Is there a strong CP problem?

Carlos Tamarit, Johannes Gutenberg-Universität Mainz

Phys.Lett.B 822 (2021) 136616,   2001.07152 [hep-th]

2403.00747 [hep-th]

Universe 2024, 10(5), 189,   2404.16026 [hep-ph]

in collaboration with

Wen-Yuan Ai
King's College London

Björn Garbrecht
TUM

Juan S. Cruz
Formerly CP3 origins

The aim:

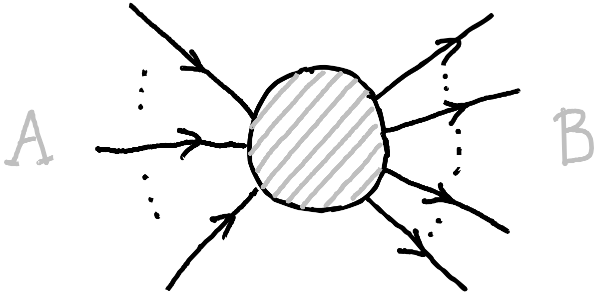
Challenge the conventional view of the strong CP problem by showing that **path integral** computations with a careful **infinite 4d volume** limit as well as calculations in **canonical quantization** imply that **QCD does not violate CP regardless** of the value of the θ **angle**

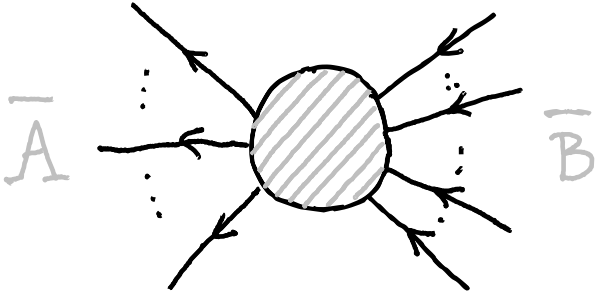
The plan:

1. The strong CP problem in the UV and the IR
2. How to compute vacuum correlators
3. Results in the infinite T method
4. Results with the wave-functional method
5. Conclusions

1. The strong CP problem in the UV and the IR

What does one need for CP violation?





$$|\mathcal{M}_{A\to B}|^2 = |c_0\hat{\mathcal{M}}_0 + c_1\hat{M}_1|^2 \qquad |\mathcal{M}_{\bar{B}\to\bar{A}}|^2 = |c_0^*\hat{\mathcal{M}}_0 + c_1^*\hat{M}_1|^2$$

$$|\mathcal{M}_{A\to B}|^2 - |\mathcal{M}_{\bar{B}\to\bar{A}}|^2 = 4\mathrm{Im}(c_0^*c_1)\mathrm{Im}(\mathcal{M}_0\mathcal{M}_1^*)$$

- Need **interfering contributions** to amplitudes with **misaligned phases**

The UV perspective: QCD Lagrangian

$$S_{\mathrm{QCD}} = \int \mathrm{d}^4x \left[-\frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{g^2\theta}{64\pi^2} \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} + \sum_{i=1}^{N_f} \overline{\psi}_i \left(i\gamma^\mu D_\mu - m_i e^{i\alpha_i\gamma_5} \right) \psi_i \right].$$

$\not{CP}$ $\not{CP}$

θ **-term** is a total derivative and thus corresponds to a **boundary term**

➡ it can never contribute in perturbation theory:

▶ effects of θ are **nonperturbative**

▶ **2 types** of **CP-odd terms**: Naively **expect CP violation**

Nonperturbative ‘t Hooft vertices in QCD

[‘t Hooft] derived an **effective Lagrangian** accounting for nonperturbative interactions arising from nontrivial saddle points (**instantons**) in the Euclidean path integral

$$\mathcal{L}^{\rm QCD}_{{\rm eff},{\rm 't\,Hooft}} \sim e^{-{\rm i}\theta}\prod_{i=1}^{N_f}\bar\psi_i P_R\psi_i + e^{{\rm i}\theta}\prod_{i=1}^{N_f}\bar\psi_i P_L\psi_i$$

According to [‘t Hooft] : **phases misaligned with fermion masses: CP violation**

- To link θ to observables, we must **match** with **low-energy theory** that **includes** relevant d.o.f.s such as **the neutron**

The IR perspective: Chiral Lagrangian

Goldstones from $U(3)_L \times U(3)_R \to U(3)_V$ $\qquad U = \langle U \rangle e^{\mathrm{i}\frac{\Pi^a \sigma^a}{\sqrt{2} f_\pi}} \sim \bar{\psi} P_R \psi$

Neutron-proton doublet $\qquad N = \begin{pmatrix} p \\ n \end{pmatrix}$

Quark masses $\qquad M = \begin{pmatrix} m_u e^{i\alpha_u} & & \\ & m_d e^{i\alpha_d} & \\ & & m_s e^{i\alpha_s} \end{pmatrix}$

