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THE MORA PROJECT
MATTER'S ORIGIN FROM RADIOACTIVITY

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Effective Field Theories (EFTs)

Lectures given at the Invisibles'24 school in Bologna

27-28 June 2024

Introduction

Brief Philosophy of EFT

Multipole expansion as EFT

Near Observer, $L \sim R$, **needs to know the position of every charge to describe the electric potential in her proximity:**

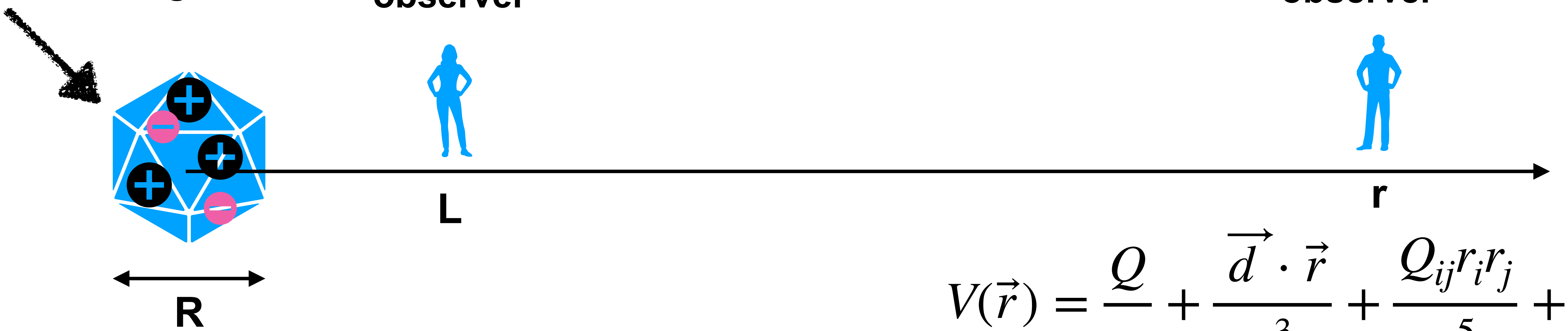
$$V(\vec{r}) = \sum_{n=1}^{N} \frac{q_n}{|\vec{r} - \vec{r}_n|}$$

Far Observer, $r \gg R$, **can instead use multipole expansion:**

$$V(\vec{r}) = \frac{Q}{r} + \frac{\vec{d} \cdot \vec{r}}{r^3} + \frac{Q_{ij} r_i r_j}{r^5} + \ldots$$

$\sim 1/r$	$\sim R/r^2$	$\sim R^2/r^3$
1	**3**	**6**
Monopole	**Dipole**	**Quadrupole**

Multipole expansion as EFT



$$V(\vec{r}) = \frac{Q}{r} + \frac{\vec{d}\cdot\vec{r}}{r^3} + \frac{Q_{ij}r_i r_j}{r^5} + \dots$$

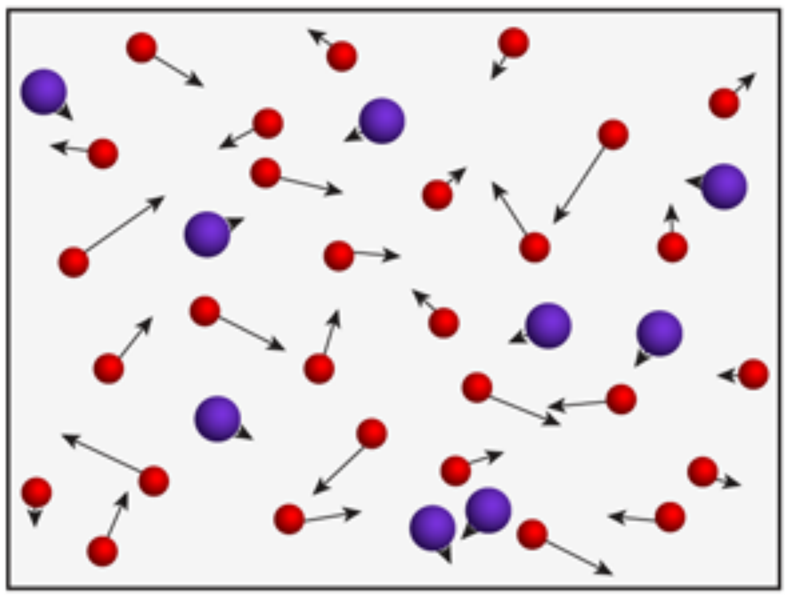
$$\sim 1/r \qquad \sim R/r^2 \qquad \sim R^2/r^3$$

1 3 6

Far Observer, perhaps unknowingly, use EFT!

- With just a handful of parameters, Far Observer is able to describe electric potential in his vicinity with a decent accuracy
- Higher order terms are suppressed by powers of the small parameter R/r
- One can truncate the expansion at some order depending on the value of R/r and experimental precision
- On the other hand, Far Observer can only guess the "fundamental" distributions of the charges, as infinitely many distinct distributions lead to the same first few moments

EFT around us



$\mapsto\;= 10^{-10}$ m

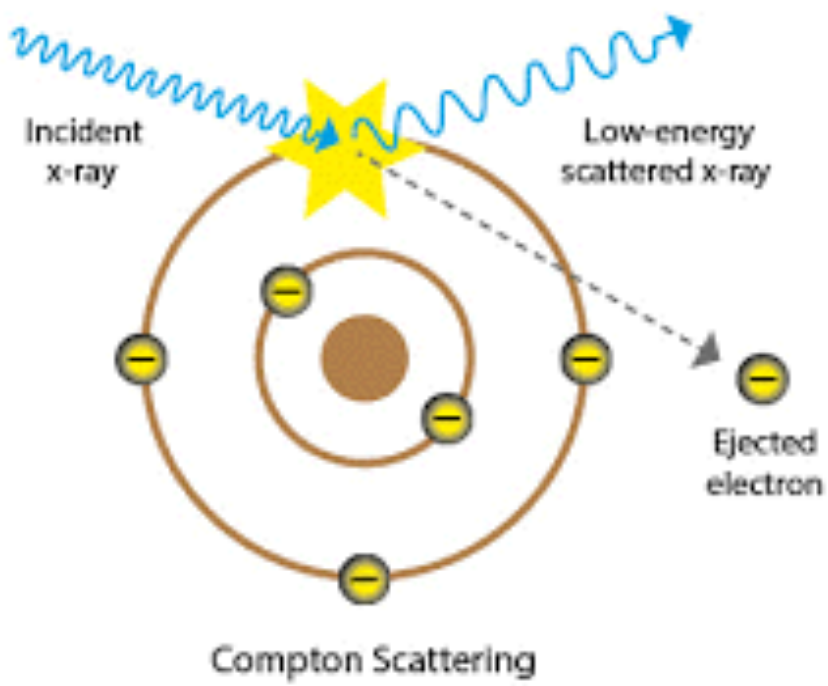
At small scales,
the degrees of freedom of gas
are positions and velocities
of its component atoms

$$\rho, p, T, s$$

$\mapsto\;= 10^{-2}$ m

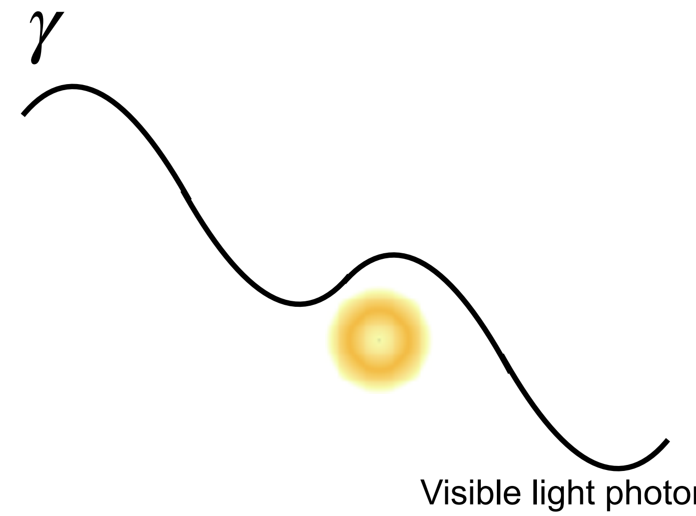
At large scales,
the useful degrees of freedom
are its macroscopic properties
like density, pressure,
temperature, or entropy

EFT around us



$$|\!-\!-\!-\!-\!| = \frac{1}{m_e \alpha}$$

X-ray photons see the atomic structure and scatter on the orbiting electrons



$$|\!-\!-\!-\!-\!| = \frac{10}{m_e \alpha}$$

Lower-energy photons see atoms as neutral objects (with multipole moments) which are basically transparent to low energy radiation

(that's how the universe becomes transparent to photons right after recombination)

