Astrophysical and cosmological aspects of gravitational waves

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Generation of GWs in linearised theory

We want to derive the GW emission by an isolated binary system

"Gravitational Waves", M. Maggiore, Oxford University Press 2008

- The source is close by and we can neglect the expansion of the universe
- We go back to *linearised theory in flat spacetime*: the gravitational field is weak and the source is described by Newtonian gravity

Weak gravitational field means low velocity for a self-gravitating system

Self-gravitating 2-body system in Newtonian theory: virial theorem applies

$$E_{\rm kin} = -\frac{1}{2}U$$

The problem is equivalent to the motion of one body with reduced mass μ, in the gravitational field given by the sum of the masses

$$m = m_1 + m_2$$

$$\mu = \frac{m_1 m_2}{m}$$

$$\frac{1}{2}\mu v^2 = -\frac{1}{2}\frac{Gm\mu}{r}$$

Schwarzschild radius $R_S = 2\frac{Gm}{c^2}$

$$\left(\frac{v}{c}\right)^2 = \frac{R_S}{2r} \ll 1$$

↓ Weak gravitational field

We perform a low velocity expansion to solve the problem

Generation of GWs in linearised theory

Solve wave equation in linearised theory

$$\Box \bar{h}_{\mu\nu} = -16\pi G \, T_{\mu\nu}$$

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int d^3x' \, \frac{T_{\mu\nu}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

Retarded time

We can go to TT gauge outside the source

$$h_{ij}(t, \mathbf{x}) = 4G \underbrace{\Lambda_{ij\ell m}(\hat{n})}_{} \int d^3x' \, \frac{T^{\ell m}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

TT projector $\quad \Lambda_{ij\ell m} = P_{i\ell} P_{jm} - \frac{1}{2} P_{ij} P_{\ell m}$

$$P_{i\ell} = \delta_{i\ell} - \hat{n}_i \hat{n}_\ell$$

Generation of GWs in linearised theory

Solve wave equation in linearised theory

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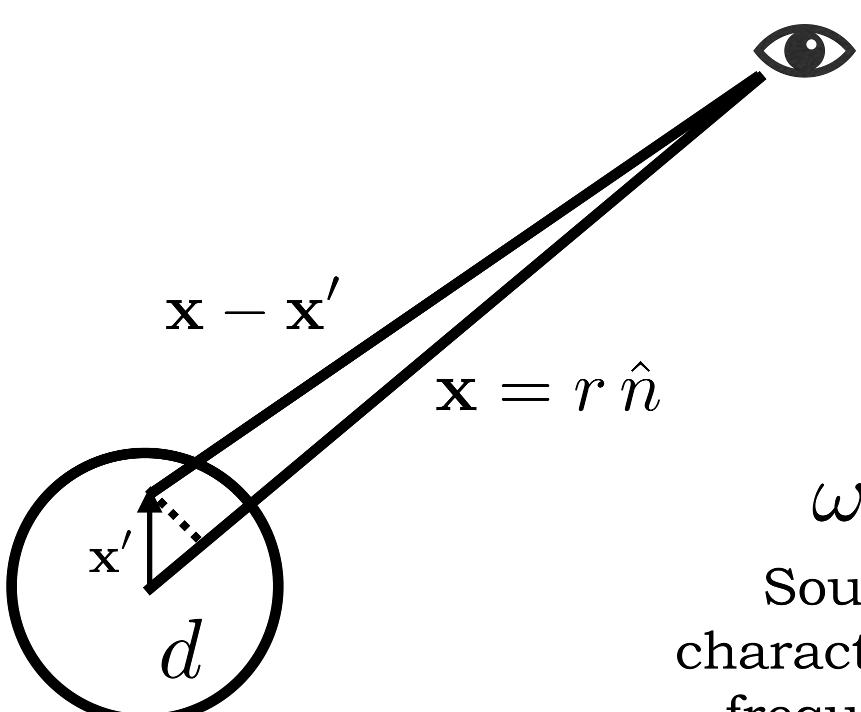
Retarded time

We can go to TT gauge outside the source

$$h_{ij}(t,\mathbf{x}) = 4G\Lambda_{ij\ell m}(\hat{n}) \int d^3x' \frac{T^{\ell m}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

Low velocity expansion in the radiation zone r » d



$$|\mathbf{x}-\mathbf{x}'| \simeq r - \mathbf{x}'\cdot\hat{n}$$

$$T_{\ell m}(t-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}') \sim T_{\ell m}(t-r,\mathbf{x}') + \underbrace{+\,\partial_t T_{\ell m}}(t-r,\mathbf{x}')(\mathbf{x}'\cdot\hat{n})$$

ω_s

Source characteristic frequency

$$\lesssim \omega_s T_{\ell m} d \sim \frac{v}{d} T_{\ell m} d \sim v\, T_{\ell m}$$

Generation of GWs in linearised theory

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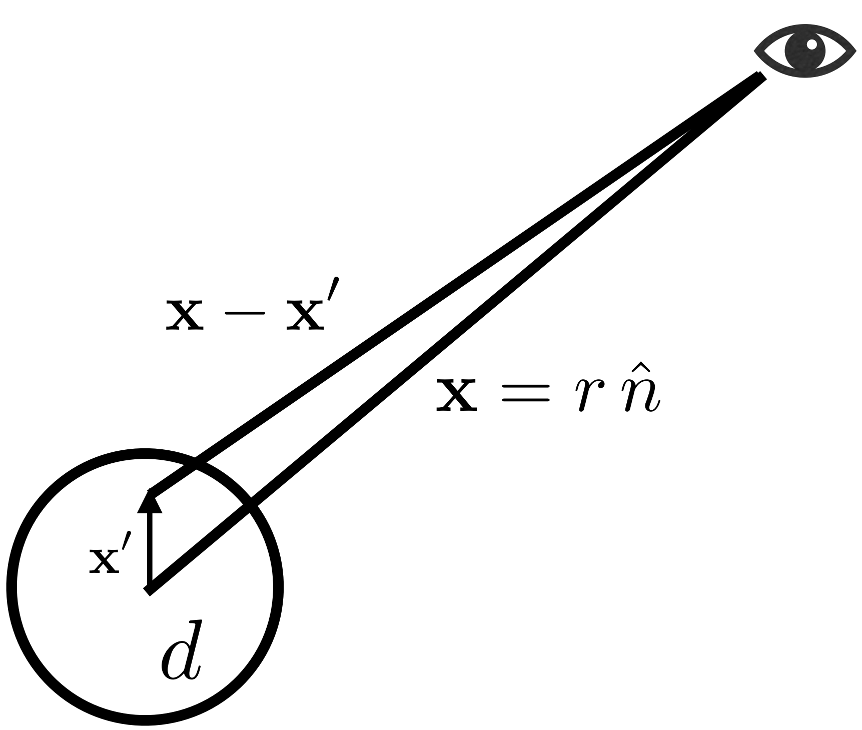
Low velocity expansion in the radiation zone r » d



radiation zone

$$r \gg \lambda\!\!\!^{-} \gg d$$

$$\Downarrow$$

$$v \sim \omega_s d$$

One cannot resolve the source with GWs

$$\lambda\!\!\!^{-} = \frac{c}{\omega} \sim \frac{c}{v}\, d \gg d$$

Generation of GWs in linearised theory

Keeping only the lowest order $h_{ij}(t,\mathbf{x}) \sim \frac{4G}{r}\Lambda_{ij\ell m}(\hat{n}) \int d^3x' T^{\ell m}(t-r,\mathbf{x}')$

Using energy momentum conservation, $\partial_\mu T^{\mu\nu} = 0$ this can be rewritten

$$\int d^3x' T^{\ell m}(t-r,\mathbf{x}') = \frac{1}{2}\int d^3x'\, \partial_t^2 T^{00}(t-r,\mathbf{x}')\, x^\ell\, x^m = \frac{1}{2}\ddot{M}^{\ell m}$$

Second mass moment

$$M^{\ell m} - \frac{1}{3}\overbrace{\delta^{\ell m} M_{jj}}^{\text{trace}} = \int d^3x\, \rho(\mathbf{x},t)\left(x^\ell x^m - \frac{1}{3}x^2\delta^{\ell m}\right) = Q^{\ell m}$$

Mass quadrupole

at the lowest order in the low velocity expansion and in the radiation zone, GW emission arises from the double time derivative of the quadrupole of the source

$$[h_{ij}(t,\mathbf{x})]_{\text{quad}} = \frac{2G}{r}\Lambda_{ij\ell m}(\hat{n})\ddot{Q}^{\ell m}(t-r)$$

C.W. Misner, K.S. Thorne, J.A. Wheeler, “Gravitation”, Freeman 1997 (chapter 35.15)

Generation of GWs in linearised theory

The quadrupole formula

Radiated GW energy energy per unit time and angle

$$\rho_{GW} = \frac{\langle \dot{h}_{ij} \dot{h}^{ij} \rangle}{32\pi G} \qquad\qquad \frac{dE_{\rm GW}}{dt\, d\Omega} = \frac{r^2}{32\pi G} \langle \dot{h}_{ij} \dot{h}_{ij} \rangle$$

Total radiated power

$$[P]_{\rm quad} = \frac{G}{5} \langle \dddot{Q}_{ij}(t-r) \dddot{Q}_{ij}(t-r) \rangle$$

$$\int d\Omega \Lambda_{ij\ell m} = \frac{2\pi}{15}(11\delta_{i\ell}\delta_{jm} - 4\delta_{ij}\delta_{\ell m} + \delta_{im}\delta_{j\ell})$$

A static source does not radiate GWs

Generation of GWs in linearised theory

- Monopole $\ell = 0$ and dipole $\ell = 1$ radiation is absent for GWs

$$h^{00}(t,\mathbf{x}) \sim \frac{4G}{r}\int d^3x' T^{00} \sim \frac{4G}{r} M(t-r)$$

h^{00} is a static component, because of mass conservation for an isolated system $\dot{M} = 0$

$$h^{0i}(t,\mathbf{x}) \sim \frac{4G}{r}\int d^3x' T^{0i} \sim \frac{4G}{r} P^i(t-r)$$

h^{0i} a static component, because of momentum conservation for an isolated system $\dot{P}^i = 0$

- HOWEVER! These conservation laws are only valid in linearised theory, for which the GW emission does not back-react on the source: is the quadrupole formula valid only in this case? NO!

- The multipole expansion of a classical radiation field has zero contribution from multipoles ℓ < S where S is the spin of the associated quantum mechanical particle

    - Comparison with early universe sources: they are relativistic and not gravitationally bound -> one needs to calculate the full tensor anisotropic stress

    - By expanding the tensor anisotropic stress in multipoles, one would find that the first moment is the quadrupole, but it doesn't dominate over the others multipoles

Generation of GWs in linearised theory

More explicit formulas

$$\Lambda_{ij\ell m}\ddot{M}_{\ell m} = (P\ddot{M}P)_{ij} - \frac{1}{2}P_{ij}\mathrm{tr}(P\ddot{M})$$

$$P_{i\ell} = \delta_{i\ell} - \hat{n}_i\hat{n}_\ell$$

The polarisation amplitudes for a wave propagating in the z direction

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} h_+(t,\hat{z}) &= \frac{G}{r}(\ddot{M}_{11} - \ddot{M}_{22})(t-r) \\ h_\times(t,\hat{z}) &= \frac{2G}{r}\ddot{M}_{21}(t-r) \end{aligned}$$

A rotation of the reference system allows to find the GW emission in a general direction

$$n_i = R_{ij} z'_j \qquad M'_{ij} = (R^T\, M\, R)_{ij}$$

In this case, the polarisation amplitudes depend on all mass moment components, and on the angles (φ,θ) of the direction n

GWs from a binary system in circular orbit

Isolated system of two point masses moving on circular trajectories determined solely by their mutual interaction

$$\partial_\mu T^{\mu\nu} = 0$$

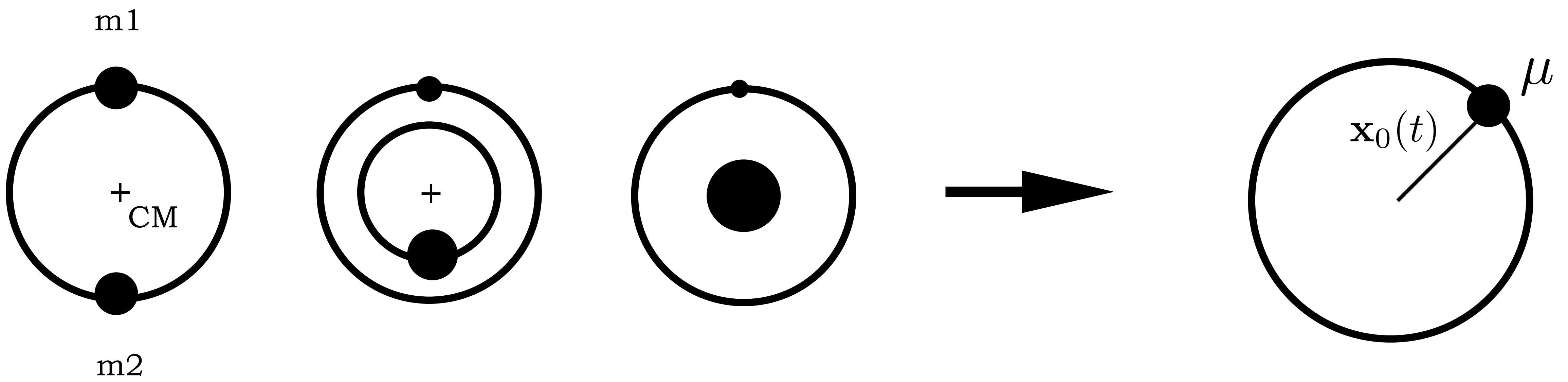
Otherwise, we cannot use linearised theory!

