The strong CP problem in the quantum rotor

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Based on:

- David Albandea, Guilherme Telo, Alberto Ramos. *"The Strong CP Problem in the Quantum Rotor"*. arXiv:2402.17518

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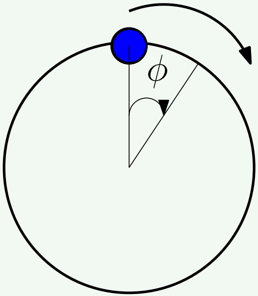
GENERALITAT VALENCIANA

The rotor

Classical

- Particle moving freely in the unit circle

$$H = \frac{1}{2I}\, p_\phi^2$$



Quantum

- Hamiltonian operator

$$\hat{H} = \frac{1}{2I}\, \hat{p}_\phi^2$$

- Wave function $\psi(\phi)$ in Hilbert space $L^2[0, 2\pi]$
- Momentum operator $\hat{p}_\phi = -\imath\partial_\phi$ unbounded
- **Self-adjoint** extensions parametrized by θ
- Domain $D(p_\phi)$

$$\psi(2\pi) = e^{\imath\theta}\psi(0)\,.$$

- Eigenfunctions / Eigenvalues

$$\psi_n(\phi) = e^{\imath\left(n+\frac{\theta}{2\pi}\right)\phi};\, \lambda_n = \left(n + \frac{\theta}{2\pi}\right)$$

The many Quantum Rotors

- **Many** possible quantum models from a **single** classical one
- Parametrized by an angle θ
- Energy levels depend on θ

$$E_n = \frac{1}{2I}\left(n + \frac{\theta}{2\pi}\right)^2$$

- Path integral representation at finite temperature.

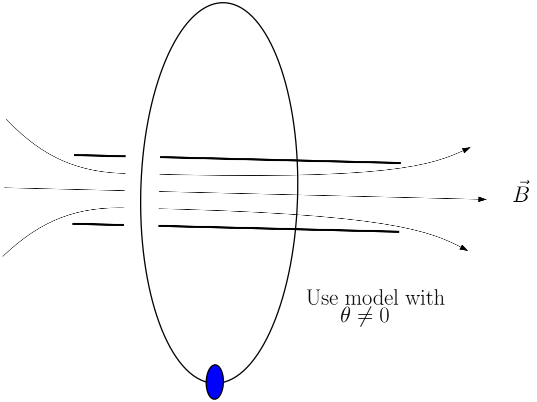
$$\mathcal{Z}(\theta) = \int \mathcal{D}\phi(t)\, e^{-S_E[\phi(t)]+\imath\theta Q}$$

Theory in the Euclidean $t \in [0, T]$ with $\beta \equiv 1/T$ and Q the **topological charge**

$$Q = \frac{1}{2\pi}\int_0^T \dot{\phi}(t)\mathrm{d}t \in \mathbb{Z}$$

Which is the "correct" value of θ?

- Question makes no sense! Depends on what you want to model



The Lattice model

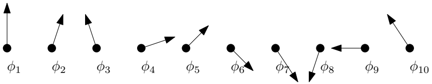
Continuum

- Euclidean action

$$S_E[\phi(t)] = \frac{I}{2}\int_0^T \mathrm{d}t\,(\dot\phi)^2$$

- Topological charge

$$Q = \frac{1}{2\pi}\int_0^T \mathrm{d}t\,\dot\phi$$



$Q = 1$

Different discretizations, same continuum limit

$$1/\hat{I} = a/I \to 0$$

Lattice (discretized with spacing a)

- Euclidean action ($\hat{I} = I/a$); $\phi_t = \phi(at)$)

$$S_{\mathrm{CP}}(\phi) = \frac{\hat{I}}{2}\sum_{t=0}^{T/a-1}((\phi_{t+1}-\phi_t) \mod 2\pi)^2$$

$$S_{\mathrm{ST}}(\phi) = \frac{\hat{I}}{2}\sum_{t=0}^{T/a-1}[1-\cos(\phi_{t+1}-\phi_t)]$$

- Different boundary conditions

$$S_{\mathrm{ST,PBC}}(\phi) \implies \phi(T) = \phi(0)$$

$$S_{\mathrm{ST,OBC}}(\phi) \implies \phi(T) = \phi(T-a)$$

$$S_{\mathrm{ST},\infty}(\phi) = \frac{\hat{I}}{2}\sum_{t=-\infty}^{\infty}[1-\cos(\phi_{t+1}-\phi_t)]$$

- Q quantized **only** with PBC

THE SUSCEPTIBILITY AND THE ORDER OF LIMITS

Task in life

$$\hat{I} \sum_{t=-\infty}^{\infty} \langle q_t q_0 \rangle_\infty = ?? \qquad (q_t \text{ is the Topological charge density, i.e.: } q_t = \sin(\phi_{t+1} - \phi_t))$$

- Finite volume with PBC. "Volume" T (Q quantized)

$$\hat{I} \sum_{t=0}^{T/a-1} \langle q_t q_0 \rangle_T = \frac{\hat{I}}{T/a} \sum_{t,t'=0}^{T/a-1} \langle q_t q_{t'} \rangle_T = \frac{I/a}{T/a} \left\langle \left(\sum_{t=0}^{T/a-1} q_t \right) \left(\sum_{t'=0}^{T/a-1} q_{t'} \right) \right\rangle_T = I \frac{\langle Q^2 \rangle_T}{T}$$