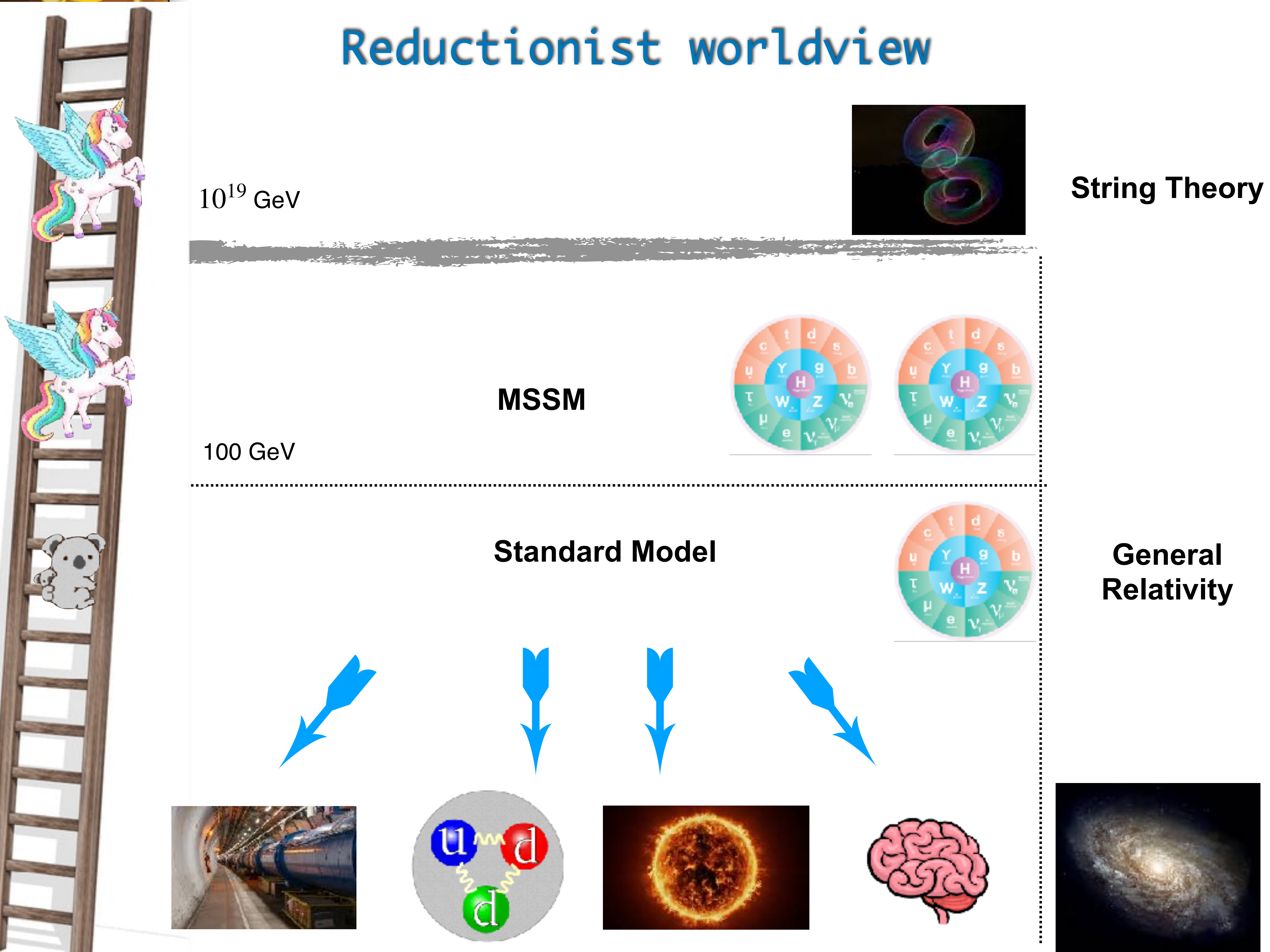
EFT and QFT

- Up to this point, one can say that EFT is just fancy dimensional analysis

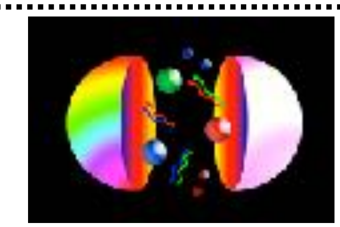
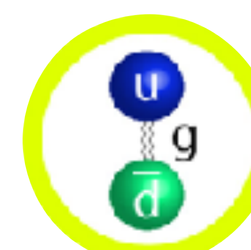
- When EFT is married with a relativistic quantum theory, additional principles are at work which make it less trivial:
  - Poincaré symmetry (particles are representation of little group)
  - Locality (constrains the structure of singularities of S matrix)
  - Unitarity (connects singularities of S-matrix to lower point amplitudes)
  - Causality (constrains the analytic structure of S-matrix)

- From this point, EFT will be discussed only in the context of relativistic QFTs. Most of the time, EFT will be decoded in a Poincaré-invariant, local, hermitian Lagrangian, where these principles are more or less automatically satisfied

Reductionist worldview



EFT worldview

UV	100 TeV ?	**Dragons**	
SMEFT	100 GeV	$\gamma, g, W, Z, \nu_i, e, \mu, \tau$ + u, d, s, c, b, t + h	
WEFT5	5 GeV	$\gamma, g, \nu_i, e, \mu, \tau$ + u, d, s, c, b	
WEFT4	2 GeV	$\gamma, g, \nu_i, e, \mu, \tau$ + u, d, s, c	
ChRT	1 GeV	γ, ν_i, e, μ + hadrons	
ChPT	100 MeV	$\gamma, \nu_i, e, \mu, \pi, K, p^{(*)}$	
QED+	1 MeV	$\gamma, \nu_i, e, p^{(*)}$	
EH+	0.01 eV	$\gamma, \nu_i, p^{(*)}, e^{(*)}$ $\gamma, p^{(*)}, e^{(*)}$	

EFT worldview

UV	10^{19} GeV	**More Dragons**
	100 TeV ?	**Dragons**
GRSMEFT	100 GeV	$h_2, h_0, \gamma, g, W, Z, \nu_i, e, \mu, \tau,$ **u, d, s, c, b, t**
GRWEFT5	5 GeV	$h_2, \gamma, g, \nu_i, e, \mu, \tau,$ **u, d, s, c, b**
GRWEFT4	2 GeV	$h_2, \gamma, g, \nu_i, e, \mu, \tau,$ **u, d, s, c**
GRChRT	1 GeV	$h_2, \gamma, \nu_i, e, \mu$ **+ hadrons**
GRChPT	100 MeV	$h_2, \gamma, \nu_i, e, \mu, \pi, K, p^{(*)}$
GRQED	1 MeV	$h_2, \gamma, \nu_i, e, p^{(*)}$
GREH+	0.01 eV	$h_2, \gamma, \nu_i, p^{(*)}, e^{(*)}$ $h_2, \gamma, p^{(*)}, e^{(*)}$

Recommended reading

General

- Kaplan [nucl-th/0510023]
- Rothstein [hep-ph/0308266]
- Manohar [1804.05863]

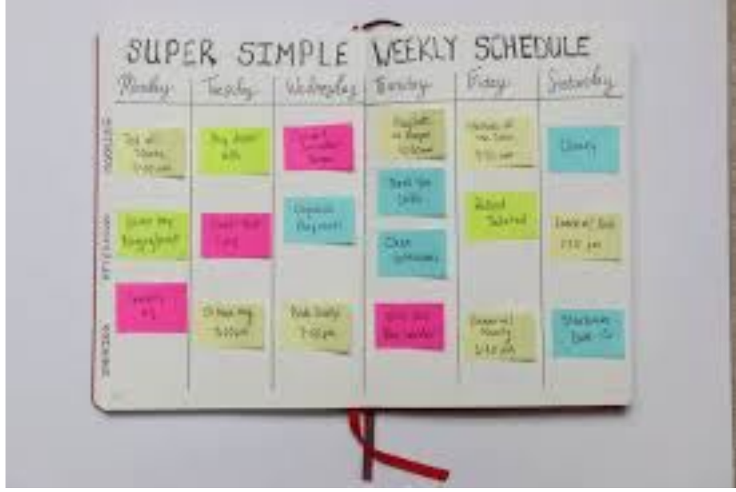
See also my lecture notes from GGI'24

Recommended reading

Specific EFTs

- EFT for superconductors: Polchinski [hep-th/9210046]
- EFT for heavy mesons: Grinstein [hep-ph/9411275]
- EFT for binary inspirals: Goldberger [hep-ph/07101129]
- EFT for low-energy QCD: Pich [1804.05664]
- EFT for nuclei: Van Kolck [1902.03141]
- EFT of the SM degrees of freedom: AA [Eur.Phys.J.C 83 (2023) 7, 656]

Timetable



- Lecture 1
  Effective toy story or an EFT of a single scalar

- Lecture 2
  EFT in action or an illustrated philosophy of EFT

- Lecture 3
  SMEFT et al. or effective theory above the electroweak scale

Lecture 1

Effective Toy Story

Settings

- We will write down a simple toy model EFT and beat it to death
- The EFT has a single degree of freedom: a real scalar
- We will also consider a renormalizable model with two scalars, one parametrically heavier than the other, and discuss the relationship between the low-energy limit of the two-scalar "fundamental" model and the one-scalar EFT
- The goal is to demonstrate, at the more quantitative level, some important EFT concepts as power counting, matching, running, reparametrization invariance, basis, naturalness …

EFT Lagrangian

Consider an EFT of a single real scalar ϕ of mass m invariant under the Z_2 symmetry $\phi \to -\phi$
From the bottom-up perspective, the EFT Lagrangian should have the form

$$\mathscr{L}_{\text{EFT}} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m^2\phi^2\right] + \mathscr{L}_{\text{int}}(\phi, \partial)$$

- An EFT is by definition a theory with some cutoff $\Lambda > m$ containing an infinite number of interactions
- Each interaction term is a function of the field ϕ and its derivatives $\partial_\mu \phi$
- We need a principle to organize these interactions so as to identify the most important ones in the energy regime $E \lesssim \Lambda$ where the EFT is valid
- Such a principle is called power counting .

For $m \ll E$, on dimensional grounds the 2-to-2 scattering amplitude should be

$$\mathscr{M}(\phi\phi \to \phi\phi) \sim \sum_i \#_i C_i E^{-[C_i]}$$

(ignoring non-analytic pieces from loops)

Order one coefficient (where $\pi \sim 1$) — $\#_i$

Wilson coefficients of EFT — C_i

Its mass dimension in natural units — $[C_i]$

Dimensional analysis

Using the unit system where $c = \hbar = 1$. Then all objects can be assigned mass dimension

$$[m] = [E] = \text{mass}^1 \;\Rightarrow\; [x] = [t] = \text{mass}^{-1} \;\Rightarrow\; [\partial_\mu] \equiv \left[\frac{\partial}{\partial x^\mu}\right] = \text{mass}^1$$

Canonical dimension of fields follow from canonically normalized action:

$$S = \int d^4x \mathscr{L} = \int d^4x \left\{ \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2}[\partial_\mu A_\nu - \partial_\nu A_\mu]\partial^\mu A^\nu \right\}$$

Action is dimensional
(because path integral contains $e^{iS/\hbar}$) $\Rightarrow$

$$[\phi] = \text{mass}^1$$

$$[\psi] = \text{mass}^{3/2}$$

$$[A] = \text{mass}^1$$

These rules allows one to determine dimensions of any interaction term, e.g.

$$\mathscr{L} \supset \lambda |H|^4 + C_H |H|^6 + C_\psi (\psi\psi)(\bar{\psi}\bar{\psi}) + \ldots \;\Rightarrow\; [\lambda] = \text{mass}^0 \quad [C_H] = \text{mass}^{-2} \quad [C_\psi] = \text{mass}^{-2}$$

Power counting

Consider an EFT of a single real scalar ϕ of mass m invariant under the Z_2 symmetry $\phi \to -\phi$
From the bottom-up perspective. the EFT Lagrangian should have the form

$$\mathscr{L}_{\text{EFT}} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m^2\phi^2\right] + \mathscr{L}_{\text{int}}(\phi, \partial)$$

- A natural power counting is to assume $C_i \sim \Lambda^{[C_i]}$
- Then, on dimensional grounds, the 2-to-2 scattering amplitude should be

$$\mathscr{M}(\phi\phi \to \phi\phi) \sim \sum_i \#_i (E/\Lambda)^{-[C_i]}$$

- For $m \ll E \ll \Lambda$ the Wilson coefficients with larger $[C_i]$ should be more relevant, while those with smaller $[C_i]$ should be less relevant
- Depending on the experimental precision, below some small enough $[C_i]$ the effects of the Wilson coefficients can be ignored whatsoever

Power counting

Consider an EFT of a single real scalar ϕ of mass m invariant under the Z_2 symmetry $\phi \to -\phi$
From the bottom-up perspective. the EFT Lagrangian should have the form

dimension 4 | dimension 6 | higher dimension

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!} + \mathcal{O}(\Lambda^{-4})$$

$C_4 \sim \Lambda^0$, $C_6 \sim \Lambda^{-2}$

- Then, on dimensional grounds, the 2-to-2 scattering amplitude should be

$$\mathscr{M}(\phi\phi \to \phi\phi) \sim \#_4 C_4 (E/\Lambda)^0 + \#_6 C_6 (E/\Lambda)^2 + \mathcal{O}(E/\Lambda)^4$$