In the centre of mass reference frame, the second mass moment is the one of a single particle with reduced mass, and trajectory given by the relative trajectory of the two point masses

$$M^{ij} = \mu\, x_0^i(t)\, x_0^j(t)$$

$$\mu = \frac{m_1 m_2}{m}$$

$$\mathbf{x}_0(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$$



GWs from a binary system in circular orbit

Orbit in the (x,y) plane

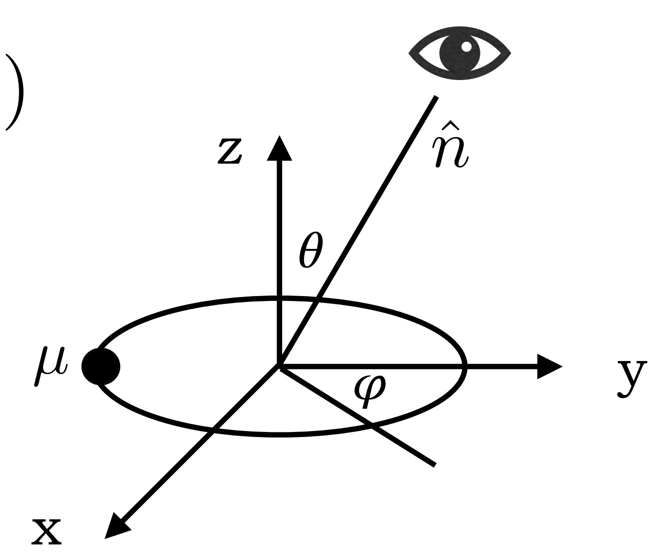
$$x_0(t) = R\cos(\omega_s t + \pi/2)$$
$$y_0(t) = R\sin(\omega_s t + \pi/2)$$
$$z_0(t) = 0$$

Double time derivative of the second mass moment

$$\ddot{M}_{11}(t) = -\ddot{M}_{22}(t) = 2\mu R^2\omega_s^2\cos(2\omega_s t)$$
$$\ddot{M}_{12}(t) = 2\mu R^2\omega_s^2\sin(2\omega_s t)$$

After performing the rotation of the reference system, the GW polarisation amplitudes in arbitrary direction n are

$$h_+(t,\theta,\varphi) = \frac{4G}{r}\,\mu R^2\omega_s^2\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{\rm ret} + 2\varphi)$$
$$h_\times(t,\theta,\varphi) = \frac{4G}{r}\,\mu R^2\omega_s^2\,\cos\theta\,\sin(2\omega_s t_{\rm ret} + 2\varphi)$$



GWs from a binary system in circular orbit

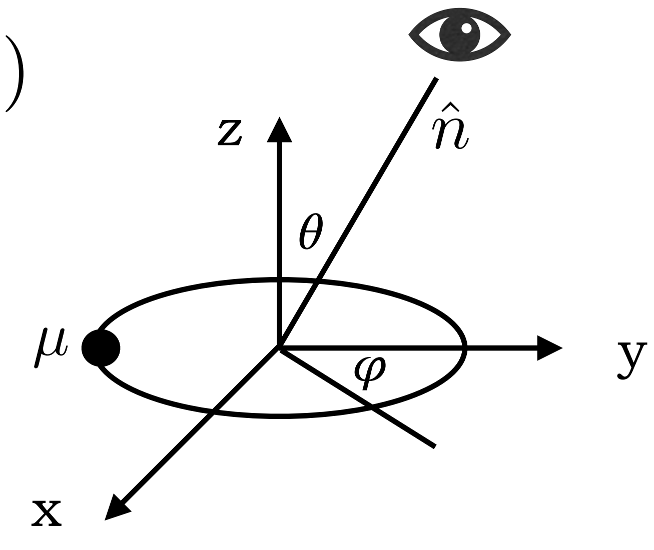
- A non-relativistic source performing harmonic oscillations with frequency ω_s emits monochromatic radiation with frequency $2\,\omega_s$

- The dependence on φ can be reabsorbed in a redefinition of the origin of time

- From the degree of polarisation observed, one can derive the inclination of the orbit

$$h_+(t,\theta,\varphi) = \frac{4G}{r}\,\mu R^2\omega_s^2\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega_s t_{\text{ret}} + 2\varphi)$$

$$h_\times(t,\theta,\varphi) = \frac{4G}{r}\,\mu R^2\omega_s^2\,\cos\theta\,\sin(2\omega_s t_{\text{ret}} + 2\varphi)$$



GWs from a binary system in circular orbit

$$\frac{dE_{\mathrm{GW}}}{dt\,d\Omega} = \frac{r^2}{32\pi G}\langle \dot{h}_{ij}\dot{h}_{ij}\rangle$$

Radiated power

$$\frac{dP_{\mathrm{GW}}}{d\Omega} = \frac{r^2}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle$$

$$\left[\frac{dP_{\mathrm{GW}}}{d\Omega}\right]_{\mathrm{quad}} = \frac{2G\mu^2 R^4\omega_s^6}{\pi}\left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right]$$

$$[P_{\mathrm{GW}}]_{\mathrm{quad}} = \frac{32}{5}G\mu^2 R^4\omega_s^6$$

$$= \frac{32}{5}\omega_s\left(\frac{G\mu^2}{R}\right)v^5$$

Energy emitted in GW in the quadrupole approximation (highest term) is suppressed with v^5 with respect to the gravitational self-energy of the system

Inspiral of compact binaries in circular motion

Up to now we have assumed that the orbit is fixed
However, there is a way to account for
the back reaction of the GW on the emitting system in the context of linearised theory

$$E_{\rm orbit} = E_{\rm kin} + E_{\rm pot} = -\frac{G\mu m}{2R} \qquad v^2 = \frac{Gm}{R}$$

Chirp mass

$$M_c \equiv \mu^{3/5} m^{2/5} \qquad \omega_s^2 = \frac{Gm}{R^3} \quad \longrightarrow \quad [P_{\rm GW}]_{\rm quad} = \frac{32}{5}\frac{(G\,M_c\,\omega_s)^{10/3}}{G}$$

- The energy of the orbit must diminish because of the GW emission, so R must decrease
- If R decreases, ω_s increases
- If ω_s increases, the emitted power increases as well
- If the emitted power increases, R decreases further
- This runaway process leads to the coalescence of the binary system

"Gravitational Waves", M. Maggiore, Oxford University Press 2008

Inspiral of compact binaries in circular motion

To account for the back reaction of the GW on the emitting system one postulates that
the energy lost by the source per unit time equals the power of the emitted GWs in the radiation zone, far away from the observer

NB! This is not so obvious beyond linear theory

$$-\frac{dE_{\rm orbit}}{dt} = [P_{\rm GW}]_{\rm quad}$$

$$\dot{f} = \frac{96\pi^{8/3}}{5}(G\,M_c)^{5/3} f^{11/3}$$

$$f = \frac{\omega_{\rm GW}}{2\pi} = \frac{\omega_s}{\pi}$$

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{(G\,M_c)^{5/8}\,\tau^{3/8}}$$

Time to coalescence $\tau = t_{\rm coal} - t$

Inspiral of compact binaries in circular motion

To calculate the GW amplitudes, one needs to account for the fact that the orbit depends on time

$$x_0(t) = R(t)\cos(\Phi(t))$$
$$y_0(t) = R(t)\sin(\Phi(t))$$
$$z_0(t) = 0$$

$$\Phi(t) = \Phi_{\rm coal} + \int_{t_c}^{t} dt' \omega_s(t')$$

We remain in the approximation that the orbit is almost circular with slowly varying radius

$$\dot{\omega}_s \ll \omega_s^2 \qquad |\dot{R}| \ll v$$

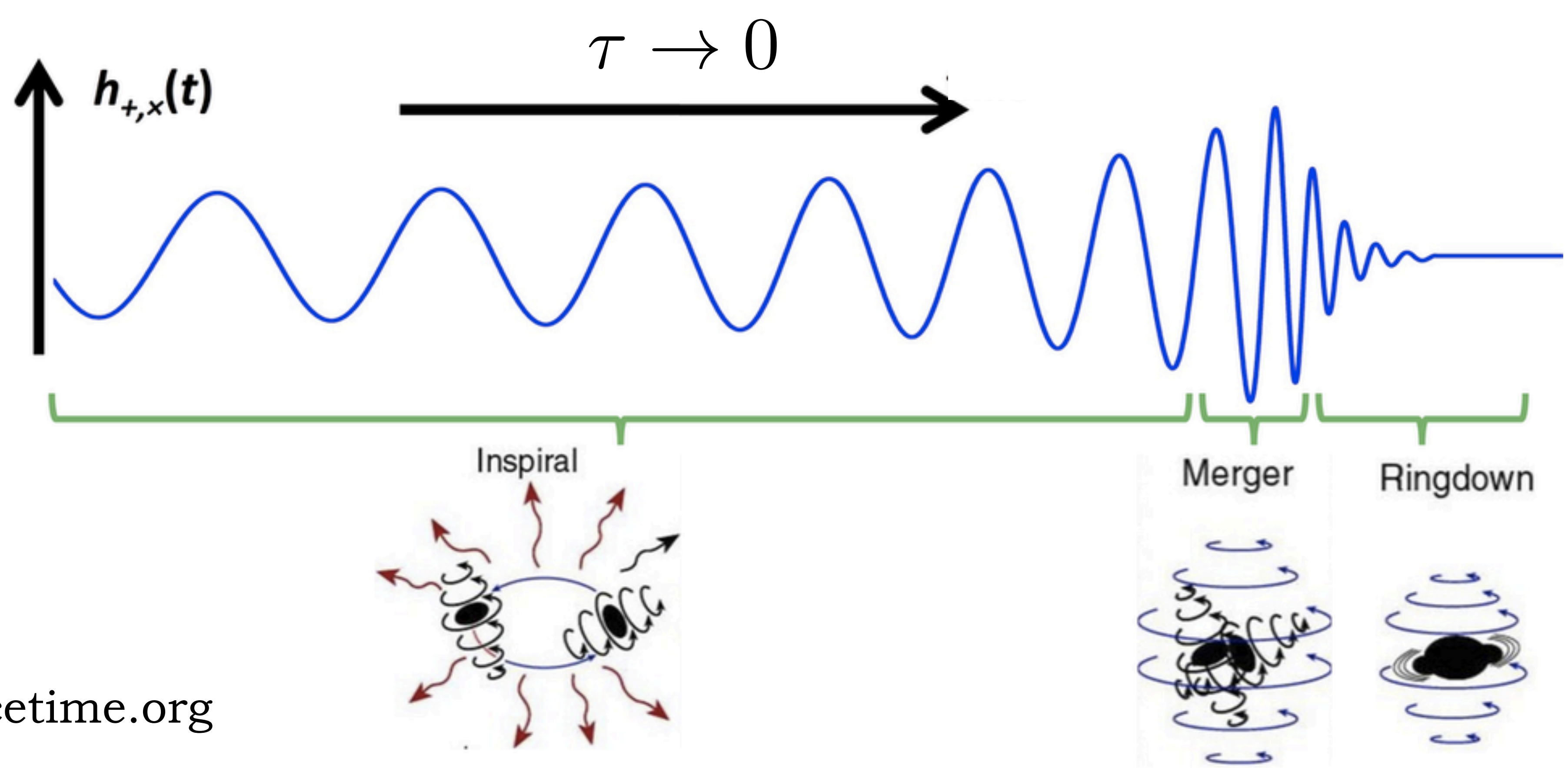
$$h_+(t,\theta,\varphi) = \frac{4}{r}(G\,M_c)^{5/3}[\pi f(t_{\rm ret})]^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\Phi(t_{\rm ret}))$$
$$h_\times(t,\theta,\varphi) = \frac{4}{r}(G\,M_c)^{5/3}[\pi f(t_{\rm ret})]^{2/3}\,\cos\theta\,\sin(2\Phi(t_{\rm ret}))$$

Inspiral of compact binaries in circular motion

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{(G\,M_c)^{5/8}\,\tau^{3/8}} \qquad \Phi(\tau) = -2\left(\frac{\tau}{5\,G\,M_c}\right)^{5/8} + \Phi_{\rm coal}$$

$$h_+(t,\theta,\varphi) = \frac{1}{r}(G\,M_c)^{5/4}\left(\frac{5}{\tau}\right)^{1/4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\Phi(\tau))$$

$$h_\times(t,\theta,\varphi) = \frac{1}{r}(G\,M_c)^{5/4}\left(\frac{5}{\tau}\right)^{1/4}\cos\theta\,\sin(2\Phi(\tau))$$

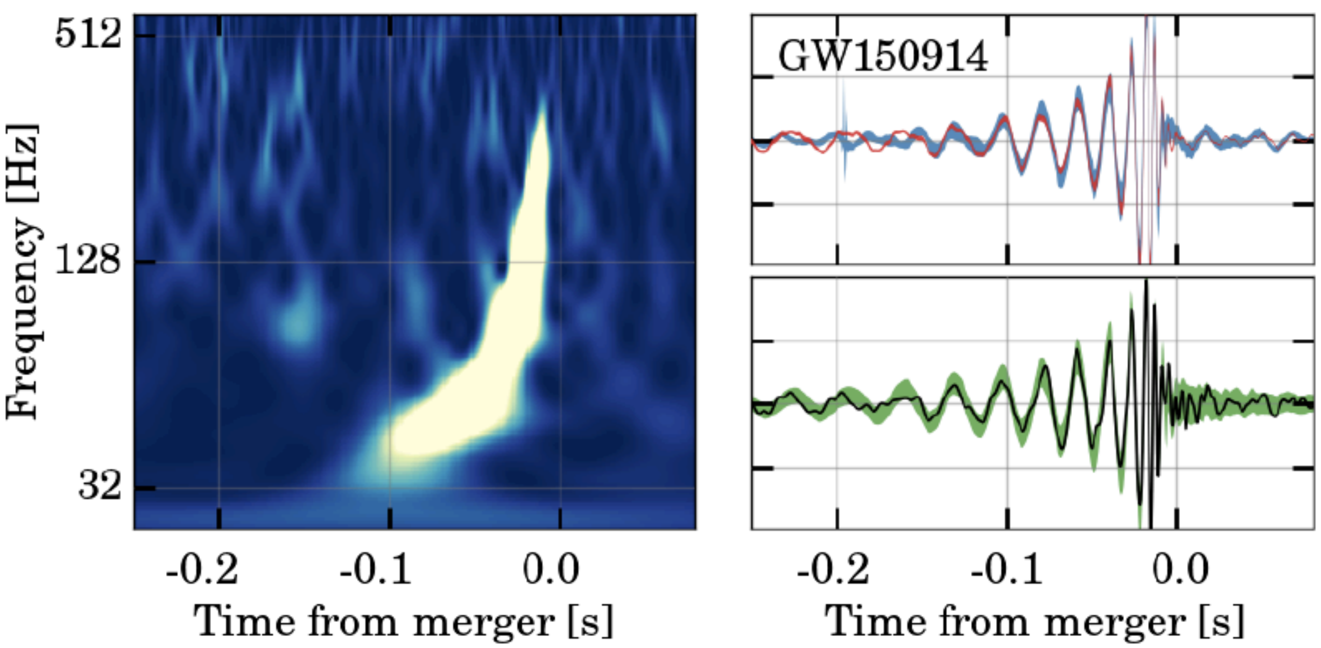


soundofspacetime.org

Inspiral of compact binaries in circular motion

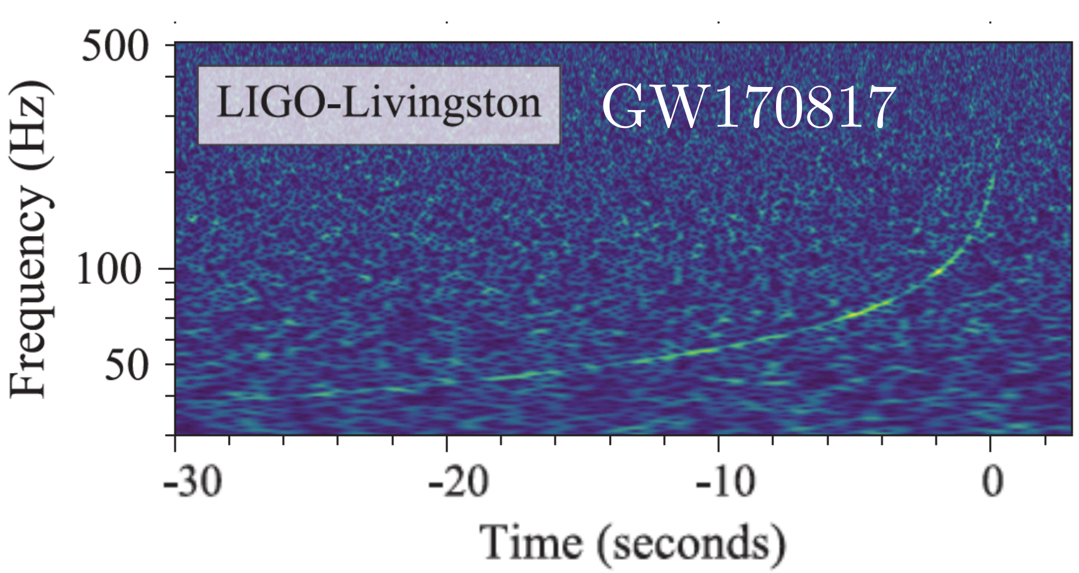
$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{(G\,M_c)^{5/8}\,\tau^{3/8}}$$

$$M_c = 25\,\mathrm{M}_\odot \quad \tau = 0.2\,\mathrm{sec} \quad \longrightarrow \quad f = 37\,\mathrm{Hz}$$
$$M_c = 1.2\,\mathrm{M}_\odot \quad \tau = 30\,\mathrm{sec} \quad \longrightarrow \quad f = 38\,\mathrm{Hz}$$



$$m_1 = 35.6^{+4.8}_{-3.0}\,M_\odot$$
$$m_2 = 30.6^{+3.0}_{-4.4}\,M_\odot$$
$$M_c = 28.6^{+1.6}_{-1.5}\,M_\odot$$
$$d_L = 430^{+150}_{-170}\,\mathrm{Mpc}$$

