Astrophysical and cosmological aspects of gravitational waves

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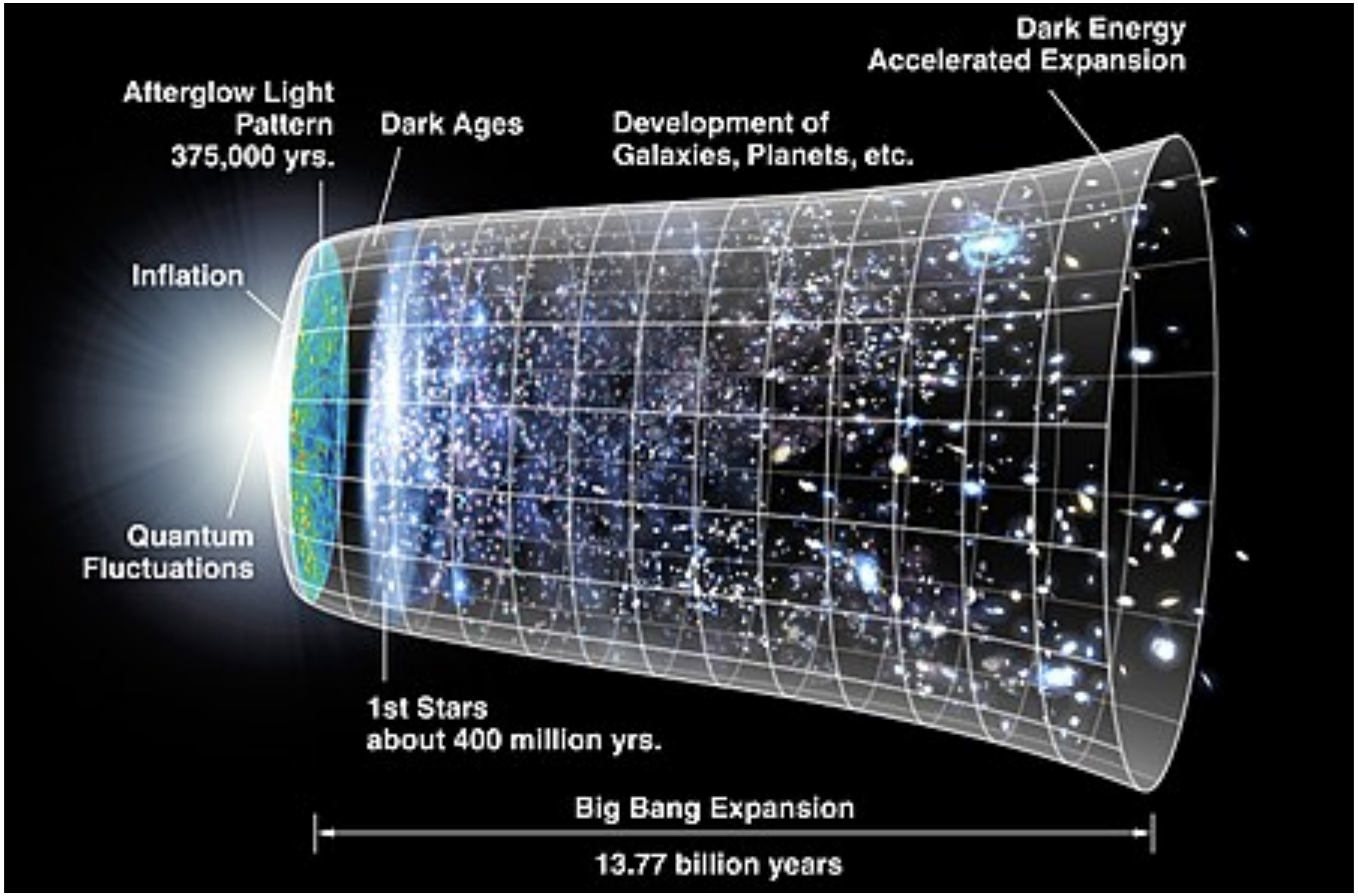
How can GW help to probe the universe?

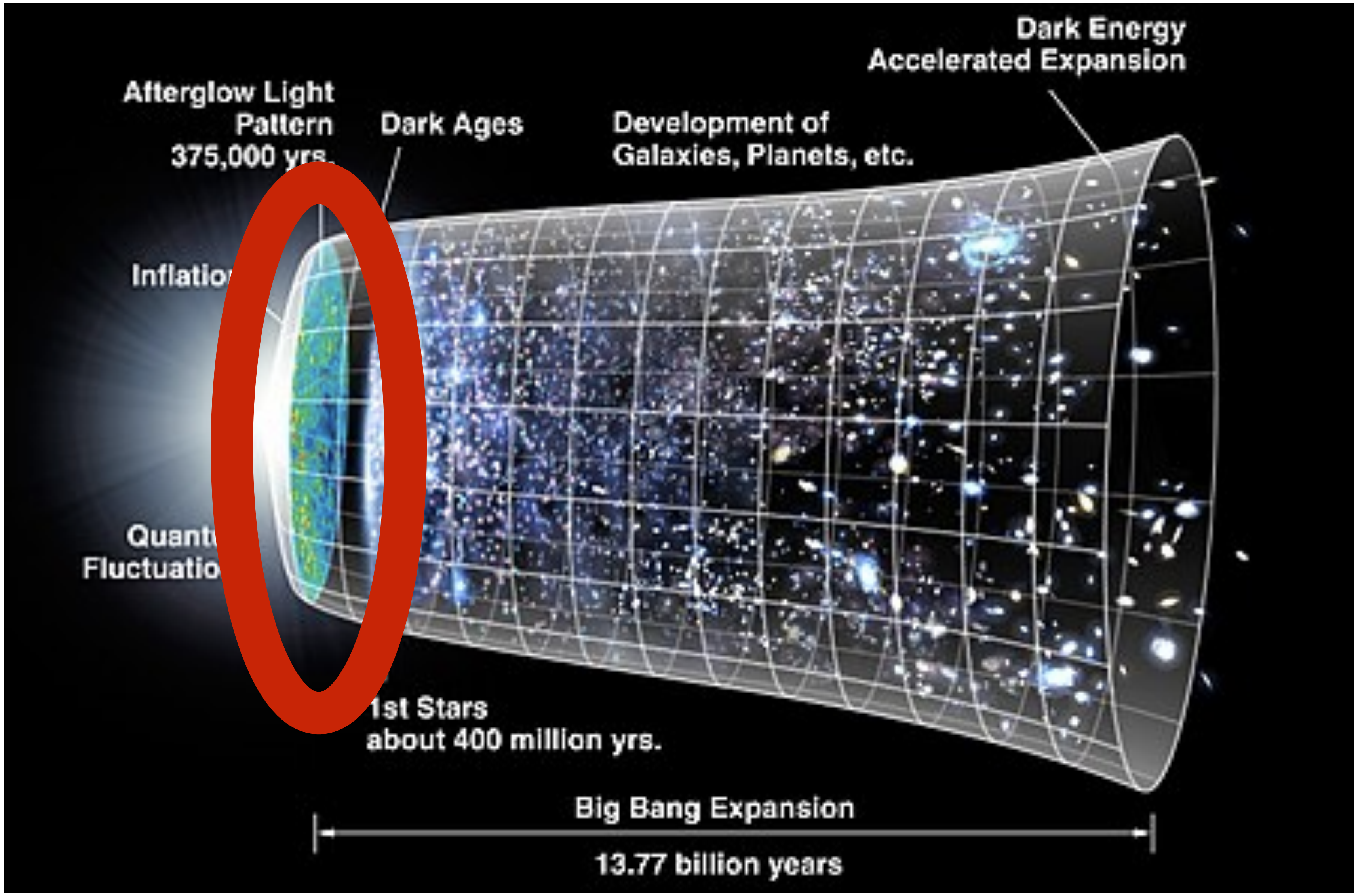


because of the weakness of the gravitational interaction the universe is “transparent” to GWs

$$\frac{\Gamma(T)}{H(T)} \sim \frac{G^2\,T^5}{T^2/M_{Pl}} \sim \left(\frac{T}{M_{Pl}}\right)^3 < 1$$

How can GW help to probe the universe?

