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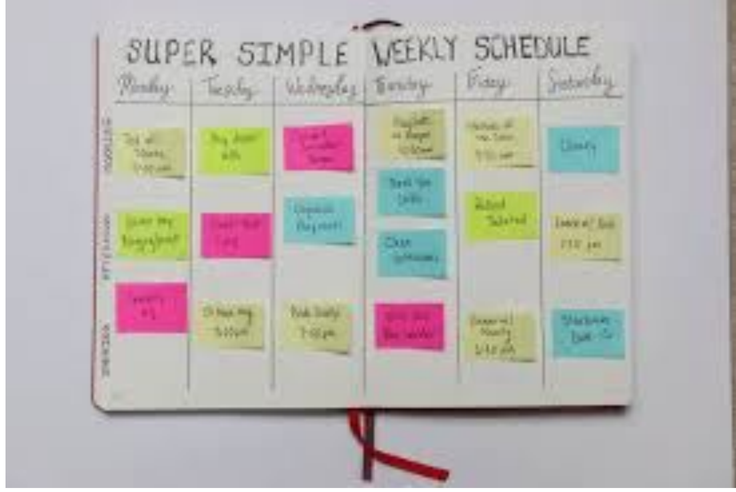
Effective Field Theories

(EFTs)

Lectures given at the Invisibles'24 school in Bologna

27-28 June 2024

Timetable



- Lecture 1
  Effective toy story or an EFT of a single scalar

- Lecture 2
  EFT in action or an illustrated philosophy of EFT

- Lecture 3
  SMEFT et al. or effective theory above the electroweak scale

Illustration #1

Euler-Heisenberg EFT

Euler-Heisenberg EFT

Consider effective theory for photons propagating in vacuum with $E_\gamma \ll 2m_e \sim 1$ MeV

- **At these energies all charged particles are integrated out, hence the effective Lagrangian must be a function of only the photon field A_μ**
- **Photons are massless, so the only explicit mass scale in this construction is the EFT cutoff scale Λ**
- **Gauge and Lorentz invariance requires the effective Lagrangian to be a function of the field strength $F_{\mu\nu}$ and its derivatives**

$$\mathscr{L}_{\rm EH} = \mathscr{L}\left(F_{\mu\nu}, \tilde{F}_{\mu\nu}, \partial_\mu, \Lambda\right)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}$$

Much as for the one-scalar toy model, we will build the effective Lagrangian as an expansion in canonical dimension

$$\mathscr{L}_{\rm EH} = \underset{\sim \Lambda^2}{\mathscr{L}_{D=2}} + \underset{\sim \Lambda^0}{\mathscr{L}_{D=4}} + \underset{\sim \Lambda^{-2}}{\mathscr{L}_{D=6}} + \underset{\sim \Lambda^{-4}}{\mathscr{L}_{D=8}} + \ldots$$

Here D denotes the canonical dimension of each term
(no odd dimensions because $[F_{\mu\nu}]=2$, and derivatives must always come in pairs)

Euler-Heisenberg EFT

$$\mathscr{L}_{\mathrm{EH}} = \mathscr{L}_{D=2} + \mathscr{L}_{D=4} + \mathscr{L}_{D=6} + \mathscr{L}_{D=8} + \ldots$$

D=2: $F_{\mu\mu} = \tilde{F}_{\mu\mu} = 0$ **No possible invariants thus** $\mathscr{L}_{D=2} = 0$

D=4: **One invariant** $F_{\mu\nu}F_{\mu\nu}$

$\tilde{F}_{\mu\nu}\tilde{F}_{\mu\nu} = F_{\mu\nu}F_{\mu\nu}$

$F_{\mu\nu}\tilde{F}_{\mu\nu}$ **is a total derivative**

$\partial_\mu \partial_\nu F_{\mu\nu} = 0$

$$\mathscr{L}_{D=4} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu}$$

the numerical coefficient is pure convention, except for the sign, which is required to avoid ghost instability

D=6: **Again, no non-trivial invariants! Hence** $\mathscr{L}_{D=6} = 0$

$$F_{\mu\nu}F_{\nu\rho}F_{\rho\mu} = 0 = F_{\mu\nu}F_{\nu\rho}\tilde{F}_{\rho\mu} = \ldots$$

$$F_{\mu\nu}\partial_\alpha F_{\mu\alpha}\partial_\beta F_{\nu\beta} = 0$$

$\mathscr{L}_{D=6} = cF_{\mu\nu}\Box F_{\mu\nu}$ **can be eliminated by the change of variables** $A_\mu \to A_\mu + \frac{2c}{\Lambda^2}\Box A_\mu$

Non-trivial interactions between photons can arise only at order 1/ Λ^4 in the EFT!

Euler-Heisenberg EFT

$$\mathscr{L}_{\mathrm{EH}} = -\frac{1}{4}\underbrace{F_{\mu\nu}F_{\mu\nu}}_{\sim\Lambda^0} + \underbrace{\mathscr{L}_{D=8}}_{\sim\Lambda^{-4}} + \ldots$$

D=8: **The most general non-redundant Lagrangian at D=8 is**

$$\boxed{\mathscr{L}_{D=8} = \frac{1}{16}\left\{ C_1(F_{\mu\nu}F_{\mu\nu})^2 + C_2(F_{\mu\nu}\tilde{F}_{\mu\nu})^2 + C_3(F_{\mu\nu}F_{\mu\nu})(F_{\alpha\beta}\tilde{F}_{\alpha\beta}) \right\}}$$

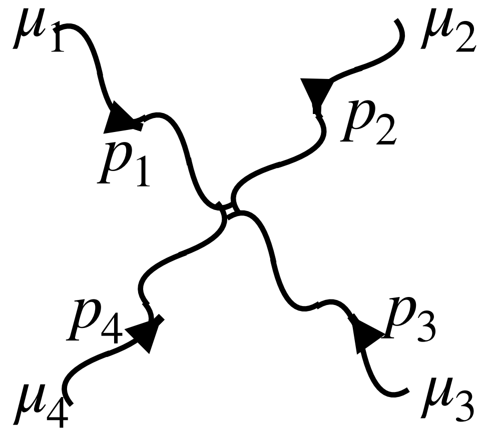
Other possible structures can be shown to be redundant, that is they can be eliminated or expressed by the three above. E.g.

$$F_{\mu\alpha}F_{\alpha\nu}F_{\mu\beta}F_{\beta\nu} = \frac{1}{4}(F_{\mu\nu}F_{\mu\nu})^2 + \frac{1}{2}(F_{\mu\nu}\tilde{F}_{\mu\nu})^2$$

Euler-Heisenberg EFT

$$\mathscr{L}_{\mathrm{EH}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{16}\left\{ C_1(F_{\mu\nu}F_{\mu\nu})^2 + C_2(F_{\mu\nu}\tilde{F}_{\mu\nu})^2 + C_3(F_{\mu\nu}F_{\mu\nu})(F_{\alpha\beta}\tilde{F}_{\alpha\beta}) \right\} + \dots$$

This Lagrangian defines a completely healthy and consistent EFT
with quartic (and higher-point) self-interactions between photons



$$2iC_1\left(p_1^{\mu_2}p_2^{\mu_1} - p_1p_2\eta^{\mu_1\mu_2}\right)\left(p_3^{\mu_4}p_4^{\mu_3} - p_3p_4\eta^{\mu_3\mu_4}\right) + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)$$
$$+\frac{iC_2}{\Lambda^4}(\dots) + \frac{iC_3}{\Lambda^4}(\dots)$$

Scattering amplitudes can be calculated in a systematic expansion in $1/\Lambda$. E.g.

$$\mathscr{M}[1_\gamma^+2_\gamma^+3_\gamma^+4_\gamma^+] = \frac{C_1 - C_2 + iC_3}{2}\left[s^2 + t^2 + u^2\right] + \mathcal{O}(\Lambda^{-6})$$

$$\mathscr{M}[1_\gamma^-2_\gamma^-3_\gamma^+4_\gamma^+] = \frac{C_1 + C_2}{2}s^2 + \mathcal{O}(\Lambda^{-6})$$

$$\mathscr{M}[1_\gamma^-2_\gamma^-3_\gamma^-4_\gamma^-] = \frac{C_1 - C_2 - iC_3}{2}\left[s^2 + t^2 + u^2\right] + \mathcal{O}(\Lambda^{-6})$$

$$\mathscr{M}[1_\gamma^-2_\gamma^-3_\gamma^-4_\gamma^+] = \mathscr{M}[1_\gamma^-2_\gamma^+3_\gamma^+4_\gamma^+] = \mathcal{O}(\Lambda^{-6})$$

$s = (p_1 + p_2)^2$
$t = (p_1 + p_3)^2$
$s = (p_1 + p_4)^2$

Note that a non-zero C_3 violates parity!

The difference between this EFT and a renormalizable QFT is that counterterms of order $1/\Lambda^n$, also with n>4, are generated at loop level

Euler-Heisenberg EFT

$$\mathscr{L}_{\rm EH} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{16}\left\{ C_1(F_{\mu\nu}F_{\mu\nu})^2 + C_2(F_{\mu\nu}\tilde{F}_{\mu\nu})^2 + C_3(F_{\mu\nu}F_{\mu\nu})(F_{\alpha\beta}\tilde{F}_{\alpha\beta}) \right\} + \dots$$

This Lagrangian defines a completely healthy and consistent EFT
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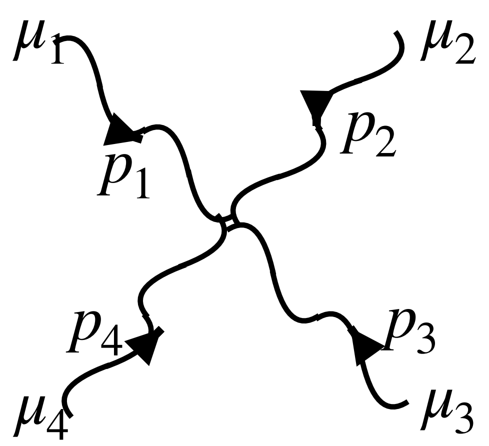


$$2iC_1\left(p_1^{\mu_2}p_2^{\mu_1} - p_1p_2\eta^{\mu_1\mu_2}\right)\left(p_3^{\mu_4}p_4^{\mu_3} - p_3p_4\eta^{\mu_3\mu_4}\right) + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)$$
$$+\frac{iC_2}{\Lambda^4}(\dots) + \frac{iC_3}{\Lambda^4}(\dots)$$

In the more proper notation

$$\mathscr{M}[1_\gamma^+2_\gamma^+3_\gamma^+4_\gamma^+] = \frac{C_1 - C_2 + iC_3}{2}\left[[12]^2[34]^2 + [13]^2[24]^2 + [14]^2[23]^2\right] + \mathcal{O}(\Lambda^{-6})$$

$$\mathscr{M}[1_\gamma^-2_\gamma^-3_\gamma^+4_\gamma^+] = \frac{C_1 + C_2}{2}\langle 12\rangle^2[34]^2 + \mathcal{O}(\Lambda^{-6})$$

$$\mathscr{M}[1_\gamma^-2_\gamma^-3_\gamma^-4_\gamma^-] = \frac{C_1 - C_2 - iC_3}{2}\left[\langle 12\rangle^2\langle 34\rangle^2 + \langle 13\rangle^2\langle 24\rangle^2 + \langle 14\rangle^2\langle 23\rangle^2\right] + \mathcal{O}(\Lambda^{-6})$$