"Usual" order of limits

$$\lim_{T\to\infty} I \frac{\langle Q^2 \rangle_T}{T} = \lim_{T\to\infty} \left(\sum_{Q=-\infty}^{\infty} p_Q(Q) I \frac{Q^2}{T} \right) = \frac{1}{4\pi^2} (1 + \dots)$$

Sum after $T \to \infty$

$$\lim_{N\to\infty} \lim_{T\to\infty} \left(\sum_{Q=-N}^{N} p_Q(Q) I \frac{Q^2}{T} \right) = 0$$

The susceptibility and the order of limits

Task in life

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The approach to infinite volume for local correlators

Local correlators ($t_1, t_2 \ll L$) show universal behavior

- With PBC, based on thermal partition function

$$\langle O(t_1)O(t_2)\rangle_T = \frac{\mathrm{Tr}\,\hat{O}(t_1)\hat{O}(t_2)e^{-TH}}{\mathrm{Tr}\,e^{-TH}} = \frac{\sum_n \langle n|\hat{O}(t_1)\hat{O}(t_2)e^{-TE_n}|n\rangle}{\sum_n \langle n|e^{-TE_n}|n\rangle} \xrightarrow[T\to\infty]{} \overbrace{\langle 0|\hat{O}(t_1)\hat{O}(t_2)|0\rangle}^{\langle O(t_1)O(t_2)\rangle_\infty} + \mathcal{O}\left(e^{-T(E_1-E_0)}\right)$$

- With PBC at fixed topological sector with $\mathcal{Z}(\theta) = \sum_n e^{-TE_n(\theta)}$ [Brower et al. Phys.Lett.B 560 (2003)]

$$\langle O(t_1)O(t_2)\rangle_Q \propto \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\alpha\, e^{\imath\alpha Q}\,\mathcal{Z}(\theta)\langle O(t_1)O(t_2)\rangle_{T,\theta} \xrightarrow[T\to\infty]{} \langle O(t_1)O(t_2)\rangle_\infty + \mathcal{O}\left(\frac{1}{T}\right)$$

NOTE: no Hamiltonian, no spectral decomposition, breaking of locality!

Boundary conditions/quantization of Q **irrelevant in local correlators**

$$\lim_{T\to\infty}\langle O(t_1)O(t_2)\rangle_T = \lim_{T\to\infty}\langle O(t_1)O(t_2)\rangle_Q = \langle O(t_1)O(t_2)\rangle_\infty$$

The approach to infnite volume for local correlators

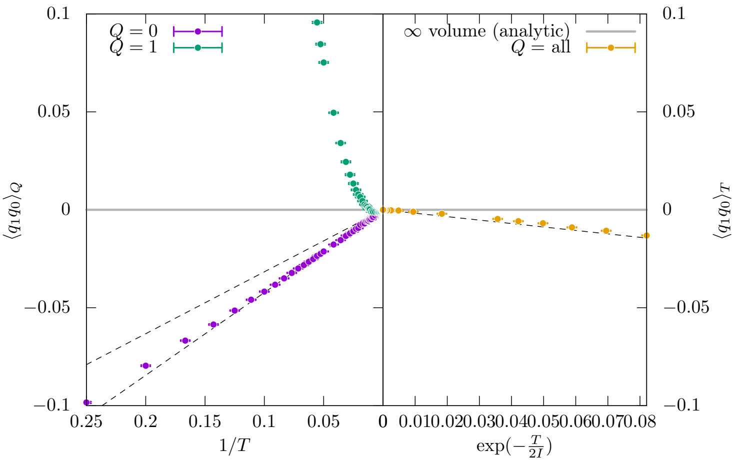
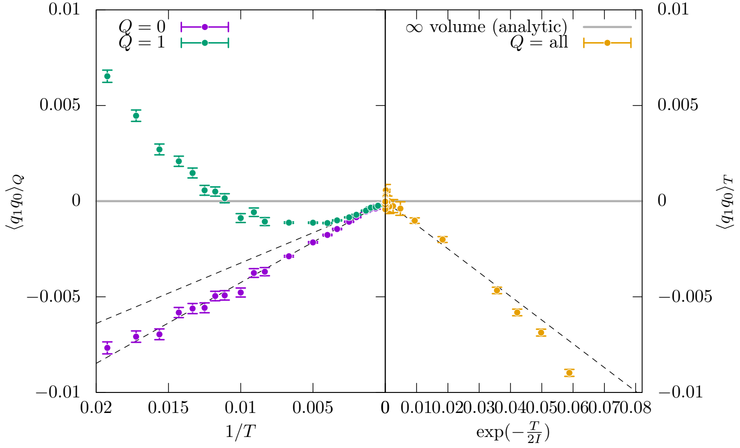


Figure: Approach to infinite volume for $\langle q_1 q_0 \rangle$ and $T/a = 4, \ldots, 2000$

The approach to infnite volume for local correlators



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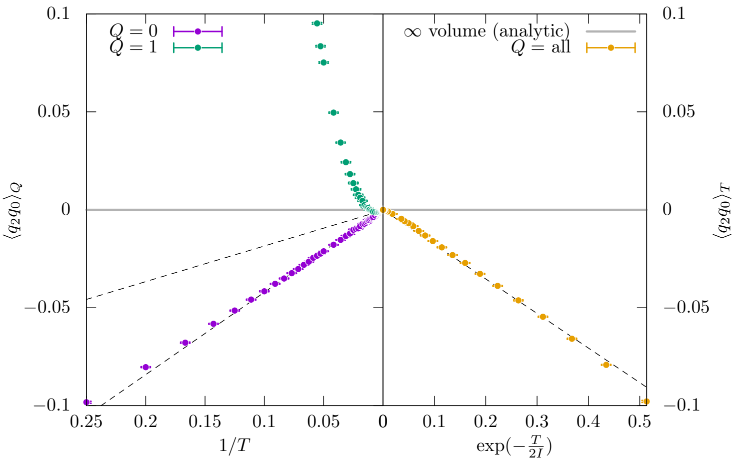

