}f^{bce} = 0$. Magically, our half-assed attempt automatically satisfies the remaining residue in Eq. (5.33). This happens thanks to the Jacobi identity, that is to say, thanks to the fact that the coefficients of the 3-point amplitudes have a geometric interpretation as structure constants of a Lie algebra. From the on-shell perspective you can see that the group structure in Yang Mills theories is not there for some esthetic reason - it is absolutely necessary for maintaining unitarity of the 4-point amplitudes! All in all, for the one-plus helicity configuration the tree-level 4-point amplitude takes the form

$$\mathcal{M}(1_a^- 2_b^- 3_c^- 4_d^+) = 6g\mathcal{C}_G(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde{\lambda}_4 p_1 \bar{\sigma} p_2 \sigma \tilde{\lambda}_4)\left\{\frac{f^{abe}f^{cde}}{su} + \frac{f^{ace}f^{bde}}{tu}\right\} + \text{contact}. \tag{5.36}$$

The contact terms start at dimension 10, thus they are subleading compared to the pole term above, which is dimension 6, and even to the contact terms in the all-plus amplitude.

The last non-trivial calculation of the 4-point function is that of $\mathcal{M}(1_a^- 2_b^- 3_c^+ 4_d^+)$. It proceeds along the same lines as the mostly minus case, with similar tricks, including the use of Jacobi identity. Therefore I'm only quoting the final result here:

$$\begin{aligned}\mathcal{M}(1_a^- 2_b^- 3_c^+ 4_d^+) =& -2g^2(\lambda_1\lambda_2)^2(\tilde\lambda_3\tilde\lambda_4)^2\left[\frac{f_{ace}f_{bde}}{st}+\frac{f_{ade}f_{bce}}{su}\right]\\ &-9|C_G|^2 f^{abe}f^{ecd}(\lambda_1\lambda_2)^2(\tilde\lambda_3\tilde\lambda_4)^2\frac{t-u}{s}+\text{contact}. \end{aligned} \tag{5.37}$$

The contact terms start at dimension eight. For example, in the $U(1)$ case there is a single dimension-8 contact term: $\mathcal{M}(1_a^- 2_b^- 3_c^+ 4_d^+) = C_0(\lambda_1\lambda_2)^2(\tilde\lambda_3\tilde\lambda_4)^2$.

The all-plus and mostly plus amplitude, $\mathcal{M}(1_a^+ 2_b^+ 3_c^+ 4_d^+)$ and $\mathcal{M}(1_a^- 2_b^+ 3_c^+ 4_d^+)$ can be recycled from previous calculations by $\lambda \leftrightarrow \tilde\lambda$. This follows from $\mathcal{M}(\alpha\to\beta)^* = \mathcal{M}(\beta\to\alpha)$. E.g. $\mathcal{M}(1_a^- 2_b^- 3_c^- 4_d^-)^* = \mathcal{M}(\to 1_a^- 2_b^- 3_c^- 4_d^-) = \mathcal{M}((-1)_a^+(-2)_b^+(-3)_c^+(-4)_d^+)$, where we used the crossing symmetry in the last step. We conclude that $\mathcal{M}(1_a^+ 2_b^+ 3_c^+ 4_d^+) = \mathcal{M}((-1)_a^-(-2)_b^-(-3)_c^-(-4)_d^-)^*$, and the effect of the complex conjugation is precisely $\lambda \leftrightarrow \tilde\lambda$ for real kinematics (flipping the sign of momentum has no effect because in this case spinors always come in pairs).