In the following discussion we will ignore interactions with dimensions 8 and higher

EFT Lagrangian

By general arguments, the EFT Lagrangian must have the following form

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!} + \mathscr{O}(\Lambda^{-4})$$

What about other dimension-6 operators, e.g.

$$\hat{O}_6 \equiv (\Box\phi)^2, \quad \tilde{O}_6 \equiv \phi\,\Box\,\phi^3, \quad \tilde{O}'_6 \equiv \phi^2\,\Box\,\phi^2, \quad \tilde{O}''_6 \equiv \phi^2\partial_\mu\phi\partial_\mu\phi, \quad \ldots$$

These are all redundant, that is to say,
they can be expressed by the operators already present in $\mathscr{L}_{\rm EFT}$ by using integration by parts and field redefinitions

Redundant operators

$$\mathscr{L}_{\mathrm{EFT}} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!} + \mathcal{O}(\Lambda^{-4})$$

$$O_6 \equiv \phi^6, \quad \hat{O}_6 \equiv (\Box\phi)^2, \quad \tilde{O}_6 \equiv \phi^3\Box\phi, \quad \tilde{O}_6' \equiv \phi^2\Box\phi^2, \quad \tilde{O}_6'' \equiv \phi^2\partial_\mu\phi\partial_\mu\phi, \quad \ldots$$

Use Leibniz rule + integration by parts:

$$\phi^2\partial_\mu\phi\partial_\mu\phi = -2\phi\partial_\mu\phi\partial_\mu\phi\phi - \phi^3\Box\phi \quad \Rightarrow \quad \tilde{O}_6'' = -\frac{1}{3}\phi^3\Box\phi = -\frac{1}{3}\tilde{O}_6$$

$$\phi^2\Box\phi^2 = 2\phi^2\partial_\mu(\phi\partial_\mu\phi) = 2\phi^3\Box\phi + 2\phi^2(\partial_\mu\phi)^2 \quad \Rightarrow \quad \tilde{O}_6' = 2\tilde{O}_6 + 2\tilde{O}_6'' = \frac{4}{3}\tilde{O}_6$$

Redundant operators

$$\mathscr{L}_{\text{EFT}} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!} + \mathcal{O}(\Lambda^{-4})$$

$$O_6 \equiv \phi^6, \quad \hat{O}_6 \equiv (\Box\phi)^2, \quad \tilde{O}_6 \equiv \phi^3\Box\phi, \quad \tilde{O}'_6 \equiv \phi^2\Box\phi^2, \quad \tilde{O}''_6 \equiv \phi^2\partial_\mu\phi\partial_\mu\phi, \quad \ldots$$

Use equations of motion: $\Box\phi = -m^2\phi - \frac{C_4}{6}\phi^3 + \mathcal{O}(\Lambda^{-2})$

This is relevant only if we want to keep track of dimension-8 operators

$$\tilde{O}_6 \equiv \phi^3\Box\phi = -m^2\phi^4 - \frac{C_4}{6}\phi^6 = -m^2 O_4 - \frac{C_4}{6}O_6$$

$$\hat{O}_6 \equiv (\Box\phi)^2 = m^4\phi^2 + \frac{m^2C_4}{3}\phi^4 + \frac{C_4^2}{36}\phi^6 = m^4 O_2 + \frac{m^2C_4}{3}O_4 + \frac{C_4^2}{36}O_6$$

$O_2 \equiv \phi^2$

$O_4 \equiv \phi^4$

Redundant operators

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!} + \mathcal{O}(\Lambda^{-4})$$

$$O_6 \equiv \phi^6, \quad \hat{O}_6 \equiv (\Box\phi)^2, \quad \tilde{O}_6 \equiv \phi^3\Box\phi, \quad \tilde{O}_6' \equiv \phi^2\Box\phi^2, \quad \tilde{O}_6'' \equiv \phi^2\partial_\mu\phi\partial_\mu\phi, \quad \ldots$$

In this case, equations of motion = field redefinitions

$$\phi \to \phi + x\phi^3 \qquad x \sim \mathcal{O}(\Lambda^{-2})$$

$$\mathscr{L}_{\rm EFT} = -\frac{1}{2}\phi\left[\Box + m^2\right]\phi - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!} + \mathcal{O}(\Lambda^{-4})$$

$$\to \mathscr{L}_{\rm EFT} - x\left\{\phi^3\left[\Box + m^2\right]\phi + C_4\frac{\phi^6}{6}\right\} + \mathcal{O}(\Lambda^{-4})$$

Chisholm Nucl. Phys. 26 (1961), no. 3 469–479

Since shifted and unshifted Lagrangian must lead to the same on-shell amplitudes,

C. Arzt, [hep-ph/9304230]

$$\phi^3\left[\Box + m^2\right]\phi + C_4\frac{\phi^6}{6} = 0 \quad \Rightarrow \quad \tilde{O}_6 = -m^2 O_4 - \frac{C_4}{6}O_6$$

$O_4 \equiv \phi^4$

Bases of operators

"Unbox basis"

$$\mathscr{L}_{\mathrm{EFT}} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4\,!} - C_6\frac{\phi^6}{6\,!} + \mathcal{O}(\Lambda^{-4})$$

We can equivalently use an EFT Lagrangian where O₆ is absent, and replaced by another equivalent operator

$O_2 \equiv \phi^2$

$O_4 \equiv \phi^4$

$O_6 \equiv \phi^6$

$$\tilde{O}_6 \equiv \phi^3 \Box \phi = -m^2 O_4 - \frac{C_4}{6} O_6 \Rightarrow O_6 = -\frac{6}{C_4}\phi^3 \Box \phi - \frac{6m^2}{C_4} O_4$$

"Box basis"

$$\tilde{\mathscr{L}}_{\mathrm{EFT}} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - \tilde{m}^2\phi^2\right] - \tilde{C}_4\frac{\phi^4}{4\,!} - \tilde{C}_6\frac{\phi^3 \Box \phi}{4\,!} + \mathcal{O}(\Lambda^{-4})$$

Map between the Wilson coefficients in the two bases

$$\tilde{C}_6 = -\frac{C_6}{5C_4}$$

$$\tilde{C}_4 = C_4 - \frac{m^2 C_6}{5C_4}$$

$$\tilde{m} = m$$

Bases of operators

"Unbox basis"

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!} + \mathcal{O}(\Lambda^{-4})$$

We can equivalently use an EFT Lagrangian where O_6 is absent, and replaced by another equivalent operators

$$O_2 \equiv \phi^2$$
$$O_4 \equiv \phi^4$$
$$O_6 \equiv \phi^6$$

$$\hat{O}_6 \equiv (\Box\phi)^2 = m^4\phi^2 + \frac{m^2 C_4}{3}\phi^4 + \frac{C_4^2}{36}\phi^6 \quad \Rightarrow \quad O_6 = \frac{36}{C_4^2}\hat{O}_6 - 12\frac{m^2}{C_4}O_4 + \frac{36 m^4}{C_4^2}O_2$$

"Double-Box basis"

$$\tilde{\mathscr{L}}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - \hat{m}^2\phi^2\right] - \hat{C}_4\frac{\phi^4}{4!} - \hat{C}_6\frac{(\Box\phi)^2}{2} + \mathcal{O}(\Lambda^{-4})$$

Map between the Wilson coefficients in the two bases

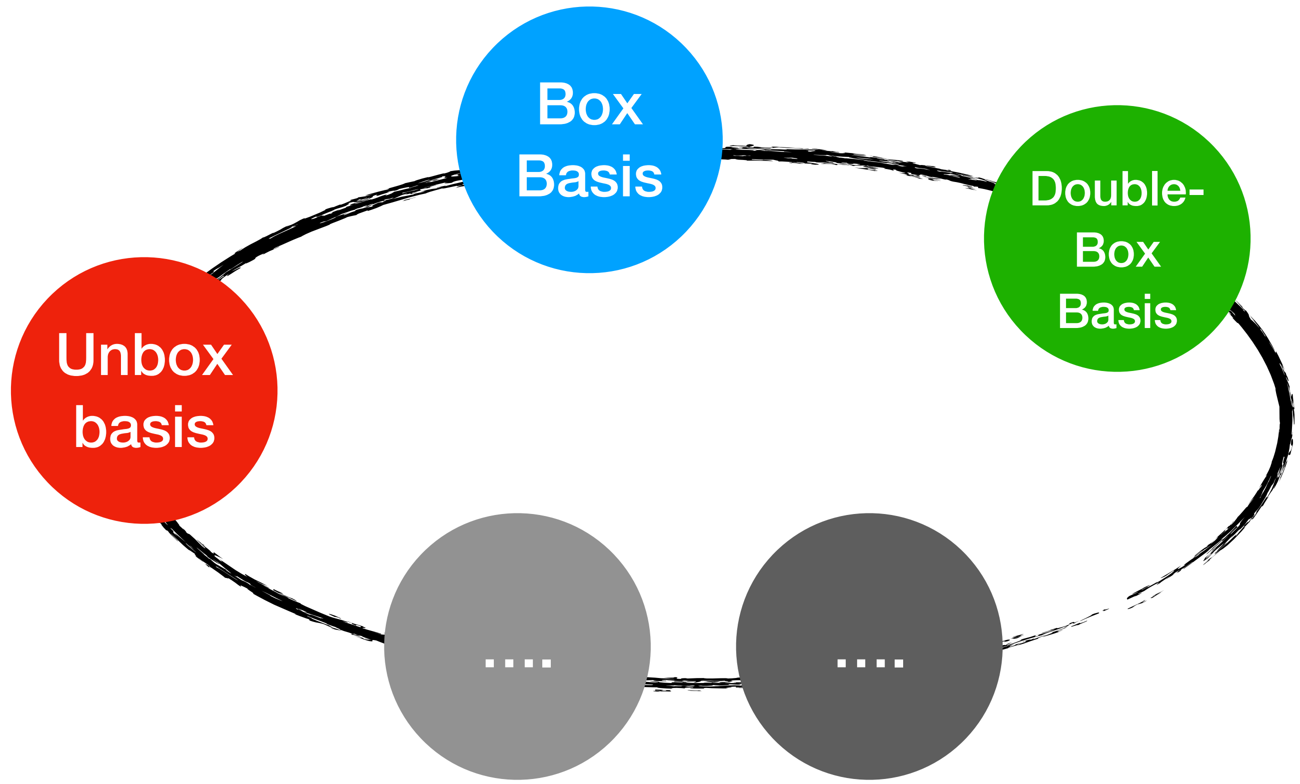
$$\hat{C}_6 = -\frac{C_6}{10 C_4^2}$$

$$\hat{C}_4 = C_4 - \frac{2m^2 C_6}{5 C_4}$$

$$\hat{m}^2 = m^2 - \frac{m^4 C_6}{30 C_4^2}$$

Bases of operators

Every EFT has an infinite number of equivalent bases



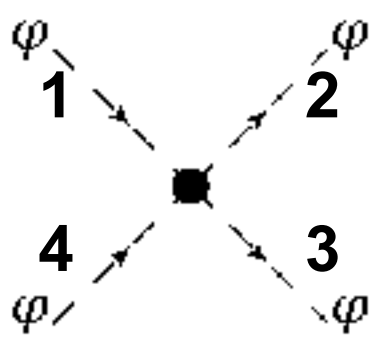
Physics is independent of which basis we use,
but the Lagrangian and intermediate calculations look different in different bases!