Figure: Approach to infinite volume for $\langle q_2 q_0 \rangle$ and $T/a = 4, \ldots, 2000$

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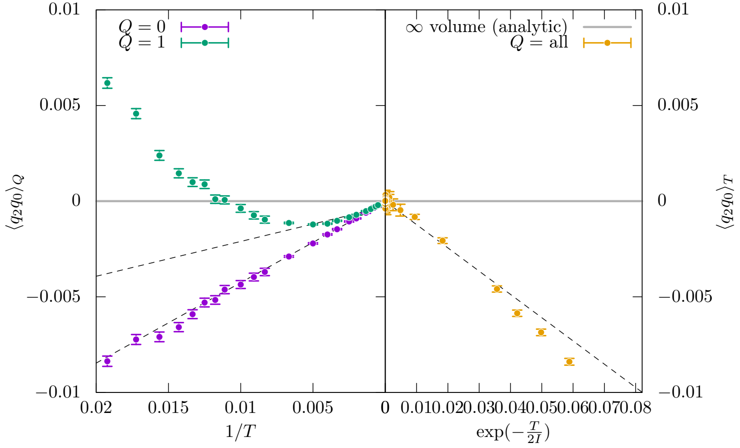

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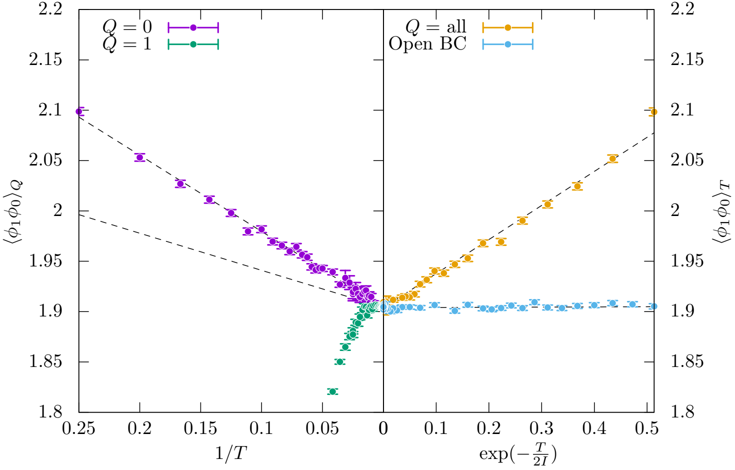

Figure: Approach to infinite volume for $\langle \phi_1 \phi_0 \rangle$ and $T/a = 4, \ldots, 2000$

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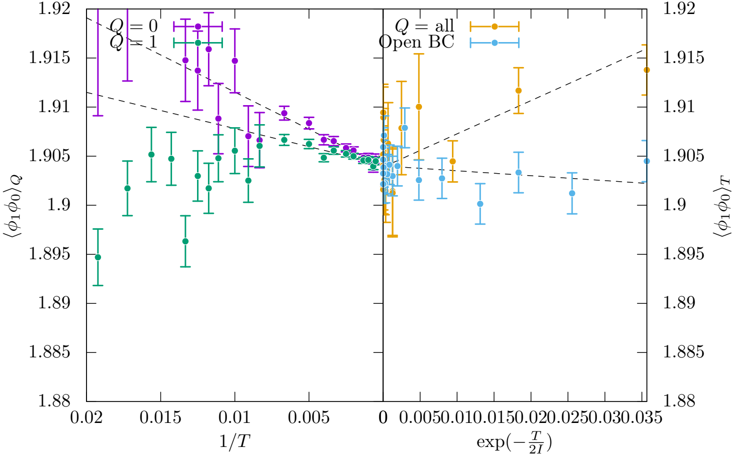

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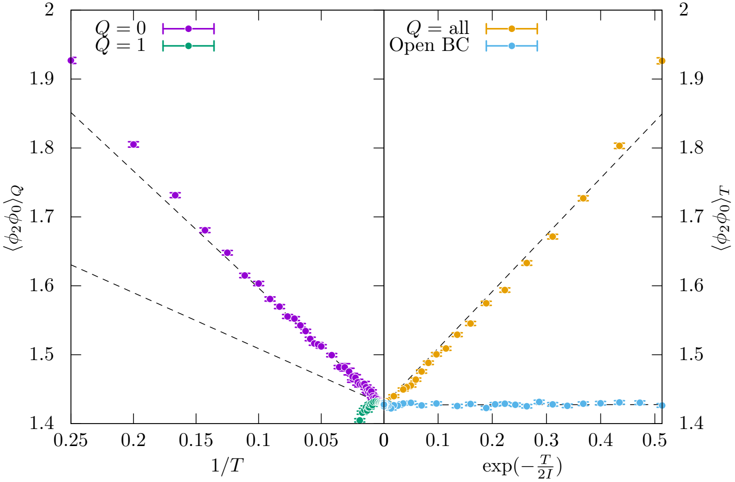