LIGO/Virgo
arXiv:1811.12907



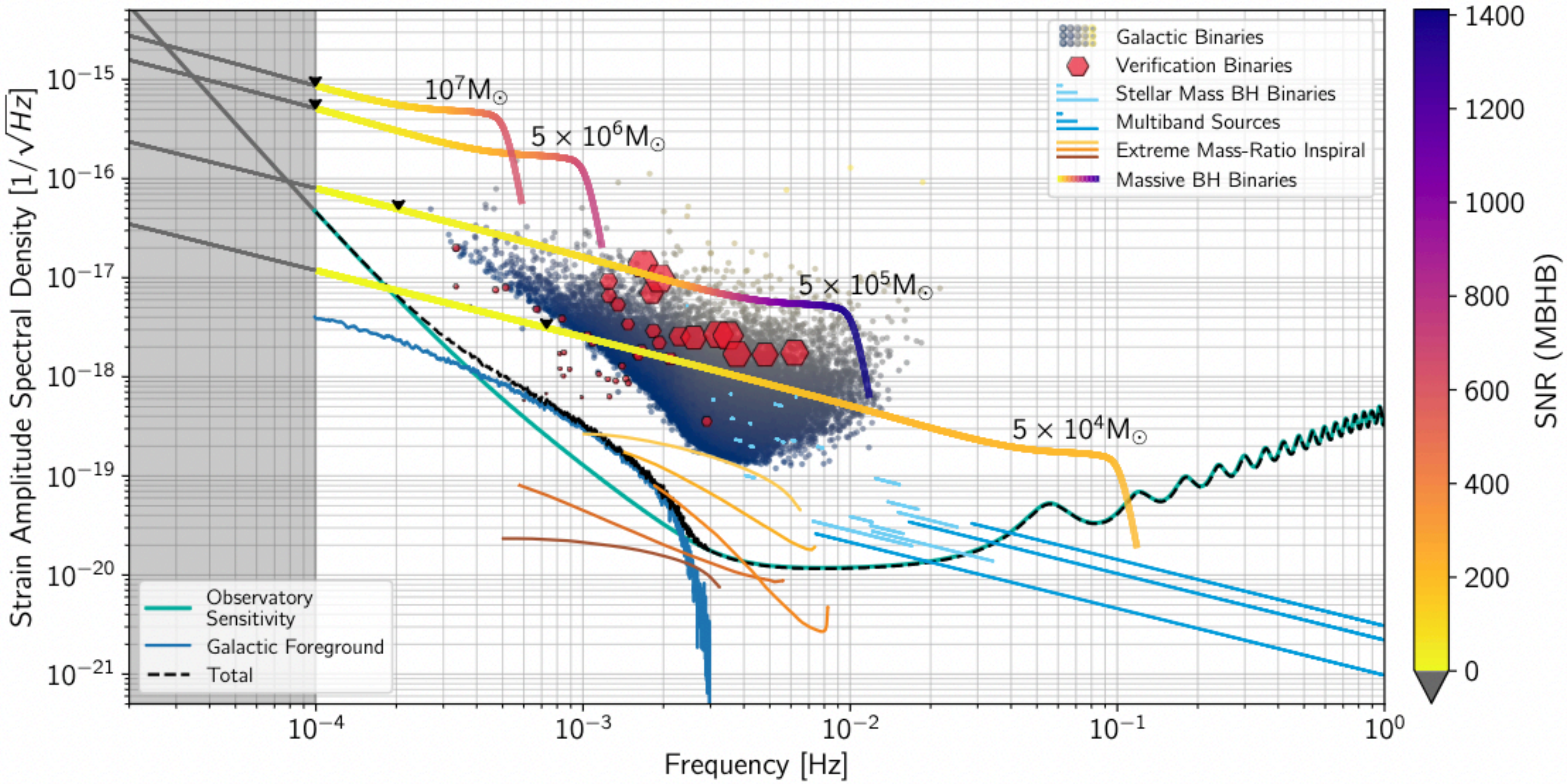
$$m_1 = 1.36 - 1.60\,M_\odot$$
$$m_2 = 1.17 - 1.36\,M_\odot$$
$$M_c = 1.188^{+0.004}_{-0.002}\,M_\odot$$
$$d_L = 40^{+8}_{-14}\,\mathrm{Mpc}$$

Inspiral of compact binaries in circular motion

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8}\frac{1}{(G\,M_c)^{5/8}\,\tau^{3/8}}$$

$$M_c = 25\,\mathrm{M}_\odot \qquad \tau = 10\,\mathrm{year} \qquad \longrightarrow \qquad f = 0.01\,\mathrm{Hz}$$

$$M_c = 10^6\,\mathrm{M}_\odot \qquad \tau = 1\,\mathrm{hour} \qquad \longrightarrow \qquad f = 1\,\mathrm{mHz}$$



M. Colpi et al, arXiv:2402.07571 - LISA Red Book

Inspiral of compact binaries in circular motion

$$f(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\right)^{3/8} \frac{1}{(G\,M_c)^{5/8}\,\tau^{3/8}}$$

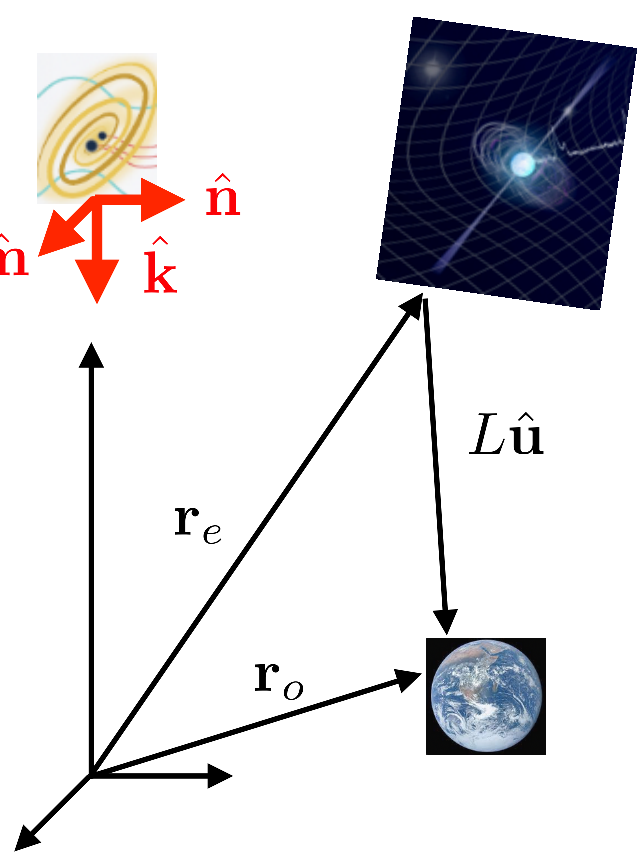
$$M_c = 10^9\,\mathrm{M}_\odot \qquad \tau = 10^5\,\text{year} \qquad \longrightarrow \qquad f = 7\cdot 10^{-9}\,\mathrm{Hz}$$

Equation from yesterday's course:

$$\frac{\Delta T}{P} \simeq \frac{1}{2}\frac{\hat{u}^i\hat{u}^j}{1-\hat{\mathbf{k}}\cdot\hat{\mathbf{u}}}\left[h_{ij}(t_e+L,\mathbf{r_o}) - h_{ij}(t_e,\mathbf{r_e})\right]$$

$$\sim 7\cdot 10^{-23}\,\frac{\mathrm{pc}}{d_L}\left(\frac{M_c}{M_\odot}\right)^{5/3}\left(\frac{f_{\mathrm{GW}}}{10^{-8}\,\mathrm{Hz}}\right)^{2/3} \simeq 7\cdot 10^{-16}$$

$$M_c \simeq 10^9\,M_\odot$$
$$d_L = 100\,\mathrm{Mpc}$$

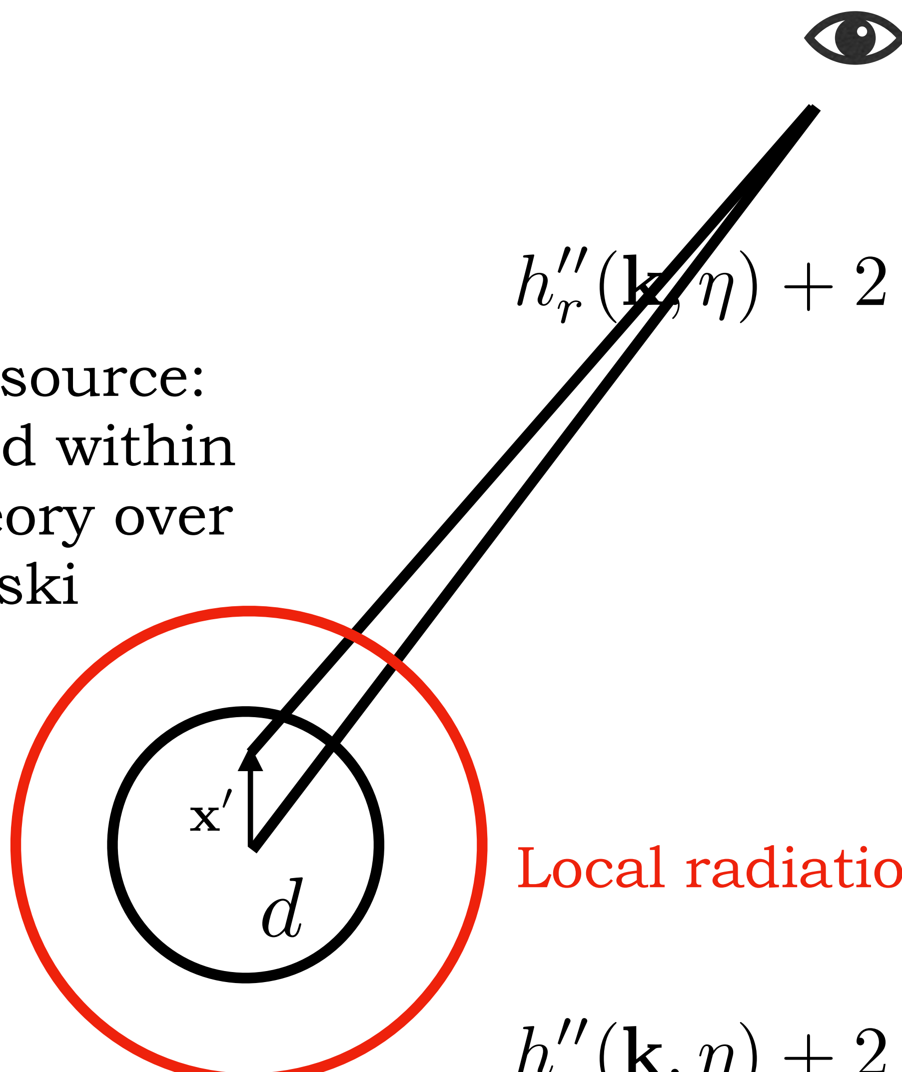


Inspiral of compact binaries at cosmological distance

To reach the observer:
free propagation in
FLRW space-time

$$h_r''(\mathbf{k},\eta) + 2\,\mathcal{H}\,h_r'(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = \cancel{16\pi G\,a^2\,\Pi_r(\mathbf{k},\eta)}$$

Close to the source:
we have solved within
linearised theory over
Minkowski



$$h_r''(\mathbf{k},\eta) + 2\,\cancel{\mathcal{H}}\,h_r'(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = 16\pi G\,\cancel{a}^2\,\Pi_r(\mathbf{k},\eta)$$

$$h_+(t,\theta,\varphi) = \frac{4}{r}(G\,M_c)^{5/3}[\pi f(t_{\rm ret})]^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\Phi(t_{\rm ret}))$$

$$h_\times(t,\theta,\varphi) = \frac{4}{r}(G\,M_c)^{5/3}[\pi f(t_{\rm ret})]^{2/3}\,\cos\theta\,\sin(2\Phi(t_{\rm ret}))$$

Inspiral of compact binaries at cosmological distance

$$h_+(t_S,\theta,\varphi) = \frac{4}{a(t_O)r}(G\,M_c)^{5/3}[\pi f(t_S^{\rm ret})]^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\Phi(t_S^{\rm ret}))$$

$$h_\times(t_S,\theta,\varphi) = \underbrace{\frac{4}{a(t_O)r}}(G\,M_c)^{5/3}[\pi f(t_S^{\rm ret})]^{2/3}\,\cos\theta\,\sin(2\Phi(t_S^{\rm ret}))$$

Propagation effect at
the observer

“Gravitational Waves”, M. Maggiore, Oxford University Press 2008

Inspiral of compact binaries at cosmological distance

$$h_+(t_S,\theta,\varphi) = \frac{4}{a(t_O)r}(G\,M_c)^{5/3}[\pi f(t_S^{\rm ret})]^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\Phi(t_S^{\rm ret}))$$

$$h_\times(t_S,\theta,\varphi) = \frac{4}{a(t_O)r}(G\,M_c)^{5/3}[\pi f(t_S^{\rm ret})]^{2/3}\,\cos\theta\,\sin(2\Phi(t_S^{\rm ret}))$$

Still measured by the source clock,
we want it in the observer's clock

$$f_S = f_O(1+z)$$

$$\Phi_S = \Phi_O$$

The phase is constant along null geodesics

$$\Phi_S(t_S) = 2\pi\int_{t_{c,S}}^{t_S} dt'_S f_S(t'_S) = 2\pi\int_{t_{c,O}}^{t_O} dt'_O f_O(t'_O) = \Phi_O(t_O)$$

$$dt_O = (1+z)dt_S$$

"Gravitational Waves", M. Maggiore, Oxford University Press 2008

Inspiral of compact binaries at cosmological distance

$$h_+(t_O,\theta,\varphi) = \frac{4}{a(t_O)r}(G\,M_c)^{5/3}[\pi f(t_O^{\rm ret})(1+z)]^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\Phi(t_O^{\rm ret}))$$

$$h_\times(t_O,\theta,\varphi) = \frac{4}{a(t_O)r}(G\,M_c)^{5/3}[\pi f(t_O^{\rm ret})(1+z)]^{2/3}\cos\theta\,\sin(2\Phi(t_O^{\rm ret}))$$

$$\frac{4}{d_L(z)}(G\,\mathcal{M}_c)^{5/3}[\pi f_O]^{2/3}$$

$$\mathcal{M}_c = (1+z)M_c$$

Redshifted chirp mass

Inspiral of compact binaries at cosmological distance

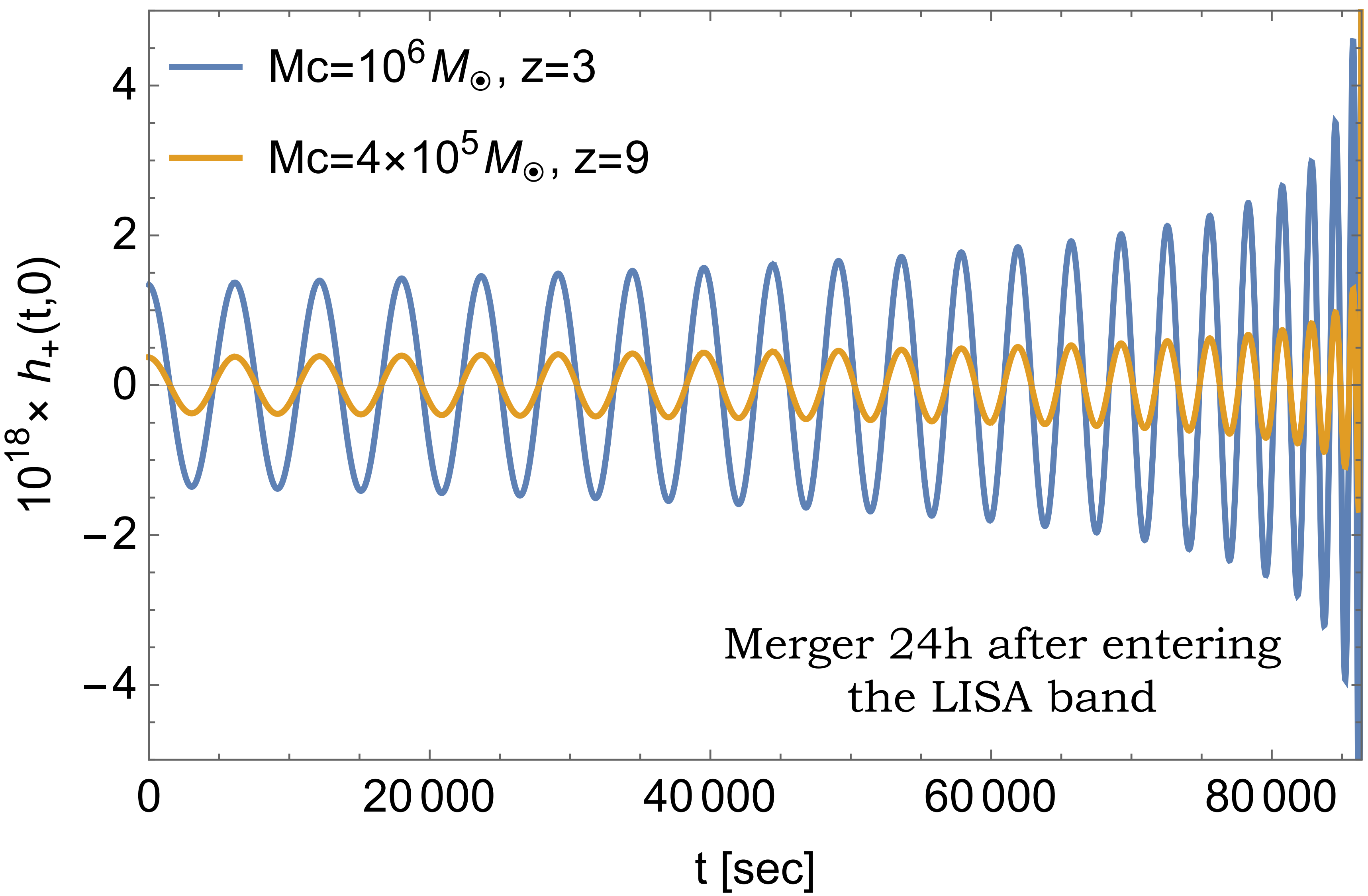
$$h_+(\tau,\theta,\varphi) = \frac{4}{d_L(z)}(G\,\mathcal{M}_c)^{5/3}[\pi f(\tau)]^{2/3}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\Phi(\tau))$$