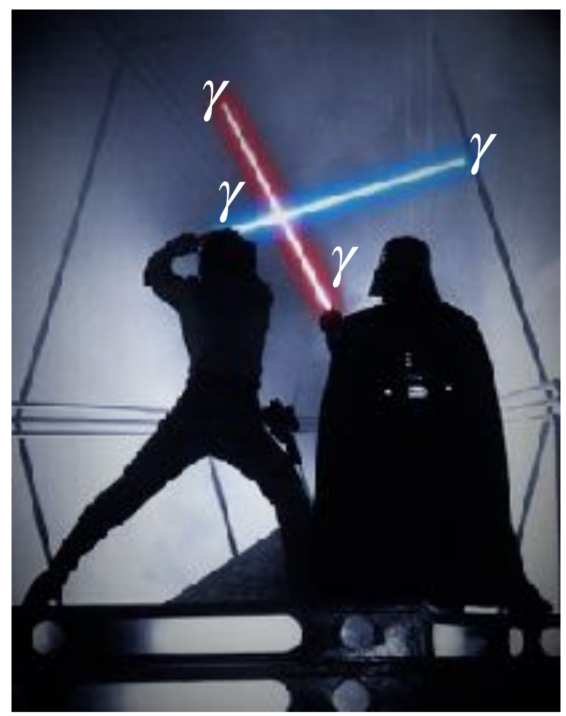
$$\mathscr{M}[1_\gamma^-2_\gamma^-3_\gamma^-4_\gamma^+] = \mathscr{M}[1_\gamma^-2_\gamma^+3_\gamma^+4_\gamma^+] = \mathcal{O}(\Lambda^{-6})$$

Euler-Heisenberg EFT

$$\mathscr{L}_{\rm EH} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{16}\left\{ C_1(F_{\mu\nu}F_{\mu\nu})^2 + C_2(F_{\mu\nu}\tilde{F}_{\mu\nu})^2 + C_3(F_{\mu\nu}F_{\mu\nu})(F_{\alpha\beta}\tilde{F}_{\alpha\beta}) \right\} + \dots$$

This Lagrangian describes the effective theory of light at low energies (UV, visible, IR, microwaves, radio) at the leading order beyond the Maxwell approximation

This is the effective theory underlying the physics of light sabers



In its validity regime, it is also appropriate to describe vacuum birefringence, photon-photon scattering at low energies, and more

Euler-Heisenberg EFT

$$\mathscr{L}_{\rm EH} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + \frac{1}{16}\left\{ C_1(F_{\mu\nu}F_{\mu\nu})^2 + C_2(F_{\mu\nu}\tilde{F}_{\mu\nu})^2 + C_3(F_{\mu\nu}F_{\mu\nu})(F_{\alpha\beta}\tilde{F}_{\alpha\beta}) \right\} + \dots$$

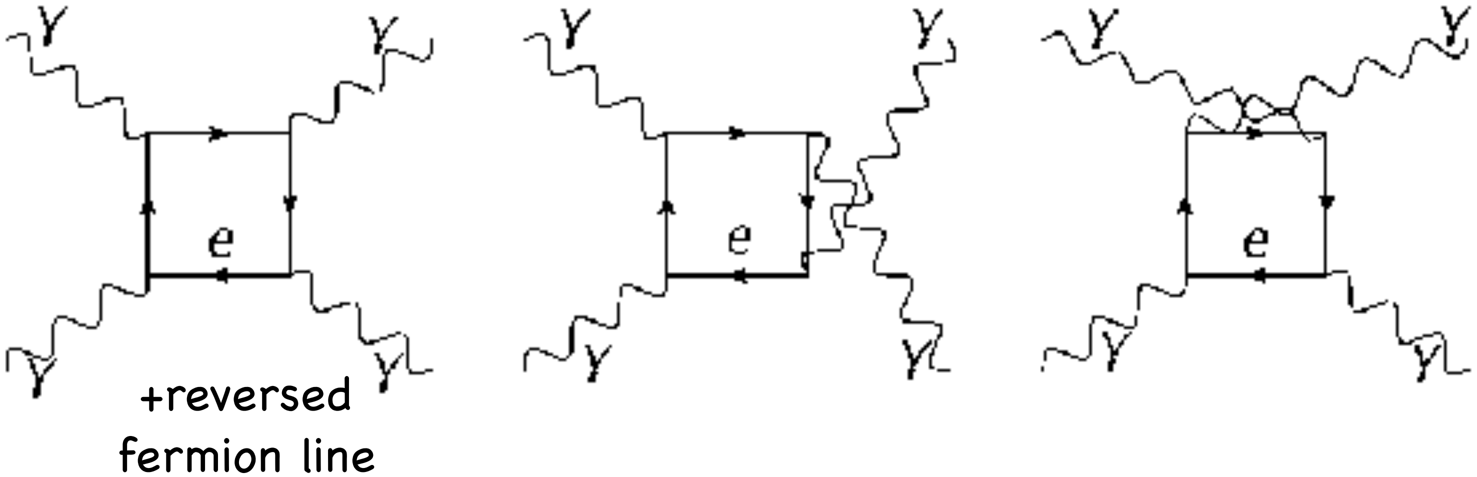
Scattered comments:

- This is the effective theory of light at low energies (UV, visible, IR, microwaves, radio) at the leading non-trivial order
- The quartic photon interaction terms in this EFT lead to non-linear field equations for the electromagnetic field. Thus, electrodynamics is really non-linear, and the superposition principle they taught you in school is not exactly true!
- One potentially observable effect of the D=8 terms is the so-called vacuum birefringence, that is rotation of light polarization propagating in vacuum in strong magnetic field. This effect was possibly observed in 2016 in a neutron star light.
- Another potentially observable effect is light-by-light scattering. This has been routinely observed in colliders, however at higher energies where this EFT is no longer valid.
- In the absence of new physics, the ordinary QED is the UV completion of this EFT, in which case the cutoff Λ can be identified with $2m_e$. However, in the presence of light axions or light milli-charged particles, this may no longer be the case.
- The Wilson coefficients c_1, c_2, c_3 can be calculated theoretically by matching this EFT to its UV completion, e.g. QED. However, I'm not aware of a systematic experimental measurement of these Wilson coefficients. A future such measurement will be a non-trivial result, as some unknown light particles could in principle contribute to it, along with the electron and other SM charged particles

Euler-Heisenberg EFT

QED UV completion

$$\mathscr{L}_{\text{UV}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m_e\bar{\psi}\psi + eA_{\mu}\bar{\psi}\gamma^{\mu}\psi$$



$$I_{\circ}^{s} \equiv \int \frac{d^4k}{i(2\pi)^4}\frac{1}{[k^2-m^2][(k+p_1+p_2)^2-m^2]}$$

$$I_{\triangleright}^{s} \equiv \int \frac{d^4k}{i(2\pi)^4}\frac{1}{[k^2-m^2][(k+p_1+p_2)^2-m^2][(k+p_1)^2-m^2]}$$

$$I_{\square}^{st} \equiv \int \frac{d^4k}{i(2\pi)^4}\frac{1}{[k^2-m^2][(k+p_1+p_2)^2-m^2][(k+p_1)^2-m^2][(k-p_3)^2-m^2]}$$

In this example, the UV completion of our effective theory is a renormalizable theory, which could in principle be valid to very high energy scales

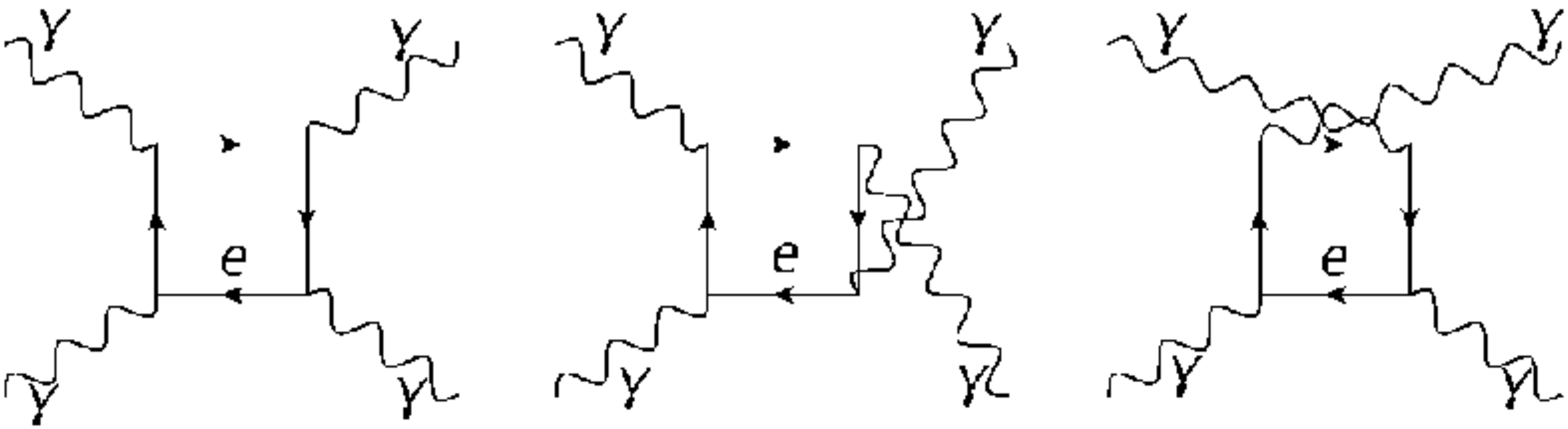
$$\mathscr{M}_{\text{UV}}[1_\gamma^- 2_\gamma^- 3_\gamma^- 4_\gamma^-] = -e^4\left\{16m^4\left[I_{\square}^{st} + I_{\square}^{su} + I_{\square}^{tu}\right] - \frac{1}{2\pi^2}\right\} = \mathscr{M}_{\text{UV}}[1_\gamma^+ 2_\gamma^+ 3_\gamma^+ 4_\gamma^+]$$

$$\mathscr{M}_{\text{UV}}[1_\gamma^- 2_\gamma^- 3_\gamma^+ 4_\gamma^+] = -e^4\left\{\left[16m^4 - 8m^2 s\right]\left[I_{\square}^{st} + I_{\square}^{su} + I_{\square}^{tu}\right] + \frac{32m^2 stu - 4tu(t^2+u^2)}{s^2}I_{\square}^{tu} - \frac{32m^2 s - 8(t^2+u^2)}{s^2}\left[tI_{\triangleright}^{t} + uI_{\triangleright}^{u}\right] - 8\frac{t-u}{s}\left[I_{\circ}^{t} - I_{\circ}^{u}\right] + \frac{1}{2\pi^2}\right\}$$

Euler-Heisenberg EFT

QED UV completion

$$\mathscr{L}_{\mathrm{UV}} = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} + i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m_e\bar{\psi}\psi + eA_{\mu}\bar{\psi}\gamma^{\mu}\psi$$



+reversed fermion line

$$\mathscr{M}_{\mathrm{UV}}[1_\gamma^- 2_\gamma^- 3_\gamma^- 4_\gamma^-] = -\frac{\alpha^2(s^2+t^2+u^2)}{15m^4} + \mathscr{O}(m^{-6}) = \mathscr{M}_{\mathrm{UV}}[1_\gamma^+ 2_\gamma^+ 3_\gamma^+ 4_\gamma^+]$$

$$\mathscr{M}_{\mathrm{UV}}[1_\gamma^- 2_\gamma^- 3_\gamma^+ 4_\gamma^+] = \frac{11\alpha^2 s^2}{45m^4} + \mathscr{O}(m^{-6})$$