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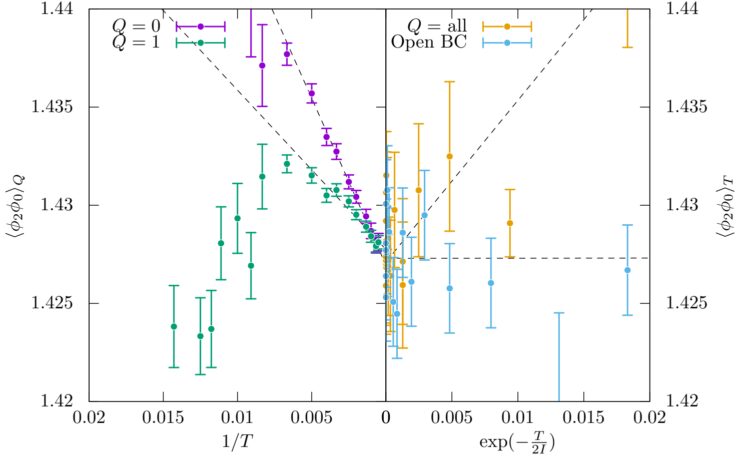


Figure: Approach to infinite volume for $\langle\phi_2\phi_0\rangle$ and $T/a = 4, \ldots, 2000$

What about non-local quantities?

$$\langle O(t)O(0)\rangle_T \xrightarrow[T\to\infty]{} \langle O(t)O(0)\rangle_\infty + \mathcal{O}\left(e^{-T(E_1-E_0)}\right)$$

$$\langle O(t)O(0)\rangle_Q \xrightarrow[T\to\infty]{} \langle O(t)O(0)\rangle_\infty + \mathcal{O}\left(\frac{1}{T}\right)$$

Since

$$\langle O(t)O(0)\rangle_\infty \xrightarrow[t\to\infty]{} e^{-t(E_1-E_0)} + \mathcal{O}(e^{-t(E_2-E_0)})$$

We have

$$\sum_{t=0}^{T}\langle O(t)O(0)\rangle_T \xrightarrow[T\to\infty]{} \sum_{t=-\infty}^{\infty}\langle O(t)O(0)\rangle_\infty + \mathcal{O}\left(Te^{-\#T}\right)$$

$$\sum_{t=0}^{T}\langle O(t)O(0)\rangle_Q \xrightarrow[T\to\infty]{} \langle O(t)O(0)\rangle_\infty + \mathcal{O}\left(\frac{T}{T}\right)$$

Main conclusion:

$\sum_{t=0}^{T}\langle O(t)O(0)\rangle_Q$ **DOES NOT NEED TO APPROXIMATE** $\sum_{t=-\infty}^{\infty}\langle O(t)O(0)\rangle_\infty$

What about non-local quantities?

$$\sum_{t=0}^{T}\langle O(t)O(0)\rangle_T \xrightarrow[t\to\infty]{} \sum_{t=-\infty}^{\infty}\langle O(t)O(0)\rangle_\infty + \mathcal{O}\left(Te^{-\#T}\right)$$

Correct order of limits

$$\sum_{t=-\infty}^{\infty}\langle O(t)O(0)\rangle_\infty = \lim_{T\to\infty}\left(\sum_{t=0}^{T}\langle O(t)O(0)\rangle_T\right) = \lim_{T\to\infty}\left(\sum_{Q=-\infty}^{\infty} p(Q,T)\sum_{t=0}^{T}\langle O(t)O(0)\rangle_Q\right)$$

Solution to the puzzle

- With correct order of limits infinite volume reproduced

$$\sum_{t=-\infty}^{\infty}\langle q_t q_0\rangle_\infty = \lim_{T\to\infty}\left(\sum_{Q=-\infty}^{\infty} p(Q,T)\sum_{t=0}^{T}\langle q_t q_0\rangle_Q\right) = \frac{1}{4\pi^2\hat{I}} \neq 0$$

THE SPECTRUM OF THE HAMILTONIAN

$$\hat{p} = -\imath\partial_\phi; \quad \psi_n(\phi) = e^{\imath(n+\frac{\theta}{2\pi})\phi}; \quad \lambda_n = \left(n + \frac{\theta}{2\pi}\right) \Longleftrightarrow \mathcal{Z} = \int \left(\prod_k \mathrm{d}\phi_t\right) e^{S(\phi_t)+\imath\theta Q}$$

- Expansion coefficients of $\langle\phi_{t+1}|e^{-a\hat{H}}|\phi_t\rangle$ in basis $\psi_n(\phi_{t+1}-\phi_t)$

$$\langle\phi_{t+1}|e^{-a\hat{H}}|\phi_t\rangle = \sum_n \langle\phi_{t+1}|n\rangle e^{-aE_n(\theta)}\langle n|\phi_t\rangle = \sum_n e^{-aE_n(\theta)}\psi_n(\phi_{t+1}-\phi_t)$$