$$h_\times(\tau,\theta,\varphi) = \frac{4}{d_L(z)}(G\,\mathcal{M}_c)^{5/3}[\pi f(\tau)]^{2/3}\cos\theta\,\sin(2\Phi(\tau))$$

time to coalescence at the observer

The signal does depend on redshift, but there is a degeneracy among the redshift and the true chirp mass

$$\mathcal{M}_c = (1+z)M_c$$



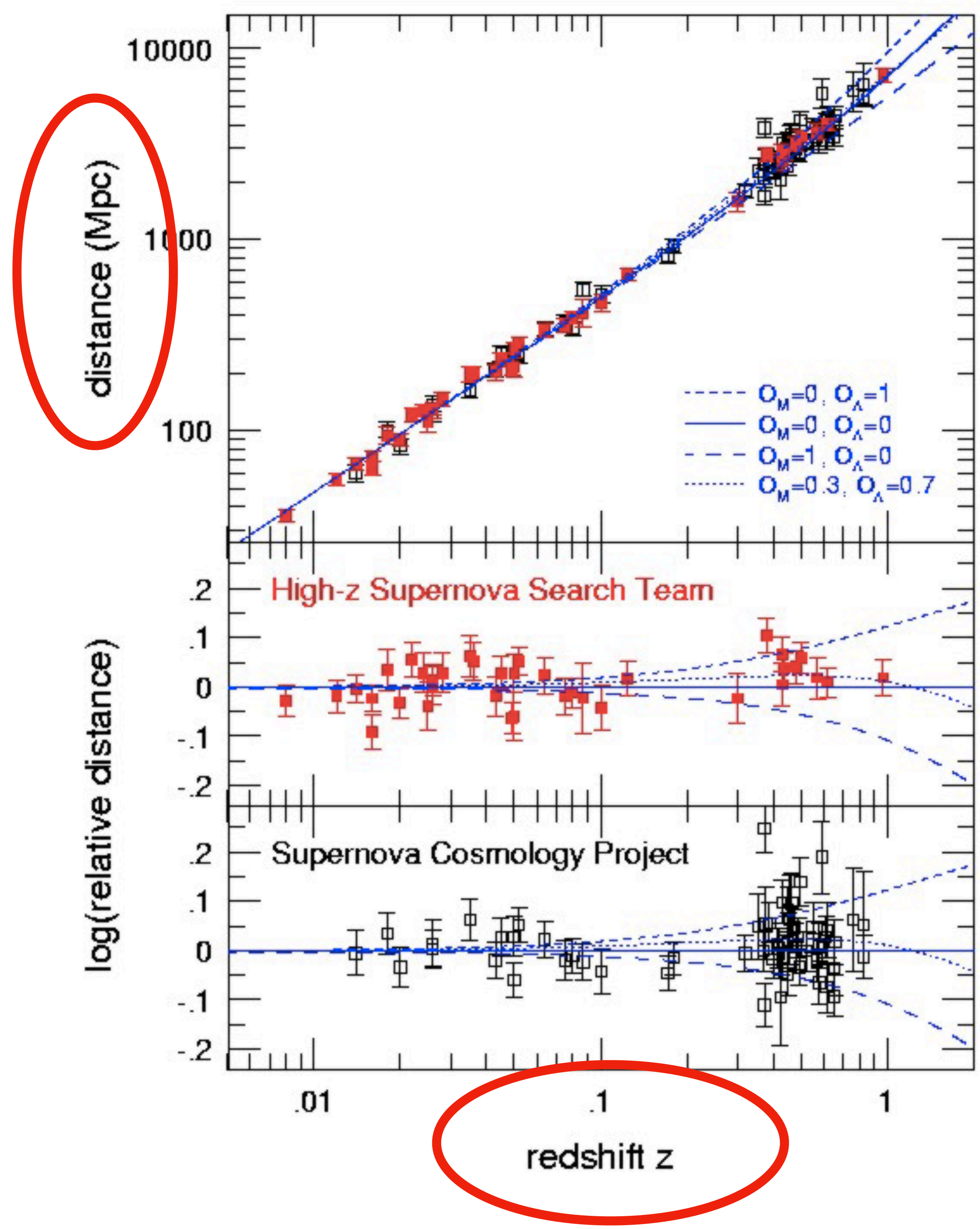
Measurement of $d_L(z)$: standard candles

Probe the luminosity distance-redshift relation to infer the universe content and its expansion

Redshift measured directly from the optical emission
EASY!

$$d_L(z) = \sqrt{\frac{L}{4\pi\mathcal{F}}}$$

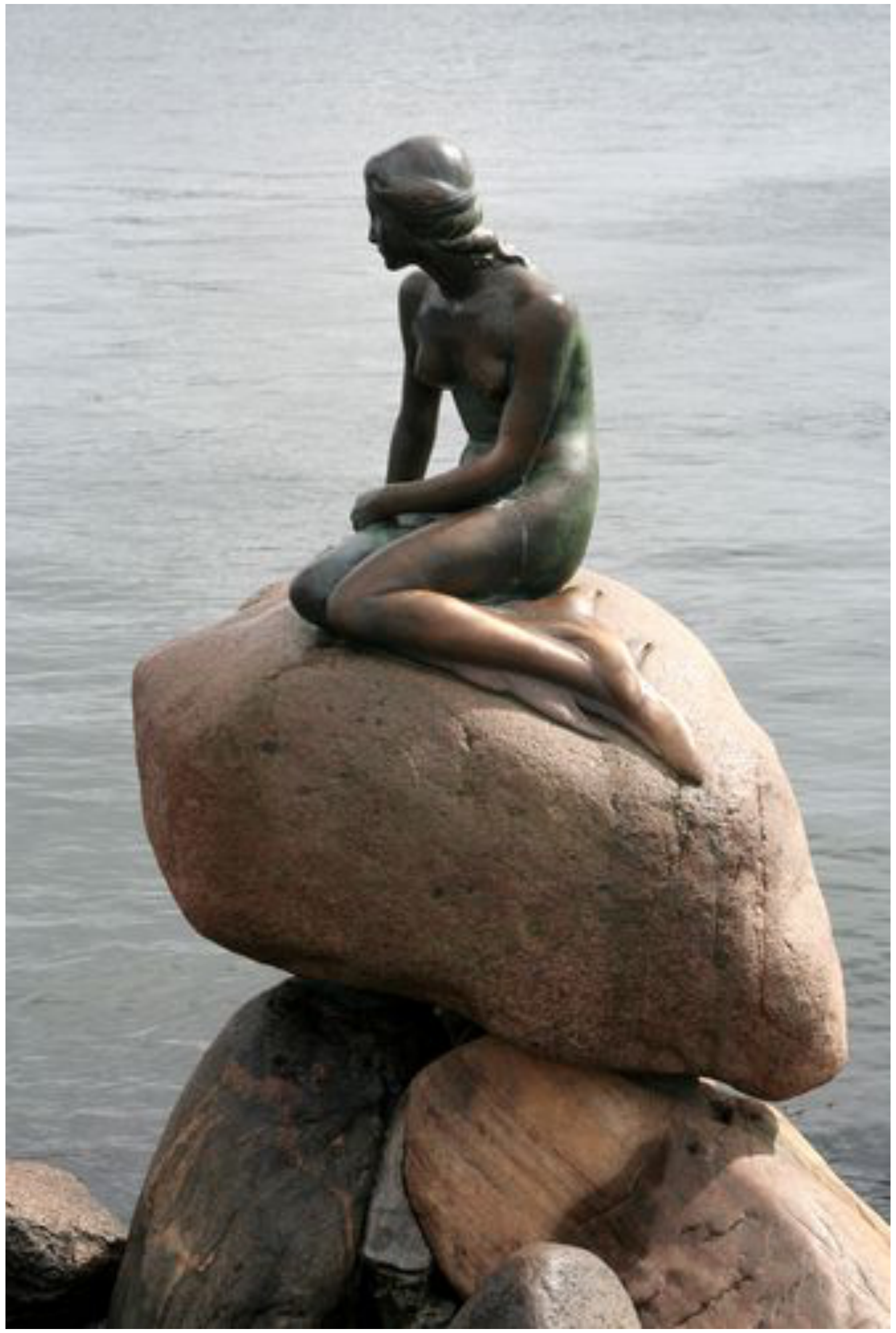
- Flux measured directly
- Intrinsic luminosity known from calibration
- Measurement of the luminosity distance is NOT SO EASY



Measurement of $d_L(z)$: standard sirens

GW emission by compact binaries can also be used to test the expansion of the universe

- Measurement of the luminosity distance: no calibration needed, EASY AND DIRECT
- Measurement of the redshift: IMPOSSIBLE!



B. Schutz, Nature 323, 310 (1986)

SGWB from a population of inspiralling binaries

In the PTA part of yesterday's course we have stated

$$h_c(f) = A\left(\frac{f}{f_{\mathrm{ref}}}\right)^{-\alpha} \text{with} \quad \alpha = \frac{2}{3}$$

Where does this slope come from?

SGWB from a population of inspiralling binaries

In the PTA part of yesterday's course we have stated

$$h_c(f) = A\left(\frac{f}{f_{\rm ref}}\right)^{-\alpha} \quad \text{with} \quad \alpha = \frac{2}{3}$$

In terms of the power spectrum of the GW energy density

$$\Omega_{\rm GW}(f) = \frac{2\pi^2}{3H_0^2} f^2 h_c^2(f) = \Omega_{\rm GW}(f_{\rm ref})\left(\frac{f}{f_{\rm ref}}\right)^{2/3}$$

$$\frac{\rho_{\rm GW}^{\rm (tot)}}{\rho_c} = \int_0^\infty \frac{{\rm d}f}{f}\,\Omega_{\rm GW}(f) = \int {\rm d}\xi \int {\rm d}V_c \int {\rm d}\tau_c\, \frac{{\rm d}^3 N(z,\tau_c,\xi,\theta)}{{\rm d}\xi {\rm d}V_c {\rm d}\tau_c}\,\frac{\rho_{\rm GW}^{\rm (event)}}{\rho_c}$$

- ${\rm d}\xi$: Parameters of the binary signal (essentially chirp mass)
- ${\rm d}V_c$: Coming volume
- ${\rm d}\tau_c$: Time to coalescence
- $\frac{{\rm d}^3 N}{{\rm d}\xi {\rm d}V_c {\rm d}\tau_c}$: Number density of GW sources (given within an astrophysical model for the binary population)
- $\frac{\rho_{\rm GW}^{\rm (event)}}{\rho_c}$: GW energy emitted by a single event

At the source

$$\frac{\rho_{\rm GW}^{\rm (event)}}{\rho_c} = \frac{1}{16\pi G\rho_c}\,\frac{\langle \dot h_+^2 + \dot h_\times^2\rangle}{(1+z)^4}$$

SGWB from a population of inspiralling binaries

$$\dot{h}_+(t_S) = \frac{4\pi^{2/3}}{a(t_S)r}(G\,M_c)^{5/3}\left(\frac{1+\cos^2\theta}{2}\right)\underbrace{\frac{\mathrm{d}[f^{2/3}(t_S)\cos(2\Phi(t_S))]}{\mathrm{d}t_S}}$$

In the limit of circular orbit with slowly varying radius

$$\dot{f}_S \ll f_S^2$$

$$\simeq -f^{2/3}(t_S)2\,\dot{\Phi}(t_S)\sin(2\Phi(t_S))$$

$$\dot{\Phi}(t_S) \to \pi f_S$$

$$\langle \dot{h}_+^2(t_S)\rangle = \frac{32}{a_S^2 r^2}(\pi G\,M_c)^{10/3}\left(\frac{1+\cos^2\theta}{2}\right) f_S^{10/3}$$

$$\frac{\rho_{\rm GW}^{(\rm tot)}}{\rho_c} = \int \mathrm{d}\xi \int \mathrm{d}\tau_c \int \mathrm{d}z \frac{d_M^2}{H(z)} \frac{\mathrm{d}^3 N(z,\tau_c,\xi,\theta)}{\mathrm{d}\xi \mathrm{d}V_c \mathrm{d}\tau_c} \frac{1}{16\pi G\rho_c(1+z)^4}$$

$$\frac{32}{a_S^2 r^2}(\pi G M_c f_S)^{10/3}\int \mathrm{d}\Omega\left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right]$$

$$d_M = a_0\, r$$

Extra factor $(1+z)^2$

$$\mathrm{d}V_c = \frac{d_M^2}{H(z)}\mathrm{d}\Omega\,\mathrm{d}z$$

$$= 16\pi/5$$

SGWB from a population of inspiralling binaries

Express the integral over time to coalescence in terms of frequency and change to frequency at the observer

$$\frac{\mathrm{d}f_S}{\mathrm{d}\tau_c} = \frac{96\pi^{8/3}}{5}(G\,M_c)^{5/3} f_S^{11/3}$$

$$f_S = f(1+z)$$

$$\frac{\rho_{\mathrm{GW}}^{(\mathrm{tot})}}{\rho_c} = \frac{\pi^{2/3}}{3\,G\rho_c}\int\frac{\mathrm{d}f}{f}\,f^{2/3}\int\mathrm{d}\xi\int\frac{\mathrm{d}z}{H(z)(1+z)^{4/3}}(GM_c)^{10/3}\frac{\mathrm{d}^3N(z,\tau_c,\xi,\theta)}{\mathrm{d}\xi\mathrm{d}V_c\mathrm{d}\tau_c}$$