Compare with

$$\mathscr{M}_{\mathrm{EFT}}[1_\gamma^+ 2_\gamma^+ 3_\gamma^+ 4_\gamma^+] = \frac{C_1 - C_2 + iC_3}{2}\left[s^2+t^2+u^2\right] + \mathscr{O}(\Lambda^{-6})$$

$$\mathscr{M}_{\mathrm{EFT}}[1_\gamma^- 2_\gamma^- 3_\gamma^+ 4_\gamma^+] = \frac{C_1 + C_2}{2}s^2 + \mathscr{O}(\Lambda^{-6})$$

$$\mathscr{M}_{\mathrm{EFT}}[1_\gamma^- 2_\gamma^- 3_\gamma^- 4_\gamma^-] = \frac{C_1 - C_2 - iC_3}{2}\left[s^2+t^2+u^2\right] + \mathscr{O}(\Lambda^{-6})$$

$$C_1 = \frac{8\alpha^2}{45m_e^4}$$

$$C_2 = \frac{14\alpha^2}{45m_e^4}$$

$$C_3 = 0$$

Euler-Heisenberg EFT

More generally, for mass m, spin S and charge Q particle minimally coupled to electromagnetic field:

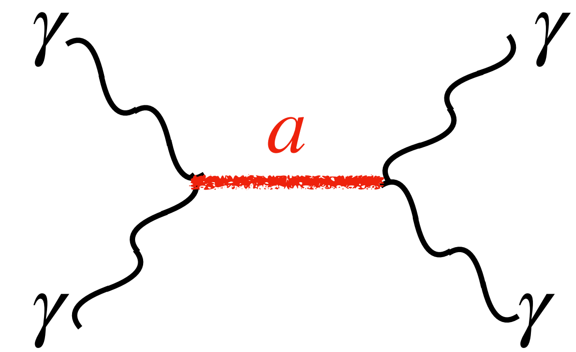
$$C_1^{(0)} = \frac{7\alpha^2 Q^4}{90 m^4} \qquad C_1^{(1/2)} = \frac{8\alpha^2 Q^4}{45 m^4} \qquad C_1^{(1)} = \frac{29\alpha^2 Q^4}{10 m^4}$$

$$C_2^{(0)} = \frac{\alpha^2 Q^4}{90 m^4} \qquad C_2^{(1/2)} = \frac{14\alpha^2 Q^4}{45 m^4} \qquad C_2^{(1)} = \frac{27\alpha^2 Q^4}{10 m^4}$$

$$C_3^{(0)} = 0 \qquad C_3^{(1/2)} = 0 \qquad C_3^{(1)} = 0$$

Btw. why no results given for higher spins ?

Euler-Heisenberg EFT



ALP UV completion

$$\mathscr{L}_{\rm UV} \supset \frac{1}{2}(\partial_\mu a)^2 - \frac{m^2}{2}a^2 + \frac{a}{f}\left\{ gF_{\mu\nu}F_{\mu\nu} + \tilde{g}F_{\mu\nu}\tilde{F}_{\mu\nu} \right\}$$

Integrating out the axion at tree-level:

$$C_1 = \frac{8g^2}{f^2m^2}, \qquad C_2 = \frac{8\tilde{g}^2}{f^2m^2}, \qquad C_3 = \frac{16g\tilde{g}}{f^2m^2}$$

Note that $C_i \sim \dfrac{1}{f^2m^2}$ **rather than** $C_i \sim \dfrac{1}{m^4}$ **as in the previous example**

Thus, C_i **does not allow one to read off the cutoff of the EFT, which is** $\Lambda \sim m$

The naive power counting is disrupted, because the UV completion of an effective theory is itself an effective theory and contains other mass parameters than m

Interlude

Analyticity constraints

Analyticity constraints

From the low-energy point of view, Wilson coefficients of an EFT are arbitrary, within perturbativity limits

However, assuming the UV completion is causal, Poincaré invariant, and local one can surprisingly find additional constraints on the Wilson coefficients

Given $\mathscr{M}_{EFT}(X_1 X_2 \to X_1 X_2) = M(s, t, u)$

and $M_{\text{forward}}(s) = M(s, 0, -s)$

$$\frac{d^2 M_{\text{forward}}(s)}{s^2}\Big|_{s\to 0} > 0$$

Proof using dispersion relations

Euler-Heisenberg EFT

$$\mathcal{M}_{\rm EFT}[1_\gamma^+ 2_\gamma^+ 3_\gamma^+ 4_\gamma^+] = \frac{C_1 - C_2 + iC_3}{2}\left[s^2 + t^2 + u^2\right] + \mathcal{O}(\Lambda^{-6})$$
$$\mathcal{M}_{\rm EFT}[1_\gamma^- 2_\gamma^- 3_\gamma^+ 4_\gamma^+] = \frac{C_1 + C_2}{2} s^2 + \mathcal{O}(\Lambda^{-6})$$
$$\mathcal{M}_{\rm EFT}[1_\gamma^- 2_\gamma^- 3_\gamma^- 4_\gamma^-] = \frac{C_1 - C_2 - iC_3}{2}\left[s^2 + t^2 + u^2\right] + \mathcal{O}(\Lambda^{-6})$$

Applying this to the Euler-Heisenberg effective Lagrangian

$$\mathcal{M}_{\rm EFT}(\gamma^-\gamma^- \to \gamma^-\gamma^-) = \mathcal{M}_{\rm EFT}(1_\gamma^- 2_\gamma^- 3_\gamma^+ 4_\gamma^+) = \frac{C_1 + C_2}{2} s^2 \qquad \Rightarrow \qquad C_1 + C_2 > 0$$

One can actually get stronger bounds, by considering amplitude in the linear polarization basis

$$\mathcal{M}_{\rm forward}(\gamma^x\gamma^x \to \gamma^x\gamma^x) = C_1 s^2 \qquad \Rightarrow \qquad C_1 > 0$$
$$\mathcal{M}_{\rm forward}(\gamma^x\gamma^y \to \gamma^x\gamma^y) = C_2 s^2 \qquad \Rightarrow \qquad C_2 > 0$$

Of course, this bound is respected in our examples

Euler-Heisenberg EFT

Classical effects due to higher-dimensional operators in Euler-Heisenberg EFT

First define engineers-friendly variables: $\vec{E} = -\vec{\nabla} A^0 - \partial_t \vec{A}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$

$$\mathscr{L}_{\mathrm{EH}} = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) + \frac{C_1}{4}\left(\vec{E}^2 - \vec{B}^2\right)^2 + C_2\left(\vec{E}\vec{B}\right)^2 + \frac{C_3}{2}\left(\vec{E}^2 - \vec{B}^2\right)\left(\vec{E}\vec{B}\right) + \ldots$$

Set $C_3 = 0$ for this discussion.

Define electric displacement $\vec{D} = \frac{\partial \mathscr{L}}{\partial \vec{E}}$ **and magnetic intensity** $\vec{H} = -\frac{\partial \mathscr{L}}{\partial \vec{B}}$

$$\vec{D} = \vec{E} + C_1\left(\vec{E}^2 - \vec{B}^2\right)\vec{E} + 2C_2\left(\vec{E}\vec{B}\right)\vec{B}$$

$$\vec{H} = \vec{B} + C_1\left(\vec{E}^2 - \vec{B}^2\right)\vec{B} - 2C_2\left(\vec{E}\vec{B}\right)\vec{E}$$

In the presence of the dimension-8 operators, vacuum behaves like a medium, that is $\vec{P} \equiv \vec{D} - \vec{E} \neq 0$, **and** $\vec{M} \equiv \vec{B} - \vec{H} \neq 0$

In these variables the equations of motion are just Maxwell equations:

$$\vec{\nabla} \cdot \vec{D} = 0 \qquad \frac{\partial \vec{D}}{\partial t} = \vec{\nabla} \times \vec{H}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$$

Euler-Heisenberg EFT

$$\vec{D} = \vec{E} + C_1\left(\vec{E}^2 - \vec{B}^2\right)\vec{E} + 2C_2\left(\vec{E}\vec{B}\right)\vec{B}$$

$$\vec{H} = \vec{B} + C_1\left(\vec{E}^2 - \vec{B}^2\right)\vec{B} - 2C_2\left(\vec{E}\vec{B}\right)\vec{E}$$

Imagine an electromagnetic wave passing through a region with constant magnetic field

$$\vec{E} = \vec{E}_w, \qquad \vec{B} = \vec{B}_0 + \vec{B}_w$$

At linear order in the wave perturbation

$$\vec{D}_w = \vec{E}_w - C_1 B_0^2 \vec{E}_w + 2C_2\left(\vec{E}_w \vec{B}_0\right)\vec{B}_0$$

$$\vec{H}_w = \vec{B}_w - C_1 B_0^2 \vec{B}_w - 2C_1\left(\vec{B}_w \vec{B}_0\right)\vec{B}_0$$

Define the electric permittivity and magnetic permeability: $\vec{D} = \epsilon\vec{E}, \quad \vec{B} = \mu\vec{H}$

$$\epsilon_\perp = 1 - C_1 B_0^2, \qquad \epsilon_\parallel = 1 - C_1 B_0^2 + 2C_2 B_0^2$$

$$\mu_\perp = 1 + 3C_1 B_0^2, \qquad \mu_\parallel = 1 + C_1 B_0^2$$

Index of refraction: $n \equiv \frac{1}{\mathrm{v}} = \sqrt{\epsilon\mu}$ $\qquad n_\perp = 1 + C_1 B_0^2, \qquad n_\parallel = 1 + C_2 B_0^2$

Causality in non-trivial backgrounds requires $C_1 \geq 0, \quad C_2 \geq 0$

This is also a practical way to measure the Wilson coefficient (e.g. in PVLAS)

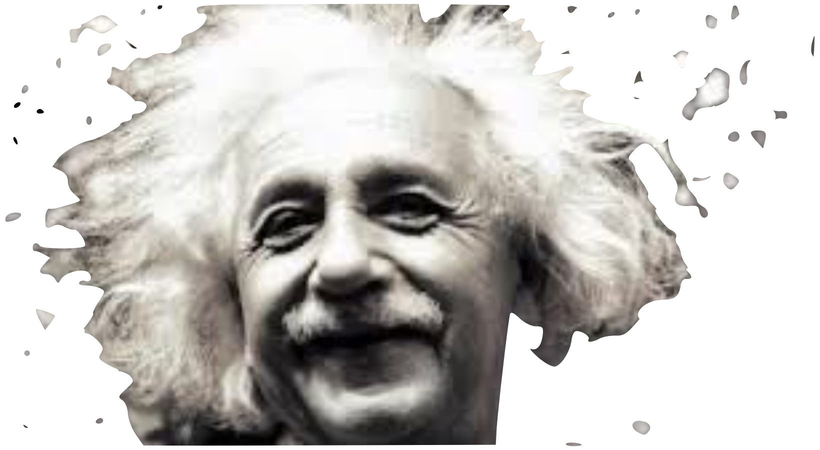
Euler-Heisenberg EFT

Summary and lessons learned

- Symmetries of a low-energy system often determine the structure of the effective theory at leading orders, up to a few unknown numerical parameters
- Furthermore, in some case even the sign of the Wilson coefficients is fixed, given some plausible assumptions about the UV theory
- The EFT Lagrangian can be used for perturbative calculations of low-energy scattering amplitudes.
- It is also a useful tool to work out subtle effects of classical field configurations