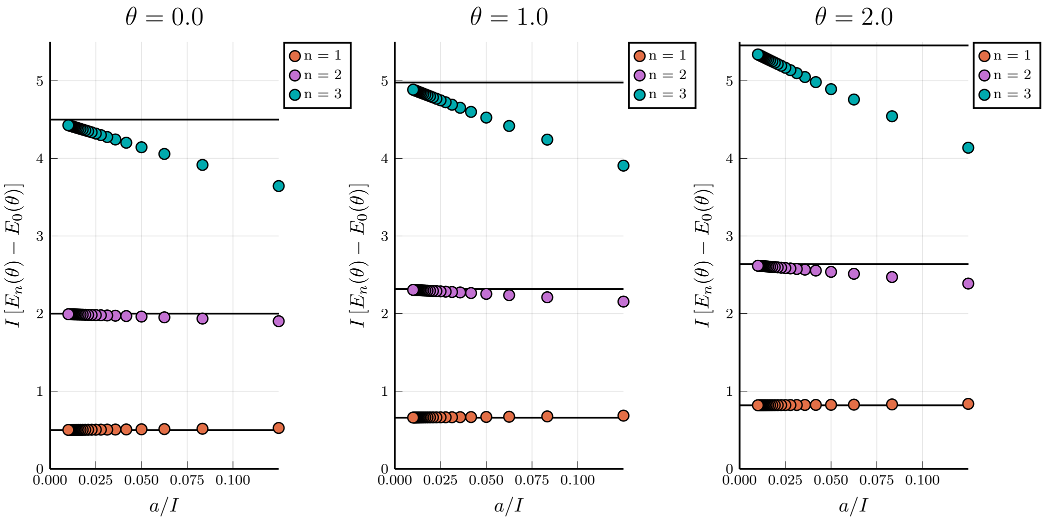
- Usual transfer matrix formalism

$$\langle\phi_{t+1}|e^{-a\hat{H}}|\phi_t\rangle = \sqrt{\frac{\hat{I}}{2\pi}}\exp\left\{-\hat{I}(1-\cos(\phi_{t+1}-\phi_t)) + \imath\frac{\theta}{2\pi}(\phi_{t+1}-\phi_t \mod 2\pi)\right\}$$

- Exact formula for the Hamiltonian spectrum on the lattice

$$e^{-aE_n(\theta)} = \sqrt{\frac{\hat{I}}{2\pi}}\int_{-\pi}^{\pi}\mathrm{d}x\, e^{-\hat{I}(1-\cos x)+\imath\frac{x\theta}{2\pi}}\, e^{-\imath nx}$$

The spectrum of the Hamiltonian



What about the strong CP problem? θ dependence on observables

What effect has T on spectrum of the Hamiltonian?

- **NONE**
- H is the same for any choice of
  - T (i.e. Euclidean Volume in QM)
  - Boundary conditions in time (H is defined at each time slice)

What effect has Q quantization on the spectrum of the Hamiltonian?

- **NONE**
- Q quantized only for some choices of boundary conditions in time
- Spectrum does not depend on boundary conditions in time

Can we determine the spectrum from simulations at fixed Q?

- No Hamiltonian! Breaking of locality!
- Lot of care is required!. θ cancels in expectation values at fixed topology

Extracting the spectrum in practice

Numerical extraction of energy levels

- Use spectral decomposition **and** large Euclidean times

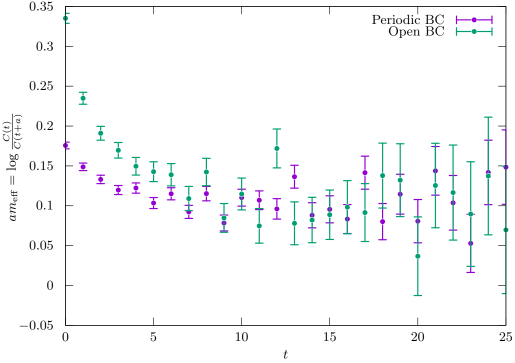
$$C_\infty(t) = \langle \phi_t \phi_0 \rangle_\infty \xrightarrow[t\to\infty]{} e^{-(E_1-E_0)t} + \mathcal{O}(e^{-t(E_2-E_0)})$$

$$C_O(t) = \langle \phi_t \phi_{T/2} \rangle_T \xrightarrow[t\to\infty]{(t\ll T/2)} e^{-(E_1-E_0)t} + \mathcal{O}(e^{-t(E_2-E_0)})$$

$$C_T(t) = \langle \phi_t \phi_0 \rangle_T \xrightarrow[t\to\infty]{(t\ll T)} e^{-(E_1-E_0)t} + \mathcal{O}(e^{-t(E_2-E_0)})$$

- For large Euclidean times (and $t \ll T$), "effective mass"

$$am_{\text{eff}} = a(E_1 - E_0) = \lim_{t\to\infty} \log \frac{C(t)}{C(t+a)}$$



Extracting the spectrum in practice

Energy levels at $\theta \neq 0$

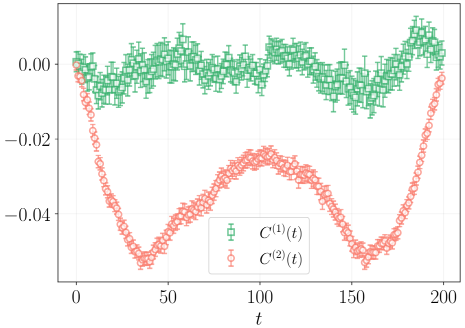
- **Sign problem:** No direct simulations at $\theta \neq 0$
- Expand for $\theta \ll 1$

$$
\begin{aligned}
C(t,\theta) &= C^{(0)}(t) + C^{(1)}(t)\theta + C^{(2)}(t)\theta^2 + \dots \\
E_n(\theta) &= E_n^{(0)} + E_n^{(1)}\theta + \dots
\end{aligned}
$$

- Example with PBC ($\Delta E_n^{(k)} = E_n^{(k)} - E_0^{(k)}$)

$$
\frac{C^{(2)}(t)}{C^{(0)}(t)} \xrightarrow[t \gg 1]{} \frac{(\Delta E_1^{(1)})^2}{2} \left[t^2 + \frac{T^2 - 2tT}{1 + e^{-\Delta E_1^{(0)}(T-2t)}} \right]
$$