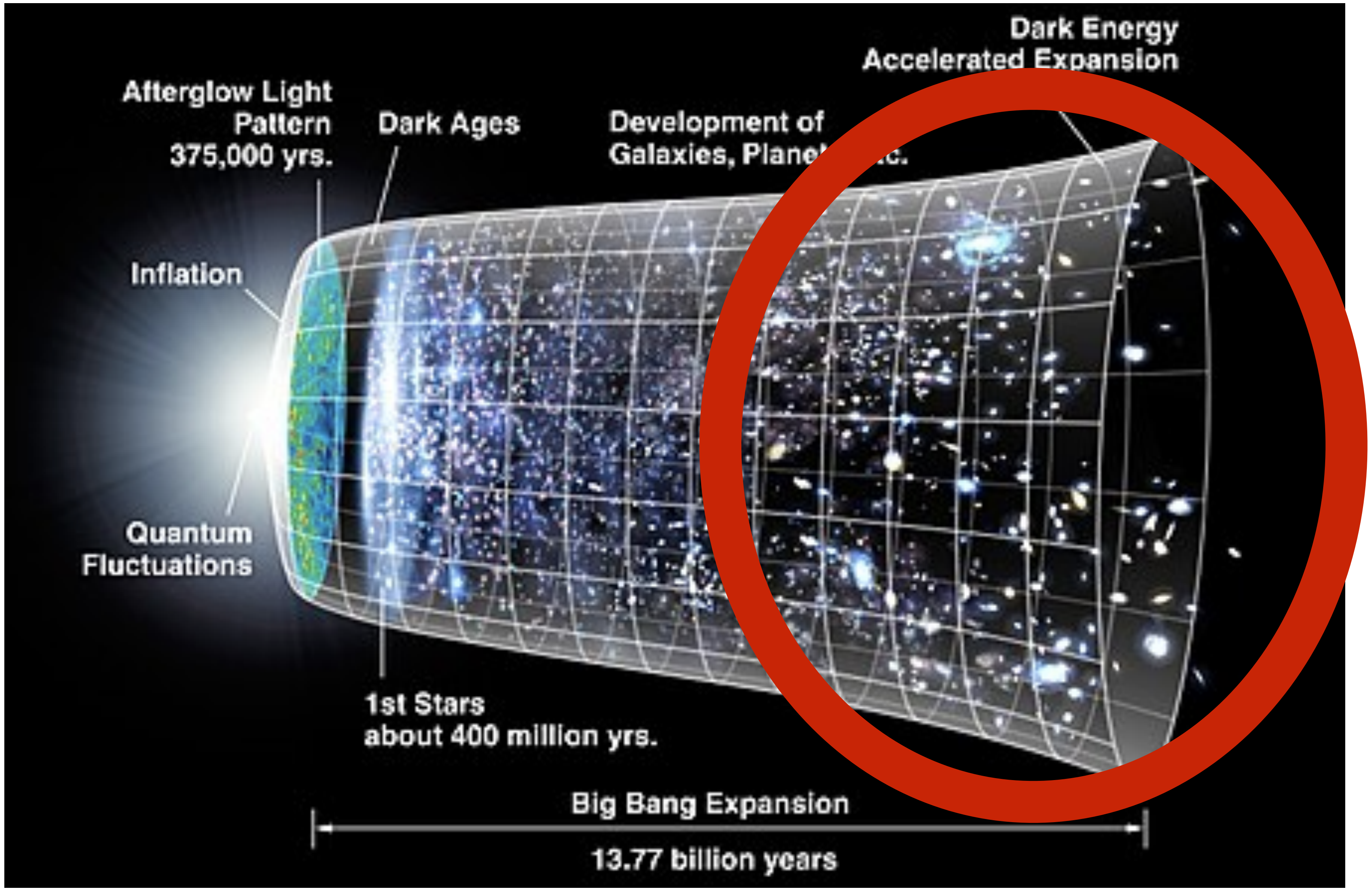
early universe



GW can bring direct information from the early universe: phenomena occurring in the early universe can produce stochastic GW backgrounds (SGWB) a fossil radiation like the CMB

tests of high energy phenomena

How can GW help to probe the universe?



late
universe

Binaries of compact objects (black holes, neutron stars…) orbiting around each other and possibly merging emit GWs

Provide information on binary formation and evolution, cosmological structure formation, black hole growth and environment, tests of General Relativity in strong and weak regime, tests of the cosmic expansion…

Summary of the course

- FIRST PART: GW definition, GW energy momentum tensor, GW in FLRW space-time, GW equation of motion, relevant solutions

- SECOND PART: Stochastic GW background from the early universe (with a digression on PTA measurement), and a few examples of SGWB sources

- THIRD PART: GW emission from binaires and their cosmological applications: standard sirens

C.C. and D.G. Figueroa, “Cosmological backgrounds of GWs”, arXiv:1801.04268

What are gravitational waves?

- GWs emerge naturally in General Relativity:

  Newtonian theory + special relativity = a causal theory of gravitation

  There must be some form of radiation propagating information causally: GWs!

- "waves" in physics are propagating perturbations over a background. In General Relativity:

  1. take a background space-time metric (the gravitational field)
  2. define a small perturbation over this background metric
  3. insert it into the equations that describe the space-time dynamics (Einstein equations)
  4. (if everything goes well) one finds a dynamical solution for the perturbation which is propagating as a wave -> GWs!

Which background metric to choose?
Simplest choice: flat space-time

GWs in linearised theory over Minkowski

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}(x)| \ll 1$$

Linearise in $h_{\mu\nu}$, raise and lower indices with $\eta_{\mu\nu}$

Affine connection
$$\Gamma^{\alpha}{}_{\mu\nu} \simeq \frac{1}{2}(\partial_\nu h^{\alpha}{}_{\mu} + \partial_\mu h^{\alpha}{}_{\nu} - \partial^{\alpha} h_{\mu\nu})$$

Riemann tensor
$$R^{\alpha}{}_{\mu\nu\beta} \simeq \frac{1}{2}(\partial_\mu\partial_\nu h^{\alpha}{}_{\beta} + \partial_\beta\partial^\alpha h_{\mu\nu} - \partial_\nu\partial^\alpha h_{\mu\beta} - \partial_\beta\partial_\mu h^{\alpha}{}_{\nu})$$

Einstein tensor
$$G_{\mu\nu} \simeq \frac{1}{2}(\partial_\alpha\partial_\nu \bar{h}^{\alpha}{}_{\mu} + \partial^\alpha\partial_\mu \bar{h}_{\nu\alpha} - \Box\bar{h}_{\mu\nu} - \eta_\mu\nu\partial_\alpha\partial^\beta \bar{h}^{\alpha}{}_{\beta})$$

$$\Box \equiv \partial_\alpha\partial^\alpha$$

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\, h$$

trace-reversed metric perturbation (OK - still small)

"Gravitational Waves", M. Maggiore, Oxford University Press 2008

GWs in linearised theory over Minkowski

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}(x)| \ll 1$$

Linearise in $h_{\mu\nu}$, raise and lower indices with $\eta_{\mu\nu}$

Affine connection $$\Gamma^{\alpha}{}_{\mu\nu} \simeq \frac{1}{2}(\partial_\nu h^{\alpha}{}_{\mu} + \partial_\mu h^{\alpha}{}_{\nu} - \partial^{\alpha} h_{\mu\nu})$$

Riemann tensor $$R^{\alpha}{}_{\mu\nu\beta} \simeq \frac{1}{2}(\partial_\mu\partial_\nu h^{\alpha}{}_{\beta} + \partial_\beta\partial^{\alpha} h_{\mu\nu} - \partial_\nu\partial^{\alpha} h_{\mu\beta} - \partial_\beta\partial_\mu h^{\alpha}{}_{\nu})$$

Einstein tensor $$G_{\mu\nu} \simeq \frac{1}{2}(\partial_\alpha\partial_\nu \bar{h}^{\alpha}{}_{\mu} + \partial^{\alpha}\partial_\mu \bar{h}_{\nu\alpha} - \Box\bar{h}_{\mu\nu} - \eta_\mu\nu\partial_\alpha\partial^{\beta}\bar{h}^{\alpha}{}_{\beta})$$

$G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ $\longrightarrow$ we would like to set $\partial^{\mu}\bar{h}_{\mu\nu}(x) = 0$

GWs in linearised theory over Minkowski

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad\qquad |h_{\mu\nu}(x)| \ll 1$$

GR is invariant under general coordinate transformation

the linearised theory is invariant under
infinitesimal (slowly varying) coordinate transformation

$$x'^{\mu} \longrightarrow x^{\mu} + \xi^{\mu} \qquad\qquad h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}$$

$$|\partial_{\alpha}\xi_{\beta}| \lesssim |h_{\alpha\beta}| \quad \longrightarrow \quad |h'_{\mu\nu}(x')| \ll 1$$

$$\partial^{\mu}\bar{h}_{\mu\nu}(x) \longrightarrow \partial'^{\mu}\bar{h}'_{\mu\nu}(x') = \partial^{\mu}\bar{h}_{\mu\nu}(x) - \Box\xi_{\nu}$$

By a suitable coordinate transformation, it is always possible to go to the LORENTZ GAUGE

$$\partial'^{\mu}\bar{h}'_{\mu\nu}(x') = 0$$

GWs in linearised theory over Minkowski

IN LORENTZ GAUGE EINSTEIN EQUATIONS TAKE THE FORM OF A WAVE EQUATION

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}$$

$T_{\mu\nu}$ source energy momentum tensor

From the Lorentz gauge condition $\partial^\mu \bar{h}_{\mu\nu}(x) = 0$ one gets

$$\partial^\mu T_{\mu\nu} = 0$$

The energy-momentum tensor of the source is conserved

the source does not loose energy and momentum by the GW emission

in linearised theory, the background space-time is flat, i.e. the source is described by Newtonian gravity

linearised theory does not describe how GW emission influences the source, but the behaviour of test masses is described in the full metric

GWs in linearised theory over Minkowski

IN LORENTZ GAUGE EINSTEIN EQUATIONS TAKE THE FORM OF A WAVE EQUATION

$$\Box \bar{h}_{\mu\nu} = -16\pi G\, T_{\mu\nu}$$

$T_{\mu\nu}$ source energy momentum tensor

$\bar{h}_{\mu\nu} = \bar{h}_{\nu\mu}$ (16-6) $\partial^{\mu}\bar{h}_{\mu\nu}(x) = 0$ (-4) $\longrightarrow$ 6 radiative components

ARE THESE ALL PHYSICAL?