Illustration #2

GREFT



GREFT

We will write down an EFT for a massless spin-2 particle, aka the graviton

Such a particle can be described by a real and symmetric tensor field

$$h_{\mu\nu}(x)$$

For a massless spin-1 particle, QFT makes sense only in the presence of gauge invariance

Likewise, for a massless spin-2 particle, QFT makes sense only in the presence of <u>general coordinate invariance</u>

Otherwise, there is no way a 10-component symmetric tensor $h_{\mu\nu}(x)$ can describe 2 components of the massless graviton

To implement the GC invariance, it is convenient to combine the graviton field with the Minkowski metric to write

$$h_{\mu\nu}(x) \to g_{\mu\nu}(x) \equiv \eta_{\mu\nu} + h_{\mu\nu}(x)$$

and demand that $g_{\mu\nu}$ transform as a tensor under GC transformations

$$x \to y \quad \Rightarrow \quad g_{\mu\nu} \to \frac{dx^\alpha}{dy^\mu}\frac{dx^\beta}{dy^\nu} g_{\alpha\beta}$$

Crash course in GR

Christoffel connection: $$\Gamma^{\mu}_{\nu\rho} \equiv \frac{1}{2} g^{\mu\alpha} \left(\partial_\rho g_{\alpha\nu} + \partial_\nu g_{\alpha\rho} - \partial_\alpha g_{\nu\rho} \right)$$

Riemann tensor: $$R^{\alpha}{}_{\mu\nu\beta} \equiv \partial_\nu \Gamma^{\alpha}_{\mu\beta} - \partial_\beta \Gamma^{\alpha}_{\mu\nu} + \Gamma^{\rho}_{\mu\beta} \Gamma^{\alpha}_{\rho\nu} - \Gamma^{\rho}_{\mu\nu} \Gamma^{\alpha}_{\rho\beta}$$

Ricci tensor: $$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\nu\alpha}$$

Ricci scalar: $$R \equiv g^{\mu\nu} R_{\mu\nu}$$

Weyl tensor: $$C_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta} - g_{\mu[\alpha} R_{\beta]\nu} + g_{\nu[\alpha} R_{\beta]\mu} + \frac{1}{3} g_{\mu[\alpha} g_{\beta]\nu} R$$

Dual Riemann tensor: $$\tilde{R}_{\mu\nu\alpha\beta} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} R_{\alpha\beta}{}^{\rho\sigma}$$

GC invariant volume: $$\sqrt{-g}\, d^4x$$

GREFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$S_{\text{GREFT}} = \int d^4x \mathscr{L}_{\text{GREFT}}$$

$$\mathscr{L}_{\text{GREFT}} = \sqrt{-g}\left\{ \underset{\sim \Lambda^4}{\mathscr{L}_{D=0}} + \underset{\sim \Lambda^2}{\mathscr{L}_{D=2}} + \underset{\sim \Lambda^0}{\mathscr{L}_{D=4}} + \underset{\sim \Lambda^{-2}}{\mathscr{L}_{D=6}} + \underset{\sim \Lambda^{-4}}{\mathscr{L}_{D=8}} + \ldots \right\}$$

At the leading order the only possible invariant under GC transformations is

$$\mathscr{L}_{D=0} = C_0$$

This is the cosmological constant.
Phenomenologically, this term is non-zero but tiny, though no one understands why….
It only plays a role at cosmological distance scale, so we ignore it in the following

GREFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$\mathcal{L}_{\mathrm{GREFT}} = \sqrt{-g}\left\{ \cancel{\mathcal{L}_{D=0}} + \underbrace{\mathcal{L}_{D=2}}_{\sim \Lambda^2} + \mathcal{L}_{D=4} + \mathcal{L}_{D=6} + \mathcal{L}_{D=8} + \ldots \right\}$$

At the next-to-leading order the only possible invariant under GC transformations is

$$\mathcal{L}_{D=2} = C_1 R$$

Let's rename variables, trading $C_1 = \frac{M_{\mathrm{Pl}}^2}{2}$ $\qquad M_{\mathrm{Pl}} \equiv (8\pi G)^{-1/2} = 2.44 \times 10^{18}\ \mathrm{GeV}$

$$\mathcal{L}_{D=2} = \frac{1}{2} M_{\mathrm{Pl}}^2 R$$

At this point we have recovered the Einstein-Hilbert Lagrangian for general relativity!

GREFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$\mathscr{L}_{\mathrm{GREFT}} = \sqrt{-g}\left\{ \frac{1}{2}M_{\mathrm{Pl}}^2 R + \underset{\sim M_{\mathrm{Pl}}^{0}}{\mathscr{L}_{D=4}} + \underset{\sim M_{\mathrm{Pl}}^{-2}}{\mathscr{L}_{D=6}} + \underset{\sim M_{\mathrm{Pl}}^{-4}}{\mathscr{L}_{D=8}} + \dots \right\}$$

Expanding the leading term around the flat Minkowski metric in powers of the graviton field:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_{\mathrm{Pl}}} h_{\mu\nu}$$

$$\sqrt{-g}\,\mathscr{L}_{D=2} = \mathscr{L}_{D=2}^{(1)} + \mathscr{L}_{D=2}^{(2)} + \mathscr{L}_{D=2}^{(3)} + \mathscr{L}_{D=2}^{(4)} + \dots$$

$$\mathscr{L}_{D=2}^{(1)} = \frac{2}{M_{\mathrm{Pl}}}\left[\Box h - \partial_\mu \partial_\nu h_{\mu\nu}\right]$$

This is a total derivative and can be dropped

$$\mathscr{L}_{D=2}^{(2)} = \frac{1}{2}(\partial_\rho h_{\mu\nu})^2 - \frac{1}{2}(\partial_\rho h)^2 - (\partial_\rho h_{\mu\rho})^2 + \partial_\mu h \partial_\rho h_{\mu\rho}$$

This is the so-called Fierz-Pauli Lagrangian. Up to normalization, this is the unique ghost-free kinetic Lagrangian for a massless spin-2 particle

GREFT

Let's build an EFT out of $g_{\mu\nu}$ according to the usual rules

$$\mathscr{L}_{\mathrm{GREFT}} = \sqrt{-g}\left\{ \frac{1}{2} M_{\mathrm{Pl}}^2 R + \underset{\sim M_{\mathrm{Pl}}^0}{\mathscr{L}_{D=4}} + \underset{\sim M_{\mathrm{Pl}}^{-2}}{\mathscr{L}_{D=6}} + \underset{\sim M_{\mathrm{Pl}}^{-4}}{\mathscr{L}_{D=8}} + \ldots \right\}$$

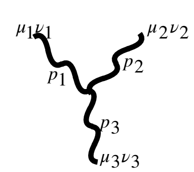
Expanding the leading term around the flat Minkowski metric in powers of the graviton field:

$$\sqrt{-g}\,\mathscr{L}_{D=2} = \mathscr{L}^{(1)}_{D=2} + \mathscr{L}^{(2)}_{D=2} + \mathscr{L}^{(3)}_{D=2} + \mathscr{L}^{(4)}_{D=2} + \ldots$$

$$\mathscr{L}^{(2)}_{D=2} = \frac{1}{2}(\partial_\rho h_{\mu\nu})^2 - \frac{1}{2}(\partial_\rho h)^2 - (\partial_\rho h_{\mu\rho})^2 + \partial_\mu h \partial_\rho h_{\mu\rho}$$

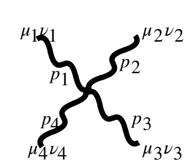
Kinetic terms

$$\mathscr{L}^{(3)}_{D=2} \sim \frac{1}{M_{\mathrm{Pl}}} h^3 \partial^2$$



Cubic interactions

$$\mathscr{L}^{(4)}_{D=2} \sim \frac{1}{M_{\mathrm{Pl}}^2} h^4 \partial^2$$



Quartic interactions

$\ldots$

and so on

These interactions encapsulate all phenomenology of general relativity

GREFT

$$\mathscr{L}_{\rm GREFT} = \sqrt{-g}\left\{ \frac{1}{2}M_{\rm Pl}^2 R + \underset{\sim M_{\rm Pl}^0}{\mathscr{L}_{D=4}} + \underset{\sim M_{\rm Pl}^{-2}}{\mathscr{L}_{D=6}} + \underset{\sim M_{\rm Pl}^{-4}}{\mathscr{L}_{D=8}} + \dots \right\}$$

- We have built a consistent quantum theory of gravity (at least for small fluctuations around flat spacetime)

- It is an EFT organized as an expansion in $1/M_{\rm Pl}$, and contains general relativity as its leading term

- The only price to pay is that we have to include an infinite series of higher order interactions, suppressed by $M_{\rm Pl}$ or a lower scale (if we do not, they will need to be added anyway as counterterms due to loop divergences). This is a problem if we want to predict processes where momentum exchange is of order $M_{\rm Pl}$. This is however not a problem if we restrict to low-energy effects (almost all of the observable ones)

- It is arguably the best EFT ever, because its validity range is the largest of known EFTs, spanning from very low-energies all the way to the Planck scale, $H_0 \ll E \ll M_{\rm Pl}$

GREFT

Going to higher orders

$$\mathscr{L}_{\mathrm{GREFT}} = \sqrt{-g}\left\{ \frac{1}{2}M_{\mathrm{Pl}}^2 R + \underset{\sim M_{\mathrm{Pl}}^0}{\mathscr{L}_{D=4}} + \underset{\sim M_{\mathrm{Pl}}^{-2}}{\mathscr{L}_{D=6}} + \underset{\sim M_{\mathrm{Pl}}^{-4}}{\mathscr{L}_{D=8}} + \ldots \right\}$$

Start with

$$\mathscr{L}_{D=4} = C_{2,1}R^2 + C_{2,2}R_{\mu\nu}^2 + C_{2,3}R_{\mu\nu\alpha\beta}^2$$

The Gauss-Bonnet theorem says that $R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\alpha\beta}^2$ is a total derivative.

Thus, we can eliminate the redundant $R_{\mu\nu\alpha\beta}^2$ term:

$$\mathscr{L}_{D=4} = \tilde{C}_{2,1}R^2 + \tilde{C}_{2,2}R_{\mu\nu}^2$$

However, we can use the lower dimensional equations of motion: $R_{\mu\nu} = 0$,

which also imply $R = 0$. All in all

$$\mathscr{L}_{D=4} = 0$$

First non-trivial EFT corrections to general relativity arise at the dimension-6 level, that is at 6-derivative level !