In our toy example, a basis of dimension-6 operators is one dimensional
(to be compared e.g. with the 3045-dimensional basis of dimension-6 operators in the SMEFT)

On-shell vs Off-shell

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!}$$
$$-C_6\frac{\phi^6}{6!} - \tilde{C}_6\frac{\phi^3\Box\phi}{4!} + \mathcal{O}(\Lambda^{-4})$$

Calculate 4-point Feynman vertex off shell



$$= i\left\{ -C_4 + \frac{\tilde{C}_6}{4}\sum_{i=1}^{4} p_i^2 \right\}$$

Off shell, the redundant operator clearly differ !

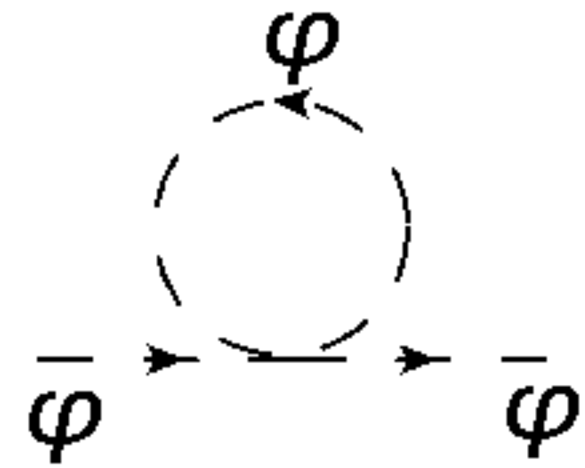
On shell: $= i\left\{ -\tilde{C}_4 + \frac{\tilde{C}_6}{4}\sum_{i=1}^{4} p_i^2 \right\} \to i\left\{ -\tilde{C}_4 + \tilde{C}_6 m^2 \right\} = i\left\{ -C_4 + \frac{m^2C_6}{5C_4} - \frac{m^2C_6}{5C_4} \right\} = -iC_4$

On shell, the box basis and the unbox basis give the same Feynman rule, taking into account the map between the Wilson coefficients

$$\tilde{C}_6 = -\frac{C_6}{5C_4}$$
$$\tilde{C}_4 = C_4 - \frac{m^2C_6}{5C_4}$$
$$\tilde{m} = m$$

One-loop corrections in EFT

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!}$$



In this EFT, there is a single diagram contributing to the ϕ mass at one loop

$$\delta M_2^{\rm EFT} = -\frac{C_4}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2}$$

$$= C_4\frac{m^2}{32\pi^2}\left[\frac{1}{\bar\epsilon} + \log\left(\frac{\mu^2}{m^2}\right) + 1\right]$$

$$1/\bar\epsilon \equiv 1/\epsilon + \gamma_E + \log(4\pi)$$

Note that we use dimensional regularization, which is very convenient in the EFT context, as it does not introduce new mass scales, so it does not mess up the EFT power counting

Furthermore, we will use the MSbar renormalization, simply dropping all $1/\bar\epsilon$ poles

The one-loop-corrected ϕ mass in the EFT at one loop in this scheme:

$$m^2_{\rm phys} = m^2 - C_4\frac{m^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m^2}\right) + 1\right]$$

Running of the mass parameter

The physical mass is an observable in this model, therefore it cannot depend on the arbitrary parameter μ

$$\frac{dm^2_{\rm phys}}{d\log\mu} = 0$$

This means that the Lagrangian mass parameter, up to higher-loop corrections, must satisfy

$$\frac{dm^2}{d\log\mu} = C_4\frac{m^2}{16\pi^2}$$

The solution is

$$m^2(\mu) = m^2(\Lambda)\left(\frac{\mu}{\Lambda}\right)^{\frac{C_4}{16\pi^2}}$$

We can interpret μ as the renormalization group scale

This also shows that naive scaling of EFT parameters with Λ is modified by loop effects therefore the exponent is called *the anomalous dimension*

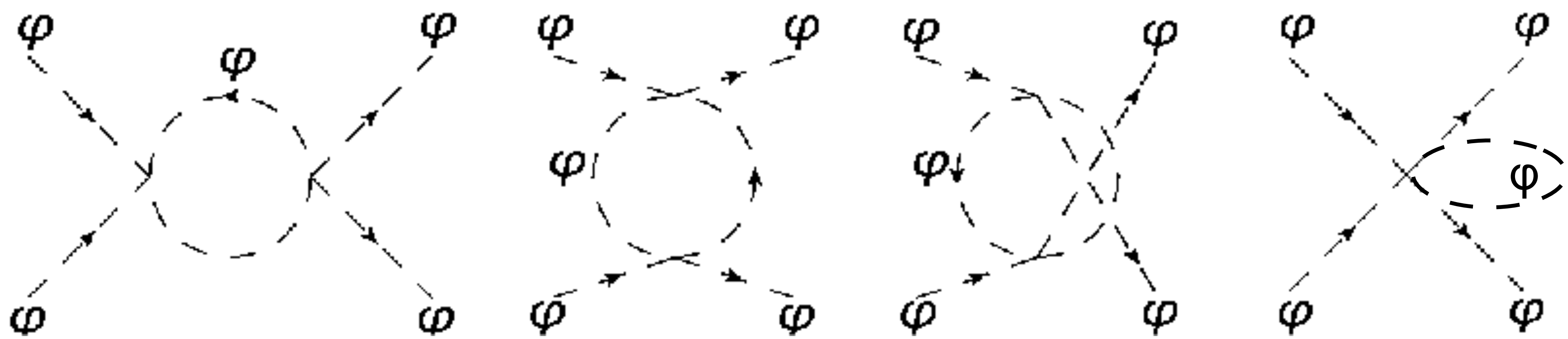
$$m^2 \sim \Lambda^{2+\gamma} \qquad \gamma = -\frac{C_4}{16\pi^2}$$

Corrections to two-to-two scattering

We move to one-loop matching of the quartic coupling

EFT calculation

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!}$$



Answer

$$f(s,m) \equiv \sqrt{1-\frac{4m^2}{s}}\log\left(\frac{2m^2 - s + \sqrt{s(s-m^2)}}{2m^2}\right)$$

$$M_4^{\rm EFT} = -C_4 + \frac{C_4^2}{32\pi^2}\left[f(s,m) + f(t,m) + f(u,m)\right]$$

$$+\frac{3C_4^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + 2\right) + \frac{C_6 m^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right) + 1\right)$$

Running of the EFT quartic coupling

$$M_4^{\text{EFT}} = -C_4 + \frac{C_4^2}{32\pi^2}\left[f(s,m) + f(t,m) + f(u,m)\right] + \frac{3C_4^2}{32\pi^2}\left(\log\left(\frac{\mu^2}{m^2}\right) + 2\right) + \frac{C_6 m^2}{32\pi^2}\left(\log\left(\frac{\mu^2}{m^2}\right) + 1\right)$$

The observable in this case is $A_4^{\text{EFT}} \equiv \dfrac{M_4^{\text{EFT}}}{(1+\delta_\phi)^2}$ **where δ_ϕ is wave function renormalization**

A_4 **can be related to the cross section, so it must not depend on** μ

One can show that $\delta_\phi = 0$ at one loop in the unbox basis

Therefore M_4 cannot depend on the arbitrary parameter μ: $\dfrac{dM_4^{\text{EFT}}}{d\log\mu} = 0$

This means that the Lagrangian parameters, up to higher-loop corrections, must satisfy

$$\frac{dC_4}{d\log\mu} = \frac{3C_4^2}{16\pi^2} + \frac{C_6 m^2}{16\pi^2}$$

Studying 6-point amplitudes, we would also obtain an RG equation for C_6:

$$16\pi^2 \frac{dC_6}{d\log\mu} = \# C_6$$

Summary of one-scalar EFT

- Relativistic EFTs can be organized according to canonical dimensions of interaction terms in the Lagrangian (here, interactions of dimension 4, 6, 8, in order of important at low energies)

- Interaction terms can be redundant if they are related by integration by parts or field redefinitions (here, one dimensional basis at dimension 6)

- Symmetries can constrain the number of allowed interactions (here Z_2 symmetry forbids interactions with odd number of fields)

- EFTs make perfect sense beyond tree level. Wilson coefficients of higher-dimensional operators exhibit running behaviour

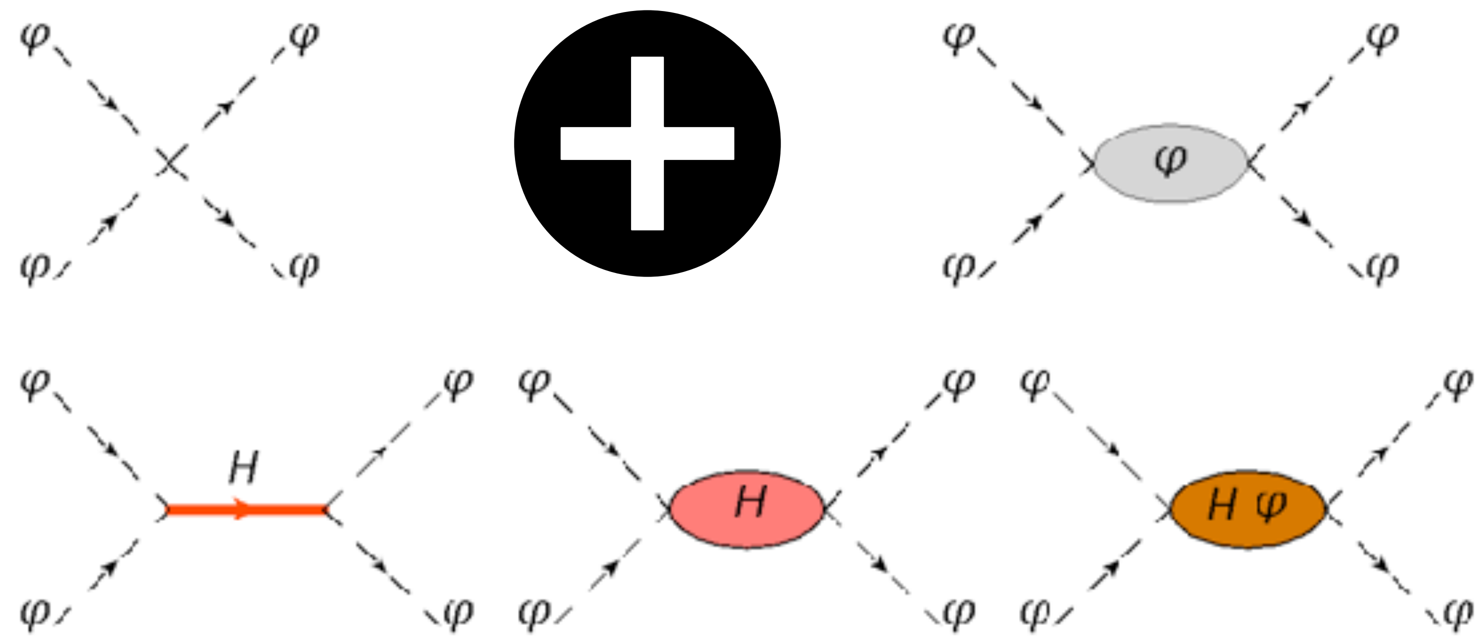
UV model

Toy model: one light scalar ϕ and one heavy scalar H

$$\mathscr{L}_{\mathrm{UV}} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m_L^2 \phi^2 + (\partial_\mu H)^2 - M^2 H^2\right] - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2} M \phi^2 H - \frac{\lambda_2}{4}\phi^2 H^2$$

In this theory we can consider scattering amplitudes for the light scalar, e.g.