$x'^{\mu} \longrightarrow x^{\mu} + \xi^{\mu}$ satisfying $\Box \xi_{\mu} = 0$ to remain in the Lorentz gauge

$\bar{h}_{\mu\nu} \longrightarrow \bar{h}_{\mu\nu} + \xi_{\mu\nu}$ with $\xi_{\nu\mu} \equiv \eta_{\nu\mu}\partial^{\alpha}\xi_{\alpha} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}$

IF IN VACUUM: $T_{\mu\nu} = 0$ $\Box' \bar{h}'_{\mu\nu} \simeq \Box(\bar{h}_{\mu\nu} + \xi_{\mu\nu}) = 0$

GWs in linearised theory over Minkowski, in vacuum

Restricting to vacuum space-time, the residual coordinate freedom can be used to fix 4 constraints

TRANSVERSE TRACELESS GAUGE

$$\bar{h}^{\mu}{}_{\mu} = 0 \qquad h_{0i} = 0 \qquad \partial^i h_{ij} = 0$$

$$h_{00} = 0$$

come for free

There are only 2 remaining *physical* degrees of freedom in the metric

$$\Box h_{ij}(\mathbf{x}, t) = 0$$

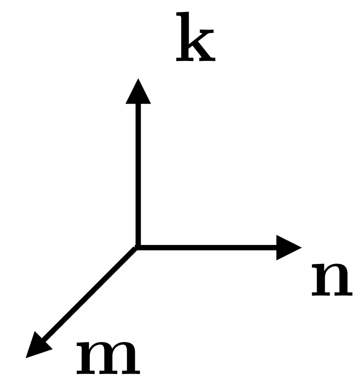
$$h_{ij}(\mathbf{x}, t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, h_r(\mathbf{k})\, e^r_{ij}(\hat{\mathbf{k}}) e^{-ik(t-\hat{k}\cdot\mathbf{x})}$$

Plane waves, transverse, moving at the speed of light $k = \omega$
with two independent polarisation components +, ×

GWs in linearised theory over Minkowski, in vacuum

- Polarisation tensors

$$e^{+}_{ij}(\hat{\mathbf{k}}) = \hat{m}_i\,\hat{m}_j - \hat{n}_i\,\hat{n}_j$$
$$e^{\times}_{ij}(\hat{\mathbf{k}}) = \hat{m}_i\,\hat{n}_j + \hat{n}_i\,\hat{m}_j$$



- Free wave traveling in the z direction

$$h_{ij}(z,t) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos\left[\omega(t-z)\right]$$

$$\mathbf{k} = \omega\hat{\mathbf{z}}$$

- Metric line element

$$ds^2 = -dt^2 + dz^2 + (1 + h_+ \cos[\omega(t-z)])dx^2 + \\ + (1 - h_+ \cos[\omega(t-z)])dy^2 + 2h_\times \cos[\omega(t-z)]dxdy$$

- Polarisation states are related to the spin of the massless particle expected upon quantisation

$$S = \frac{2\pi}{\theta}$$

Misner, Thorne, Wheeler
"Gravitation"
Chapter 35.6

Where θ is the rotation angle under which the polarisation modes are invariant

GWs in linearised theory over Minkowski, in vacuum

invariant under rotation around the z-axis of $\theta = \pi$

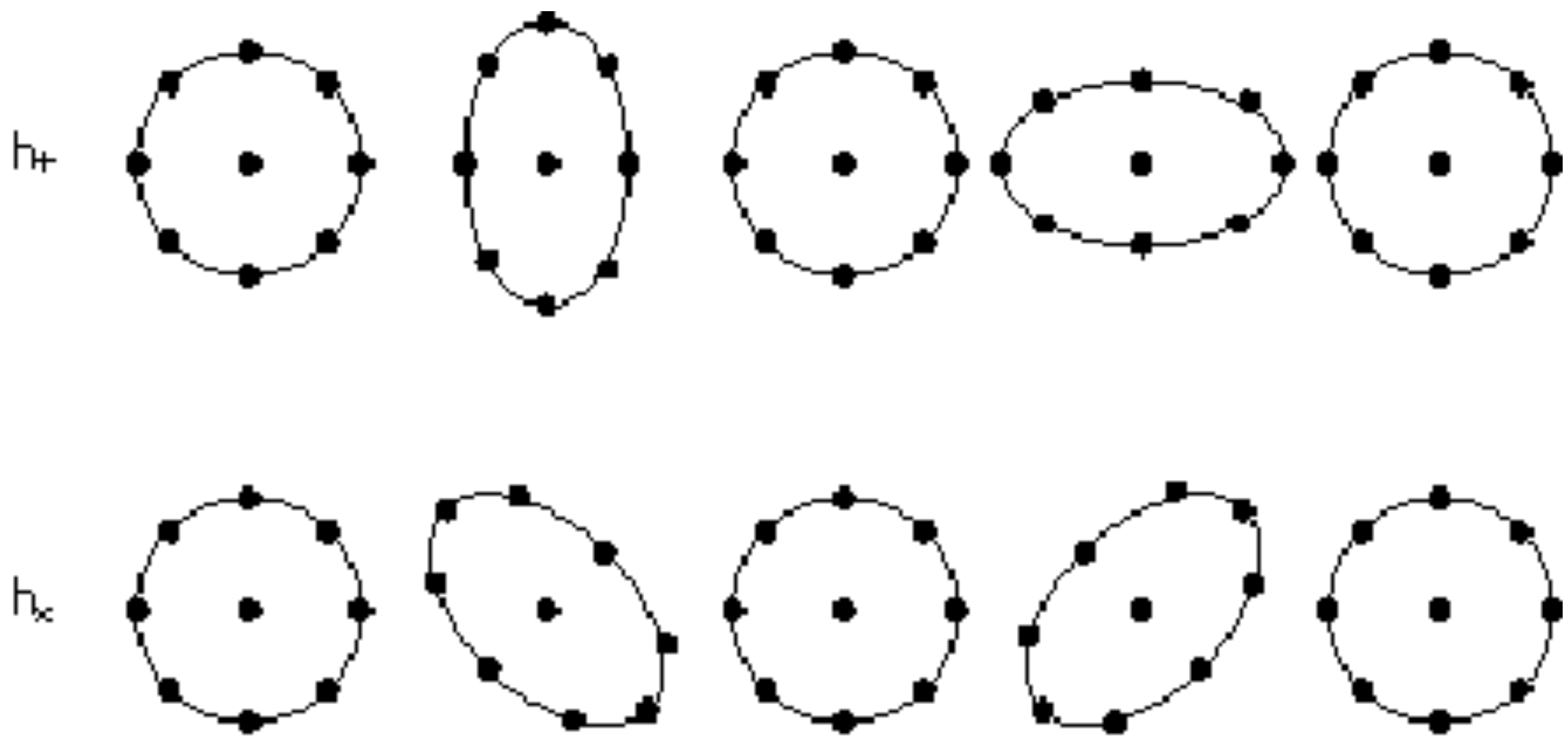
$$h_+^\theta = h_+ \cos 2\theta - h_\times \sin 2\theta$$
$$h_\times^\theta = h_\times \cos 2\theta + h_+ \sin 2\theta$$

Graviton is a spin 2 particle

- Effect of GW on a ring of test masses

Geodesic deviation equation $\ddot{\xi}^i = -R^i{}_{0j0}\xi^j = \frac{1}{2}\ddot{h}_{ij}\xi^j$

$h_+, h_\times$



GWs in linearised theory over Minkowski, with matter

To exhibit the two physical d.o.f. of GWs we had to restrict to vacuum

However, the fact that GWs have only two physical components is a manifestation of the intrinsic nature of the gravitational interaction, mediated by the graviton, a spin-two massless field that has only two independent helicity states

It should be true also in space-time with matter

WHAT WE DO NEXT:

1. Drop the condition of vacuum
2. Exploit the invariance of Minkowski space-time under spatial rotations, and split the metric perturbation into irreducible components under rotations (scalar, vector, tensor)
3. Construct metric perturbation variables that are invariant under infinitesimal coordinate transformations
4. Find the metric perturbation variable that obeys a wave equation -> *we define GWs without restricting to vacuum*

GWs in linearised theory over Minkowski, with matter

2. Exploit the invariance of Minkowski space-time under spatial rotations, and split the metric perturbation into irreducible components under rotations

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \qquad\qquad |h_{\mu\nu}(x)| \ll 1$$

$$\begin{aligned} h_{00} &= -2\phi \\ h_{0i} &= \partial_i B + S_i \quad (\partial_i S_i = 0) \\ h_{ij} &= -2\psi\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E + \partial_i F_j + \partial_j F_i + H_{ij} \\ &\quad (\partial_i F_i = 0,\ \partial_i H_{ij} = 0,\ H_{ii} = 0) \end{aligned}$$

scalars	vectors	tensor
$\phi,\ B,\ \psi,\ E$	$S_i,\ F_i$	H_{ij}

E.E. Flanagan and S.A. Hughes, “The basics of GW theory”, arXiv:gr-qc/0501041

“Space-time and geometry: an introduction to GWs”, S. Carroll, Pearson Education Limited, 2014

“The Cosmic Microwave Background”, R. Durrer, Cambridge University Press, 2008

GWs in linearised theory over Minkowski, with matter

3. Construct metric perturbation variables that are invariant under infinitesimal coordinate transformations

$$x'^{\mu} \longrightarrow x^{\mu} + \xi^{\mu} \qquad h'_{\mu\nu}(x') = h_{\mu\nu}(x) - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu}$$

$$\xi_{\mu} = (\xi_0, \xi_i) \equiv (d_0, \partial_i d + d_i) \qquad \text{with} \quad \partial_i d_i = 0$$

Two scalars, one vector and one tensor gauge invariant variables

$$\Phi \equiv \phi + \dot{B} - \ddot{E}/2$$
$$\Theta \equiv -2\psi - \nabla^2 E/3$$
$$\Sigma_i \equiv S_i - \dot{F}_i \qquad \text{with} \quad \partial_i \Sigma_i = 0$$
$$H_{ij} \qquad \text{with} \quad \partial_i H_{ij} = 0 \;\; H^i_i = 0$$

Automatically gauge invariant (no tensor gauge transformation)

16 free functions - 6 contraints - 4 constraints =
6 physical degrees of freedom

GWs in linearised theory over Minkowski, with matter

1. Drop the condition of empty space-time

$$
\begin{aligned}
T_{00} &= \rho \\
T_{0i} &= \partial_i u + u_i \quad (\partial_i u_i = 0) \\
T_{ij} &= p\delta_{ij} + (\partial_i \partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + \partial_i v_j + \partial_j v_i + \Pi_{ij} \\
& \qquad (\partial_i v_i = 0,\ \partial_i \Pi_{ij} = 0,\ \Pi_{ii} = 0)
\end{aligned}
$$

scalars	vectors	tensor
$\rho,\ u,\ p,\ \sigma$	$u_i,\ v_i$	Π_{ij}

Gauge invariant automatically but four further constraints given by energy-momentum conservation
(OK since we are still in linearised theory!)

$$\partial_\mu T^{\mu\nu} = 0$$

16 free functions - 6 contraints - 4 constraints =
6 physical degrees of freedom

GWs in linearised theory over Minkowski, with matter

4. Find the metric perturbation variable that obeys a wave equation -> *we define GWs in non-vacuum space-times*

Write Einstein equations in terms of the 6 gauge invariant variables

$$\nabla^2\Theta = -8\pi G\rho \qquad \nabla^2\Phi = 4\pi G\left(\rho + 3p - 3\dot{u}\right)$$

$$\nabla^2\Sigma_i = -16\pi G S_i \qquad \Box H_{ij} = -16\pi G\,\Pi_{ij}$$

Three Poisson-like equations, one wave equation
Only the TT metric components are radiative

Cosmological case: the same procedure, exploiting the symmetries (homogeneity and isotropy) of FLRW spacetime, but:

- There is a energy-momentum tensor also in the background
- The equation of the tensor perturbations contains Hubble friction (see later)

GWs in linearised theory over Minkowski, with matter

4. Find the metric perturbation variable that obeys a wave equation -> *we define GWs in non-vacuum space-times*

Write Einstein equations in terms of the 6 gauge invariant variables

$$\nabla^2\Theta = -8\pi G\rho \qquad \nabla^2\Phi = 4\pi G\,(\rho + 3p - 3\dot{u})$$

$$\nabla^2\Sigma_i = -16\pi G S_i \qquad \Box H_{ij} = -16\pi G\,\Pi_{ij}$$

Three Poisson-like equations, one wave equation

Only the TT metric components are radiative

Cosmological case: the same procedure, exploiting the symmetries (homogeneity and isotropy) of FLRW spacetime, but:

- One finds a wave equation also for the Bardeen potential (sound waves in the fluid)

$$\ddot{\Phi} + 3\mathcal{H}(1+c_s^2)\dot{\Phi} + [\mathcal{H}^2(1+3c_s^2) - \mathcal{H}^2(1+3w) + k^2 c_s^2]\Phi = 0$$

GW energy-momentum tensor and GW propagation

According to GR, any form of energy contributes to space-time curvature
Are GWs a source of space-time curvature?

- One needs to go beyond linearisation over Minkowski, otherwise one excludes from the beginning the presence of any background space-time curvature

$$g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x)\,, \qquad |h_{\mu\nu}(x)| \ll 1$$

- In this new setting, how to decide what is the background and what is the fluctuation?

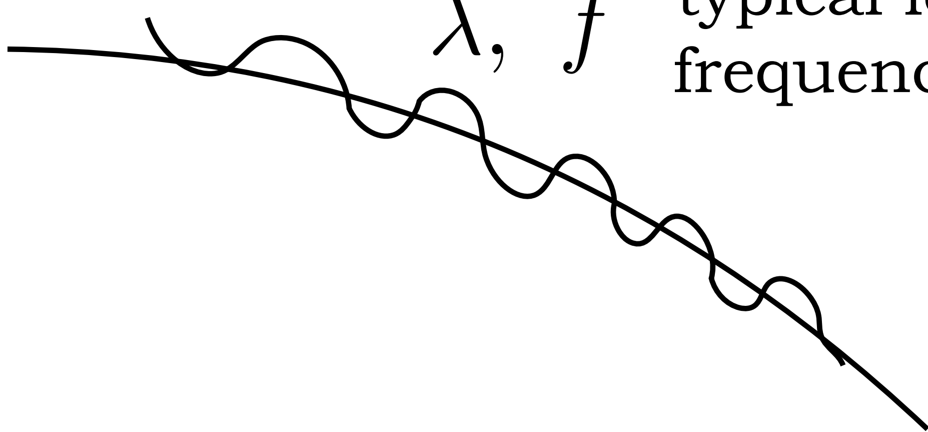
1. The background space-time has a clear symmetry (static, FLRW…)
2. It is possible to resort to a clear separation of scales/frequencies

“Gravitational Waves”, M. Maggiore, Oxford University Press 2008

E.E. Flanagan and S.A. Hughes, “The basics of GW theory”, arXiv:gr-qc/0501041

R.A. Isaacson, Physical Review, Volume 166, number 5, pages 1263 and 1272, 1968

GW energy-momentum tensor and GW propagation



$\lambda, \ f$ typical length-scale / frequency of the GWs $\qquad \lambda = \dfrac{c}{f}$

$L_B, \ f_B$ typical scale of spatial variations / frequency of time variations of the background (needn't be related among themselves)

- There are two expansions in the game:

1. $|h_{\mu\nu}| \ll 1$ $\qquad$ 2. $\dfrac{\lambda}{L_B} \ll 1, \quad \dfrac{f_B}{f} \ll 1$

- In order to effectively implement the distinction among background and GWs, one needs to average physical quantities