These implies that corrections from higher-dimension operators are even more suppressed
Btw. it also explains why pure gravity is one-loop finite

GREFT

Going to higher orders

$$\mathscr{L}_{\text{GREFT}} = \sqrt{-g}\left\{ \frac{1}{2} M_{\text{Pl}}^2 R + \underset{\sim M_{\text{Pl}}^{-2}}{\mathscr{L}_{D=6}} + \underset{\sim M_{\text{Pl}}^{-4}}{\mathscr{L}_{D=8}} + \ldots \right\}$$

$$\mathscr{L}_{D=6} = \frac{1}{3\,!}\left\{ C_3 R_{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} R_{\rho\sigma\mu\nu} + \tilde{C}_3 R_{\mu\nu\alpha\beta} R_{\alpha\beta\rho\sigma} \tilde{R}_{\rho\sigma\mu\nu} \right\}$$

often written equivalently as

$$C_{\mu\nu\alpha\beta} \equiv R_{\mu\nu\alpha\beta} - g_{\mu[\alpha} R_{\beta]\nu} + g_{\nu[\alpha} R_{\beta]\mu} + \frac{1}{3} g_{\mu[\alpha} g_{\beta]\nu} R$$

$$\mathscr{L}_{D=6} = \frac{1}{3\,!}\left\{ C_3 C_{\mu\nu\alpha\beta} C_{\alpha\beta\rho\sigma} C_{\rho\sigma\mu\nu} + \tilde{C}_3 C_{\mu\nu\alpha\beta} C_{\alpha\beta\rho\sigma} \tilde{C}_{\rho\sigma\mu\nu} \right\}$$

This is the same because we can use the lower order equations of motion $R_{\mu\nu} = 0$

We do not know what is the UV completion of GREFT
so we do not know the numerical value coefficients C_3 and $\tilde{C}_3$
At this point, they parametrize our ignorance about nature.
Maybe one day we will measure them experimentally,
and that will give us a hint about the underlying, more fundamental theory of gravity

GREFT

Going to higher orders

$$\mathscr{L}_{\text{GREFT}} = \sqrt{-g}\left\{ \frac{1}{2} M_{\text{Pl}}^2 R + \underset{\sim M_{\text{Pl}}^{-2}}{\mathscr{L}_{D=6}} + \underset{\sim M_{\text{Pl}}^{-4}}{\mathscr{L}_{D=8}} + \ldots \right\}$$

$$\mathscr{L}_{D=8} = \frac{1}{8}\left\{ C_{4,1}(R_{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta})^2 + C_{4,2}(R_{\mu\nu\alpha\beta}\tilde{R}_{\mu\nu\alpha\beta})^2 + \tilde{C}_4(R_{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta})(R_{\mu\nu\alpha\beta}\tilde{R}_{\mu\nu\alpha\beta}) \right\}$$

We do not know what is the UV completion of GREFT
so we do not know the numerical value coefficients $C_{4,1}$ and $\tilde{C}_{4,2}$
At this point, they parametrize our ignorance about nature.
Maybe one day we will measure them experimentally,
and that will give us a hint about the underlying, more fundamental theory of gravity

GREFT

The Wilson coefficient C_3 and $C_{4,k}$ are generated by string UV completions.
In that case, their magnitude is set by the string scale

Otherwise, for mass m, spin S and charge Q particle minimally coupled to gravity:

$$C_3^{(S)} = \frac{(-)^{2S+1}}{16\pi^2}\frac{(2S+1)}{2520m^2}, \quad \tilde{C}_3 = 0$$

$$C_{4,1}^{(S)} - C_{4,2}^{(S)} = \frac{(-)^{2S}}{16\pi^2}\frac{(2S+1)}{3780m^4}$$

Bern, Kosmopoulos, Zhiboedo
[arXiv:2103.12728]

$$C_{4,1}^{(0)} = \frac{1}{16\pi^2}\frac{11}{37800m^4} \quad C_{4,1}^{(1/2)} = \frac{1}{16\pi^2}\frac{47}{151200m^4} \quad C_{4,1}^{(1)} = \frac{1}{16\pi^2}\frac{1}{350m^4} \quad C_{4,1}^{(3/2)} = \frac{1}{16\pi^2}\frac{1217}{75600m^4} \quad C_{4,1}^{(2)} = \frac{1}{16\pi^2}\frac{1009}{3780m^4}$$

$$C_{4,2}^{(0)} = \frac{1}{16\pi^2}\frac{1}{37800m^4} \quad C_{4,2}^{(1/2)} = \frac{1}{16\pi^2}\frac{127}{151200m^4} \quad C_{4,2}^{(1)} = \frac{1}{16\pi^2}\frac{13}{6300m^4} \quad C_{4,2}^{(3/2)} = \frac{1}{16\pi^2}\frac{1297}{75600m^4} \quad C_{4,2}^{(2)} = \frac{1}{16\pi^2}\frac{251}{945m^4}$$

Positivity: $C_{4,1} > 0, \qquad C_{4,2} > 0$

Bellazzini, Cheung, Remmen
[arXiv:1509.00851]

GREFT

Summary and lessons learned

- Gravity is (to a large extent) like any other QFT, and can be treated by EFT methods. As usual, symmetry is the key to building the EFT.

- GREFT is not only a good classical theory. It is a consistent EFT at a quantum level, describing a self-interacting massless spin-2 particle. The theory is valid in the very broad energy regime up to the Planck scale

- Corrections from higher dimensional operators added to the Einstein-Hilbert Lagrangian are probably very small. But it is not excluded they are suppressed by a lower scale (e.g. if there are light particles interacting only gravitationally)

- Graviton can be coupled to matter, leading to a more complicated EFT. E.g. at the electroweak scale the corresponding EFT is called GRSMEFT

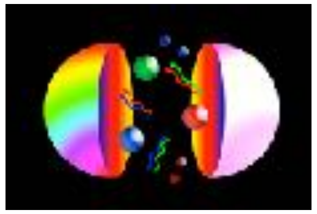
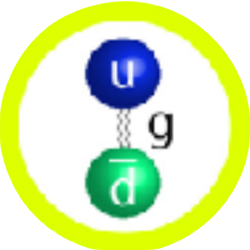
Illustration #3

Fermi EFT

Fermi EFT

- Fermi EFT is the theory of beta decay
- In the SM beta decay corresponds to up quark mutating into down quark (or the other way around), while emitting electron and neutrino
- This is however not a practically useful description. For that we need to descend a ladder of EFTs down to the scales relevant for nuclear physics

EFT worldview

UV	100 TeV ?	**Dragons**	
SMEFT	100 GeV	$\gamma, g, W, Z, \nu_i, e, \mu, \tau$ + **u, d, s, c, b, t** + **h**	
WEFT5	5 GeV	$\gamma, g, \nu_i, e, \mu, \tau$ + **u, d, s, c, b**	
WEFT4	2 GeV	$\gamma, g, \nu_i, e, \mu, \tau$ + **u, d, s, c**	
ChRT	1 GeV	γ, ν_i, e, μ + **hadrons**	
ChPT	100 MeV	$\gamma, \nu_i, e, \mu, \pi, K, p^{(*)}$	
QED+	1 MeV	$\gamma, \nu_i, e, p^{(*)}$	
EH+	0.01 eV	$\gamma, \nu_i, p^{(*)}, e^{(*)}$ $\gamma, p^{(*)}, e^{(*)}$	

Fermi EFT

Starting point: SM Lagrangian with charged current weak interactions:

$$\mathscr{L}_{\mathrm{SM}} \supset -W_\mu^+(\Box - m_W^2)W_\mu^- - \left\{ \frac{g_L}{\sqrt{2}} \left[V_{ud}\bar{u}\bar{\sigma}^\mu d + \bar{\nu}\bar{\sigma}^\mu e \right] W_\mu^+ + \mathrm{h.c.} \right\}$$

e.o.m:

$$-(\Box - m_W^2)W_\rho^- - \frac{g_L}{\sqrt{2}} \left[V_{ud}\bar{u}\bar{\sigma}^\mu d + \bar{\nu}\bar{\sigma}^\mu e \right] = 0$$

solution:

$$W_\rho^- = -\frac{g_L}{\sqrt{2}} (\Box - m_W^2)^{-1} \left[V_{ud}\bar{u}\bar{\sigma}^\mu d + \bar{\nu}\bar{\sigma}^\mu e \right]$$

(Non-local) Effective Lagrangian:

$$\mathscr{L}_{\mathrm{eff}} = \frac{g_L^2}{2} \left[V_{ud}\bar{d}\bar{\sigma}^\mu u + \bar{e}\bar{\sigma}^\mu \nu \right] (\Box - m_W^2)^{-1} \left[V_{ud}\bar{u}\bar{\sigma}^\mu d + \bar{\nu}\bar{\sigma}^\mu e \right]$$

Leading (local) Effective Lagrangian:

$$\frac{1}{\Box - m_W^2} = -\frac{1}{m_W^2} - \frac{\Box}{m_W^4} - \frac{\Box^2}{m_W^6} - \ldots$$

$$\mathscr{L}_{\mathrm{WEFT}} \supset -\frac{g_L^2}{2m_W^2} \left[V_{ud}\bar{d}\bar{\sigma}^\mu u + \bar{e}\bar{\sigma}^\mu \nu \right] \left[V_{ud}\bar{u}\bar{\sigma}^\mu d + \bar{\nu}\bar{\sigma}^\mu e \right]$$

Note on fermion conventions

I am using the 2-component spinor formalism

A Dirac fermion is described by a pair of spinor fields f and $\bar{f}^c$ **with the kinetic and mass terms**

$$\mathscr{L} = i\bar{f}\bar{\sigma}^\mu D_\mu f + i f^c \sigma^\mu D_\mu \bar{f}^c - m f^c f - m \bar{f}\bar{f}^c$$

$$\sigma^\mu = (1, \boldsymbol{\sigma})$$
$$\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$$
$$\bar{f} \equiv f^*$$

To translate to 4-component Dirac notation use

$$F = \begin{pmatrix} f \\ \bar{f}^c \end{pmatrix}, \qquad \bar{F} = \begin{pmatrix} f^c & \bar{f} \end{pmatrix}, \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\bar{F} \equiv F^\dagger \gamma^0$$

For example

$$\bar{f}\bar{\sigma}^\mu \partial_\mu f = \bar{F}_L \gamma^\mu \partial_\mu F_L$$

$$f^c \sigma^\mu \partial_\mu \bar{f}^c = \bar{F}_R \gamma^\mu \partial_\mu F_R$$

$$f^c f = \bar{F}_R F_L$$

$$\bar{f}\bar{f}^c = \bar{F}_L F_R$$

See the spinor bible [arXiv:0812.1594] for more details

Fermi EFT

$$\mathscr{L}_{\mathrm{WEFT}} \supset -\frac{g_L^2}{2m_W^2}\left[V_{ud}\bar{d}\bar{\sigma}^\mu u + \bar{e}\bar{\sigma}^\mu \nu\right]\left[V_{ud}\bar{u}\bar{\sigma}^\mu d + \bar{\nu}\bar{\sigma}^\mu e\right]$$

What is relevant for beta decay is

$$\mathscr{L}_{\mathrm{WEFT}} \supset -\frac{2V_{ud}}{\mathrm{v}^2}(\bar{u}\bar{\sigma}^\mu d)(\bar{e}\bar{\sigma}^\mu \nu) + \mathrm{h.c.}$$

$$\mathrm{v} \equiv \frac{2m_{\mathrm{W}}}{g_L} \approx 246 \text{ GeV}$$

$$= -\frac{V_{ud}}{\mathrm{v}^2}[\bar{u}\gamma_\mu(1-\gamma_5)d][\bar{e}\bar{\sigma}^\mu \nu] + \mathrm{h.c.}$$

This interaction governs beta decay at the quark level
However what is need in practice is nucleon level description

In the following we much the quark level EFT to the nucleon level EFT

Fermi EFT

$$\mathscr{L}_{\mathrm{WEFT}} \supset -\frac{V_{ud}}{\mathrm{v}^2}[\bar{u}\gamma_\mu(1-\gamma_5)d][\bar{e}\bar{\sigma}^\mu\nu] + \mathrm{h.c.}$$