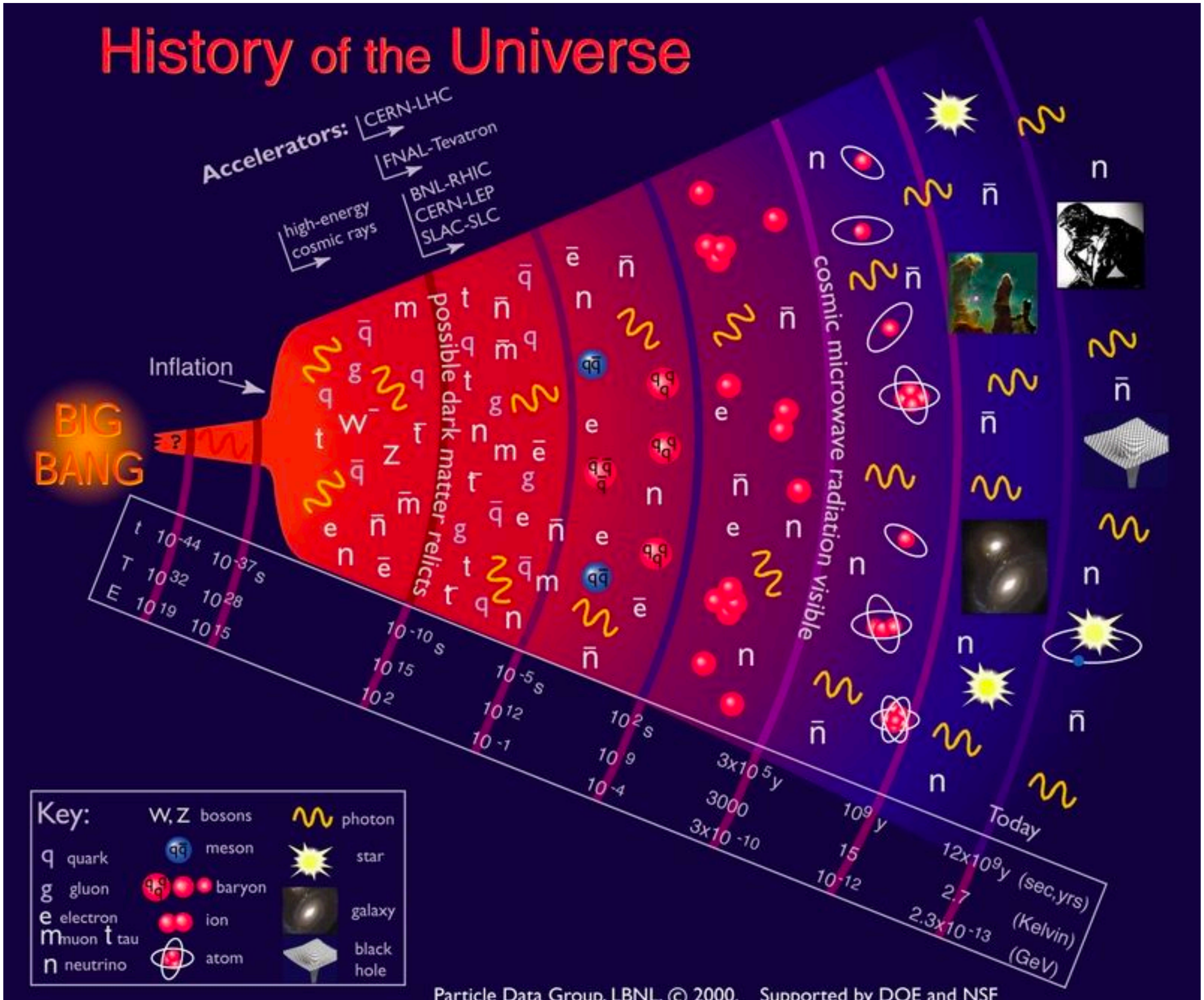
$$\frac{\rho_{\mathrm{GW}}^{(\mathrm{tot})}}{\rho_c} = \int_0^\infty\frac{\mathrm{d}f}{f}\,\Omega_{\mathrm{GW}}(f)$$

$$\Omega_{\mathrm{GW}}(f) = \Omega_{\mathrm{GW}}(f_{\mathrm{ref}})\left(\frac{f}{f_{\mathrm{ref}}}\right)^{2/3}$$

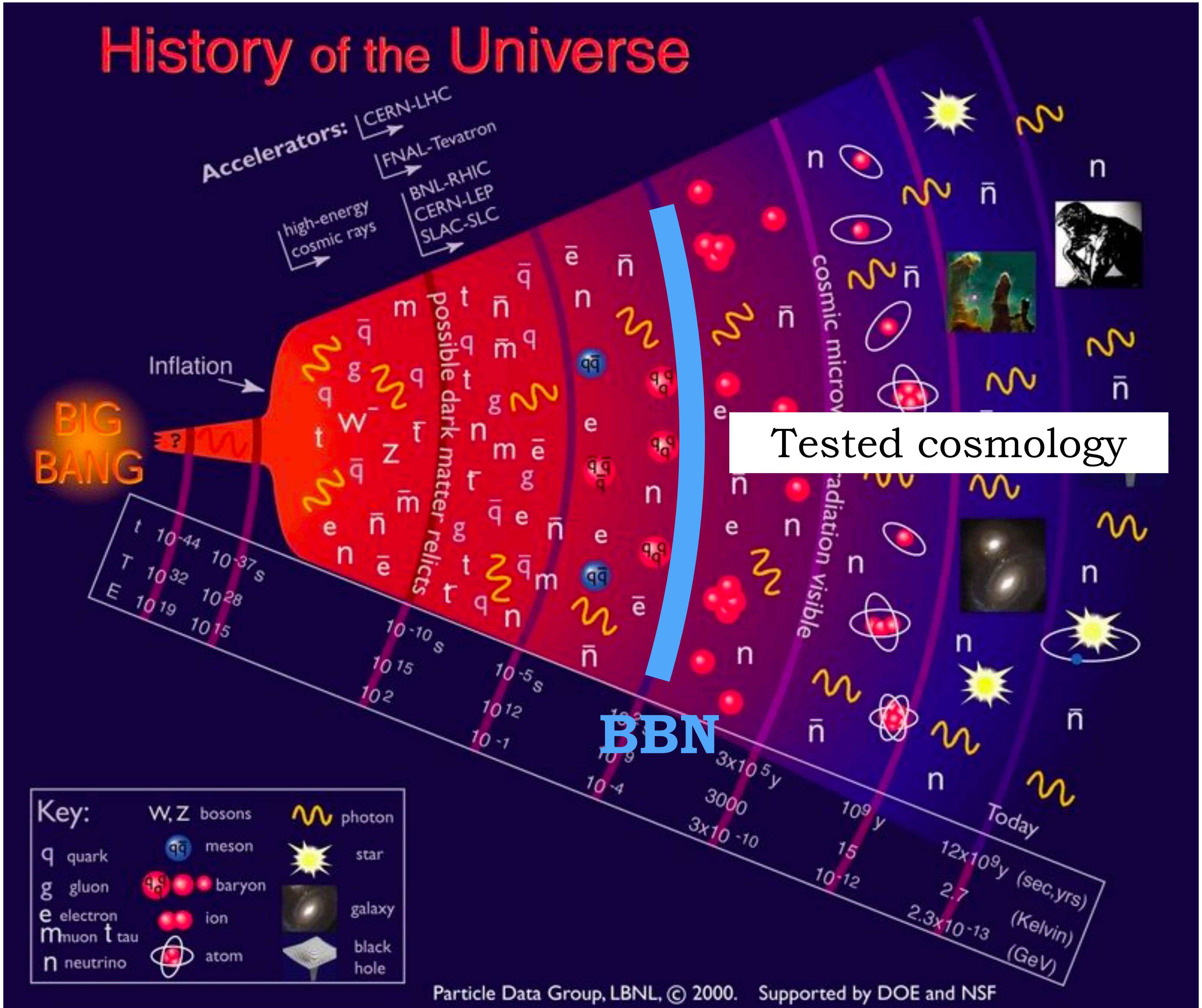
↓

SGWB amplitude determined by the population characteristics and the cosmology

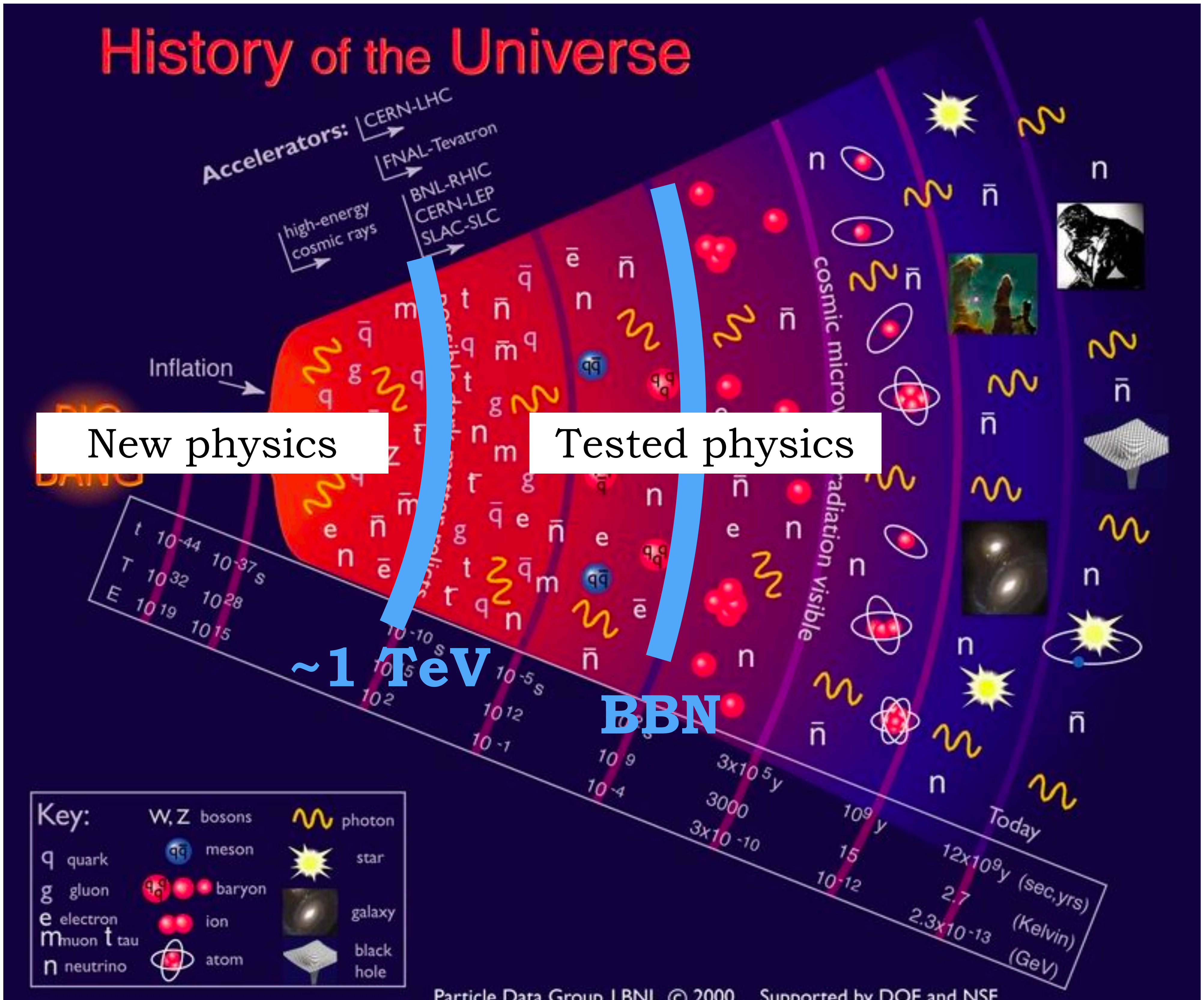
Examples of SGWB sources in the early universe



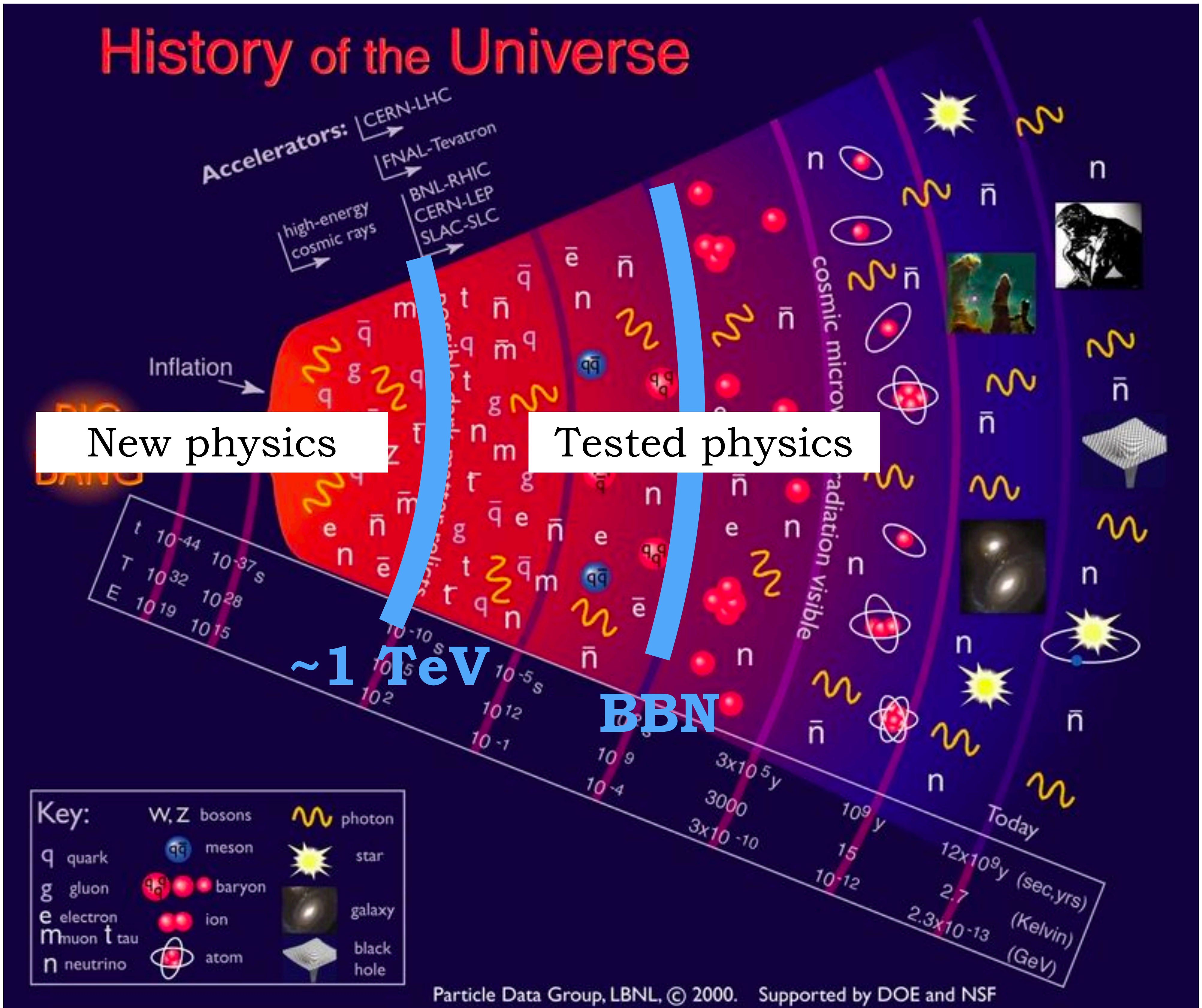
GWs can bring direct information from very early stages of the universe evolution, to which we have no direct access through em radiation —> amazing discovery potential