- Extraction of the linear dependence $\Delta E_1^{(1)}$



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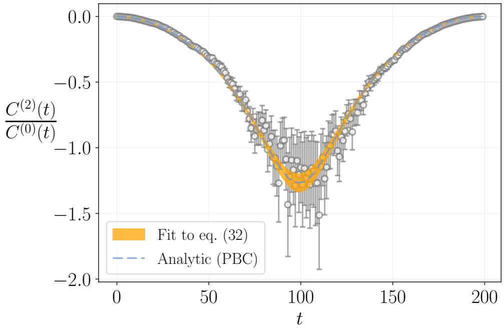

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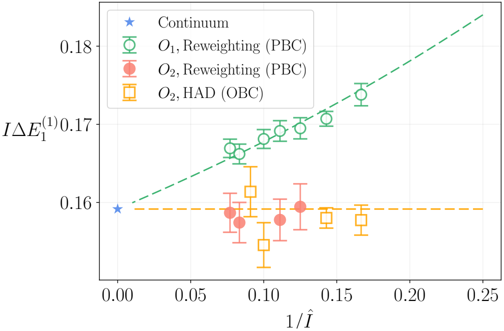

Figure: $O_1(t) = \phi_t$; $O_2(t) = \sin \phi_t$

Conclusions

- Approach of **local correlators** to the infinite volume in path integral formulation of QM

$$\langle O(t)O(0)\rangle_{T,BC} \xrightarrow[T\to\infty]{} \langle O(t)O(0)\rangle_{\infty} + \mathcal{O}\left(e^{-T(E_1-E_0)}\right)$$

$$\langle O(t)O(0)\rangle_{Q,\theta=0} \xrightarrow[T\to\infty]{} \langle O(t)O(0)\rangle_{\infty,\theta=0} + \mathcal{O}\left(\frac{1}{T}\right)$$

- Non-local correlators/correlators at $\theta \neq 0$ require special care at fixed Q

$$I\chi_t = \hat{I}\sum_{t=-\infty}^{\infty}\langle q_t q_0\rangle_{\infty} = \lim_{T\to\infty}\left(\hat{I}\sum_{t=0}^{T}\langle q_t q_0\rangle_T\right) = \frac{1}{4\pi^2} \neq 0 \quad \text{but} \quad I\sum_{t=0}^{T}\langle q_t q_0\rangle_{Q=0} = 0$$

- Spectrum of Hamiltonian insensitive to Euclidean time boundary conditions/topology quantization
- Detailed numerical study in the Quantum Rotor. Expectations tested with high accuracy!
  - Topology freezing
  - Sign problem
- Model for strong CP problem:
  - Different quantum models from a single classical one
  - θ dependence present in various correlators
  - Extraction techniques (sign problem, topology freezing)

Volume dependence in QCD

Main relation expected to hold in QCD

- Approach of **local correlators** to the infinite volume on L^4 lattice

$$\langle O(x)O(y)\rangle_{L,BC} \xrightarrow[L\to\infty]{} \langle O(x)O(y)\rangle_{\infty} + \mathcal{O}\left(e^{-Lm}\right)$$

$$\langle O(x)O(y)\rangle_{Q,\theta=0} \xrightarrow[L\to\infty]{} \langle O(x)O(y)\rangle_{\infty,\theta=0} + \mathcal{O}\left(\frac{1}{V}\right)$$

- Correct order of limits deduced from this behavior

$$\langle O(x)O(y)\rangle_{\infty} = \lim_{L\to\infty} \langle O(x)O(y)\rangle_{L,BC} = \lim_{L\to\infty} \sum_{Q=-\infty}^{\infty} p(Q,L)\langle O(x)O(y)\rangle_{Q}$$

- At $\theta = 0$ one can also use

$$\langle O(x)O(y)\rangle_{\infty,\theta=0} = \lim_{L\to\infty} \langle O(x)O(y)\rangle_{Q,\theta=0}$$

- Special care needed at $\theta \neq 0$ and for Global correlators

Volume dependence in QCD

Spectral quantities in QCD

- Hamiltonian does not depend on Euclidean time boundary conditions
- Spectral quantities are the same for any choice of
  - T (i.e. Euclidean time length)
  - Boundary conditions in Euclidean time
- Masses of stable particles have **exponential** finite volume effects **[Lüscher '86]**

Spectral quantities do not depend on Q quantization

- Choose boundary conditions in Euclidean time where Q is not quantized
- Useful to avoid topology freezing **[Lüscher, Schaefer, '11; CLS collaboration]**

Direct calculation in $T = \infty$

- Action in infinite volume

$$S_{\mathrm{ST},\infty}(\phi) = \frac{\hat{I}}{2} \sum_{t=-\infty}^{\infty} [1 - \cos(\phi_{t+1} - \phi_t)]$$

- Define $q_t = \sin(\phi_{t+1} - \phi_t)$
- Infinite volume direct calculation. All integrals decouple $v_k = \phi_{k+1} - \phi_k$

$$\hat{I}\langle q_t q_0 \rangle_\infty = \frac{1}{\mathcal{Z}} \int \left(\prod_t \mathrm{d}v_k \right) \sin(v_t) \sin(v_0) e^{-\frac{\hat{I}}{2} \sum_{t=-\infty}^{\infty} [1-\cos(v_k))]} = \frac{1}{4\pi^2} \delta_{t,0} \left(1 + \ldots\right) \xrightarrow[a \to 0]{} \frac{1}{4\pi^2} \delta_{t,0}$$