This interaction leads to beta decays, in particular to the neutron decay

$$d \to u e^- \bar{\nu} \quad \Rightarrow \quad n \to p e^- \bar{\nu}$$

Amplitude for the latter process is

$$\begin{aligned} M(n \to p e^- \bar{\nu}_e) &= -\frac{V_{ud}}{\mathrm{v}^2}\langle p e^- \bar{\nu}_e | [\bar{u}\gamma_\mu(1-\gamma_5)d][\bar{e}\bar{\sigma}^\mu\nu] | n > \\ &= -\frac{V_{ud}}{\mathrm{v}^2}\langle e^- \bar{\nu}_e | \bar{e}\bar{\sigma}^\mu\nu | 0 > \langle p | \bar{u}\gamma_\mu(1-\gamma_5)d | n > \\ &= -\frac{V_{ud}}{\mathrm{v}^2}\left(\bar{x}(p_e)\bar{\sigma}_\mu\, y(p_\nu)\right)\left\{ \langle p | (\bar{u}\gamma_\mu d) | n > - \langle p | (\bar{u}\gamma_\mu\gamma_5 d) | n > \right\} \end{aligned}$$

$$P_L \equiv \frac{1-\gamma_5}{2}$$

where x(p), y(p) are 2-component spinor wave functions for particle and antiparticles

Fermi EFT

$$M(n \to p e^- \bar{\nu}_e) = -\frac{V_{ud}}{v^2}\left(\bar{x}(p_e)\bar{\sigma}_\mu y(p_\nu)\right)\left\{ \langle p\,|\,(\bar{u}\gamma_\mu d)\,|\,n > - \langle p\,|\,(\bar{u}\gamma_\mu\gamma_5 d)\,|\,n > \right\}$$

Due to strong QCD interaction, the quark matrix element cannot be calculated perturbatively

However, with the input from dimensional analysis and QCD (approximate) symmetries they can be reduced to a few unknowns, which can be subsequently calculated on the lattice or using phenomenological models

Lorentz invariance + Parity of QCD implies

$$q \equiv p_n - p_p$$

$$\langle p\,|\,(\bar{u}\gamma_\rho d)\,|\,n > = \bar{u}(p_p)\left[g_V(q^2)\gamma_\rho + \frac{\tilde{g}_{TV}(q^2)}{2m_n}\sigma_{\rho\nu}q^\nu + \frac{\tilde{g}_S(q^2)}{2m_n}q_\rho\right]u(p_n)$$

$$\langle p\,|\,(\bar{u}\gamma_\rho\gamma_5 d)\,|\,n > = \bar{u}(p_p)\left[g_A(q^2)\gamma_\rho + \frac{\tilde{g}_{TA}(q^2)}{2m_n}\sigma_{\rho\nu}q^\nu + \frac{\tilde{g}_P(q^2)}{2m_n}q_\rho\right]\gamma_5 u(p_n)$$

Fermi EFT

$$M(n \to p e^- \bar{\nu}_e) = -\frac{V_{ud}}{\mathrm{v}^2}\left(\bar{x}(p_e)\bar{\sigma}_\mu\, y(p_\nu)\right)\left\{\langle p|(\bar{u}\gamma_\mu d)|n> - \langle p|(\bar{u}\gamma_\mu\gamma_5 d)|n>\right\}$$

For beta decay processes, and especially for neutron decay, recoil is much smaller than nucleon mass. Therefore at the leading order one can approximate

$$\langle p|(\bar{u}\gamma_\mu d)|n> = g_V \bar{u}(p_p)\gamma_\mu u(p_n) + \mathcal{O}(q)$$

$$\langle p|(\bar{u}\gamma_\mu\gamma_5 d)|n> = g_A \bar{u}(p_p)\gamma_\mu\gamma_5 u(p_n) + \mathcal{O}(q)$$

$$q \equiv p_n - p_p$$

where $g_V=g_V(0)$ and $g_A=g_A(0)$ are now numbers, called the vector and axial charges

Furthermore, in the isospin symmetric $g_V=1$, because the quark current is the isospin current One can prove that departures of g_V from one are second order in isospin breaking, thus tiny

All in all

$$M(n \to p e^- \bar{\nu}_e) = -\frac{V_{ud}}{\mathrm{v}^2}\left(\bar{x}(p_e)\bar{\sigma}_\mu\, y(p_\nu)\right)\left\{\bar{u}(p_p)\gamma_\mu u(p_n) - g_A \bar{u}(p_p)\gamma_\mu\gamma_5 u(p_n) + \mathcal{O}(q)\right\}$$

Fermi EFT

$$M(n \to p e^- \bar{\nu}_e) = -\frac{V_{ud}}{\mathrm{v}^2}\left(\bar{x}(p_e)\bar{\sigma}_\mu y(p_\nu)\right)\left\{ \langle p\,|\,(\bar{u}\gamma_\mu d)\,|\,n > - \langle p\,|\,(\bar{u}\gamma_\mu\gamma_5 d)\,|\,n > \right\}$$

$$\mathscr{L}_{\mathrm{WEFT}} \supset -\frac{V_{ud}}{\mathrm{v}^2}[\bar{u}\gamma_\mu(1-\gamma_5)d][\bar{e}\bar{\sigma}^\mu\nu] + \mathrm{h.c.}$$

Matching

$$\mathscr{L}_{\mathrm{LY}} \supset -\frac{V_{ud}}{\mathrm{v}^2}[\bar{e}\bar{\sigma}^\mu\nu]\left\{ (\bar{p}\gamma_\mu n) - g_A(\bar{p}\gamma_\mu\gamma_5 n) \right\} + \mathrm{h.c.} + \mathcal{O}\left(\frac{\partial}{m_n}\right)$$

as our n→p e ν amplitude can be obtained from this effective Lagrangian

The non-perturbative parameter g_A appearing in this matching has to be calculated on the lattice or measured in experiment

Lattice

$$g_A = 1.246 \pm 0.028$$

Fermi EFT

Customarily the nucleon-level Lagrangian is written in a different form

$$\mathscr{L}_{\mathrm{WEFT}} \supset -\frac{V_{ud}}{\mathrm{v}^2}[\bar{u}\gamma_\mu(1-\gamma_5)d][\bar{e}\bar{\sigma}^\mu\nu] + \mathrm{h.c.}$$

Matching

$$\mathscr{L}_{\mathrm{LY}} \supset -[\bar{e}\bar{\sigma}^\mu\nu]\left\{ C_V^+(\bar{p}\gamma_\mu n) + C_A^+(\bar{p}\gamma_\mu\gamma_5 n) \right\} + \mathrm{h.c.} + \mathcal{O}\left(\frac{\partial}{m_n}\right)$$

$$C_V^+ = \frac{V_{ud}}{\mathrm{v}^2} g_V \sqrt{1+\Delta_R^V}\left(1+\epsilon_L+\epsilon_R\right)$$

$$C_A^+ = -\frac{V_{ud}}{\mathrm{v}^2} g_A \sqrt{1+\Delta_R^A}\left(1+\epsilon_L-\epsilon_R\right)$$

From this Lagrangian one can calculate all observables in neutron decay using the standard QFT techniques.
For example, the differential decay spectrum can be found to be

$$\frac{d\Gamma}{dE_e} = \frac{(C_V^+)^2 + 3(C_A^+)^2}{4\pi^3} p_e E_e (E_e^{\max} - E_e)^2$$

Electron momentum **Electron energy** **Endpoint energy**

Fermi EFT

Customarily the nucleon-level Lagrangian is written in a different form

$$\mathscr{L}_{\mathrm{WEFT}} \supset -\frac{V_{ud}}{\mathrm{v}^2}[\bar{u}\gamma_\mu(1-\gamma_5)d][\bar{e}\bar{\sigma}^\mu\nu] + \mathrm{h.c.}$$

Matching

$$\mathscr{L}_{\mathrm{LY}} \supset -[\bar{e}\bar{\sigma}^\mu\nu]\left\{ C_V^+(\bar{p}\gamma_\mu n) + C_A^+(\bar{p}\gamma_\mu\gamma_5 n) \right\} + \mathrm{h.c.} + \mathscr{O}\left(\frac{\partial}{m_n}\right)$$

$$C_V^+ = \frac{V_{ud}}{\mathrm{v}^2} g_V \sqrt{1+\Delta_R^V}\left(1+\epsilon_L+\epsilon_R\right)$$

$$C_A^+ = -\frac{V_{ud}}{\mathrm{v}^2} g_A \sqrt{1+\Delta_R^A}\left(1+\epsilon_L-\epsilon_R\right)$$

Furthermore, one also includes interactions which are not predicted by the SM

$$\begin{aligned}
\mathscr{L}_{\mathrm{LY}} = & -C_V^+ \bar{e}\gamma_\mu\nu_L \cdot \bar{p}\gamma^\mu n & & -C_V^- \bar{e}\gamma_\mu\nu_R \cdot \bar{p}\gamma^\mu n \\
& -C_A^+ \bar{e}\gamma_\mu\nu_L \cdot \bar{p}\gamma^\mu\gamma_5 n & & +C_A^- \bar{e}\gamma_\mu\nu_R \cdot \bar{p}\gamma^\mu\gamma_5 n \\
& -C_S^+ \, \bar{e}\nu_L \cdot \bar{p}n & & -C_S^- \, \bar{e}\nu_R \cdot \bar{p}n \\
& -\frac{1}{2}C_T^+ \bar{e}\sigma_{\mu\nu}\nu_L \cdot \bar{p}\sigma^{\mu\nu} n & & -\frac{1}{2}C_T^- \bar{e}\sigma_{\mu\nu}\nu_R \cdot \bar{p}\sigma^{\mu\nu} n \\
& +C_P^+ \, \bar{e}\nu_L \cdot \bar{p}\gamma_5 n & & -C_P^- \, \bar{e}\nu_R \cdot \bar{p}\gamma_5 n \\
& +\mathrm{hc}
\end{aligned}$$

Lee-Yang EFT

One can continue to use nucleon level effective Lagrangian to calculate beta decay for nuclei

$$
\begin{aligned}
\mathscr{L}_{\rm LY} = & -C_V^+ \bar{e}\gamma_\mu \nu_L \cdot \bar{p}\gamma^\mu n & & -C_V^- \bar{e}\gamma_\mu \nu_R \cdot \bar{p}\gamma^\mu n \\
& -C_A^+ \bar{e}\gamma_\mu \nu_L \cdot \bar{p}\gamma^\mu \gamma_5 n & & +C_A^- \bar{e}\gamma_\mu \nu_R \cdot \bar{p}\gamma^\mu \gamma_5 n \\
& -C_S^+ \,\bar{e}\nu_L \cdot \bar{p}n & & -C_S^- \,\bar{e}\nu_R \cdot \bar{p}n \\
& -\frac{1}{2} C_T^+ \bar{e}\sigma_{\mu\nu}\nu_L \cdot \bar{p}\sigma^{\mu\nu} n & & -\frac{1}{2} C_T^- \bar{e}\sigma_{\mu\nu}\nu_R \cdot \bar{p}\sigma^{\mu\nu} n \\
& +C_P^+ \,\bar{e}\nu_L \cdot \bar{p}\gamma_5 n & & -C_P^- \,\bar{e}\nu_R \cdot \bar{p}\gamma_5 n \\
& +\mathrm{hc}
\end{aligned}
$$

For nuclear decay one needs matrix elements of nucleon operators between nuclear states

$$
\begin{aligned}
\mathscr{M}(\mathscr{N} \to \mathscr{N}' e^- \bar{\nu}) = & \left(C_V^+ [\bar{u}(k_e)\gamma^\mu P_L \mathrm{v}(k_\nu)] + C_V^- [\bar{u}(k_e)\gamma^\mu P_R \mathrm{v}(k_\nu)] \right) \langle \mathscr{N}'(k', J', s' \ldots) | \bar{p}\gamma^\mu n | \mathscr{N}(p, J, s \ldots) \rangle \\
& + \left(C_A^+ [\bar{u}(k_e)\gamma^\mu P_L \mathrm{v}(k_\nu)] - C_A^- [\bar{u}(k_e)\gamma^\mu P_R \mathrm{v}(k_\nu)] \right) \langle \mathscr{N}'(k', J', s' \ldots) | \bar{p}\gamma^\mu \gamma_5 n | \mathscr{N}(p, J, s \ldots) \rangle \\
& + \ldots
\end{aligned}
$$

As we did when matching from quarks to nucleons, we need to parametrize the matrix elements using Lorentz symmetry, parity, isospin, etc. The additional complication is that in general we deal with higher spin states.