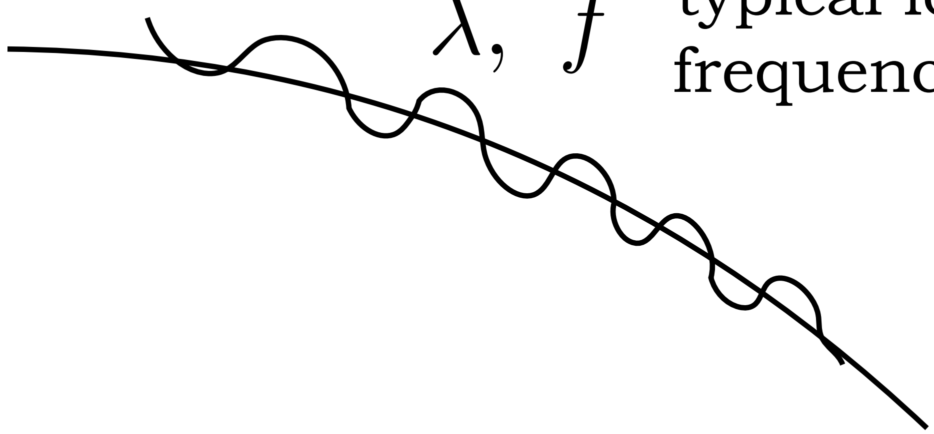
$$\bar{g}_{\mu\nu} \equiv \langle g_{\mu\nu} \rangle$$

$$\lambda \ll \bar{\ell} \ll L_B \qquad f_B \ll \bar{f} \ll f$$

$$\langle h_{\mu\nu} \rangle = 0$$

"Gravitational Waves", M. Maggiore, Oxford University Press 2008

GW energy-momentum tensor and GW propagation



$\lambda, \; f$ typical length-scale / frequency of the GWs $\qquad \lambda = \frac{c}{f}$

$L_B, \; f_B$ typical scale of spatial variations / frequency of time variations of the background
(needn't be related among themselves)

By expanding the Einstein equations to second order in $|h_{\mu\nu}| \ll 1$ and separating the background and first order components by averaging, one finds

- The expression for the GW energy momentum tensor (how GWs influence the background)
- The equation representing GW propagation on a curved background

GW energy-momentum tensor and GW propagation

Expand up to second order in $|h_{\mu\nu}| \ll 1$ $\qquad R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$

The linear term averages to zero

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$$

The quadratic term can influence the background, as it contains both high and low modes

$\langle \ldots \rangle = [\ldots]^{\text{low}}$

Background Einstein equation

$$\bar{R}_{\mu\nu} = [-R^{(2)}_{\mu\nu}]^{\text{low}} + 8\pi G\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]^{\text{low}}$$

GWs sourcing the bckg curvature

Matter sourcing the bckg curvature

$\mathcal{O}\left(\frac{1}{L_B}\right)^2 \qquad \mathcal{O}\left(\frac{h}{\lambda}\right)^2$

$$h \lesssim \frac{\lambda}{L_B}$$

Necessary condition for GW to make sense

GW energy-momentum tensor and GW propagation

Rearranging the Einstein equations and performing the average leads to:

$$\underbrace{\bar{G}_{\mu\nu}}_{} = \langle R_{\mu\nu} \rangle - \frac{1}{2}\bar{g}_{\mu\nu}\langle R \rangle = 8\pi G(\underbrace{\langle T_{\mu\nu} \rangle}_{} + \underbrace{T^{\mathrm{GW}}_{\mu\nu}}_{})$$

Dynamics of the bckg space-time

Low-mode part of the matter component

GWs

not separately conserved!

GW energy-momentum tensor

$$T^{GW}_{\mu\nu} = -\frac{1}{8\pi G}\langle R^{(2)}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu} R^{(2)} \rangle$$

calculating $R^{(2)}_{\mu\nu}$ and reducing to the TT gauge

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi G}\langle \nabla_\mu h_{\alpha\beta} \nabla_\nu h^{\alpha\beta} \rangle$$

GW energy density:

$$\rho_{GW} = \frac{\langle \dot{h}_{ij} \dot{h}^{ij} \rangle}{32\pi G}$$

C.W. Misner, K.S. Thorne, J.A. Wheeler, “Gravitation”, Freeman 1997 (chapter 35.15)

GW energy-momentum tensor and GW propagation

Expand up to second order in $|h_{\mu\nu}| \ll 1$ $\quad R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$

Focus on the linear term: $\quad R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$

Perturbed Einstein equation

$$R^{(1)}_{\mu\nu} = [-R^{(2)}_{\mu\nu}]^{\text{high}} + 8\pi G \left[T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right]^{\text{high}}$$

$$\mathcal{O}\left(\frac{h}{\lambda^2}\right)$$

$$\mathcal{O}\left(\frac{h}{\lambda}\right)^2$$

Matter possibly sourcing GWs

Negligible
(non-linear interaction of
the wave with itself)

GW energy-momentum tensor and GW propagation

Expand up to second order in $|h_{\mu\nu}| \ll 1$ $\qquad R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$

Focus on the linear term: $\qquad R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$

Perturbed Einstein equation

$$R^{(1)}_{\mu\nu} - \frac{1}{2}(\bar{g}_{\mu\nu}R^{(1)} + h_{\mu\nu}\bar{R}) \simeq 8\pi G\,[T_{\mu\nu}]^{\text{high}}$$

Evolution of GWs on a curved but smooth / slowly evolving background such as gravitational redshift and lensing

Possible source of GWs

calculating $R^{(1)}_{\mu\nu}$:

$$-\frac{1}{2}\Box\bar{h}_{\mu\nu} + R^{\lambda}{}_{\mu\nu}{}^{\sigma}\bar{h}_{\lambda\sigma} + \nabla_{(\nu}\nabla^{\sigma}\bar{h}_{\mu)\sigma} - \frac{1}{2}\bar{g}_{\mu\nu}\nabla^{\alpha}\nabla^{\beta}\bar{h}_{\alpha\beta} +$$

$$+ R^{\alpha\beta}\left[\frac{1}{2}\bar{g}_{\mu\nu}\bar{h}_{\alpha\beta} - \frac{1}{2}\bar{h}_{\mu\nu}\bar{g}_{\alpha\beta} + \bar{g}_{\beta(\mu}\bar{h}_{\nu)\alpha}\right] = 8\pi\,G\,\delta T_{\mu\nu}$$

GW energy-momentum tensor and GW propagation

Expand up to second order in $|h_{\mu\nu}| \ll 1$ $\qquad R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$

Focus on the linear term: $\qquad R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$

Perturbed Einstein equation

$$R^{(1)}_{\mu\nu} - \frac{1}{2}(\bar{g}_{\mu\nu} R^{(1)} + h_{\mu\nu}\bar{R}) \simeq 8\pi G \left[T_{\mu\nu} \right]^{\text{high}}$$

In a FLRW universe, equation of sourcing and propagation of GWs

$$ds^2 = -dt^2 + a^2(t)\,(\delta_{ij} + h_{ij})\, dx^i dx^j$$

Neglecting scalar and vector perturbations

$$\partial_i h_{ij} = h_{ii} = 0$$

$$\ddot{h}_{ij}(\mathbf{x},t) + 3\,H\,\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\, h_{ij}(\mathbf{x},t) = 16\pi G\, \Pi_{ij}(\mathbf{x},t)$$

GW propagation equation in FLRW

COMMENTS

- In the rest of the course, we will be dealing with solutions of the above equation
- It can be derived also from cosmological perturbation theory, here I presented the connection with a more general approach
- In cosmology, the FLRW space-time is homogeneous and isotropic, so tensor modes can be defined also when $\lambda \sim L_B$ (exemple: horizon re-entry after inflation), but one cannot say these are GWs, unless modes are well within the horizon ($\lambda \ll L_B$)

GW propagation equation in FLRW

$$\ddot{h}_{ij}(\mathbf{x},t) + 3\,H\,\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\,h_{ij}(\mathbf{x},t) = 16\pi G\,\Pi_{ij}(\mathbf{x},t)$$

Source: tensor anisotropic stress

Perfect fluid

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$$

In the cosmological context:
energy momentum tensor of the matter content of the universe (background + perturbations)

$$\delta T_{ij} = \bar{p}\,\delta g_{ij} + a^2[\delta p\,\delta_{ij} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\sigma + 2\partial_{(i}v_{j)} + \Pi_{ij}]$$

$$(\partial_i v_i = 0,\;\; \partial_i \Pi_{ij} = 0,\;\; \Pi_{ii} = 0)$$

NO GWs FROM THE HOMOGENEOUS MATTER COMPONENT

GW propagation equation in FLRW

$$\ddot{h}_{ij}(\mathbf{x},t) + 3\,H\,\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\,h_{ij}(\mathbf{x},t) = 16\pi G\,\Pi_{ij}(\mathbf{x},t)$$

Source: tensor anisotropic stress

Perfect fluid

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$$

In the cosmological context:
energy momentum tensor of the matter content of the universe (background + perturbations)

One exploits the translational invariance and performs a F.T. in space

$$h_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, h_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

$$\Pi_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \Pi_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

GW propagation equation in FLRW

$$\ddot{h}_{ij}(\mathbf{x},t) + 3\,H\,\dot{h}_{ij}(\mathbf{x},t) - \frac{\nabla^2}{a^2}\,h_{ij}(\mathbf{x},t) = 16\pi G\,\Pi_{ij}(\mathbf{x},t)$$

Source: tensor anisotropic stress

Perfect fluid

$$T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu}$$

In the cosmological context:
energy momentum tensor of the matter content of the universe (background + perturbations)