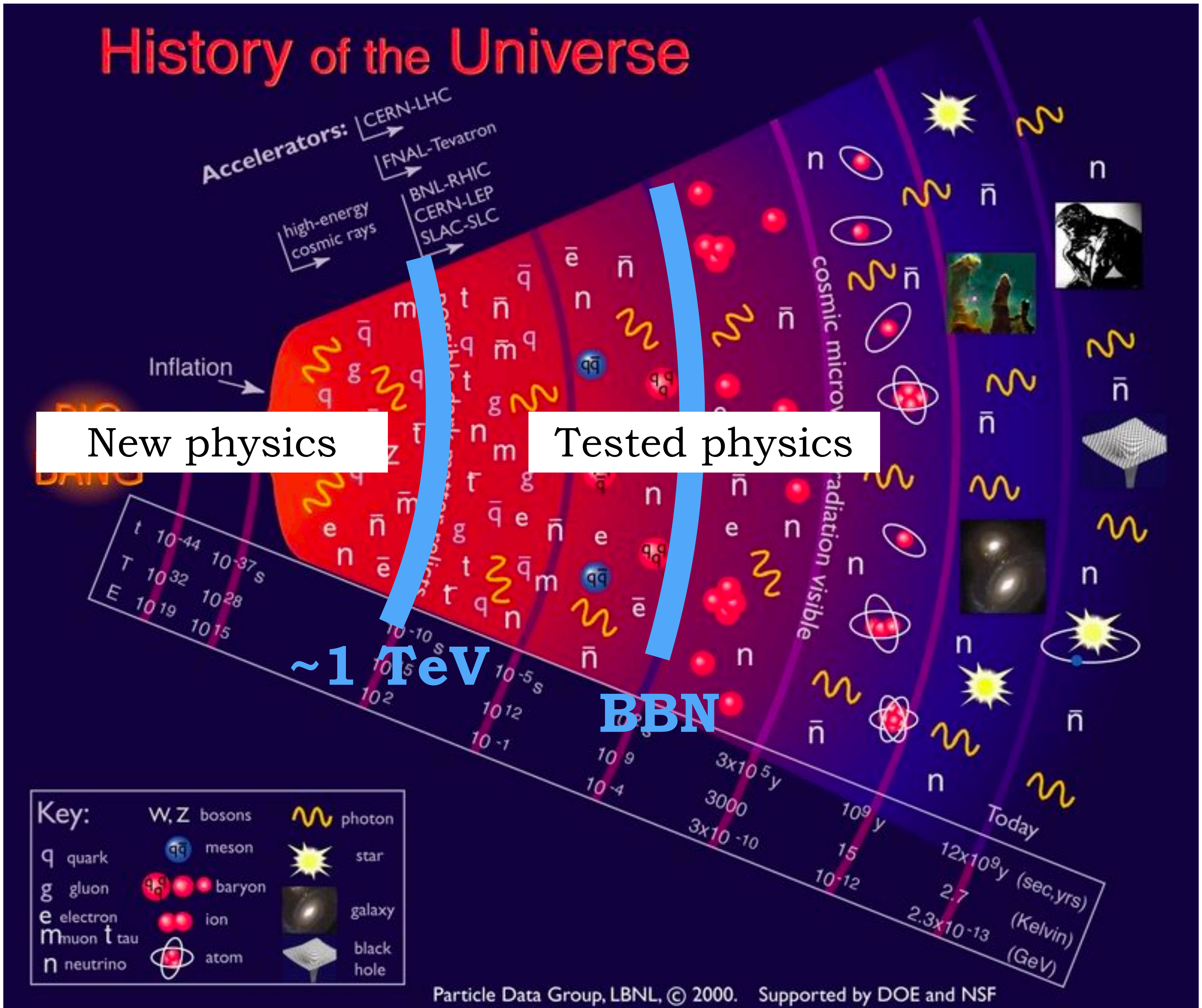
No guaranteed GW signal: predictions rely on untested phenomena, and are often difficult to estimate (non-linear dynamics, strongly coupled theories…)



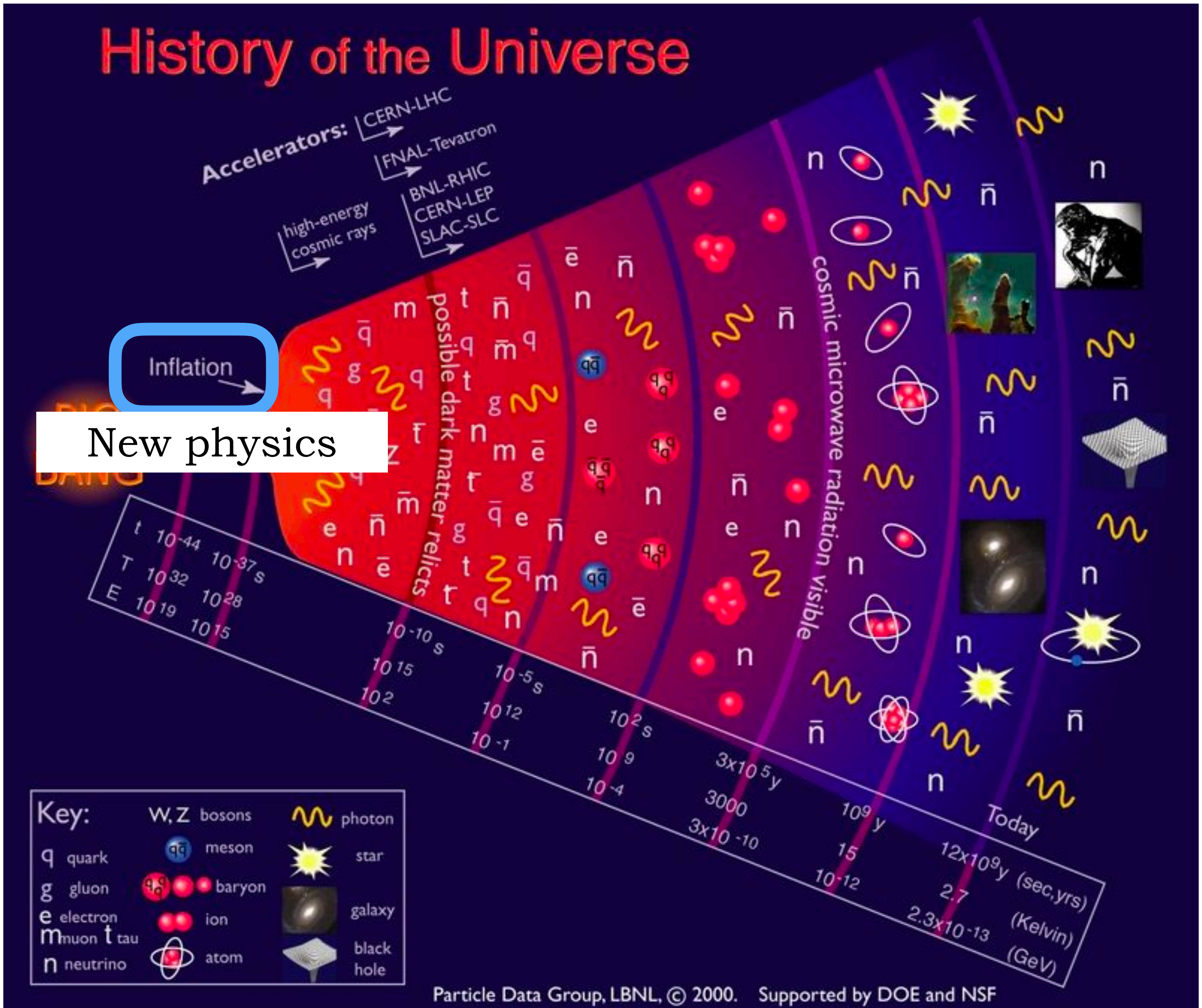
Many GW generation processes are related to PHASE TRANSITIONS



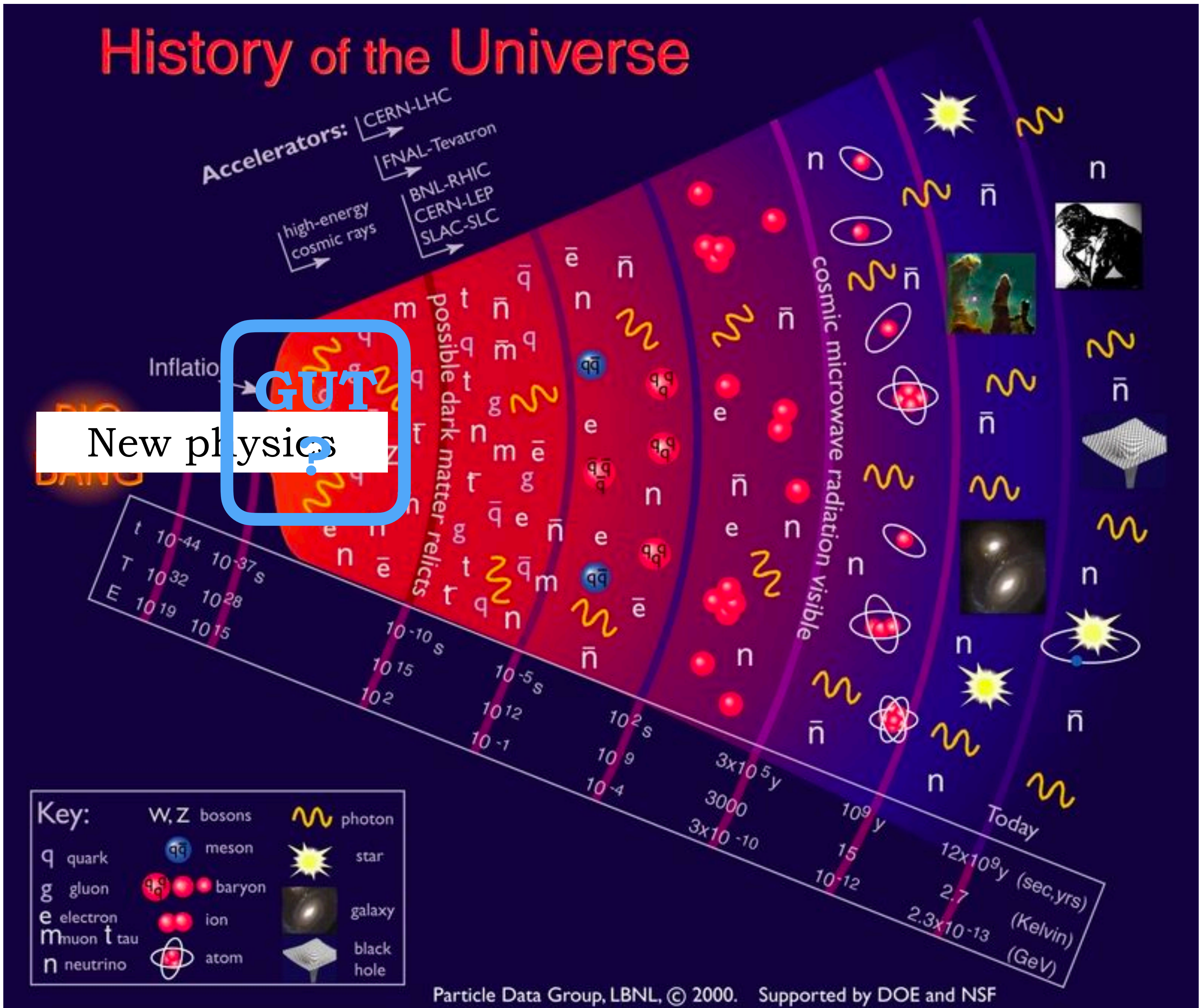
Phase transition: some field in the universe changes from one state to another, which has become more energetically favourable due to a change in external conditions (e.g. a change in temperature)



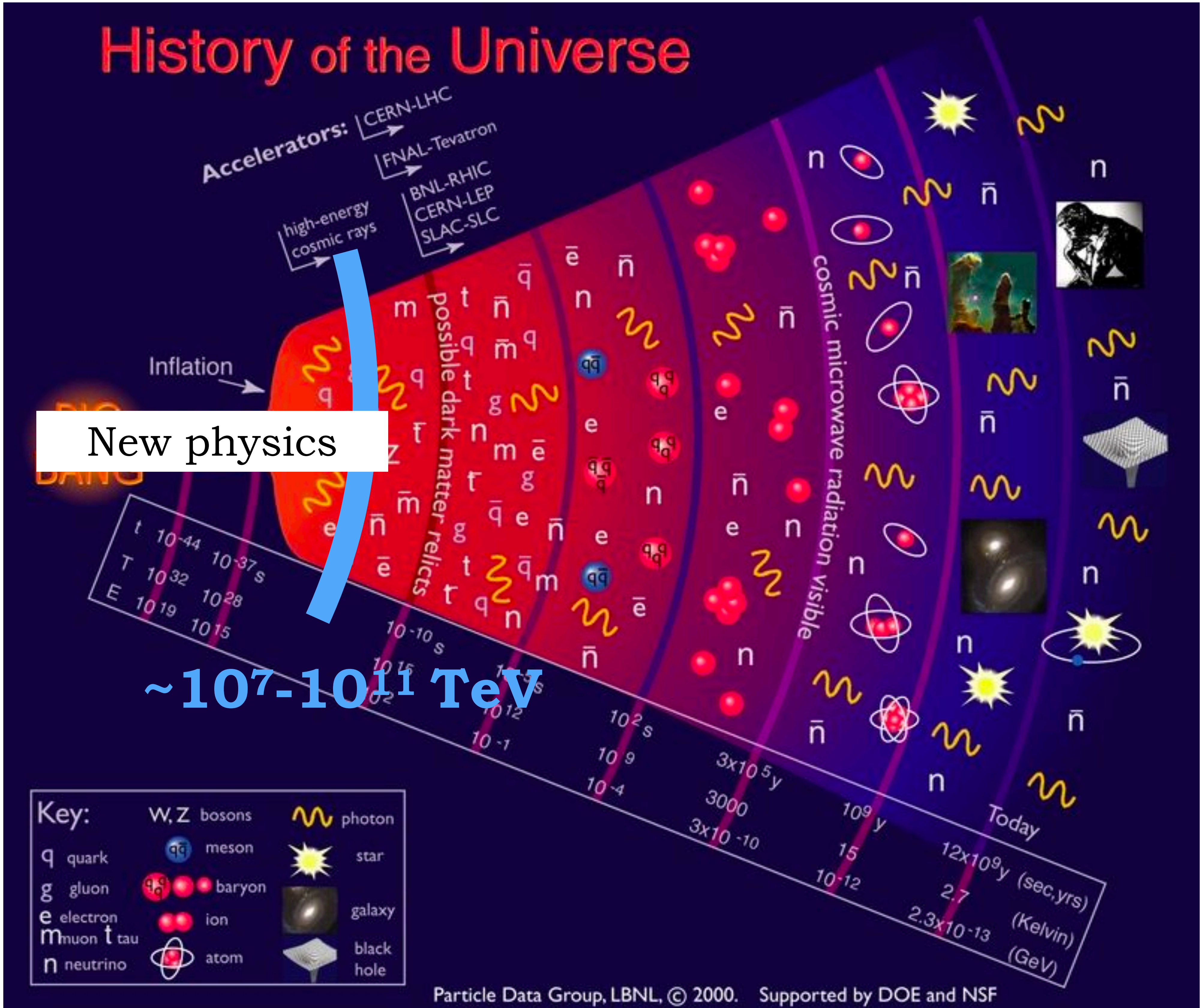
Inflation: phase transition of the Inflaton field