Fermi EFT

Summary and lesson learned

- Matching of the Wilson coefficients cannot always be calculated analytically if the UV theory is strongly coupled at the matching scale
- In those cases, it pays off to use the arguments based on symmetries and dimensional analysis, to reduce the number of unknown parameters in the EFT
- The remaining unknown parameters can be taken from the lattice, phenomenological models, or from experiment
- One can continue to use nucleon level effective Lagrangian to calculate beta decay for nuclei as well, given input of nuclear matrix elements

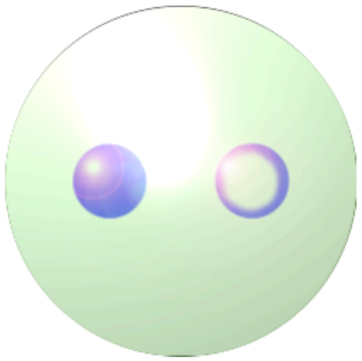
Illustration #4

Chiral Perturbation Theory

Hadrons in low-energy QCD

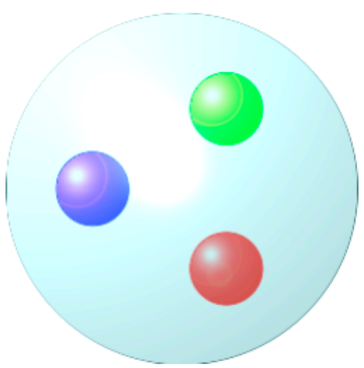
Hadron

Meson (quark and anti-quark)

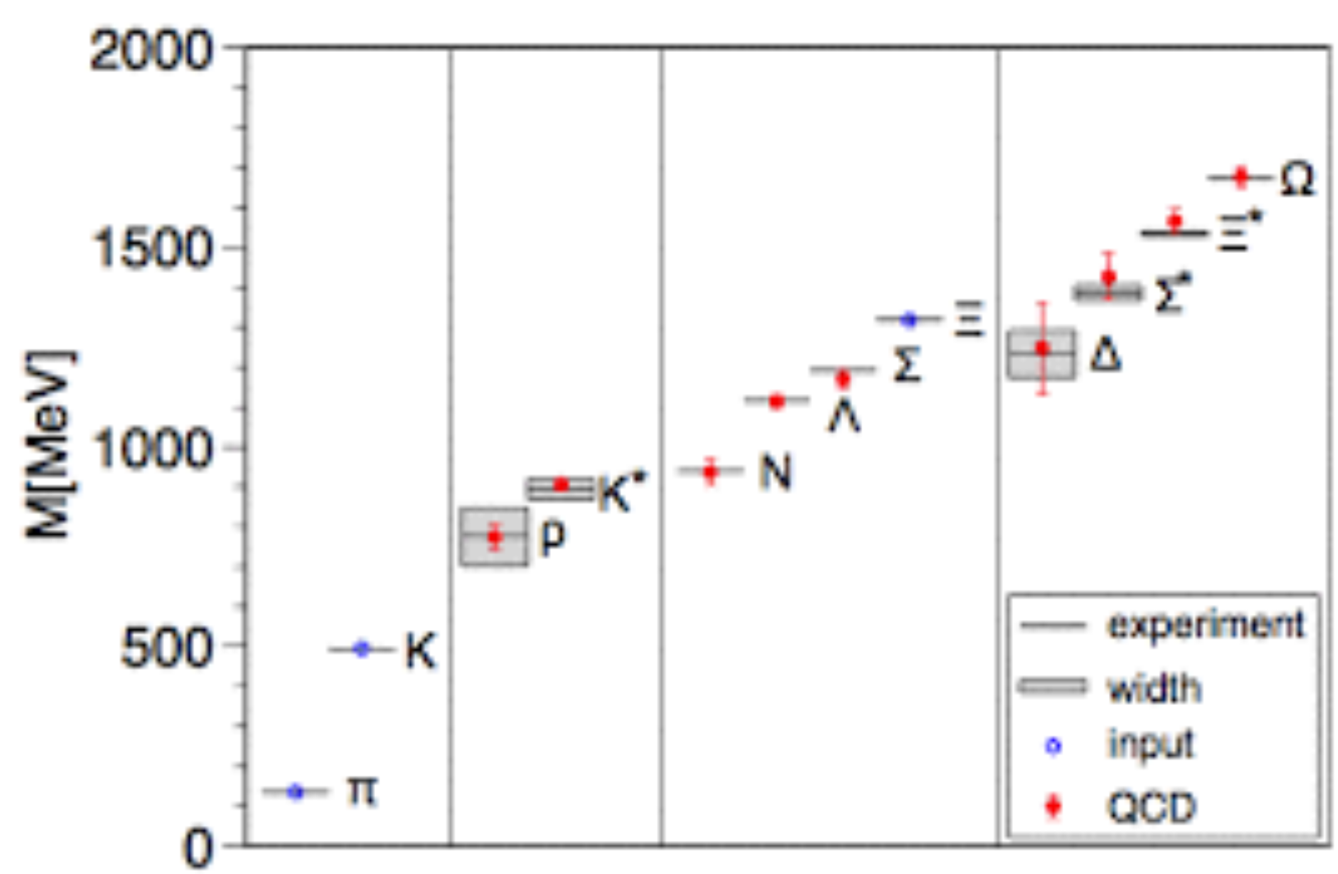


π, K, K∗, ρ, ω, η, φ, a, b, f, D, B, ...

Baryon (3 quarks)



p, n, Δ, Λ, Σ, Σ∗, Ξ, Ξ∗. Ω. Λc. Ξc. Λc. ...

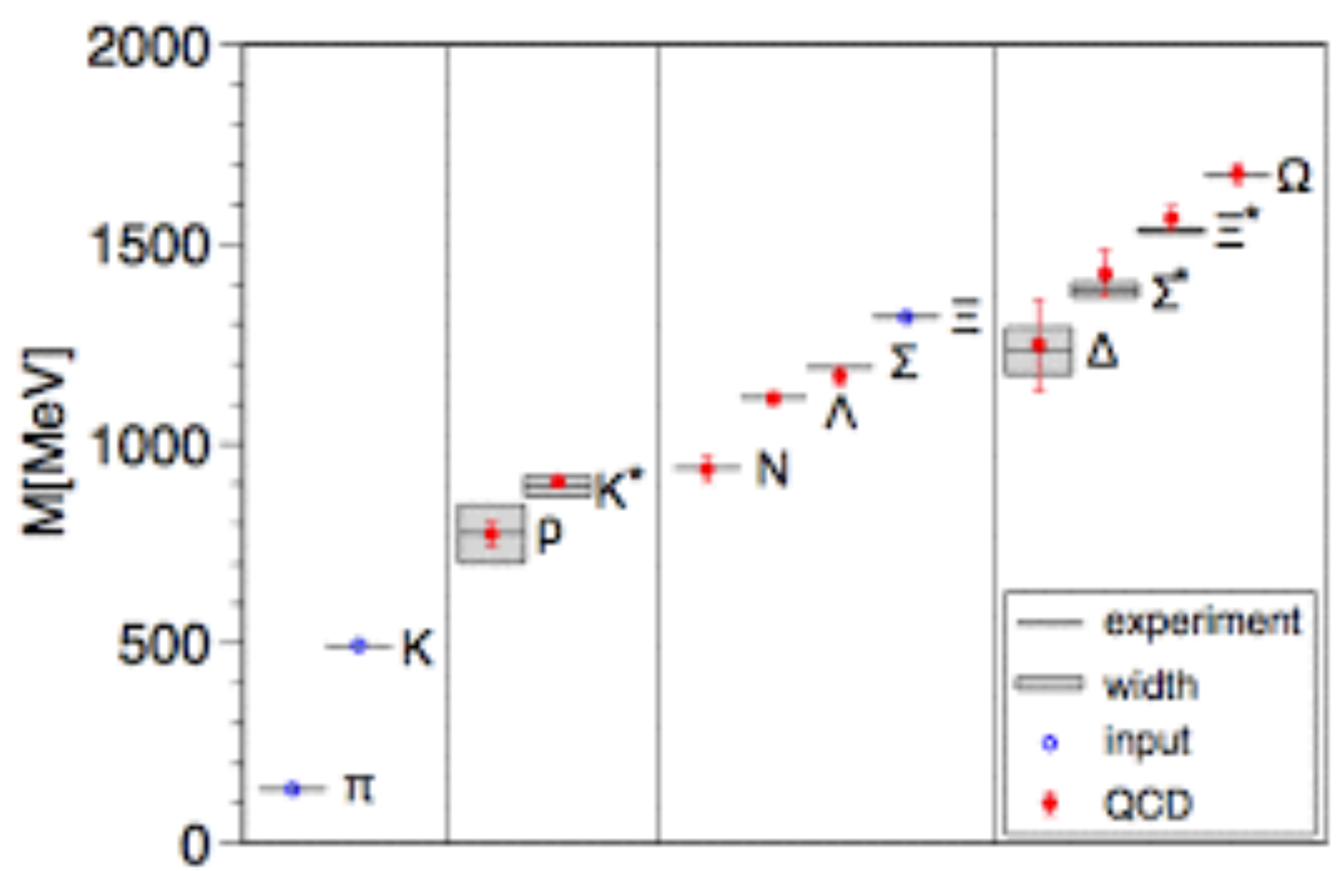


ChPT

- The goal here is to describe the interactions of lightest existing hadrons - so called pions - using EFT techniques

- Underlying theory - QCD - is known, but coefficients of EFT operators cannot be calculated analytically

- We will write the effective theory of pions using symmetries as the main principle in addition to EFT power counting