For future reference, let us summarize the pole terms of all 4-point tree-level helicity amplitudes:

$$\begin{aligned}
\mathcal{M}(1_a^- 2_b^- 3_c^- 4_d^-) =&\text{contact},\\
\mathcal{M}(1_a^- 2_b^- 3_c^- 4_d^+) =& -6g\mathcal{C}_G(\lambda_1\lambda_2)(\lambda_2\lambda_3)(\lambda_3\lambda_1)(\tilde\lambda_4 p_1\bar\sigma p_2\sigma\tilde\lambda_4)\left[\frac{f^{ace}f^{bde}}{st}+\frac{f^{ade}f^{bce}}{su}\right]+\text{contact},\\
\mathcal{M}(1_a^- 2_b^- 3_c^+ 4_d^+) =& -2g^2(\lambda_1\lambda_2)^2(\tilde\lambda_3\tilde\lambda_4)^2\left[\frac{f^{abe}f^{cde}}{st}-\frac{f^{ade}f^{bce}}{tu}\right]\\
&-9|C_G|^2 f^{abe}f^{ecd}(\lambda_1\lambda_2)^2(\tilde\lambda_3\tilde\lambda_4)^2\frac{t-u}{s}+\text{contact},\\
\mathcal{M}(1_a^+ 2_b^+ 3_c^+ 4_d^-) =&6g\mathcal{C}_G(\tilde\lambda_1\tilde\lambda_2)(\tilde\lambda_2\tilde\lambda_3)(\tilde\lambda_3\tilde\lambda_1)(\lambda_4 p_1\sigma p_2\bar\sigma\lambda_4)\left[\frac{f^{ace}f^{bde}}{st}+\frac{f^{ade}f^{bce}}{su}\right]+\text{contact},\\
\mathcal{M}(1_a^+ 2_b^+ 3_c^+ 4_d^+) =&\text{contact}.
\end{aligned}\tag{5.38}$$

Compared to the previously displayed equations, here I again used the Jacobi identity to reshuffle the amplitudes into a form more convenient for the calculation to come.

5.5 RG running on shell

Perhaps the most spectacular application of on-shell methods is for calculating RG running of higher-dimensional operators in EFTs. Working on shell greatly simplifies the calculations and makes them more transparent. It also elucidates the structure of the anomalous dimension matrix, in particular it allows one to understand "magic zeros", that appear mysterious from the point of view of standard calculations.

Let us restrict our discussion to 4-point amplitudes for concreteness, although it can be generalized easily to higher-point amplitudes. A more consequential assumption we also make here is that we deal with massless particles only, both on the external and internal legs. Then, up to one loop, the amplitude can be expanded in the basis of scalar integrals

$$\mathcal{M}_4 = \mathcal{M}_4^{(0)} + \sum_{x=s,t,u} c_2^x I_2^x + \text{triangles} + \text{boxes} + \text{rational}. \tag{5.39}$$

Here $\mathcal{M}_4^{(0)}$ denotes a tree level amplitude, like the ones discussed in the previous subsection, which in particular depends on contact terms. The sum goes over bubble integrals evaluated in dimensional regularization:

$$I_2^{p^2} \equiv \int \frac{d^d k}{i(2\pi)^d} \frac{1}{k^2(k+p)^2} = \frac{1}{16\pi^2}\left[\frac{1}{\epsilon} + \log\left(-\frac{\mu^2}{p^2}\right) + 2\right], \tag{5.40}$$

where $d = 4 - 2\epsilon$, and some irrelevant pieces have been absorbed into the $1/\epsilon$ pole. The coefficients c_2^x will be crucial for determining RG equations. The triangle and box scalar integrals are defined in a similar way but with three and four propagators in the denominator, respectively. These, as well as the rational terms, do not have any UV divergences, therefore their coefficients are not important for RG running.

The bubble coefficients can be determined by using on-shell methods and unitarity. To this end we take the double cut discontinuity of both sides with respect to the kinematic variable p^2. On the left-hand side, the discontinuity is related by unitarity to a product of tree-level amplitudes, for example

$$\text{Disc}_2^s \mathcal{M}_4(1234) = i \sum \int d\Pi_{XY} \mathcal{M}(12XY)\mathcal{M}((-X)(-Y)34), \tag{5.41}$$

where the integral is over the 2-body phase space of the intermediate particle X, Y pair, and the sum goes over their helicity, color, or flavor indices. On the right-hand side, the discontinuity picks up the corresponding bubble coefficients, $\text{Disc}_2^x I_2^x = \frac{i}{8\pi}$. One subtlety here is that the cut is non-zero on triangle and box integrals as well. However, by a direct calculation you can convince yourself that the triangle and box cuts either pick up logarithmic IR divergences, or they yield logarithms of kinematic variables. On the other hand, the bubble discontinuities are pure numbers without any logarithms or other non-analytic pieces. Therefore we can isolate the bubble coefficients by simply dropping all logarithms from the results on both sides. All in all, we arrive at the expression for the bubble coefficients:

$$c_2^s = 8\pi \mathcal{R}\big[\sum \int d\Pi_{XY} \mathcal{M}(12XY)\mathcal{M}((-X)(-Y)34)\big], \tag{5.42}$$

where the $\mathcal{R}$ operator instructs as to drop all logarithms and IR divergences. The analogous expression for $c_2^{t,u}$ is obtained by replacing $2 \leftrightarrow 3, 4$. Once the bubble coefficients are determined this way, the RG running equation for the contact follows from the independence of the amplitude on the regulating parameter μ:

$$0 = \frac{\partial}{\partial \log \mu} \mathcal{M}_4 = \frac{\partial}{\partial \log \mu} \mathcal{M}_4^{(0)} + \frac{c_2^s + c_2^t + c_2^u}{8\pi^2}. \tag{5.43}$$

Note that in general the tree level amplitudes as well as the bubble coefficients may contain independent spinor and group structures, therefore the above equation may include information about running of several distinct contact terms. There is one more subtlety that is relevant for calculating self-renormalization of EFT parameters. Namely, the discussion so far has ignored what in the standard approach is called wave function renormalization, which corresponds to cut through external legs of the amplitude. Taking that into account modifies the running equation by a piece proportional to the tree-level amplitude itself:

$$\frac{\partial}{\partial \log \mu} \mathcal{M}_4^{(0)} = -\frac{c_2^s + c_2^t + c_2^u}{8\pi^2} + \gamma_{\rm coll} \mathcal{M}_4^{(0)}, \tag{5.44}$$

where $\gamma_{\rm coll}$ is called the IR collinear anomalous dimension. In fact, if $\mathcal{M}_4^{(0)}$ contains poles, $\gamma_{\rm coll}$ does not have to be calculated separately, but in fact it can be fixed from consistency of the theory, namely by demanding UV infinities in the full amplitude $\mathcal{M}_4$ are local - they do not have poles.

In the following I will apply these techniques to our Yang-Mills EFT. I will showcase one particular example: renormalization of the dimension-8 contact terms in the all-minus 4-point amplitude in Eq. (5.38) by the dimension-6 C_G parameter defined by the 3-point amplitude in Eq. (5.24). I've been advertising on-shell methods for their simplicity, but there is no free lunch - the calculation is technical and you'll have to get your hands dirty to go through all the steps. Casual readers are perfectly excused to skip to the final discussion. However, if you wish to use on-shell techniques for one loop calculations in your research, below is exactly the kind of exercise to get you up to speed.

To make things slightly easier I restrict here to $SU(2)$ Yang Mills, where $f^{abc} = \epsilon^{abc}$. The all-minus amplitude $\mathcal{M}^{----} \equiv \mathcal{M}(1_a^- 2_b^- 3_c^- 4_d^-)$ up to one loop has the structure

$$\begin{aligned}\mathcal{M}^{----} =& \left\{ C_-^{(1)} (\lambda_1\lambda_2)^2 (\lambda_3\lambda_4)^2 \delta^{ab}\delta^{cd} + C_-^{(2)} \left[(\lambda_1\lambda_3)(\lambda_2\lambda_4) + (\lambda_1\lambda_4)(\lambda_2\lambda_3) \right]^2 \delta^{ab}\delta^{cd} + \text{sym} \right\} \\ &+ \sum_{x=s,t,u} c_2^x I_2^x + \ldots . \end{aligned} \tag{5.45}$$

The first line contains the two independent dimension-8 contact terms at tree level, for which I would like to study the running equations. In the second line I displayed the one-loop scalar bubble contributions, and the dots stand for triangle, boxes, rational, and higher-order contributions. We now isolate the bubble coefficient c_2^s via the s-channel two-particle cut. Using Eq. (5.42),

$$c_2^s = 8\pi \mathcal{R} \int d\Pi_{XY} \mathcal{M}(1_a^- 2_b^- X_e^- Y_f^+) \mathcal{M}(3_c^- 4_d^- (-Y)_f^- (-X)_e^+) + \ldots . \tag{5.46}$$