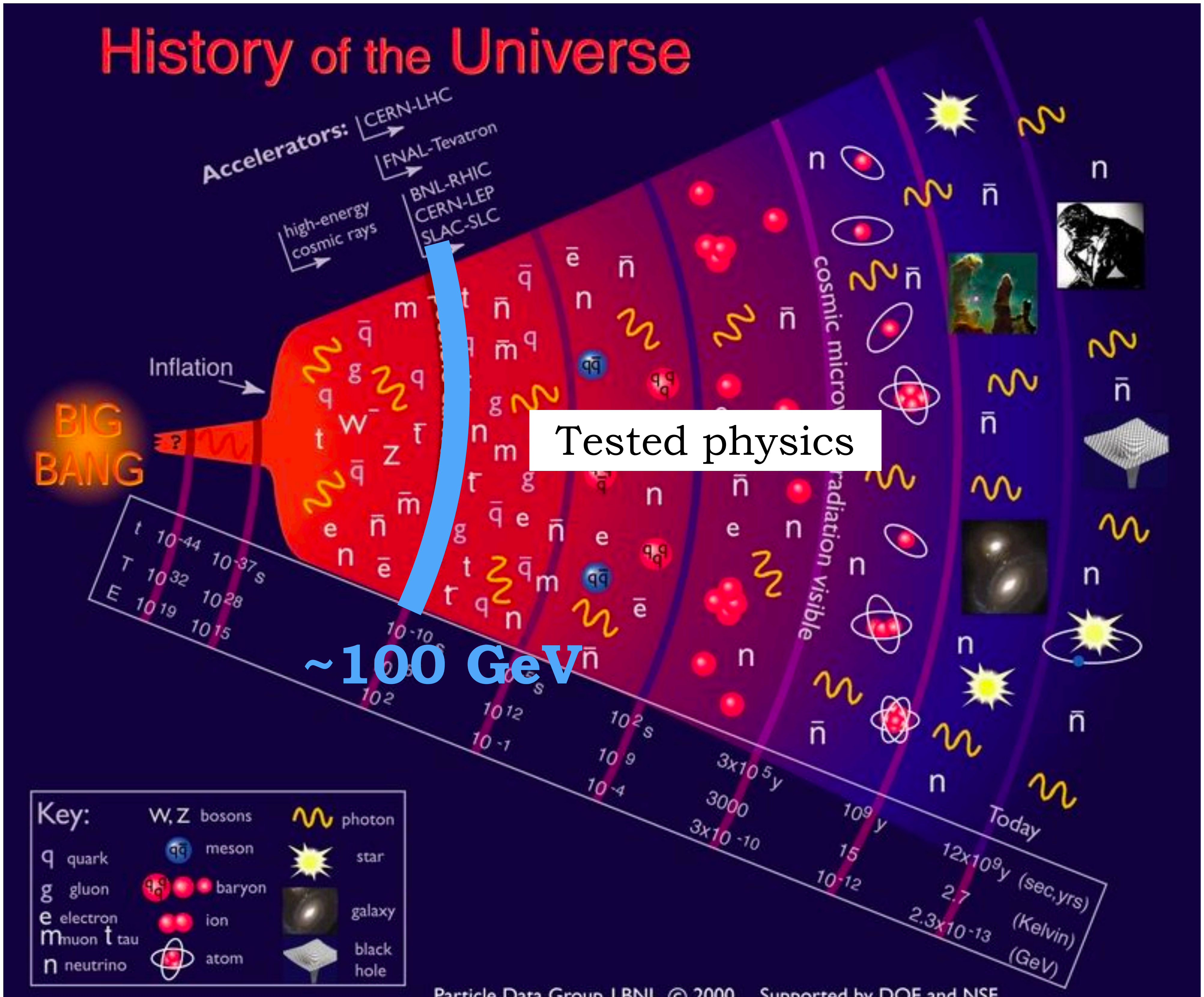
GUT phase transition or similar: related to the breaking of the symmetries of the high-energy theory describing the universe



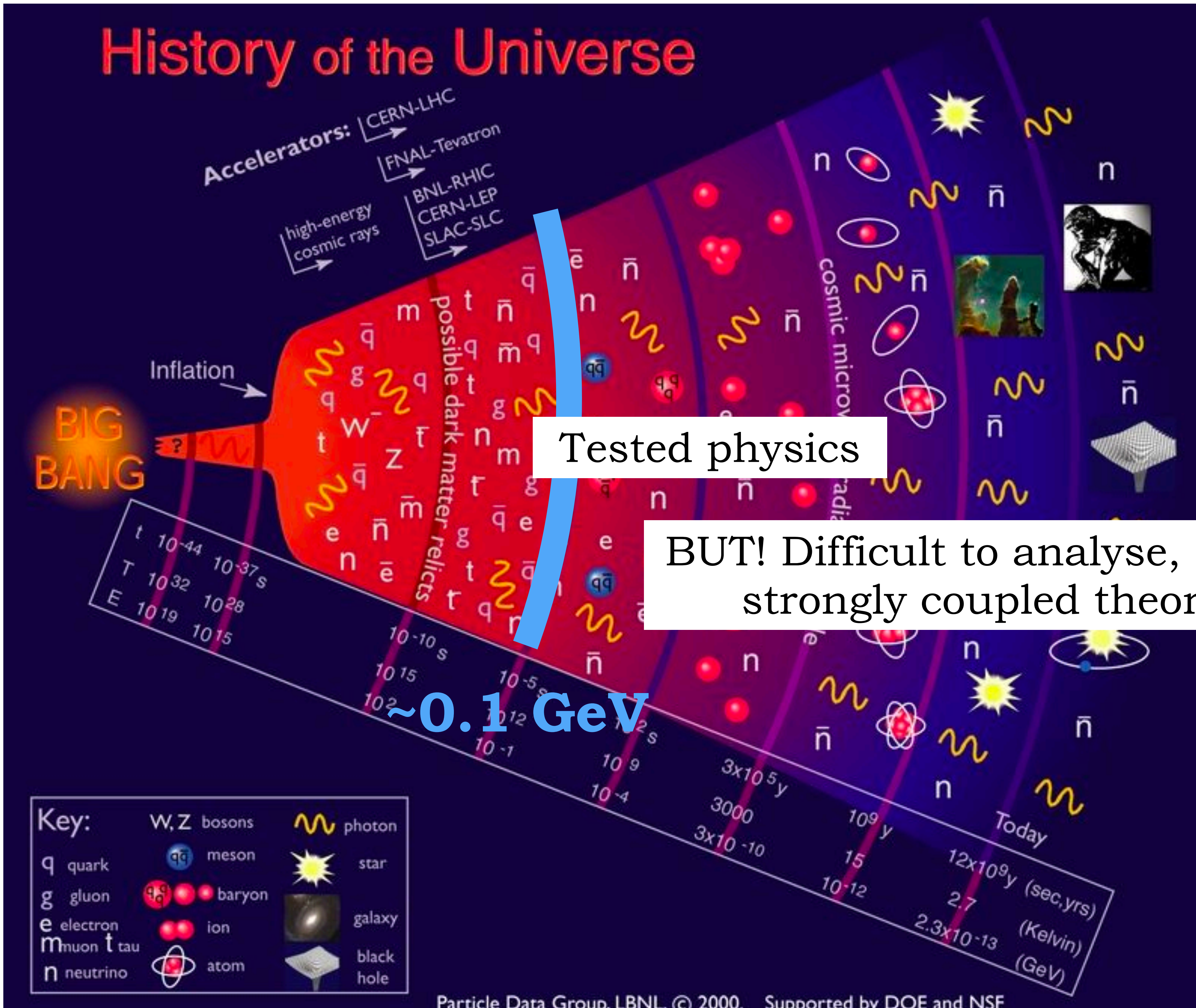
Peccei-Quinn phase transition: invoked to solve the strong CP problem



Electroweak phase transition: phase transition of the Higgs field, driven by the temperature decrease as the universe expands



QCD phase transition: phase transition related to the strong interaction, confinement of quarks into hadrons



Tested physics

BUT! Difficult to analyse, since strongly coupled theory

~0.1 GeV

Examples of GW sources in the early universe :

- irreducible SGWB from inflation
    - also sourced by second order scalar perturbations
- beyond the irreducible SGWB from inflation
    - particle production during inflation (scalar, gauge fields… coupled to the inflaton)
    - spectator fields
    - breaking symmetries (space-dependent inflaton, massive graviton)
    - modified gravity during inflation (massive GWs with $c \neq 1$)
    - primordial black holes
    - …
- preheating and non-perturbative phenomena
    - parametric amplification of bosons/fermions
    - symmetry breaking in hybrid inflation
    - decay of flat directions
    - oscillons
    - …
- first order phase transition
    - true vacuum bubble collision
    - sound waves
    - (M)HD turbulence
    - …
- cosmic topological defects
    - irreducible SGWB from topological defect networks
    - decay of cosmic string loops
    - …

CC and Figueroa arXiv:1801.04268

SGWB from a stochastic source in the radiation era

$$h_r''(\mathbf{k},\eta) + 2\,\mathcal{H}\,h_r'(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = 16\pi G\,a^2\,\Pi_r(\mathbf{k},\eta)$$

Possible sources of tensor anisotropic stress in the early universe:

- Scalar field gradients $\Pi_{ij} \sim [\partial_i\phi\partial_j\phi]^{TT}$
- Bulk fluid motion $\Pi_{ij} \sim [\gamma^2(\rho+p)v_i v_j]^{TT}$
- Gauge fields $\Pi_{ij} \sim [-E_iE_j - B_iB_j]^{TT}$
- Second order scalar perturbations, Π_{ij} from a combination of $\partial_i\Psi, \partial_i\Phi$
- ...

The components of the anisotropic stress must be treated as
random variables
because we cannot access the detailed properties of the generation processes at the moment they operated

SGWB from a stochastic source in the radiation era

unequal time correlator of the anisotropic stress

Anisotropic stress power spectral density at unequal time

$$\langle \Pi_r(\mathbf{k},\tau)\,\Pi_p^*(\mathbf{q},\zeta)\rangle = \frac{(2\pi)^3}{4}\,\delta^{(3)}(\mathbf{k}-\mathbf{q})\,\delta_{rp}\,\Pi(k,\tau,\zeta)$$

We now proceed with two approximate analytical solutions of the GW propagation equation:

- Fast source operating for less than one Hubble time -> peaked SGWB power spectrum
- Continuous source operating for several Hubble times -> flat SGWB power spectrum

---

Fast source operating in a time interval η_{fin} - η_{in} in the radiation dominated era

Typical example: first order phase transition

$$H_r^{\rm rad}(\mathbf{k},\eta>\eta_{\rm fin}) = A_r^{\rm rad}(\mathbf{k})\cos(k\eta) + B_r^{\rm rad}(\mathbf{k})\sin(k\eta)$$

$$A_r^{\rm rad}(\mathbf{k}) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\tau\, a(\tau)^3\, \sin(-k\tau)\,\Pi_r(\mathbf{k},\tau),$$

$$B_r^{\rm rad}(\mathbf{k}) = \frac{16\pi G}{k}\int_{x_{\rm in}}^{x_{\rm fin}} d\tau\, a(\tau)^3\, \cos(k\tau)\,\Pi_r(\mathbf{k},\tau)$$

SGWB from a **FAST** stochastic source in the radiation era

GW amplitude power spectrum today for modes $k\eta_0 \gg 1$

$$\langle h_r(\mathbf{k},\eta_0)\, h_p^*(\mathbf{q},\eta_0)\rangle = \frac{1}{a_0^2}[\langle A_r(\mathbf{k})\, A_p^*(\mathbf{q})\rangle + \langle B_r(\mathbf{k})\, B_p^*(\mathbf{q})\rangle]$$
$$= 8\pi^5\, \delta^{(3)}(\mathbf{k}-\mathbf{q})\, \delta_{rp}\, \frac{h_c^2(k,\eta_0)}{k^3}$$

GW energy density power spectrum today for modes $k\eta_0 \gg 1$

$$\frac{d\rho_{\rm GW}}{d\log k} = \frac{k^2\, h_c^2(k,\eta_0)}{16\pi G\, a_0^2}$$

(freely propagating sub-Hubble modes)

$$\frac{d\rho_{\rm GW}}{d\log k}(k,\eta_0) = \frac{4}{\pi}\,\frac{G}{a_0^4} k^3 \int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\tau\, a^3(\tau) \int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\zeta\, a^3(\zeta)\, \cos[k(\eta-\zeta)]\, \Pi(k,\tau,\zeta)$$

SGWB from a **FAST** stochastic source in the radiation era

GW amplitude power spectrum today for modes $k\eta_0 \gg 1$

$$\langle h_r(\mathbf{k},\eta_0)\, h_p^*(\mathbf{q},\eta_0)\rangle = \frac{1}{a_0^2}[\langle A_r(\mathbf{k})\, A_p^*(\mathbf{q})\rangle + \langle B_r(\mathbf{k})\, B_p^*(\mathbf{q})\rangle]$$
$$= 8\pi^5\, \delta^{(3)}(\mathbf{k}-\mathbf{q})\, \delta_{rp}\, \frac{h_c^2(k,\eta_0)}{k^3}$$

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(freely propagating sub-Hubble modes)

$$\frac{d\rho_{\rm GW}}{d\log k}(k,\eta_0) = \frac{4}{\pi}\,\frac{G}{a_0^4}k^3\int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\tau\, a^3(\tau)\int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\zeta\, a^3(\zeta)\, \cos[k(\eta-\zeta)]\,\Pi(k,\tau,\zeta)$$

$a^3(\tau) \to a_*^3$, $a^3(\zeta) \to a_*^3$, $\cos[k(\eta-\zeta)] \simeq 1$, $\Pi(k,\tau,\zeta) \to \Pi(k)$

SUPPOSE:

$\Delta\eta = \eta_{\rm fin} - \eta_{\rm in} \ll \mathcal{H}_*^{-1}$ $\qquad k\eta_{\rm in} \ll 1$ $\qquad \Pi(k,\tau,\eta)$ constant over $\Delta\eta$

SGWB from a **FAST** stochastic source in the radiation era

GW energy density parameter today for modes $1/\eta_0 \ll k \ll 1/\eta_{in}$

$$h^2\Omega_{\rm GW}(k,\eta_0) = \frac{3}{2\pi^2} h^2\Omega^0_{\rm rad} \left(\frac{g_0}{g_*}\right)^{\frac{1}{3}} (\Delta\eta\mathcal{H}_*)^2 \left(\frac{\rho_\Pi}{\rho_{\rm rad}}\right)^2 (k\ell_*)^3 \tilde{P}_{\rm GW}(k)$$

$$\Pi(k) = \rho_\Pi^2 \tilde{P}_{\rm GW}(k)$$

From the time integrals

SGWB from a **FAST** stochastic source in the radiation era

GW energy density parameter today for modes $1/\eta_0 \ll k \ll 1/\eta_{in}$

$$h^2\Omega_{GW}(k,\eta_0) = \frac{3}{2\pi^2} h^2\Omega^0_{rad} \left(\frac{g_0}{g_*}\right)^{\frac{1}{3}} (\Delta\eta\mathcal{H}_*)^2 \left(\frac{\rho_\Pi}{\rho_{rad}}\right)^2 (k\ell_*)^3 \tilde{P}_{GW}(k)$$

$\mathcal{O}(10^{-9})$ $\mathcal{O}(10^{-6})$ $\mathcal{O}(10^{-3})$

Value detected at PTA

Factor depending slightly on the generation epoch through the number of relativistic d.o.f.

Only slow, very anisotropic processes have the chance to generate detectable SGWB signals for sub-Hubble sources

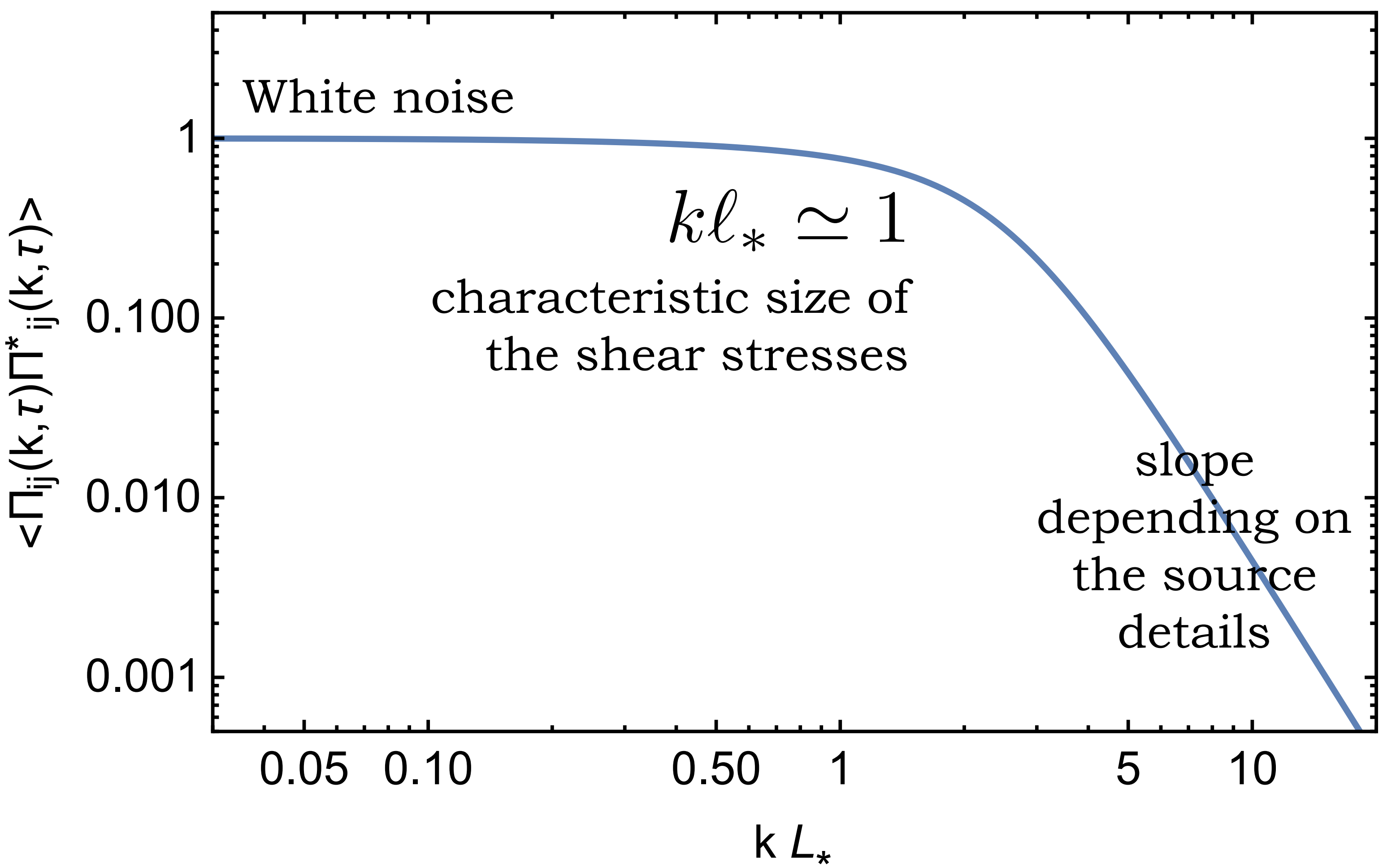
Value for detection at LISA

$\mathcal{O}(10^{-11})$ $\mathcal{O}(10^{-6})$ $\mathcal{O}(10^{-5})$

SGWB from a **FAST** stochastic source in the radiation era

GW energy density parameter today for modes $1/\eta_0 \ll k \ll 1/\eta_{in}$

$$h^2\Omega_{\rm GW}(k,\eta_0) = \frac{3}{2\pi^2} h^2 \Omega^0_{\rm rad} \left(\frac{g_0}{g_*}\right)^{\frac{1}{3}} (\Delta\eta\mathcal{H}_*)^2 \left(\frac{\rho_\Pi}{\rho_{\rm rad}}\right)^2 (k\ell_*)^3 \tilde{P}_{\rm GW}(k)$$