One exploits the translational invariance and performs a F.T. in space

$$h_{ij}(\mathbf{x},t) = \sum_{r=+,\times} \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, h_r(\mathbf{k},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, e^r_{ij}(\hat{\mathbf{k}})$$

not just plane waves as before

The evolution equation decouples for each polarisation mode

$$h_r''(\mathbf{k},\eta) + 2\,\mathcal{H}\,h_r'(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = 16\pi G\,a^2\,\Pi_r(\mathbf{k},\eta)$$

conformal time, Hubble factor and comoving wavenumber

GW propagation equation in FLRW

$$h_r''(\mathbf{k},\eta) + 2\,\mathcal{H}\,h_r'(\mathbf{k},\eta) + k^2\,h_r(\mathbf{k},\eta) = 16\pi G\,a^2\,\Pi_r(\mathbf{k},\eta)$$

Solution of the homogeneous equation

Power-law scale factor $a(\eta) = a_n \eta^n$

Covering matter (n=2) and radiation domination (n=1), and De Sitter inflation n=-1)

$$h_r(\mathbf{k},\eta) = \frac{A_r(\mathbf{k})}{a_n \eta^{n-1}}\, j_{n-1}(k\eta) + \frac{B_r(\mathbf{k})}{a_n \eta^{n-1}}\, y_{n-1}(k\eta)$$

Two notable limiting cases: sub-Hubble and super-Hubble modes

$$H_r(\mathbf{k},\eta) = a\,h_r(\mathbf{k},\eta) \qquad H_r''(\mathbf{k},\eta) + \left(k^2 - \frac{a''}{a}\right) H_r(\mathbf{k},\eta) = 0$$

$$a''/a \propto \mathcal{H}^2$$

GW propagation equation in FLRW

CASE 1: Sub-Hubble modes, relevant for propagation after the source stops

$$k^2 \gg \mathcal{H}^2 \qquad h_r(\mathbf{k},\eta) = \frac{A_r(\mathbf{k})}{a(\eta)}\, e^{ik\eta} + \frac{B_r(\mathbf{k})}{a(\eta)}\, e^{-ik\eta}$$

In this limit, GWs are plane waves with redshifting amplitude

What are the coefficients $A_r(\mathbf{k})$ and $B_r(\mathbf{k})$ from the initial condition?

Suppose the source operates in a time interval $\eta_{\rm fin}$ - $\eta_{\rm in}$ in the radiation dominated era

$$H_r^{\rm rad}(\mathbf{k},\eta<\eta_{\rm fin}) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta} d\tau\, a(\tau)^3\, \sin[k(\eta-\tau)]\, \Pi_r(\mathbf{k},\tau)$$

Matching at $\eta_{\rm fin}$ with the homogeneous solution to find the GW signal today

$$H_r^{\rm rad}(\mathbf{k},\eta>\eta_{\rm fin}) = A_r^{\rm rad}(\mathbf{k})\cos(k\eta) + B_r^{\rm rad}(\mathbf{k})\sin(k\eta)$$

$$A_r^{\rm rad}(\mathbf{k}) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\tau\, a(\tau)^3\, \sin(-k\tau)\, \Pi_r(\mathbf{k},\tau)$$

$$B_r^{\rm rad}(\mathbf{k}) = \frac{16\pi G}{k}\int_{\eta_{\rm in}}^{\eta_{\rm fin}} d\tau\, a(\tau)^3\, \cos(k\tau)\, \Pi_r(\mathbf{k},\tau)$$

GW propagation equation in FLRW

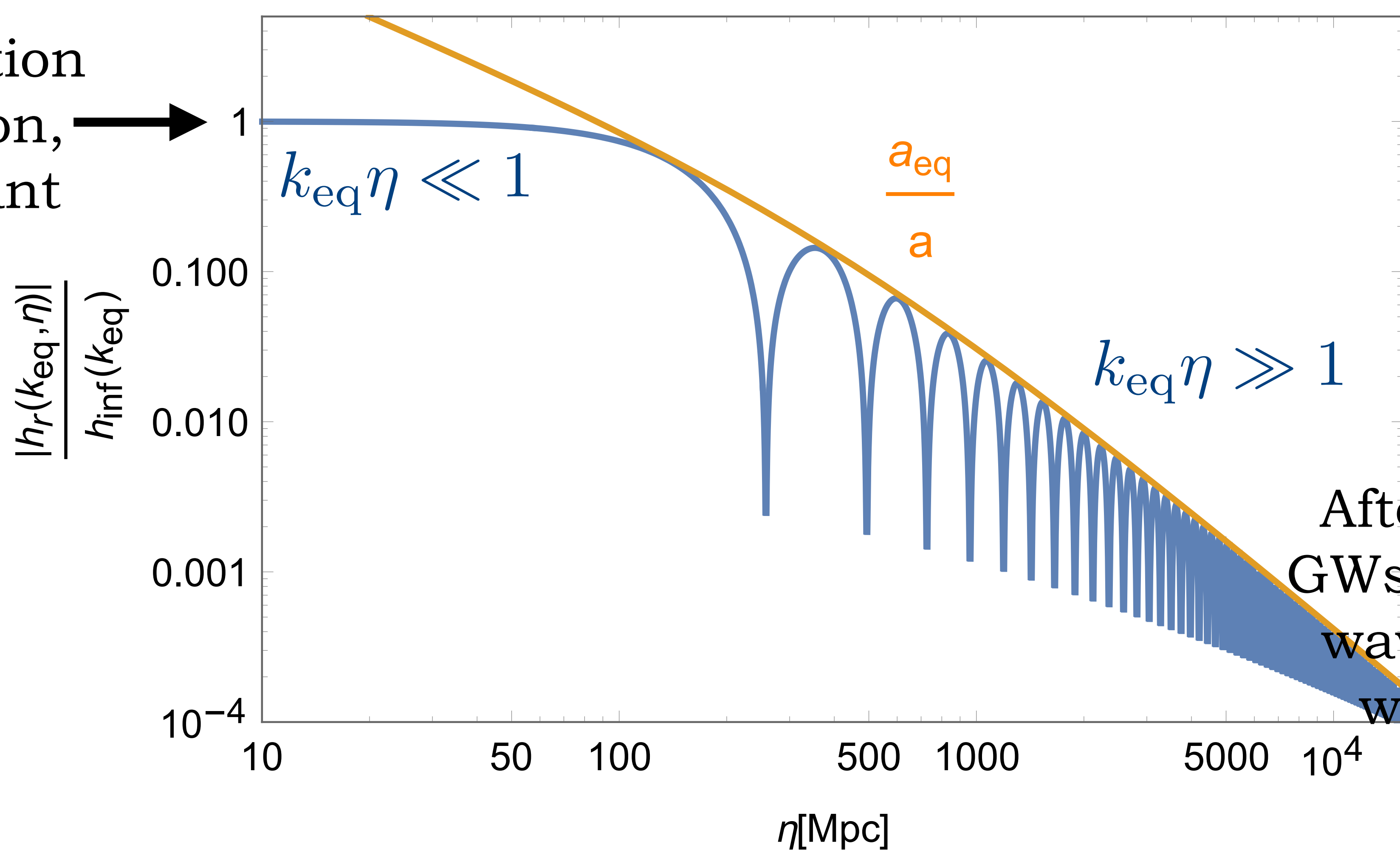
CASE 2: Super-Hubble modes, relevant for inflationary tensor perturbations

$$k^2 \ll \mathcal{H}^2 \qquad h_r(\mathbf{k},\eta) = A_r(\mathbf{k}) + B_r(\mathbf{k}) \int^{\eta} \frac{d\eta'}{a^2(\eta')}$$

Decaying mode, negligible

Full solution with inflationary initial conditions
Hubble re-entry at the radiation-matter transition

Initial condition from inflation, $A_r(\mathbf{k})$ constant



After Hubble re-entry, GWs oscillating as plane waves and redshifting with the expansion