Here, the dots stand for the analogous terms where X and Y have the same helicity; these will give self-renormalization of $C_-^{(1,2)}$ and are ignored here. Instead I focus on the renormalization of $C_-^{(1,2)}$ by terms proportional to C_G^2, provided by the helicity configuration above. Plugging in the appropriate amplitude from Eq. (5.38) I get

$$\begin{aligned} c_2^s =& -8\pi \frac{36 g^2 \mathcal{C}_G^2 (\lambda_1\lambda_2)^2 (\lambda_3\lambda_4)^2}{s^2} \mathcal{R} \int d\Pi_{XY} (\lambda_1\lambda_X)(\lambda_2\lambda_X)(\tilde\lambda_1\tilde\lambda_Y)(\tilde\lambda_2\tilde\lambda_Y)(\lambda_3\lambda_Y)(\lambda_4\lambda_Y)(\tilde\lambda_3\tilde\lambda_X)(\tilde\lambda_4\tilde\lambda_X) \\ &\times \left[\frac{\epsilon^{aeg}\epsilon^{bfg}}{(2p_1p_X)} + \frac{\epsilon^{afg}\epsilon^{beg}}{(2p_2p_X)} \right] \left[\frac{\epsilon^{cfh}\epsilon^{deh}}{(2p_4p_X)} + \frac{\epsilon^{ceh}\epsilon^{dfh}}{(2p_3p_X)} \right]. \end{aligned} \tag{5.47}$$

With a little bit of ϵ and spinor algebra this simplifies to

$$c_2^s = -8\pi \frac{36g^2\mathcal{C}_G^2(\lambda_1\lambda_2)^2(\lambda_3\lambda_4)^2}{s^2}\mathcal{R}\int d\Pi_{XY}(\lambda_3\lambda_Y)(\lambda_4\lambda_Y)(\tilde\lambda_1\tilde\lambda_Y)(\tilde\lambda_2\tilde\lambda_Y)\Bigg\{ \delta^{ad}\delta^{bc}\left[\frac{(\lambda_2\lambda_X)(\tilde\lambda_3\tilde\lambda_X)}{(\lambda_4\lambda_X)(\tilde\lambda_1\tilde\lambda_X)} + \frac{(\lambda_1\lambda_X)(\tilde\lambda_4\tilde\lambda_X)}{(\lambda_3\lambda_X)(\tilde\lambda_2\tilde\lambda_X)}\right] + \delta^{ac}\delta^{bd}\left[\frac{(\lambda_2\lambda_X)(\tilde\lambda_4\tilde\lambda_X)}{(\lambda_3\lambda_X)(\tilde\lambda_1\tilde\lambda_X)} + \frac{(\lambda_1\lambda_X)(\tilde\lambda_3\tilde\lambda_X)}{(\lambda_4\lambda_X)(\tilde\lambda_2\tilde\lambda_X)}\right]\Bigg\}. \tag{5.48}$$

Now we have to work on the phase space. A convenient trick is to change variables as

$$\begin{aligned} p_X =& -\alpha p_1 - (1-\alpha)p_2 + \sqrt{\alpha(1-\alpha)}\big[zq + z^{-1}\bar q\big], \\ p_Y =& -(1-\alpha)p_1 - \alpha p_2 - \sqrt{\alpha(1-\alpha)}\big[zq + z^{-1}\bar q\big], \end{aligned} \tag{5.49}$$

with $z = e^{i\phi}$, $\alpha \in [0,1]$, $\phi \in [0, 2\pi]$, and $p_1\sigma = \lambda_1\tilde\lambda_1$, $p_2\sigma = \lambda_2\tilde\lambda_2$, $q\sigma = \lambda_2\tilde\lambda_1$, $\bar q\sigma = \lambda_1\tilde\lambda_2$. The phase space integration becomes

$$\int d\Pi_{XY} = \frac{1}{16\pi^2}\int_0^1 d\alpha \int_0^{2\pi} d\phi = \frac{1}{8\pi}\int_0^1 d\alpha \int_{|z|=1}\frac{dz}{2\pi i z}. \tag{5.50}$$