- Finally

$$I\chi_t = \sum_{t=-\infty}^{\infty} \hat{I}\langle q_t q_0 \rangle_\infty = \sum_{t=-\infty}^{\infty} \frac{1}{4\pi^2} \delta_{t,0} = \frac{1}{4\pi^2} \neq 0$$

Hilbert space $\mathcal{H} = L^2[0, 2\pi]$ and self-adjoint operators

- Defined as functions $f : [0, 2\pi] \to \mathbb{C}$ s.t.

$$\forall f, g \in \mathcal{H} \Longrightarrow \langle f, g \rangle = \int_0^{2\pi} \overline{f(x)} g(x) \mathrm{d}x < \infty$$

- This space is very big!
  - No diferentiability condition
  - No continuity condition
- i.e. Choose $f : [0, 2\pi] \to \mathbb{R}$

$$f(x) = \sum_{n \neq 0} \frac{\imath}{2n\pi} e^{\imath n x} \Longrightarrow f(x) = \frac{x}{2\pi} - \frac{1}{2} \qquad (1)$$

- The Hilbert space of "differentiable functions" **does not exist** ($\mathcal{H}$ is complete $\Longrightarrow f \in \mathcal{H}$)
- Instead operators $\hat{T}$ only well defined on a subset $D(\hat{T}) \subset \mathcal{H}$

- Operator = "formula" + Domain
- Physical observables $\Longrightarrow$ self-adjoint operators
- Momentum operator $\hat{p} = \imath\partial$ is **unbounded**
- What domain $D(\hat{p})$ makes $\hat{p}$ self-adjoint?

An example gone wrong

- Imagine that I define

$$D(\hat{p}) = \{f \text{ Diff. } \in \mathcal{H} \,|\, f(0) = f(2\pi) = 0\}$$

- $\hat{p}$ is symmetric

$$\int_0^{2\pi} \overline{f(x)} [\imath\partial_x g(x)] \mathrm{d}x = \int_0^{2\pi} \overline{[\imath\partial_x f(x)]} g(x) \mathrm{d}x$$

- $\hat{p}$ is <u>**NOT**</u> self-adjoint (imaginary Eigenvalues!)

$$D(\hat{p}^\star) = \{f \text{ Diff. } \in \mathcal{H}\} \Longrightarrow \hat{p}^\star e^x = -\imath e^x$$

Hilbert space $\mathcal{H} = L^2[0, 2\pi]$ and self-adjoint operators

- Defined as functions $f : [0, 2\pi] \to \mathbb{C}$ s.t.

$$\forall f, g \in \mathcal{H} \Longrightarrow \langle f, g \rangle = \int_0^{2\pi} \overline{f(x)} g(x) \mathrm{d}x < \infty$$

- This space is very big!
  - No diferentiability condition
  - No continuity condition
- i.e. Choose $f : [0, 2\pi] \to \mathbb{R}$

$$f(x) = \sum_{n \neq 0} \frac{\imath}{2n\pi} e^{\imath n x} \Longrightarrow f(x) = \frac{x}{2\pi} - \frac{1}{2} \qquad (1)$$

- The Hilbert space of "differentiable functions" **does not exist** ($\mathcal{H}$ is complete $\Longrightarrow f \in \mathcal{H}$)
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An good example

- Imagine that I define

$$D(\hat{p}) = \{f \text{ Diff. } \in \mathcal{H} \,|\, f(0) = f(2\pi)\}$$

- $\hat{p}$ is symmetric

$$\int_0^{2\pi} \overline{f(x)} [\imath \partial_x g(x)] \mathrm{d}x = \int_0^{2\pi} \overline{[\imath \partial_x f(x)]} g(x) \mathrm{d}x$$

- $\hat{p}$ **<u>IS</u>** self-adjoint

$$D(\hat{p}^\star) = D(\hat{p})$$

$U(1)$ gauge theory in 2D

Continuum

- Abelian gauge field A_μ,
- Field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
- Usual action

$$S_E = \frac{1}{4g} \int d^2x \, F_{\mu\nu} F_{\mu\nu}$$

- Q quantized with PBC

$$Q = \frac{1}{2\pi} \int d^2x \, \varepsilon_{\mu\nu} F_{\mu\nu} \in \mathbb{Z}$$

Lattice

- Link variables $U_\mu(x)" = "e^{\imath A_\mu(x)}$ and plaquettes

$$P(x) = U_1(x) U_2(x + a\hat{1}) U_1^\dagger(x + a\hat{2}) U_2^\dagger(x)$$

- Action

$$S = -\frac{\beta}{2} \sum_x P(x) + P^\dagger(x)$$

- Topological charge

$$q(x) = \frac{-\imath}{2\pi} \log P(x)$$

$$Q = \sum_x q(x)$$

The susceptibility and the order of limits

$$\sum_x \langle q(x)q(0)\rangle_\infty = ??$$ Calculation in infinite volume

- Infinite volume direct calculation. All integrals decouple $q(x) = A_1(x) + A_2(x+a\hat{1}) - A_1(x+a\hat{2}) - A_2(x)$

$$\langle q(x)q(0)\rangle_\infty = \frac{1}{\mathcal{Z}}\int \left(\prod_t \mathrm{d}q(y)\right) q(x)q(0)e^{-\frac{\hat{I}}{2}\sum_y[1-\cos(q(y)))]} = \frac{1}{4\pi^2\beta}\delta_{t,0}\,(1+\dots) \xrightarrow[a\to 0]{} \frac{1}{4\pi^2\beta}\delta_{t,0}$$