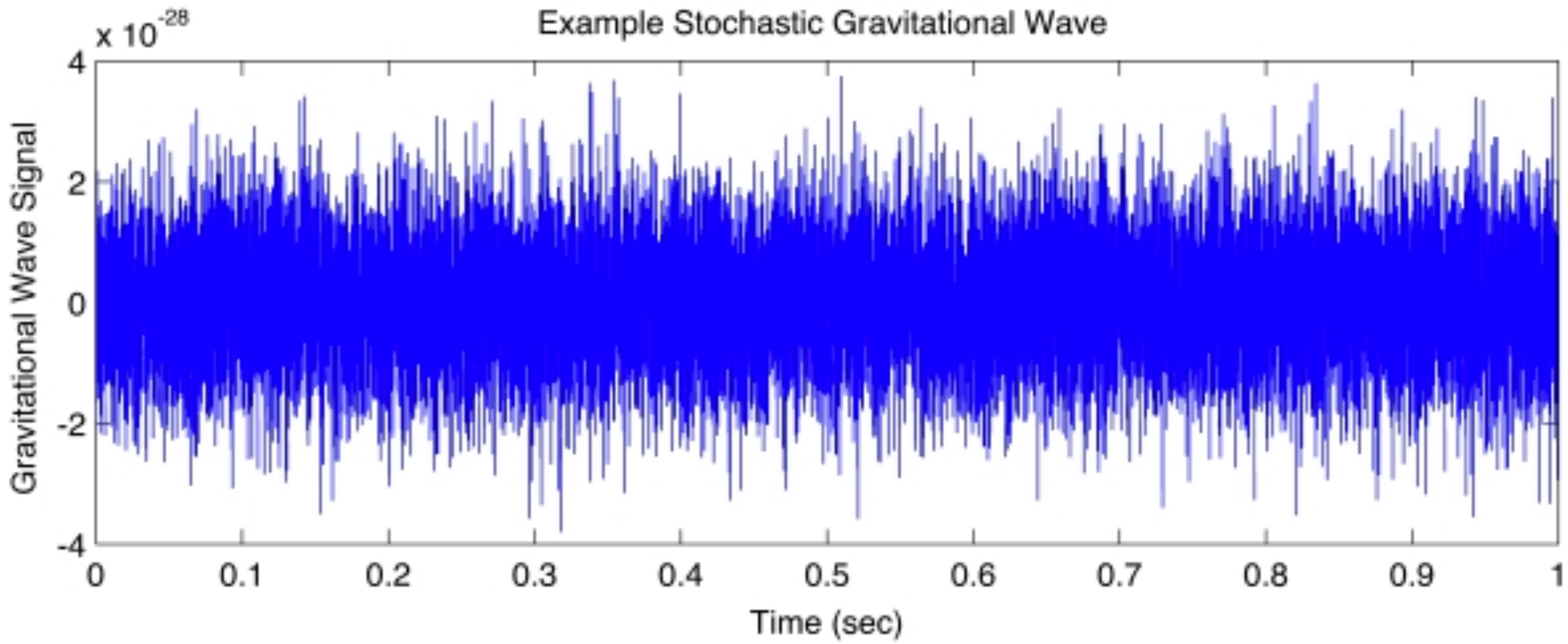
Summary up to here:

- We have defined GWs and GW energy density without ambiguity in the FLRW spacetime, making the connection with linearised gravity
- After their generation by some sourcing process, GWs in the FLRW space-time oscillate and decay with the expansion of the universe
- The sourcing process can be connected to the presence of anisotropic stresses at first order in cosmological perturbation theory, and/or to an inflationary phase
- We have gone as far as we could in all generality; to continue solving the equation one needs to specify more the characteristics of the sourcing process
- However, before analysing examples of GW sourcing processes in the early universe, we proceed with presenting some general features of signals from the early universe, and with describing present and future GW detectors with a particular focus on PTAs

Why sources in the early universe produce SGWBs?

A stochastic GW background is a signal for which *only the statistical properties can be accessed* because it is given by the incoherent superposition of sources that cannot be individually resolved

LIGO website



- For example, the superposition of deterministic GW signals from astrophysical binary sources with too low signal-to-noise ratio, or too much overlap in time and frequency -> confusion noise (Examples: LVK, LISA, PTAs…)

- Early universe GW sources produce SGWBs because they are homogeneously and isotropically distributed over the entire universe, and/or correlated on scales much smaller than the detector resolution

Why sources in the early universe produce SGWBs?

A GW source acting at time t_* in the early universe cannot produce a signal correlated on length/time scales larger than the causal horizon at that time

$$\ell_* \leq H_*^{-1}$$

ℓ_* characteristic length-scale of the source
(typical size of variation of the tensor anisotropic stresses)

Why sources in the early universe produce SGWBs?

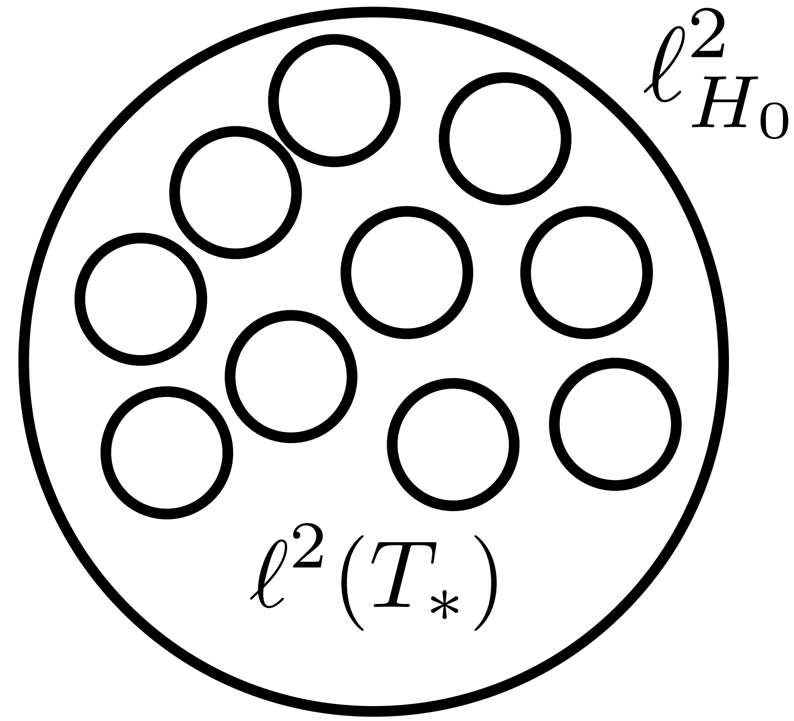
A GW source acting at time t_* in the early universe cannot produce a signal correlated on length/time scales larger than the causal horizon at that time

$$\ell_* \leq H_*^{-1}$$

Angular size on the sky today of a region in which the SGWB signal is correlated

$$\Theta_* = \frac{\ell_*}{d_A(z_*)}$$

Angular diameter distance



Number of uncorrelated regions accessible today $\sim \Theta_*^{-2}$

Suppose a GW detector angular resolution of 10 deg $\longrightarrow$ $z_* \lesssim 17$

$$\Theta(z_* = 1090) \simeq 0.9\,\text{deg} \qquad \Theta(T_* = 100\,\text{GeV}) \simeq 10^{-12}\text{deg}$$

Only the statistical properties of the signal can be accessed

Why sources in the early universe produce SGWBs?

- We access today the GW signal from many independent horizon volumes: $h_{ij}(\mathbf{x},t)$ must be treated as a random variable, only its statistical properties can be accessed, e.g. its correlator $\langle h_r(\mathbf{x}, \eta_1)\, h_s(\mathbf{y}, \eta_2)\rangle$

  where <…> is an ensemble average

- The universe is homogeneous and isotropic, so the GW source is operating everywhere at the same time with the same average properties ("a-causal" initial conditions from Inflation)

- Under the ergodic hypothesis, the ensemble average can be substituted with volume / time averages: we identify this average with the volume / time one necessary to define the GW energy momentum tensor

- Notable exception: *SGWB from Inflation* (intrinsic quantum fluctuations that become classical (stochastic) outside the horizon)

Characterisation of a primordial SGWB

The SGWB is in general homogenous and isotropic, unpolarised and Gaussian

As the FLRW space-time

$$\langle h_{ij}(\mathbf{x},\eta_1)\, h_{lm}(\mathbf{y},\eta_2)\rangle = F_{ijlm}(|\mathbf{x}-\mathbf{y}|,\eta_1,\eta_2)$$

Certainly some *induced anisotropy*, e.g. the dipole with respect to the cosmological frame

More challenging to detect than the "monopole"

If the sourcing process preserves parity

$$\langle h_{+2}(\mathbf{k},\eta)h_{+2}(\mathbf{k},\eta) - h_{-2}(\mathbf{k},\eta)h_{-2}(\mathbf{k},\eta)\rangle = \langle h_{+}(\mathbf{k},\eta)h_{\times}(\mathbf{k},\eta)\rangle = 0$$

Helicity basis $e_{ij}^{\pm 2} = \frac{e_{ij}^{+} \pm i\, e_{ij}^{\times}}{2}$

There are exceptions!