Inserting the parametrization in Eq. (5.49),

$$\begin{aligned} c_2^s = &-\frac{36g^2\mathcal{C}_G^2(\lambda_1\lambda_2)^2(\lambda_3\lambda_4)^2}{2\pi i s}\mathcal{R}\int d\alpha\sqrt{\alpha}\frac{dz}{z^2}\big[\sqrt{1-\alpha}(\lambda_1\lambda_3)+\sqrt{\alpha}z(\lambda_2\lambda_3)\big]\big[\sqrt{1-\alpha}(\lambda_1\lambda_4)+\sqrt{\alpha}z(\lambda_2\lambda_4)\big] \\ &\Bigg\{\delta^{ad}\delta^{bc}\left[\frac{\sqrt{\alpha}\big[z\sqrt{\alpha}(\tilde\lambda_1\tilde\lambda_3)-\sqrt{1-\alpha}(\tilde\lambda_2\tilde\lambda_3)\big]}{\sqrt{1-\alpha}(\lambda_2\lambda_4)\left[z-\sqrt{\frac{\alpha}{1-\alpha}}\frac{(\lambda_1\lambda_4)}{(\lambda_2\lambda_4)}\right]} + \frac{\sqrt{1-\alpha}\big[z\sqrt{\alpha}(\tilde\lambda_1\tilde\lambda_4)-\sqrt{1-\alpha}(\tilde\lambda_2\tilde\lambda_4)\big]}{\sqrt{\alpha}(\lambda_2\lambda_3)\left[z-\sqrt{\frac{\alpha}{1-\alpha}}\frac{(\lambda_1\lambda_3)}{(\lambda_2\lambda_3)}\right]}\right] \\ &+\delta^{ac}\delta^{bd}\left[\frac{\sqrt{\alpha}\big[z\sqrt{\alpha}(\tilde\lambda_1\tilde\lambda_4)-\sqrt{1-\alpha}(\tilde\lambda_2\tilde\lambda_4)\big]}{\sqrt{1-\alpha}(\lambda_2\lambda_3)\left[z-\sqrt{\frac{\alpha}{1-\alpha}}\frac{(\lambda_1\lambda_3)}{(\lambda_2\lambda_3)}\right]} + \frac{\sqrt{1-\alpha}\big[z\sqrt{\alpha}(\tilde\lambda_1\tilde\lambda_3)-\sqrt{1-\alpha}(\tilde\lambda_2\tilde\lambda_3)\big]}{\sqrt{\alpha}(\lambda_2\lambda_4)\left[z-\sqrt{\frac{\alpha}{1-\alpha}}\frac{(\lambda_1\lambda_4)}{(\lambda_2\lambda_4)}\right]}\right]\Bigg\}. \end{aligned} \tag{5.51}$$

This looks a bit scary, but the integrals can be easily done e.g. in Mathematica. The z integral is most easily approached as a contour integral, in which case we need to simply calculate the three different residues of the integrand. Doing so, one has to keep track for which range of α the pole sits within the $|z| = 1$ integration contour (the $z = 0$ pole always does, but for the other two this condition produces non-trivial $\theta(\alpha - \alpha_0)$ step functions in the result). The α integral is elementary. To apply the $\mathcal{R}$ operation, it is easiest to first perform the indefinite integrals, drop all logs, and then evaluate the result on the integration boundaries. When the smoke clears one is left with a compact expression:

$$c_2^s = 6g^2\mathcal{C}_G^2(\lambda_1\lambda_2)^2(\lambda_3\lambda_4)^2\left\{\delta^{ad}\delta^{bc}\frac{s-t}{s} + \delta^{ac}\delta^{bd}\frac{s-u}{s}\right\}. \tag{5.52}$$