- Finite volume with PBC. "Volume" $(L/a)^2$ (Q quantized)

$$\sum_x \langle q(x)q(0)\rangle_{L^2} = \left(\frac{a}{L}\right)^2 \sum_{x,x'}\langle q(x)q(x')\rangle_{L^2} = \frac{\langle Q^2\rangle_{L^2}}{L^2}$$

"Usual" order of limits

$$\lim_{L\to\infty}\frac{\langle Q^2\rangle_{L^2}}{L^2} = \lim_{L\to\infty}\left(\sum_{Q=-\infty}^{\infty} p_Q(Q)\frac{Q^2}{L^2}\right) = \frac{1}{4\pi^2\beta}\,(1+\dots)$$

Sum after $T\to\infty$

$$\lim_{N\to\infty}\left(\lim_{L\to\infty}\sum_{Q=-N}^{N} p_Q(Q)\frac{Q^2}{L^2}\right) = 0$$

The susceptibility and the order of limits

$$\chi_t = \sum_x \langle q(x)q(0)\rangle_\infty = ?? \qquad \text{Calculation in infinite volume}$$

- Infinite volume direct calculation. All integrals decouple $q(x) = A_1(x) + A_2(x + a\hat{1}) - A_1(x + a\hat{2}) - A_2(x)$

$$\chi_t = \langle q(x)q(0)\rangle_\infty = \frac{1}{\mathcal{Z}} \int \left(\prod_t \mathrm{d}q(y)\right) q(x)q(0)e^{-\frac{\hat{I}}{2}\sum_y [1-\cos(q(y)))]} = \frac{1}{4\pi^2\beta}\delta_{t,0}\,(1 + \dots) \xrightarrow[a\to 0]{} \frac{1}{4\pi^2\beta}\delta_{t,0}$$

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"Usual" order of limits

$$\lim_{L\to\infty} \frac{\langle Q^2\rangle_{L^2}}{L^2} = \lim_{L\to\infty}\left(\sum_{Q=-\infty}^{\infty} p_Q(Q)\frac{Q^2}{L^2}\right) = \frac{1}{4\pi^2\beta}\,(1 + \dots)$$

Sum after $T \to \infty$

$$\lim_{N\to\infty}\left(\lim_{L\to\infty}\sum_{Q=-N}^{N} p_Q(Q)\frac{Q^2}{L^2}\right) = 0$$

Breaking of locality and approach $L \to \infty$

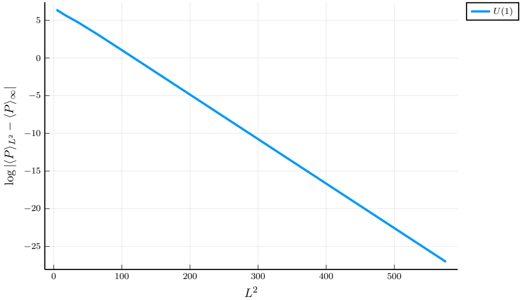

Figure: Exponential approach to infinite volume

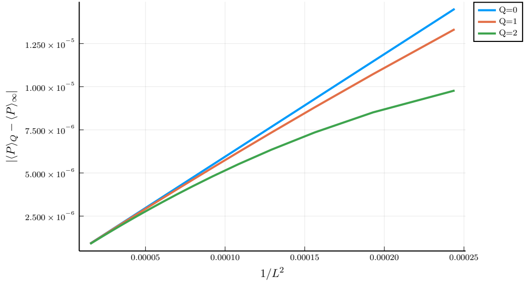

Figure: Power-like approach to infinite volume at fixed Q

Model analitically solved [Kovacs et. al. Nuclear Physics B 454, 45–58 (1995); Bonati, Rossi, Phys. Rev. D 100, 054502 (2019)]

- Infinite volume susceptibility **not** reproduced correctly
- Same approach to infinite volume for local correlators
- Results available for all groups $U(N)$

Master Field approach: Calculations at fixed Q

Global correlators from simulations at fixed Q?

- Master field approach: Only **ONE** configuration (the master field) [Lüscher '17]
- Expectation values computed as **volume averages**

$$\langle O(x)\rangle = \frac{1}{V}\sum_x O(x)$$

- What about global correlators? Define

$$f_Q(R) = \sum_{|t|<R} \langle q_t q_0\rangle_Q$$

- Extrapolate to infinite volume **first**

$$\lim_{T\to\infty} f_Q(R) = f_\infty(R)$$

- **Then**, use spectral decomposition

$$f_\infty(R) = f_\infty(\infty) + \mathcal{O}(e^{-Rm})$$

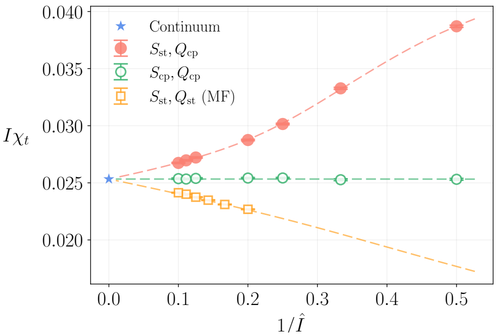

Figure: Values of $Q = 83, 61, 233, 11, 24, 118$