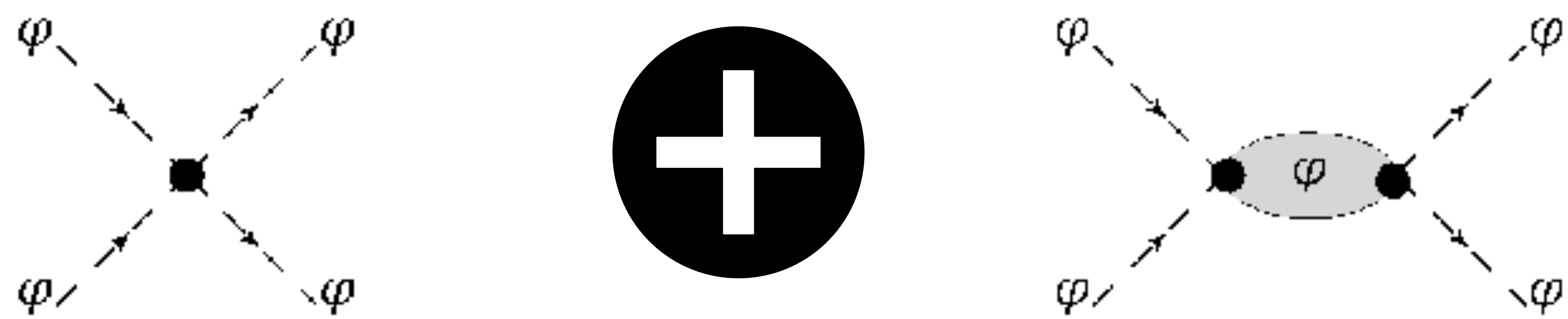
$$\mathscr{M}_{\mathrm{UV}}(\phi\phi \to \phi\phi) =$$



The goal is to write down a local effective Lagrangian, with cutoff Λ identified as M, such that the same ϕ amplitudes are recovered

$$\mathscr{L}_{\mathrm{UV}}(\phi, H) \to \mathscr{L}_{\mathrm{EFT}}(\phi)$$

$$\mathscr{M}_{\mathrm{EFT}}(\phi\phi \to \phi\phi) =$$



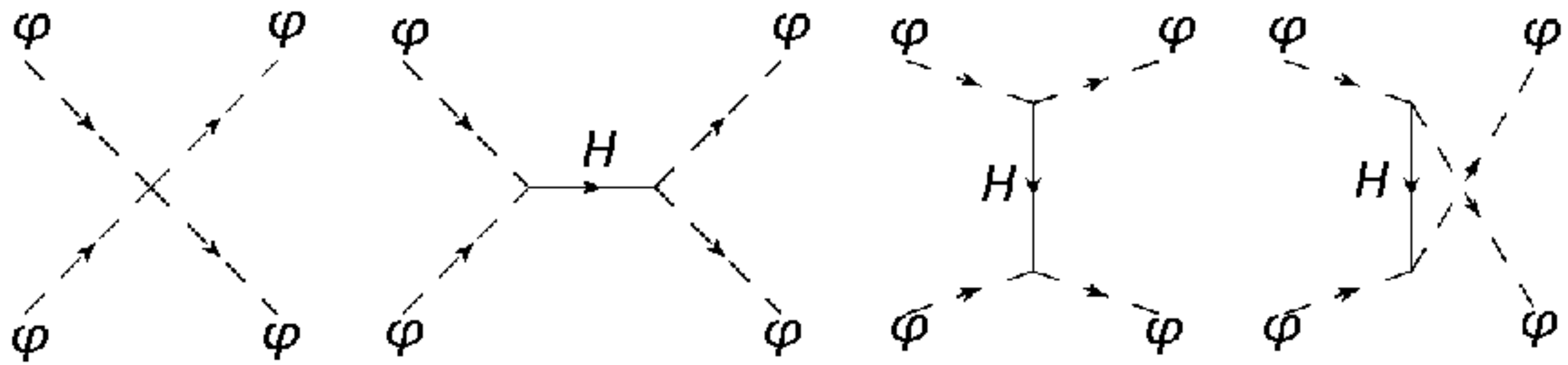
We want $\mathscr{M}_{\mathrm{EFT}}(\phi\phi \to \phi\phi) = \mathscr{M}_{\mathrm{UV}}(\phi\phi \to \phi\phi) + \mathcal{O}\left(M^{-n}(16\pi^2)^{-m}\right)$ **for some chosen n and m**

Tree level matching

Step 1: demand $\mathscr{M}_{\rm UV}(\phi\phi \to \phi\phi)$ and $\mathscr{M}_{\rm EFT}(\phi\phi \to \phi\phi)$ are equal up to order M^{-2}

UV amplitude

$$\mathscr{L}_{\rm UV} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m_L^2 \phi^2 + (\partial_\mu H)^2 - M^2 H^2\right] - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2} M \phi^2 H - \frac{\lambda_2}{4}\phi^2 H^2$$



$$\mathscr{M}_{UV} = -\lambda_0 - \lambda_1^2 M^2 \left[\frac{1}{s - M^2} + \frac{1}{t - M^2} + \frac{1}{u - M^2}\right]$$

$$= -\lambda_0 + 3\lambda_1^2 + \frac{\lambda_1^2}{M^2}(s + t + u) + \mathscr{O}(M^{-4})$$

$$= -\lambda_0 + 3\lambda_1^2 + \frac{4 m_L^2 \lambda_1^2}{M^2} + \mathscr{O}(M^{-4})$$

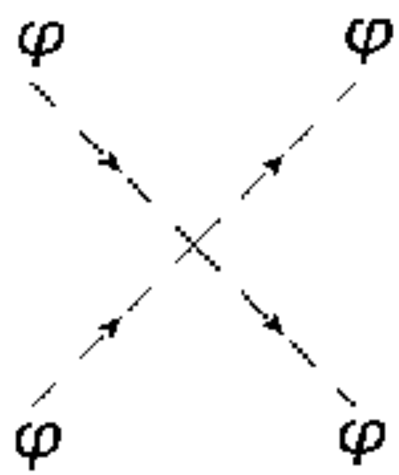
Tree level matching

Step 1: demand $\mathcal{M}_{\rm UV}(\phi\phi \to \phi\phi)$ and $\mathcal{M}_{\rm EFT}(\phi\phi \to \phi\phi)$ are equal up to order M^{-2}

$$\mathcal{M}_{UV} = -\lambda_0 + 3\lambda_1^2 + \frac{4m_L^2\lambda_1^2}{M^2} + \mathcal{O}(M^{-4})$$

EFT amplitude

$$\mathcal{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!} + \mathcal{O}(M^{-4})$$



$$\mathcal{M}_{EFT} = -C_4 + \mathcal{O}(M^{-4})$$

Matching

$$C_4 = \lambda_0 - 3\lambda_1^2 - 4\lambda_1^2\frac{m_L^2}{M^2} + \mathcal{O}(M^{-4})$$

Tree level matching

Step 2: demand $\mathscr{M}_{\rm UV}(\phi\phi \to \phi\phi\phi\phi)$ and $\mathscr{M}_{\rm EFT}(\phi\phi \to \phi\phi\phi\phi)$ are equal up to order M^{-2}

This is feasible but more pesky to do at the amplitude level.
Let's use a trick....

Interlude

EFT and path integrals

EFT and path integrals

Integrating out heavy particles is particularly transparent using the path integral formulation of QFT, because then it's literally integrating over the heavy fields…

The generating functional in the UV theory of light fields φ and heavy fields H

$$Z_{\mathrm{UV}}[J_\phi, J_H] = \int [D\phi][DH] \exp\left[i \int d^4x \left(\mathscr{L}_{\mathrm{UV}}(\phi, H) + J_\phi \phi + J_H H \right) \right]$$

The generating functional in the EFT of light fields φ

$$Z_{\mathrm{EFT}}[J_\phi] = \int [D\phi] \exp\left[i \int d^4x \left(\mathscr{L}_{\mathrm{EFT}}(\phi) + J_\phi \phi \right) \right]$$

Matching consists in imposing the condition

$$Z_{\mathrm{EFT}}[J_\phi] = Z_{\mathrm{UV}}[J_\phi, 0]$$

At leading order (tree-level), the field configurations contributing to the path integral are the ones that extremize the action:

$$Z_{\mathrm{UV}}[J_\phi, 0] = \int [D\phi] \exp\left[i \int d^4x \left(\mathscr{L}_{\mathrm{UV}}(\phi, H_{\mathrm{cl}}(\phi)) + J_\phi \phi \right) \right] \qquad 0 = \frac{\delta S}{\delta H}\Big|_{H = H_{\mathrm{cl}}(\phi)}$$

that is, $H_{cl}(\varphi)$ solves the classical equations of motion in the UV Lagrangian

Hence $$\mathscr{L}_{\mathrm{EFT}}(\phi) = \mathscr{L}_{\mathrm{UV}}(\phi, H_{\mathrm{cl}}(\phi))$$

Tree level matching

Step 2: demand $\mathscr{M}_{\rm UV}(\phi\phi \to \phi\phi\phi\phi)$ **and** $\mathscr{M}_{\rm EFT}(\phi\phi \to \phi\phi\phi\phi)$ **are equal up to order** M^{-2}

$$\mathscr{L}_{\rm UV} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m_L^2\phi^2 + (\partial_\mu H)^2 - M^2 H^2\right] - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2}M\phi^2 H - \frac{\lambda_2}{4}\phi^2 H^2$$

eom: $$\left[\Box + M^2 + \frac{\lambda_2}{2}\phi^2\right]H + \frac{\lambda_1 M}{2}\phi^2 = 0 \quad \Rightarrow \quad H(\phi) = -\frac{\lambda_1 M}{2}\left[M^2 + \Box + \frac{\lambda_2}{2}\phi^2\right]^{-1}\phi^2$$

$$\mathscr{L}_{\rm EFT}(\phi) = \mathscr{L}_{\rm UV}(\phi, H(\phi)) =$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2}M\phi^2 H(\phi) - \frac{1}{2}H(\phi)\left[\Box + M^2 + \frac{\lambda_2}{2}\phi^2\right]H(\phi)$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4 + \frac{\lambda_1^2 M^2}{8}\phi^2\left[M^2 + \Box + \frac{\lambda_2}{2}\phi^2\right]^{-1}\phi^2$$