ChPT

QCD with two flavours

$$\mathscr{L}_{\text{QCD}} = i\bar{u}\bar{\sigma}_\mu D_\mu u + iu^c\sigma_\mu D_\mu \bar{u}^c + i\bar{d}\bar{\sigma}_\mu D_\mu d + id^c\sigma_\mu D_\mu \bar{d}^c - m_u\left(u^c u + \bar{u}\bar{u}^c\right) - m_d\left(d^c d + \bar{d}\bar{d}^c\right)$$
$$= i\bar{q}\bar{\sigma}_\mu D_\mu q + iq^c\sigma_\mu D_\mu \bar{q}^c - q^c M_q q - \bar{q}M_q^\dagger \bar{q}^c$$

$$q = (u, d), \quad q^c = (u^c, d^c), \quad M_q = \text{diag}(m_u, m_d)$$

In the limit of massless quarks, the theory has an $U(2)_L \times U(2)_R$ global symmetry

$$q \to Lq, \qquad q^c \to qR^\dagger$$

This can be also written as $SU(2)_V \times SU(2)_A \times U(1)_V \times U(1)_A$ where the vector part correspond to $L = R$ and the axial part correspond $R^\dagger = L$
It turns out that $U(1)_A$ is anomalous and does not play a role in this particular story.
Moreover $U(1)_V$ is just the baryon number which does not act on pions
We focus here on the $SU(2)_L \times SU(2)_R$ part

ChPT

Two sources of breaking of $SU(2)_L \times SU(2)_R$

1. **Explicit breaking by quark masses**

$$q^c M_q q \to q^c R^\dagger M_q L q \neq q^c M_q q$$

2. **Spontaneous breaking by QCD vacuum**

$$< 0 | q^c q | 0 > \neq 0 \qquad q^c q \to q^c R^\dagger L q$$

The latter breaks $SU(2)_L \times SU(2)_R \to SU(2)_V$
Goldstone theorem then predicts three massless degrees of freedom in the limit $M_q \to 0$, one for each broken generator
These can be identified with the the 3 pion states $\pi^{\pm}, \pi^0$

ChPT

The same symmetry pattern should be reflected in the low-energy EFT describing pions
Two options:

1. **Symmetry is realized linearly, but that would lead to another state in addition to the Goldstone boson pions.**
2. **Symmetry is realized non-linearly, so that only the three pions are present. This is the option used in chiral perturbation theory**

The common formalism is to introduce a unitary matrix U transforming linearly under $SU(2)_L \times SU(2)_R$ but depending in a non-linear way on the pion fields:

$$U \to LUR^\dagger, \qquad U = \exp\left(i\frac{\pi^k \sigma^k}{F_\pi} \right)$$

Under the surviving $SU(2)_V$ pions transform linearly as a triplet, but they transform non-linear under the broken $SU(2)_A$

ChPT

The EFT Lagrangian is written in the derivative expansion

$$\mathscr{L}_{\chi\mathrm{PT}} = \mathscr{L}^{(2)}_{\chi\mathrm{PT}} + \mathscr{L}^{(4)}_{\chi\mathrm{PT}} + \mathscr{L}^{(6)}_{\chi\mathrm{PT}} + \dots$$

The leading order term is

$$\mathscr{L}^{(2)}_{\chi\mathrm{PT}} = \frac{F_\pi^2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U]\,.$$

It gives kinetic terms to pions, but also quartic and higher two derivative self-interactions

The next-to-leading order term is

$$\mathscr{L}^{(4)}_{\chi\mathrm{PT}} = \frac{l_1}{4}\left(\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U]\right)^2 + \frac{l_2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U]\mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U]\,.$$

It gives quartic and higher four-derivative self-interactions

ChPT

The EFT Lagrangian is written in the derivative expansion

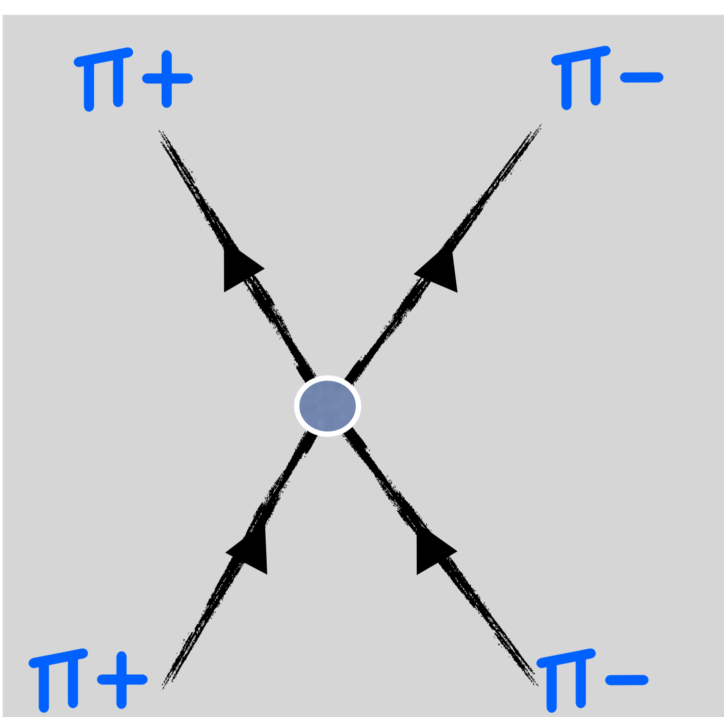
$$\mathscr{L}_{\chi\mathrm{PT}} = \mathscr{L}^{(2)}_{\chi\mathrm{PT}} + \mathscr{L}^{(4)}_{\chi\mathrm{PT}} + \mathscr{L}^{(6)}_{\chi\mathrm{PT}} + \ldots$$

The leading order term is

$$\mathscr{L}^{(2)}_{\chi\mathrm{PT}} = \frac{F_\pi^2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U]\,.$$

$$\begin{aligned}\mathcal{L}^{(2)}_{\text{eff}} =&\partial_\mu\pi^+\partial_\mu\pi^- + \frac{1}{2}\partial_\mu\pi^0\partial_\mu\pi^0\\ &+\frac{1}{2f^2}(\partial_\mu\pi^+\pi^- + \partial_\mu\pi^-\pi^+ + \partial_\mu\pi^0\pi^0)^2\\ &+\ldots\end{aligned}$$

It gives kinetic terms to pions, but also quartic and higher two derivative self-interactions



$$M^{(2)}[\pi^a\pi^b \to \pi^c\pi^d] = \frac{1}{F_\pi^2}\left\{s\delta^{ab}\delta^{cd} + t\delta^{ac}\delta^{bd} + u\delta^{ad}\delta^{bc}\right\}.$$

ChPT

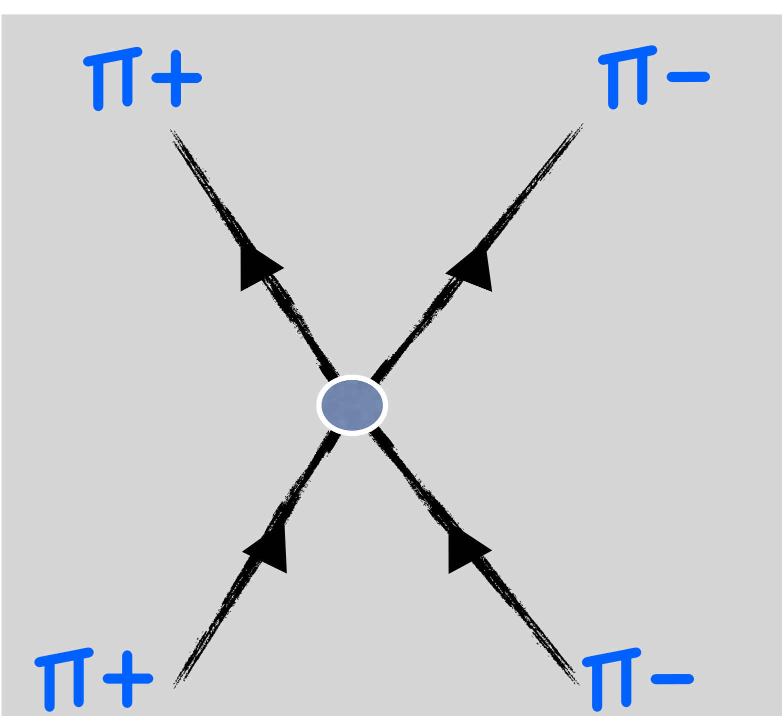
The EFT Lagrangian is written in the derivative expansion

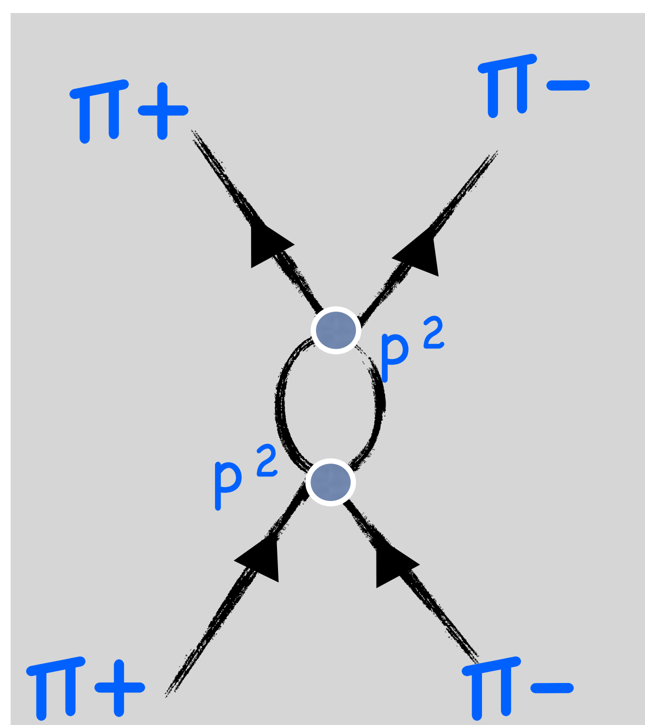
$$\mathscr{L}_{\chi\mathrm{PT}} = \mathscr{L}^{(2)}_{\chi\mathrm{PT}} + \mathscr{L}^{(4)}_{\chi\mathrm{PT}} + \mathscr{L}^{(6)}_{\chi\mathrm{PT}} + \dots$$

The next-to-leading order term is

$$\mathscr{L}^{(4)}_{\chi\mathrm{PT}} = \frac{l_1}{4}\left(\mathrm{Tr}[\partial_\mu U^\dagger \partial_\mu U]\right)^2 + \frac{l_2}{4}\mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U]\mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U] \, .$$

It gives quartic and higher four-derivative self-interactions





$$M[\pi^a\pi^b \to \pi^c\pi^d] \sim \frac{E^2}{F_\pi^2}\left\{1 + \frac{E^2 \log E}{(4\pi F_\pi)^2} + \dots\right\} .$$

ChPT

How to include pion masses:

$$q^c M_q q \to q^c R^\dagger M_q L q \neq q^c M_q q$$

But we can treat the mass term as a spurion so that formally the Lagrangian term remains invariant

$$M_q \to R M_q L^\dagger .$$

$$q^c M_q q \to q^c M_q q$$

Formally invariant Lagrangian term

$$\Delta \mathscr{L}_{\chi\mathrm{PT}} = \tilde{\Lambda} F_\pi^2 \mathrm{Tr}[M_q U] + \mathrm{h.c.}$$

$$m_\pi^2 = 2\tilde{\Lambda}(m_u + m_d) .$$

Chiral perturbation theory

Lessons learned:

- It is often advantageous to work with EFT even when matching with UV theory cannot be calculated. Then one needs to write down all possible non-redundant interaction terms consistent with EFT symmetries in some systematic expansion, and determine their coefficients from experiment

- EFT is not renormalizable, therefore it formally has infinite number of parameter. However, at a fixed order in EFT expansion it is renormalizable. As soon as all coefficients are fixed at a given order from experiment, other observables can be predicted at that order