Central limit theorem: the signal comes from the superposition of many independent regions

Characterisation of a primordial SGWB

Power spectrum of the GW amplitude $h_c(k,t)$

$$\langle h_r(\mathbf{k},\eta)\, h_p^*(\mathbf{q},\eta)\rangle = \frac{8\pi^5}{k^3}\,\delta^{(3)}(\mathbf{k}-\mathbf{q})\,\delta_{rp}\,h_c^2(k,\eta)$$

Gaussianity: the two-point correlation function is enough to fully describe the SGWB

Statistical homogeneity and isotropy ($\delta^{(3)}(\mathbf{k}-\mathbf{q})$)

Unpolarised (δ_{rp})

$$\langle h_{ij}(\mathbf{x},\eta)\, h_{ij}(\mathbf{x},\eta)\rangle = 2\int_0^{+\infty}\frac{dk}{k}\,h_c^2(k,\eta)$$

Related to the variance of the GW amplitude in real space

For *freely propagating sub-Hubble modes*, and taking the time-average:

$$h_r(\mathbf{k},\eta) = \frac{A_r(\mathbf{k})}{a(\eta)}\,e^{ik\eta} + \frac{B_r(\mathbf{k})}{a(\eta)}\,e^{-ik\eta}$$

$$\langle h_r(\mathbf{k},\eta)\, h_p^*(\mathbf{q},\eta)\rangle = \frac{1}{a^2(\eta)}\left[\langle A_r(\mathbf{k})\,A_p^*(\mathbf{q})\rangle + \langle B_r(\mathbf{k})\,B_p^*(\mathbf{q})\rangle\right] \qquad h_c(k,\eta)\propto\frac{1}{a^2(\eta)}$$

Characterisation of a primordial SGWB

Power spectrum of the GW energy density $\frac{d\rho_{\rm GW}}{d\log k}$

$$\rho_{\rm GW} = \frac{\langle \dot{h}_{ij}(\mathbf{x},t)\,\dot{h}_{ij}(\mathbf{x},t)\rangle}{32\pi G} = \frac{\langle h'_{ij}(\mathbf{x},\eta)\,h'_{ij}(\mathbf{x},\eta)\rangle}{32\pi G\,a^2(\eta)} = \int_0^{+\infty}\frac{dk}{k}\,\frac{d\rho_{\rm GW}}{d\log k}$$

$$\langle h'_r(\mathbf{k},\eta)\,h'^*_p(\mathbf{q},\eta)\rangle = \frac{8\pi^5}{k^3}\,\delta^{(3)}(\mathbf{k}-\mathbf{q})\,\delta_{rp}\,h'^2_c(k,\eta)$$

For *freely propagating sub-Hubble modes*, and taking the time-average:

$$h'^2_c(k,\eta) \simeq k^2\,h^2_c(k,\eta) \qquad\qquad \frac{d\rho_{\rm GW}}{d\log k} = \frac{k^2\,h^2_c(k,\eta)}{16\pi G\,a^2(\eta)}$$

$$\rho_{\rm GW} \propto \frac{1}{a(\eta)^4}$$

GW energy density scales like radiation for freely propagating sub-Hubble modes (free massless particles)

Characterisation of a primordial SGWB

GW energy density parameter

Evaluated today, for a source that operated at time η_*

$$h^2\,\Omega_{\rm GW}(k,\eta_0)=\frac{h^2\rho_*}{\rho_c}\left(\frac{a_*}{a_0}\right)^4\left(\frac{1}{\rho_*}\frac{d\rho_{\rm GW}}{d\log k}(k,\eta_*)\right)$$

To make connection with the detection process one assumes that

- The source has stopped operating so the waves are freely propagating
- The expansion of the universe is negligible over the time of the measurement so that the SGWB appears stationary
- One can F.T. in time as well $f=\frac{1}{2\pi}\frac{k}{a_0}$

$$\langle\bar{h}_r(f,\hat{\mathbf{k}})\bar{h}^*_p(g,\hat{\mathbf{q}})\rangle=a_0^4\,f^2\,g^2\langle A_r(\mathbf{k})\,A^*_p(\mathbf{q})\rangle=$$
$$=\frac{1}{8\pi}\,\delta(f-g)\,\delta^{(2)}(\hat{\mathbf{k}}-\hat{\mathbf{q}})\,\delta_{rp}\,S_h(f)$$

Power spectral density

$$\Omega_{\rm GW}(f)=\frac{4\pi^2}{3H_0^2}\,f^3\,S_h(f)$$

Characterisation of a primordial SGWB

characteristic frequency of the SGWB signal

$$f_* = \frac{1}{\ell_*} \geq H_*$$

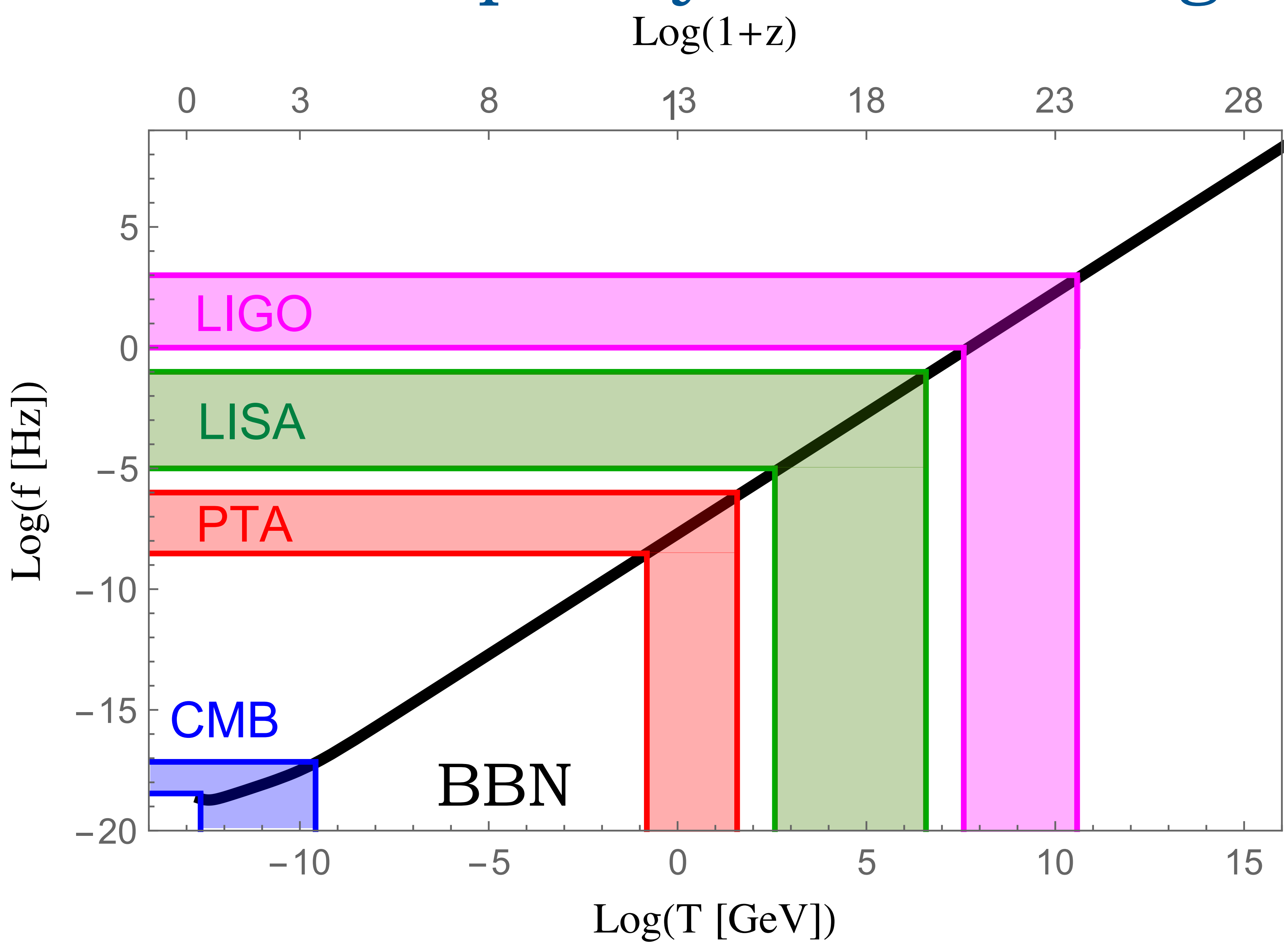
$$\epsilon_* = \ell_* H_*$$

Ratio of the typical length-scale of the GW sourcing process (size of the anisotropic stresses) and the Hubble scale at the generation time

$$f = f_* \frac{a_*}{a_0} = \frac{1.65 \times 10^{-7}}{\epsilon_*} \left(\frac{g(T_*)}{100} \right)^{1/6} \frac{T_*}{\mathrm{GeV}} \, \mathrm{Hz}$$

Characteristic frequency of the GW signal

$\epsilon_* = 1$



$T_{\rm QCD} \sim 100\,{\rm MeV}$ $\quad \ell_* H_* \sim 1$ $\longrightarrow$ $f \sim 10\,{\rm nHz}$ **PTA**

$T_{\rm EW} \sim 100\,{\rm GeV}$ $\quad \ell_* H_* \sim 0.01$ $\longrightarrow$ $f \sim {\rm mHz}$ **LISA**

$T_{\rm PQ} \sim 10^9\,{\rm GeV}$ $\quad \ell_* H_* \sim 1$ $\longrightarrow$ $f \sim 100\,{\rm Hz}$ **LVK**

Discovery potential of primordial SGWB detection