Expanding in $1/M$ **:**

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4 + \frac{\lambda_1^2}{8}\phi^2\left[1 - \frac{\Box}{M^2} - \frac{\lambda_2}{2M^2}\phi^2\right]\phi^2 + \mathcal{O}(M^{-4})$$

Tree level matching

Step 2: demand $\mathscr{M}_{\rm UV}(\phi\phi \to \phi\phi\phi\phi)$ and $\mathscr{M}_{\rm EFT}(\phi\phi \to \phi\phi\phi\phi)$ are equal up to order M^{-2}

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0}{4!}\phi^4 + \frac{\lambda_1^2}{8}\phi^2\left[1 - \frac{\Box}{M^2} - \frac{\lambda_2}{2M^2}\phi^2\right]\phi^2 + \mathscr{O}(M^{-4})$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0 - 3\lambda_1^2}{4!}\phi^4 - \frac{\lambda_1^2}{8M^2}\phi^2\Box\phi^2 - \frac{\lambda_1^2\lambda_2}{16M^2}\phi^6 + \mathscr{O}(M^{-4})$$

This should be matched to:

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!}$$

But at this point the 2 Lagrangians have a different form because the former contains redundant operators

Eliminate redundancy using

$$\phi^2\Box\phi^2 = \frac{4}{3}\phi^3\Box\phi = -\frac{4m_L^2}{3}\phi^4 - \frac{2(\lambda_0 - 3\lambda_1^2)}{9}\phi^6$$

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0 - 3\lambda_1^2}{4!}\phi^4 + \frac{\lambda_1^2 m_L^2}{6M^2}\phi^4 + \frac{\lambda_1^2\lambda_0}{36M^2}\phi^6 - \frac{\lambda_1^2\lambda_2}{16M^2}\phi^6 - \frac{\lambda_1^4}{12M^2}\phi^6 + \mathscr{O}(M^{-4})$$

$$= \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0 - 3\lambda_1^2 - 4\lambda_1^2 m_L^2/M^2}{4!}\phi^4 - \frac{\lambda_1^2(-20\lambda_0 + 45\lambda_2 + 60\lambda_1^2)}{6!M^2}\phi^6 + \mathscr{O}(M^{-4})$$

Tree level matching

Step 2: demand $\mathcal{M}_{\rm UV}(\phi\phi \to \phi\phi\phi\phi)$ and $\mathcal{M}_{\rm EFT}(\phi\phi \to \phi\phi\phi\phi)$ are equal up to order M^{-2}

$$\mathcal{L}_{\rm EFT} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m_L^2}{2}\phi^2 - \frac{\lambda_0 - 3\lambda_1^2 - 4\lambda_1^2 m_L^2/M^2}{4!}\phi^4 - \frac{\lambda_1^2(45\lambda_2 - 20\lambda_0 + 60\lambda_1^2)}{6!M^2}\phi^6 + \mathcal{O}(M^{-4})$$

$$= \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!}$$

Tree-level matching

Agrees with the result from amplitude matching →

$$m^2 = m_L^2$$

$$C_4 = \lambda_0 - 3\lambda_1^2 - \frac{4m_L^2}{M^2}\lambda_1^2$$

$$C_6 = \frac{\lambda_1^2}{M^2}\left(45\lambda_2 - 20\lambda_0 + 60\lambda_1^2\right)$$

One-loop matching

The story so far

$$\mathscr{L}_{\rm UV} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m_L^2\phi^2 + (\partial_\mu H)^2 - M^2H^2\right] - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2}M\phi^2 H - \frac{\lambda_2}{4}\phi^2H^2$$

$$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - \frac{C_6}{M^2}\frac{\phi^6}{6!}$$

$$m^2 = m_L^2 + \mathcal{O}\left(\frac{1}{16\pi^2}\right) + \mathcal{O}\left(\frac{1}{M^4}\right)$$

$$C_4 = \lambda_0 - 3\lambda_1^2 - \frac{4m_L^2}{M^2}\lambda_1^2 + \mathcal{O}\left(\frac{1}{16\pi^2}\right) + \mathcal{O}\left(\frac{1}{M^4}\right)$$

$$C_6 = \frac{\lambda_1^2}{M^2}\left(45\lambda_2 - 20\lambda_0 + 60\lambda_1^2\right) + \mathcal{O}\left(\frac{1}{16\pi^2}\right) + \mathcal{O}\left(\frac{1}{M^4}\right)$$

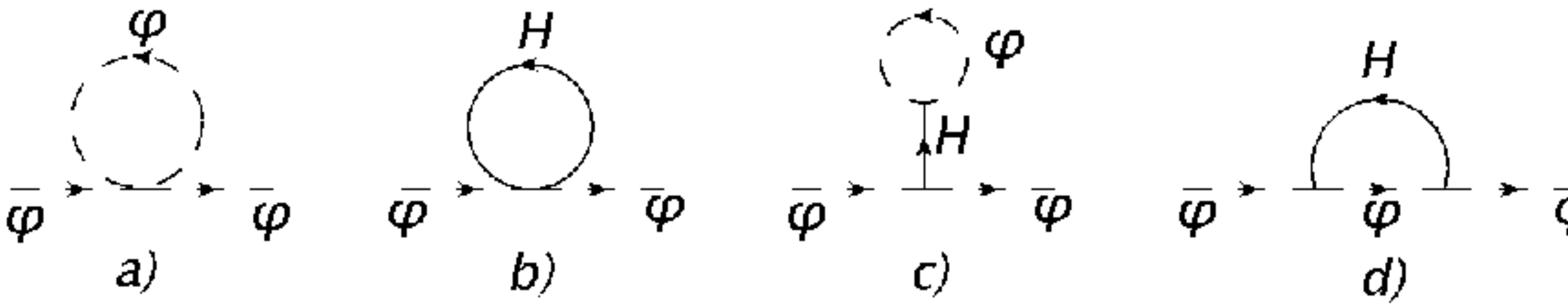
Next step will be to determine the 1-loop suppressed terms in this matching equation

Start with 1-loop matching of the mass parameters

One-loop matching

$$\mathscr{L}_{\rm UV} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m_L^2\phi^2 + (\partial_\mu H)^2 - M^2 H^2\right] - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2}M\phi^2 H - \frac{\lambda_2}{4}\phi^2 H^2$$

Start with 1-loop matching of the mass parameters



$$\delta_a M_2^{\rm UV} = \lambda_0 \frac{m_L^2}{32\pi^2}\left[\frac{1}{\bar\epsilon} + \log\left(\frac{\mu^2}{m_L^2}\right) + 1\right]$$

Same as in EFT but with a different quartic

$$\delta_b M_2^{\rm UV} = \lambda_2 \frac{M^2}{32\pi^2}\left[\frac{1}{\bar\epsilon} + \log\left(\frac{\mu^2}{M^2}\right) + 1\right]$$

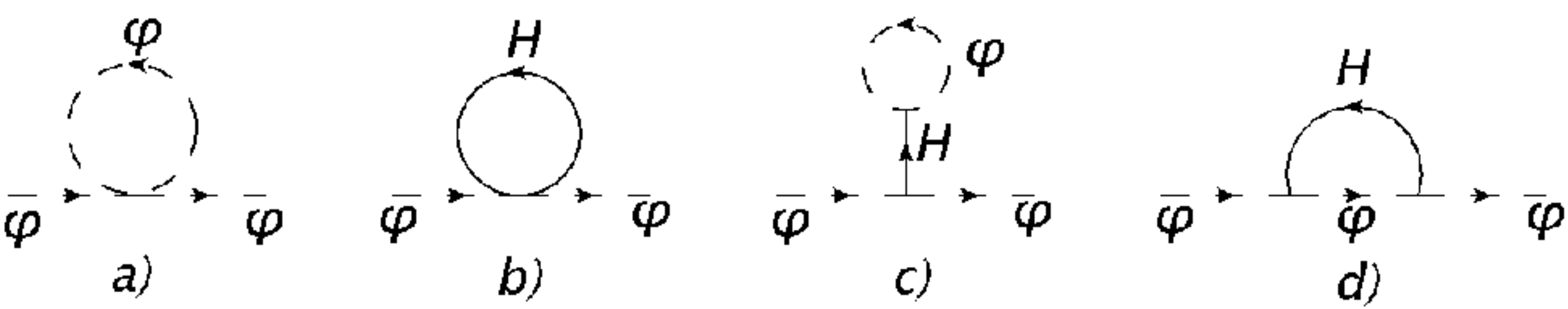
$$\delta_c M_2^{\rm UV} = -\lambda_1^2 \frac{m_L^2}{32\pi^2}\left[\frac{1}{\bar\epsilon} + \log\left(\frac{\mu^2}{m_L^2}\right) + 1\right]$$

$$\delta_d M_2^{\rm UV} = \lambda_1^2 \frac{M^2}{16\pi^2}\left[\frac{1}{\bar\epsilon} + \log\left(\frac{\mu^2}{M^2}\right) + 1\right]$$

$$+\lambda_1^2 \frac{m_L^2}{32\pi^2}\left[-2\log\left(\frac{M^2}{m_L^2}\right) + 1\right] + \lambda_1^2\frac{m_L^4}{48\pi^2 M^2}\left[-6\log\left(\frac{M^2}{m_L^2}\right) + 5\right]$$

Diagrams with a heavy particle loop lead to quadratic sensitivity to M

One-loop matching

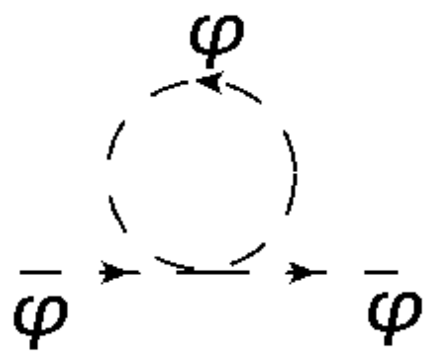


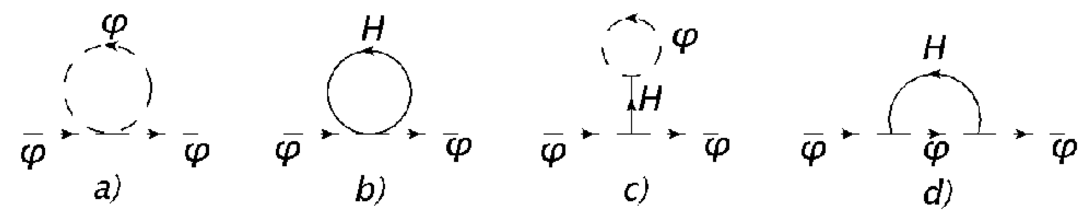
All in all, the physical mass of ϕ calculated in the UV theory is

$$
\begin{aligned}
m_{\text{phys}}^2 = m_L^2 \\
&-\left(\lambda_0 - 3\lambda_1^2 - 4\lambda_1^2 \frac{m_L^2}{M^2}\right)\frac{m_L^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m_L^2}\right)+1\right] \\
&-\frac{1}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right)\left[M^2\left(\lambda_2+2\lambda_1^2\right)+2\lambda_1^2 m_L^2+4\lambda_1^2\frac{m_L^4}{M^2}\right] \\
&-\frac{1}{32\pi^2}\left[M^2\left(\lambda_2+2\lambda_1^2\right)+3\lambda_1^2 m_L^2+\frac{22}{3}\lambda_1^2\frac{m_L^4}{M^2}\right]
\end{aligned}
$$