By the same token

$$c_2^t = 6g^2\mathcal{C}_G^2(\lambda_1\lambda_3)^2(\lambda_2\lambda_4)^2\left\{\delta^{ad}\delta^{bc}\frac{t-s}{t} + \delta^{ab}\delta^{cd}\frac{t-u}{t}\right\},$$

$$c_2^u = 6g^2\mathcal{C}_G^2(\lambda_1\lambda_4)^2(\lambda_2\lambda_3)^2\left\{\delta^{ab}\delta^{cd}\frac{u-t}{u} + \delta^{ac}\delta^{bd}\frac{u-s}{u}\right\}. \tag{5.53}$$

We can write the all minus amplitude up to one loop as

$$\begin{aligned}\mathcal{M}^{----} =&\delta^{ab}\delta^{cd}\Big\{C_-^{(1)}(\lambda_1\lambda_2)^2(\lambda_3\lambda_4)^2 + C_-^{(2)}\left[(\lambda_1\lambda_3)(\lambda_2\lambda_4)+(\lambda_1\lambda_4)(\lambda_2\lambda_3)\right]^2 \\ &+6g^2\mathcal{C}_G^2\left[(\lambda_1\lambda_3)^2(\lambda_2\lambda_4)^2\frac{t-u}{t}I_2^t + (\lambda_1\lambda_4)^2(\lambda_2\lambda_3)^2\frac{u-t}{u}I_2^u\right]\Big\} + \text{sym} + \ldots\end{aligned} \tag{5.54}$$

The amplitude contains UV divergences inside the bubble integrals I_2^x. At this point it is not yet transparent that these divergences are local, due to the poles in the expression above. To see this, one needs to derive the identity

$$(\lambda_1\lambda_3)^2(\lambda_2\lambda_4)^2\frac{u}{t} + (\lambda_1\lambda_4)^2(\lambda_2\lambda_3)^2\frac{t}{u} = -2(\lambda_1\lambda_3)(\lambda_2\lambda_4)(\lambda_1\lambda_4)(\lambda_2\lambda_3). \tag{5.55}$$

The independence of the amplitude on μ implies the running equations

$$\begin{aligned}&\frac{\partial}{\partial\log\mu}\Big\{C_-^{(1)}(\lambda_1\lambda_2)^2(\lambda_3\lambda_4)^2 + C_-^{(2)}\left[(\lambda_1\lambda_3)(\lambda_2\lambda_4)+(\lambda_1\lambda_4)(\lambda_2\lambda_3)\right]^2\Big\} \\ =&-6g^2\mathcal{C}_G^2\frac{\partial}{\partial\log\mu}\Big\{(\lambda_1\lambda_3)^2(\lambda_2\lambda_4)^2\frac{t-u}{t}I_2^t + (\lambda_1\lambda_4)^2(\lambda_2\lambda_3)^2\frac{u-t}{u}I_2^u\Big\} \\ =&-\frac{3g^2\mathcal{C}_G^2}{4\pi^2}\left[(\lambda_1\lambda_3)(\lambda_2\lambda_4)+(\lambda_1\lambda_4)(\lambda_2\lambda_3)\right]^2\end{aligned} \tag{5.56}$$

In the second step I used that $I_2^x = \frac{\log\mu}{8\pi^2} + \ldots$ and the identity in Eq. (5.55). At the end of the day we obtain the RG equations

$$\begin{aligned}\frac{\partial C_-^{(1)}}{\partial\log\mu} =&0, \\ \frac{\partial C_-^{(2)}}{\partial\log\mu} =&-\frac{3g^2\mathcal{C}_G^2}{4\pi^2}.\end{aligned} \tag{5.57}$$

The fact that $C_-^{(1)}$ does not run with C_G is just due to my choice of the basis.

Exercise: Generalize this calculation to an arbitrary Yang Mills group.

The one loop structure of $\mathcal{M}^{--++}$ encodes the running of the gauge coupling g, as well as that of the dimension-8 contact terms present in that amplitude. Focusing on $\mathcal{M}^{---+}$ instead, we could obtain RG equation for the dimension-6 parameter C_G. These are left as exercises for the long winter nights.

Acknowledgements

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