Fast source: independent on k for large enough scales (uncorrelated)

$$\ell_* \leq H_*^{-1}$$

SGWB from a **FAST** stochastic source in the radiation era

GW energy density parameter today for modes $1/\eta_0 \ll k \ll 1/\eta_{in}$

$$h^2\Omega_{\rm GW}(k,\eta_0) = \frac{3}{2\pi^2} h^2\Omega^0_{\rm rad}\left(\frac{g_0}{g_*}\right)^{\frac{1}{3}} (\Delta\eta\mathcal{H}_*)^2 \left(\frac{\rho_\Pi}{\rho_{\rm rad}}\right)^2 (k\ell_*)^3 \tilde{P}_{\rm GW}(k)$$

$$1/\eta_0 \ll \underbrace{k \ll \mathcal{H}_*}_{\text{Range of validity of the solution}} \ll \underbrace{1/(a_*\ell_*)}_{\text{Causality of the sourcing process}}$$

Range of validity of the solution

Causality of the sourcing process

$$\Omega_{\rm GW}(k) \propto (k\ell_*)^3$$

SGWB from a **FAST** stochastic source in the radiation era

- Characteristic time of the source evolution $$\delta t_c = \frac{\ell_*}{v_{\rm rms}}$$
- Characteristic time of the GW production from the Green's function: $$\delta t_{\rm gw} \sim \frac{1}{k}$$
- GW production goes faster than source evolution for all relevant wave-numbers including the spectrum peak $$k > \frac{v_{\rm rms}}{\ell_*}$$
- One assumes that the source is constant in time for a finite time interval (which can be larger than the Hubble time) $$\delta t_{\rm fin} \sim \mathcal{N} \delta t_c$$
- One can then easily integrate to find the GW spectrum

$$h^2 \Omega_{\rm GW}(k, \eta_0) \propto h^2 \Omega^0_{\rm rad} \left(\frac{g_0}{g_*}\right)^{\frac{1}{3}} \left(\frac{\rho_\Pi}{\rho_{\rm rad}}\right)^2 (k\ell_*)^3 \tilde{P}_{\rm GW}(k) \begin{cases} \ln^2[1 + \mathcal{H}_* \delta t_{\rm fin}] & \text{if } k\,\delta t_{\rm fin} < 1 \\ \ln^2[1 + (k/\mathcal{H}_*)^{-1}] & \text{if } k\,\delta t_{\rm fin} \geq 1 \end{cases}$$

A. Roper Pol et al, arXiv:2201.05630

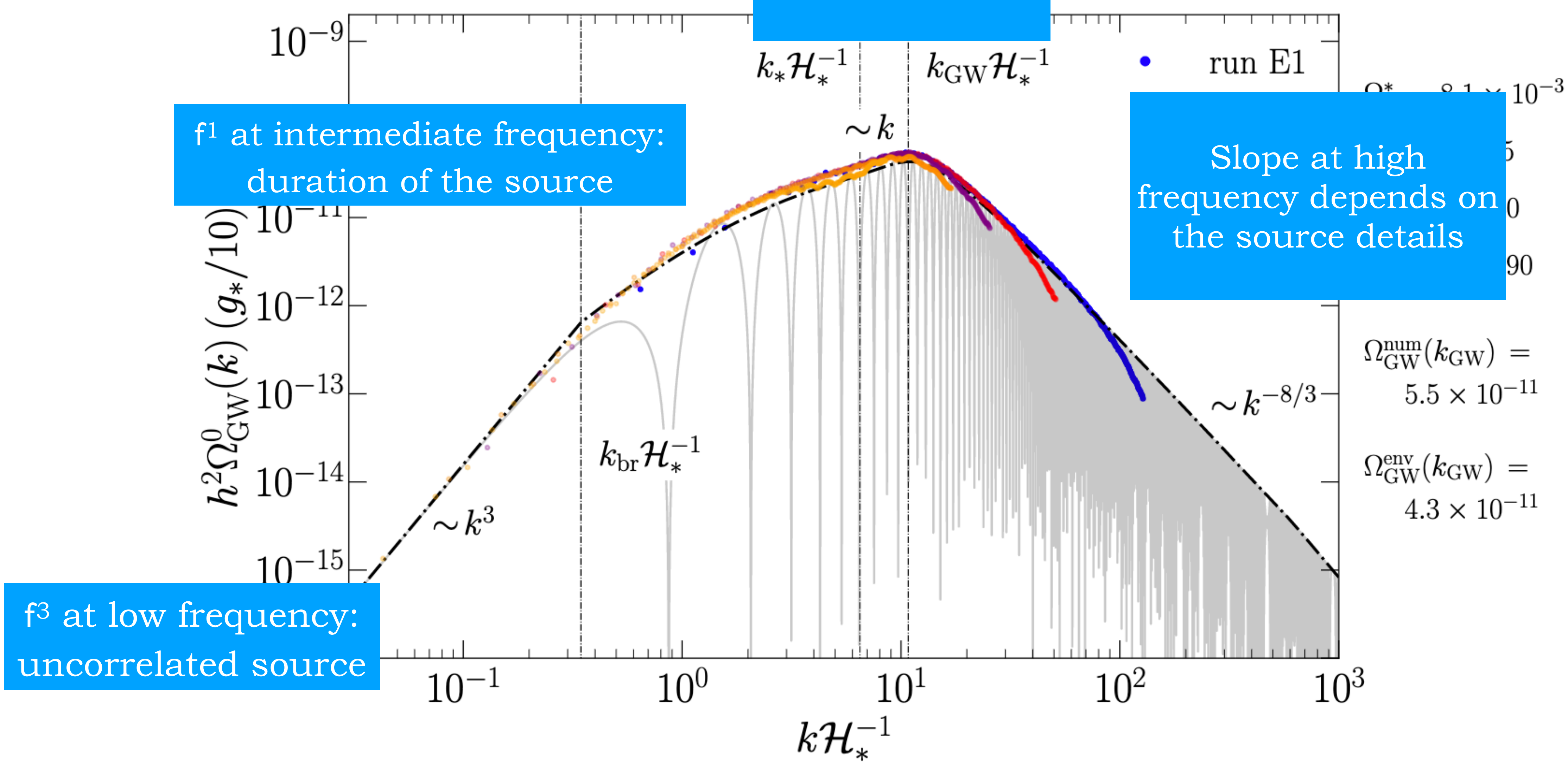
SGWB from a **FAST** stochastic source in the radiation era

$$k_{\rm peak} \simeq 4\pi/\ell_*$$

$$\Omega_{\rm gw,\,peak} \propto \left(\frac{\rho_\Pi}{\rho_{\rm rad}}\right)^2 (\mathcal{H}_*\ell_*)^2$$

Transition from k^3 to k^1 at $k \simeq 1/\delta t_{\rm fin}$

Can be smoother if $\delta t_{\rm fin} > 1/\mathcal{H}_*$



A. Roper Pol et al, arXiv:2201.05630

SGWB from a **CONTINUOUS** stochastic source in the radiation era

Typical example: topological defects

Suppose the source is operating continuously in the radiation dominated era

- No matching at the end time of the source
- No *free* sub-Hubble modes

$$H_r^{\rm rad}(\mathbf{k},\eta) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta} d\tau\, a(\tau)^3\, \sin[k(\eta-\tau)]\,\Pi_r(\mathbf{k},\tau)$$

$H_r(\mathbf{k},\eta) = a\, h_r(\mathbf{k},\eta)$

$$\rho_{\rm GW} = \frac{\langle \dot{h}_{ij}(\mathbf{x},t)\,\dot{h}_{ij}(\mathbf{x},t)\rangle}{32\pi G} = \int_0^{+\infty}\frac{dk}{k}\,\frac{d\rho_{\rm GW}}{d\log k}$$

$$\frac{d\rho_{\rm GW}}{d\log k}(k,\eta) \sim \frac{G}{a^4}\,k^3\int_{\eta_{\rm in}}^{\eta} d\tau\, a(\tau)\int_{\eta_{\rm in}}^{\eta} d\zeta\, a(\zeta)\,\mathcal{G}(k,\eta,\tau,\zeta)\,\Pi(k,\tau,\zeta)$$

D. Figueroa et al, arXiv:1212.5458

SGWB from a **CONTINUOUS** stochastic source in the radiation era

Typical example: topological defects

Suppose the source is operating continuously in the radiation dominated era

- Scaling (property of the topological defects network)
- Decays very fast in off-diagonal $k\tau \neq k\zeta$
- Decays as a power law on the diagonal $k\tau = k\zeta$

$$\Pi(k,\tau,\zeta) \sim \frac{v^4}{\sqrt{\tau\zeta}}\,\mathcal{U}(k\tau,k\zeta)$$

$$\frac{d\rho_{\rm GW}}{d\log k}(k,\eta) \sim \frac{G}{a^4}\,k^3 \int_{\eta_{\rm in}}^{\eta} d\tau\, a(\tau) \int_{\eta_{\rm in}}^{\eta} d\zeta\, a(\zeta)\, \mathcal{G}(k,\eta,\tau,\zeta)\, \Pi(k,\tau,\zeta)$$

D. Figueroa et al, arXiv:1212.5458

SGWB from a **CONTINUOUS** stochastic source in the radiation era

Typical example: topological defects

Suppose the source is operating continuously in the radiation dominated era

$$h^2\Omega_{\rm GW}(f) \sim h^2\Omega_{\rm rad}\left(\frac{v}{M_{\rm Pl}}\right)^4 F_{\rm RD}^{[\mathcal{U}]}(\infty)$$

TODAY FLAT SPECTRUM AT SUB-HORIZON MODES IN THE RADIATION ERA

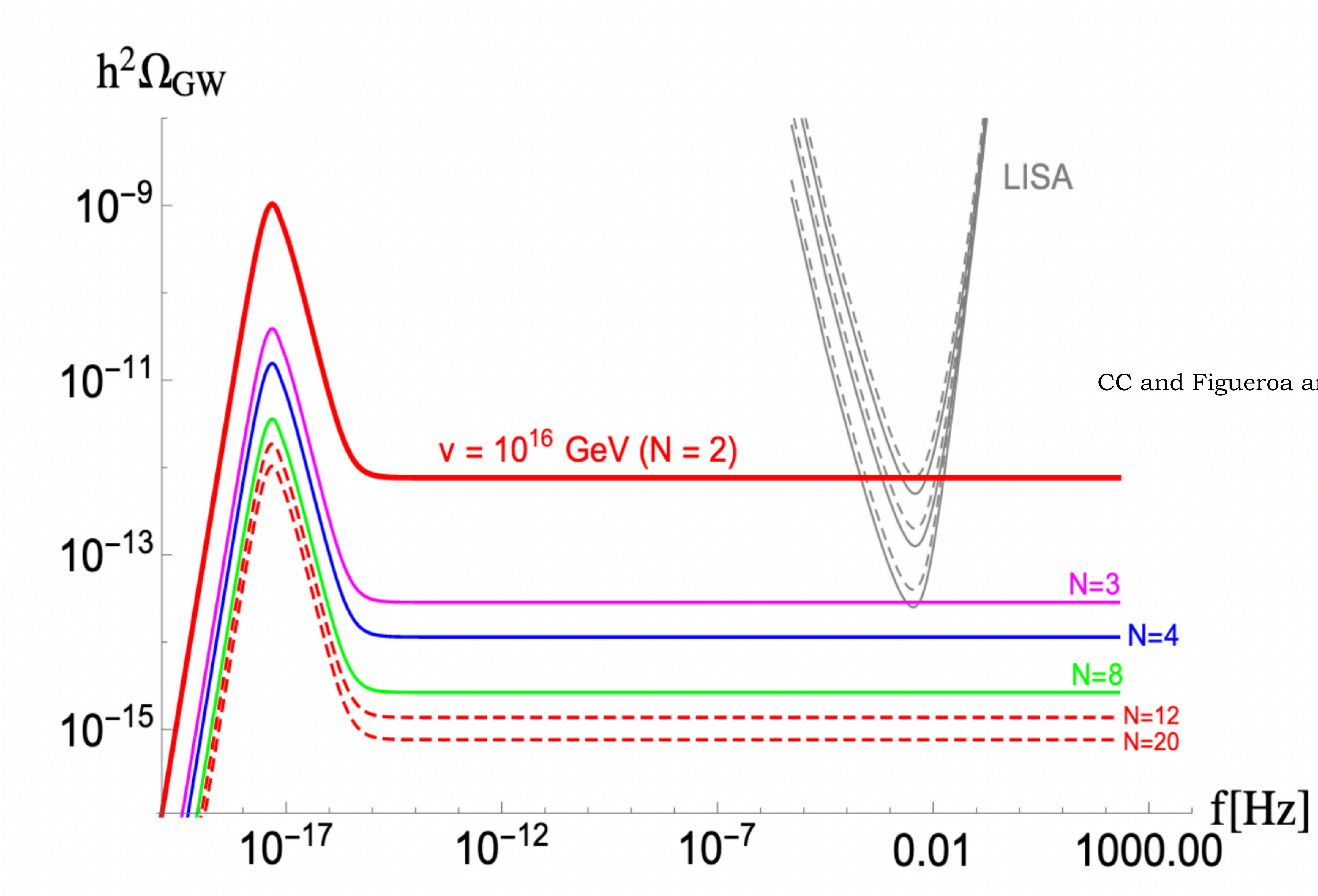
Progressively independent on the upper bound

$$\frac{d\rho_{\rm GW}}{d\log k}(k,\eta) \sim \Omega_{\rm rad}\frac{\rho_c}{a^4}\left(\frac{v}{M_{\rm Pl}}\right)^4 \int_{x_{\rm in}}^{x} dx_1 \int_{x_{\rm in}}^{x} dx_2\, \sqrt{x_1 x_2}\,\mathcal{G}(x,x_1,x_2)\,\mathcal{U}(x_1,x_2)$$

D. Figueroa et al, arXiv:1212.5458

SGWB from a **CONTINUOUS** stochastic source in the radiation era

Typical example: topological defects



CC and Figueroa arXiv:1801.04268