One-loop matching

EFT	UV
$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - C_6\frac{\phi^6}{6!}$	$\mathscr{L}_{\rm UV} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m_L^2\phi^2 + (\partial_\mu H)^2 - M^2H^2\right] - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2}M\phi^2 H - \frac{\lambda_2}{4}\phi^2H^2$





$$m^2_{\rm phys} = m^2 - C_4\frac{m^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m^2}\right)+1\right]$$

$$m^2_{\rm phys} = m_L^2 - \left(\lambda_0 - 3\lambda_1^2 - 4\lambda_1^2\frac{m_L^2}{M^2}\right)\frac{m_L^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m_L^2}\right)+1\right] - \frac{1}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right)\left[M^2\left(\lambda_2+2\lambda_1^2\right)+2\lambda_1^2m_L^2+4\lambda_1^2\frac{m_L^4}{M^2}\right] - \frac{1}{32\pi^2}\left[M^2\left(\lambda_2+2\lambda_1^2\right)+3\lambda_1^2m_L^2+\frac{22}{3}\lambda_1^2\frac{m_L^4}{M^2}\right]$$

Tree level matching

$$m^2 = m_L^2$$

$$C_4 = \lambda_0 - 3\lambda_1^2 - \frac{4m_L^2}{M^2}\lambda_1^2$$

$\log(\mu/m_L)$ **piece in the 1st line match automatically, given the tree-level matching of C₄!**

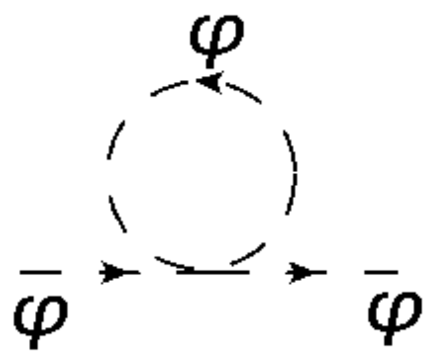
This is mathematically non-trivial, but physically it must be so, because they are both IR contributions

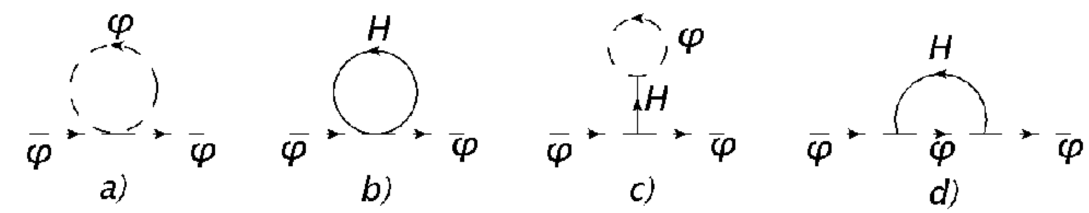
One-loop matching

$$m^2(\mu) = m_L^2(\mu) - \frac{1}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right)\left[M^2\left(\lambda_2+2\lambda_1^2\right)+2\lambda_1^2m_L^2+4\lambda_1^2\frac{m_L^4}{M^2}\right] - \frac{1}{32\pi^2}\left[M^2\left(\lambda_2+2\lambda_1^2\right)+3\lambda_1^2m_L^2+\frac{22}{3}\lambda_1^2\frac{m_L^4}{M^2}\right]$$

One-loop matching

EFT	UV
$\mathscr{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - \frac{C_6}{M^2}\frac{\phi^6}{6!}$	$\mathscr{L}_{\rm UV} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m_L^2\phi^2 + (\partial_\mu H)^2 - M^2H^2\right] - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_1}{2}M\phi^2 H - \frac{\lambda_2}{4}\phi^2H^2$





EFT:

$$m_{\rm phys}^2 = m^2 - C_4\frac{m^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m^2}\right) + 1\right]$$

UV:

$$m_{\rm phys}^2 = m_L^2 - \left(\lambda_0 - 3\lambda_1^2 - 4\lambda_1^2\frac{m_L^2}{M^2}\right)\frac{m_L^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m_L^2}\right) + 1\right] - \frac{1}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right)\left[M^2\left(\lambda_2 + 2\lambda_1^2\right) + 2\lambda_1^2 m_L^2 + 4\lambda_1^2\frac{m_L^4}{M^2}\right] - \frac{1}{32\pi^2}\left[M^2\left(\lambda_2 + 2\lambda_1^2\right) + 3\lambda_1^2 m_L^2 + \frac{22}{3}\lambda_1^2\frac{m_L^4}{M^2}\right]$$

Tree level matching

$$m^2 = m_L^2$$

$$C_4 = \lambda_0 - 3\lambda_1^2 - \frac{4m_L^2}{M^2}\lambda_1^2$$

We can perform the matching at any scale μ. But it will be simplest, both conceptually and algebraically, if we we perform the matching at $\mu = M$, as then 1) the second line in the UV term vanishes, and 2) there are no large logarithms in the matching equation

One-loop matching

$$m^2(M) = m_L^2(M) - \frac{1}{32\pi^2}\left[M^2\left(\lambda_2 + 2\lambda_1^2\right) + 3\lambda_1^2 m_L^2 + \frac{22}{3}\lambda_1^2\frac{m_L^4}{M^2}\right]$$

One-loop matching and running

At loop level, we match the parameters of the UV and EFT theories at a high scale that is at least roughly, or better exactly, the mass of the particle being integrated out

$$m^2(M) = m_L^2(M) - \frac{1}{32\pi^2}\left[M^2\left(\lambda_2 + 2\lambda_1^2\right) + 3\lambda_1^2 m_L^2 + \frac{22}{3}\lambda_1^2\frac{m_L^4}{M^2}\right]$$

To use the EFT at a lower scale, $\mu \ll M$,
we should evolve the EFT parameters using the RG running equation

$$\frac{dm^2}{d\log\mu} = C_4\frac{m^2}{16\pi^2} \quad \Rightarrow \quad m^2(\mu) = m^2(M)\left(\frac{\mu}{M}\right)^{\frac{C_4}{16\pi^2}}$$

Potentially large logarithms appearing in loop calculations are resummed in the RG evolution!

$$m^2\left[1 + \frac{C_4}{16\pi^2}\log\left(\frac{\mu}{M}\right)\right] \quad \rightarrow \quad m^2(M)\left(\frac{\mu}{M}\right)^{\frac{C_4}{16\pi^2}}$$

Hierarchy problem

The hierarchy problem is often presented as a quadratic dependence of a scalar (Higgs) mass on the cutoff of the theory, $m^2 \sim \Lambda^2/16\pi^2$

This does not make sense, as conclusions depend on the regularization procedure

EFTs and matching offer a modern and robust way to understand the hierarchy problem

$$m^2(M) = m_L^2(M) - \frac{1}{32\pi^2}\left[M^2\left(\lambda_2 + 2\lambda_1^2\right) + 3\lambda_1^2 m_L^2 + \frac{22}{3}\lambda_1^2\frac{m_L^4}{M^2}\right]$$

Even if in the UV theory we had a hierarchy of scalar masses, $m_L \ll M$, matching equations between the EFT and the UV theory push the mass of the light scalar to $m^2 \sim M^2/16\pi^2$

The hierarchy problem consists in sensitivity of scalar masses in an EFT to the masses of the heavy particles that have been integrated out

This sensitivity necessitates fine-tuning between m_L and M, or constructing a theory where this sensitivity is avoided

However, once this step is achieved, one way or another, then the EFT with a light scalar is consistent and natural, as $m(\mu) \sim m$ for any $\mu \ll M$

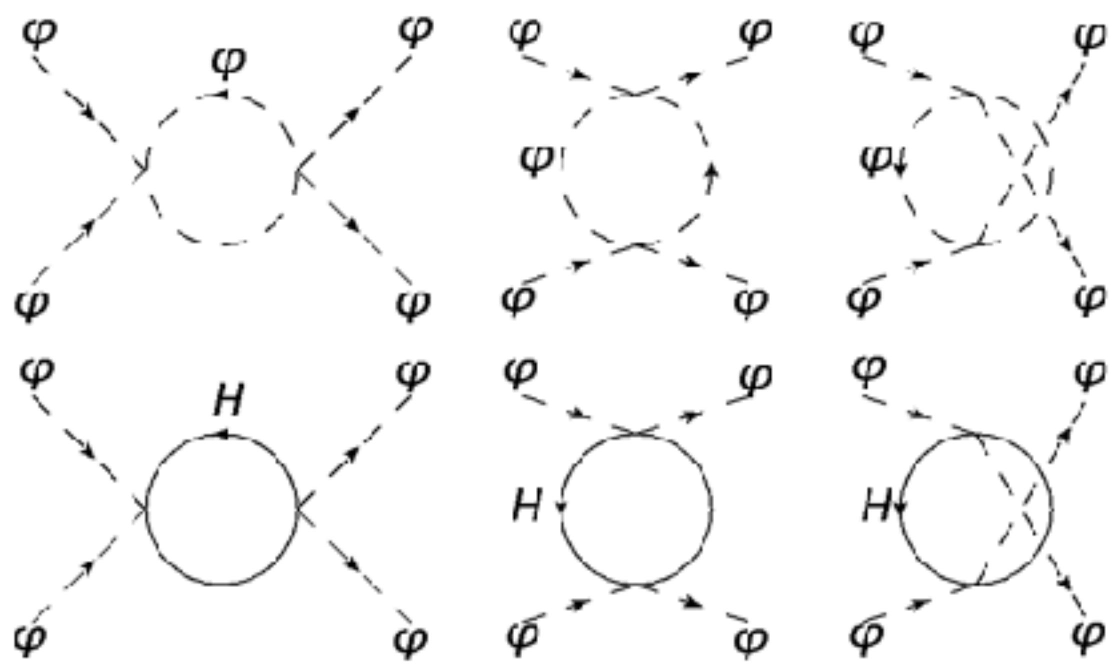
$$m^2(\mu) = m^2(M)\left(\frac{\mu}{M}\right)^{\frac{C_4}{16\pi^2}}$$

One-loop matching

$$\mathscr{L}_{\rm UV} = \frac{1}{2}\left[(\partial_\mu \phi)^2 - m_L^2\phi^2 + (\partial_\mu H)^2 - M^2H^2\right] - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_2}{4}\phi^2H^2$$

For simplicity, we set $\lambda_1 = 0$ in the UV Lagrangian in the following, to reduce the number of diagrams

UV calculation



$$f(s,m) \equiv \sqrt{1 - \frac{4m^2}{s}} \log\left(\frac{2m^2 - s + \sqrt{s(s-m^2)}}{2m^2}\right)$$

Answer

$$M_4^{\rm UV} = -\lambda_0 + \frac{3\lambda_0^2}{32\pi^2}\left(\frac{1}{\bar\epsilon} + \log\left(\frac{\mu^2}{m^2}\right) + 2\right) + \frac{3\lambda_2^2}{32\pi^2}\left(\frac{1}{\bar\epsilon} + \log\left(\frac{\mu^2}{M^2}\right) + 2\right)$$

$$+\frac{\lambda_0^2}{32\pi^2}\left[f(s,m) + f(t,m) + f(u,m)\right] + \frac{\lambda_2^2}{32\pi^2}\left[f(s,M) + f(t,M) + f(u,M)\right].$$

One-loop matching

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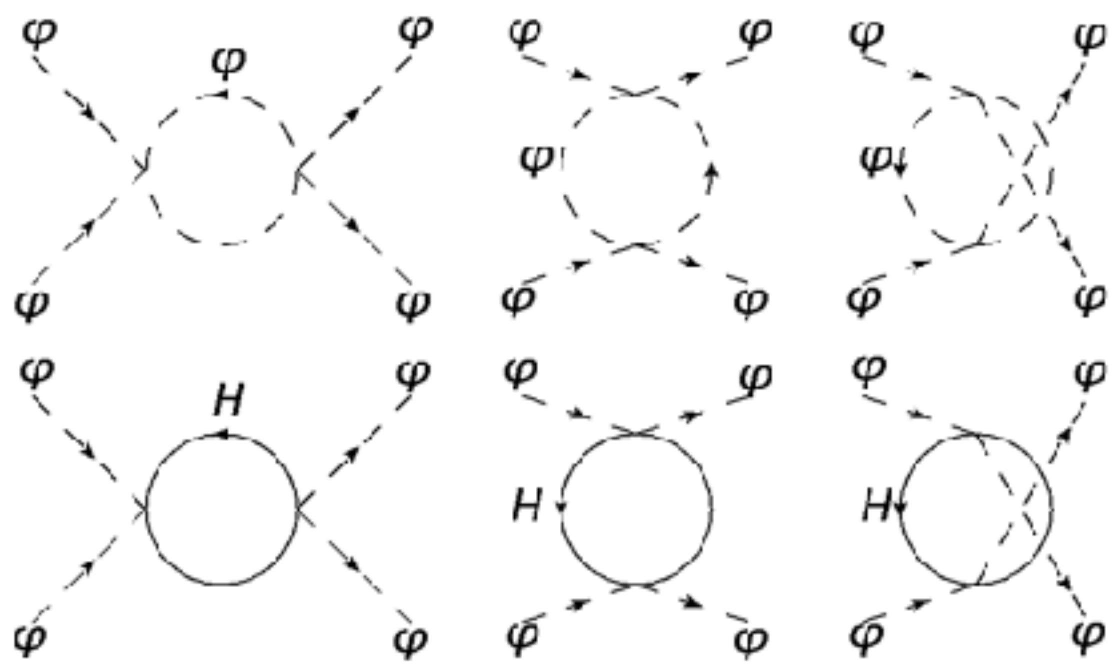


Answer expanded in $1/M$

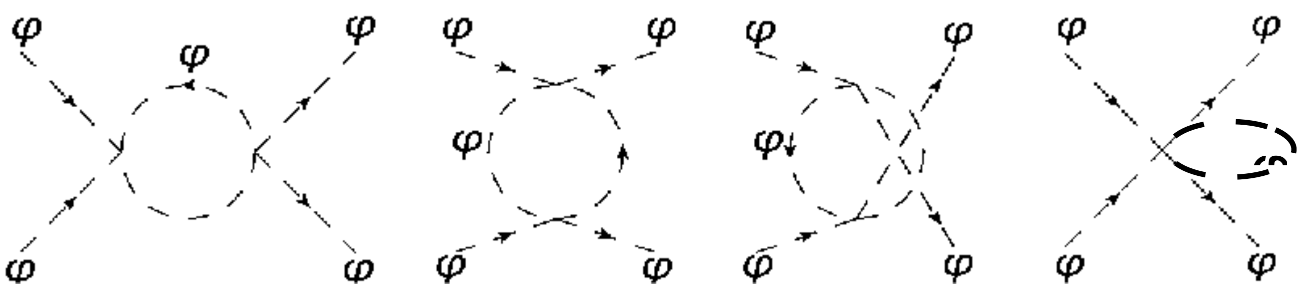
$$f(s,m) \equiv \sqrt{1-\frac{4m^2}{s}}\log\left(\frac{2m^2 - s + \sqrt{s(s-m^2)}}{2m^2}\right)$$

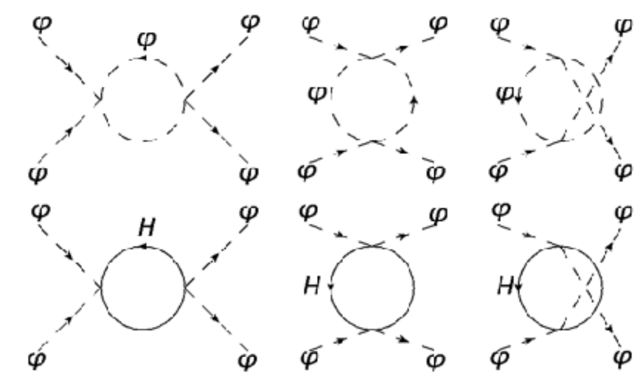
$$M_4^{\rm UV} = -\lambda_0 + \frac{3\lambda_0^2}{32\pi^2}\left(\frac{1}{\bar\epsilon} + \log\left(\frac{\mu^2}{m^2}\right) + 2\right) + \frac{3\lambda_2^2}{32\pi^2}\left(\frac{1}{\bar\epsilon} + \log\left(\frac{\mu^2}{M^2}\right)\right)$$

$$+\frac{\lambda_0^2}{32\pi^2}\left[f(s,m) + f(t,m) + f(u,m)\right] + \frac{m^2\lambda_2^2}{48\pi^2M^2} + \mathcal{O}(1/M^4)$$

One-loop matching

EFT	UV
$\mathcal{L}_{\rm EFT} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m^2\phi^2\right] - C_4\frac{\phi^4}{4!} - \frac{C_6}{M^2}\frac{\phi^6}{6!}$	$\mathcal{L}_{\rm UV} = \frac{1}{2}\left[(\partial_\mu\phi)^2 - m_L^2\phi^2 + (\partial_\mu H)^2 - M^2H^2\right] - \frac{\lambda_0}{4!}\phi^4 - \frac{\lambda_2}{4}\phi^2H^2$





EFT:

$$M_4^{\rm EFT} = -C_4 + \frac{3C_4^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m^2}\right)+2\right] + \frac{C_4^2}{32\pi^2}\left[f(s,m)+f(t,m)+f(u,m)\right] + \frac{C_6 m^2}{32\pi^2M^2}\left[\log\left(\frac{\mu^2}{m^2}\right)+1\right]$$

UV:

$$M_4^{\rm UV} = -\lambda_0 + \frac{3\lambda_0^2}{32\pi^2}\left[\log\left(\frac{\mu^2}{m^2}\right)+2\right] + \frac{\lambda_0^2}{32\pi^2}\left[f(s,m)+f(t,m)+f(u,m)\right] + \frac{m^2\lambda_2^2}{48\pi^2M^2} + \frac{3\lambda_2^2}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right)$$

Matching so far

$$m^2 = m_L^2(M) - \lambda_2\frac{M^2}{32\pi^2}$$

$$C_4 = \lambda_0$$

$$C_6 = 0$$

Once again, matching is simpler and potentially large logarithms are avoided, if matching is performed at $\mu = M$

$$C_4(M) = \lambda_0(M) - \frac{\lambda_2^2 m^2}{48\pi^2M^2}$$

One-loop matching and running

UV-EFT matching at a high scale

$$m^2(M) = m_L^2(M) - \lambda_2 \frac{M^2}{32\pi^2}$$

$$C_4(M) = \lambda_0(M) - \frac{\lambda_2^2 m^2}{48\pi^2 M^2}$$

To use the EFT at a lower scale, $\mu \ll M$,
we should evolve the EFT parameters using the RG running equation

$$\frac{dm^2}{d\log\mu} = C_4 \frac{m^2}{16\pi^2}$$

$$\frac{dC_4}{d\log\mu} = \frac{3C_4^2}{16\pi^2} + \frac{C_6 m^2}{16\pi^2}$$

$C_6 = 0$

$$m^2(\mu) = m^2(M)\left(\frac{\mu}{M}\right)^{\frac{C_4(\mu)}{16\pi^2}}$$

$$\frac{1}{C_4(\mu)} = \frac{1}{C_4(M)} - \frac{3}{16\pi^2}\log\left(\frac{\mu}{M}\right)$$

One-loop matching and running

It is important to stress that, for calculations at the energy scale below M,
EFT is superior to the UV theory

$$M_4^{\mathrm{UV}} = -\lambda_0 + \frac{3\lambda_0^2}{32\pi^2}\left(\log\left(\frac{\mu^2}{m^2}\right)+2\right) + \frac{\lambda_0^2}{32\pi^2}\left[f(s,m)+f(t,m)+f(u,m)\right]$$

$$+\frac{m^2\lambda_2^2}{48\pi^2 M^2} + \frac{3\lambda_2^2}{32\pi^2}\log\left(\frac{\mu^2}{M^2}\right)$$

UV theory contains multiple logarithms,
and perturbative control will be lost when $\lambda^2\log(M/m)$ is of order $16\pi^2$

$$M_4^{\mathrm{EFT}} = -C_4(\mu) + \frac{3C_4^2}{32\pi^2}\left(\log\left(\frac{\mu^2}{m^2}\right)+2\right) + \frac{C_4^2}{32\pi^2}\left[f(s,m)+f(t,m)+f(u,m)\right] + \frac{C_6 m^2}{32\pi^2}\left(\log\left(\frac{\mu^2}{m^2}\right)+1\right)$$

$$= -C_4(m) + \frac{C_4^2}{32\pi^2}\left[f(s,m)+f(t,m)+f(u,m)+6\right] + \frac{C_6 m^2}{32\pi^2}$$

No large logarithms in the EFT!
The potentially problematic $\log(M/m)$ terms are all hidden (resummed)
in the running Wilson coefficient $C_4(m)$

Summary

Obtaining an EFT from a UV theory consists in the following algorithm

- Take a UV theory containing light degrees of freedom, with the characteristic mass scale m, and heavy degrees of freedom, with the characteristic mass scale Λ, where $\Lambda \ll m$
- In the UV theory, calculate scattering amplitudes for light particles up to a desired order in loop expansion, and expand them to a desired order in 1/Λ
- Write down the most general EFT Lagrangian for the light particles respecting the symmetries of the UV theory. Organize the Lagrangian in a systematic expansion in 1/Λ
- In that EFT, calculate the same amplitudes as is the UV theory, up to the same loop order and up to the same order in the 1/Λ expansion
- Fix the Wilson coefficients in the EFT at the scale Λ by demanding that the amplitudes calculated in the two theories are equal at that scale
- Run the Wilson coefficients from Λ down to the low scale (e.g. mass scale m of the light particles, or the characteristic energy scale E of the process of interest, $m \ll E \ll M$)