CP-odd phases

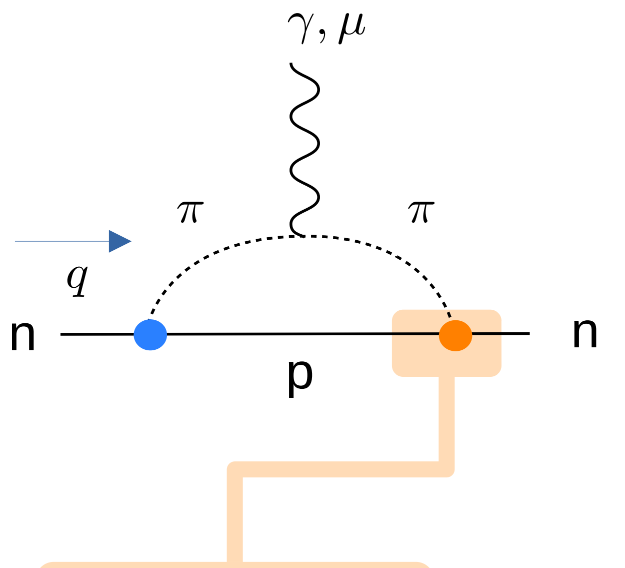
Lagrangian

$$\mathcal{L}_{\pi,\mathrm{p},\mathrm{n}} \supset \frac{1}{4} f_\pi^2 \mathrm{Tr} D_\mu U D^\mu U^\dagger + (a f_\pi^3 \mathrm{Tr} M U + |b| e^{-i\xi} f_\pi^4 \mathrm{det} U + \mathrm{h.c.})$$

$$+ i\bar{N} \not{D} N - (m_N \bar{N} \tilde{U} P_L N + ic\, \bar{N} \gamma^\mu \tilde{U}^\dagger D_\mu \tilde{U} P_L N + d\, \bar{N} \tilde{M}^\dagger P_L N + e\, \bar{N} \tilde{U} \tilde{M} \tilde{U} P_L N + \mathrm{h.c.})$$

($\tilde{U}$: projection into u,d flavours)

Neutron dipole moment



$$\mathcal{L}_{\text{eff}} \supset (\xi + \alpha_u + \alpha_d + \alpha_s) f(q^2) \bar{N} (\vec{S} \cdot \vec{E}) N$$

$d_n \propto (\xi + \alpha_u + \alpha_d + \alpha_s)$ neutron dipole moment

Matching the UV and the IR a la ‘t Hooft

UV: ‘t Hooft vertices

$$\mathcal{L}^{\rm QCD}_{\rm eff,'t\,Hooft} \sim e^{-i\theta}\prod_{i=1}^{N_f}\bar\psi_i P_R\psi_i + e^{i\theta}\prod_{i=1}^{N_f}\bar\psi_i P_L\psi_i$$

IR: Chiral Lagrangian

$$\mathcal{L}_{\rm pion} \supset |b|e^{-i\xi}f_\pi^4 {\rm det}U + {\rm h.c.}, \qquad U\sim\bar\psi P_R\psi$$

Matching leads to

$$\xi = \theta$$

Neutron dipole moment:

$$|d_n| \propto (\xi+\alpha_u+\alpha_d+\alpha_s) = \theta + \sum_i \alpha_i \equiv \bar\theta$$

Experimental bounds:

$$\bar\theta < 10^{-10}$$

Our work

- We have noted that $\xi = \theta$ is not the only option compatible with chiral symmetries

Using **path integral methods**:

- We have **recomputed Green's functions in the dilute instanton gas**, in Euclidean and Minkowski spacetime, and found $\xi = -\sum \alpha_i$ → **no CPV**

  This talk

- We also have a UV **computation** of fermion correlators **which does not rely on instantons**, yielding the same conclusion

- Using **canonical quantization**, we have rederived how θ drops out of observables and P is conserved

  This talk

2. How to compute vacuum correlators

Towards correlators: vacuum path integral

$$\int_{\phi_i,\phi_f,T}\left(\prod\mathcal{D}\phi\right)\,e^{\mathrm{i}S_T}=\langle\phi_f|e^{-\mathrm{i}HT}|\phi_i\rangle=\sum_n e^{-\mathrm{i}E_nT}\langle\phi_f|n\rangle\langle n|\phi_i\rangle$$

To get a **vacuum transition amplitude** we can take the **infinite *T* limit,**

$$Z=\lim_{T\to\infty e^{-i0+}}\int_T\left(\prod\mathcal{D}\phi\right)e^{\mathrm{i}S_T}\sim\lim_{T\to\infty e^{-i0+}}\langle 0|e^{-\mathrm{i}HT}|0\rangle$$

To recover the vacuum amplitude for **finite *T***, one **needs to know the wave functional of the vacuum**

$$\langle 0|e^{-iHT}|0\rangle=\int[\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\langle 0|\phi_f\rangle\langle\phi_f|e^{-iHT}|\phi_i\rangle\langle\phi_i|0\rangle$$

$$=\int[\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\langle 0|\phi_f\rangle\langle\phi_i|0\rangle\int_{\phi_i,\phi_f,T}\left(\prod\mathcal{D}\phi\right)e^{\mathrm{i}S}$$

Wrap-up: the importance of boundary conditions

Infinite *T* method

$$Z = \lim_{T\to\infty e^{-i0_+}} \int_T \left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S_T} \sim \lim_{T\to\infty e^{-i0_+}} \langle 0|e^{-\mathrm{i}HT}|0\rangle$$

Boundary conditions remain arbitrary!

Wave functional method

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\langle 0|\phi_f\rangle\langle\phi_i|0\rangle \int_{\phi_i,\phi_f,T} \left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S}$$

Boundary conditions are fixed by wave functional, need additional reweighting

Wrap-up: the importance of boundary conditions

- To ensure projection into vacuum, we first use the Euclidean **path integral for infinite V** ***T*****,** without the need to enforce particular b.c.s

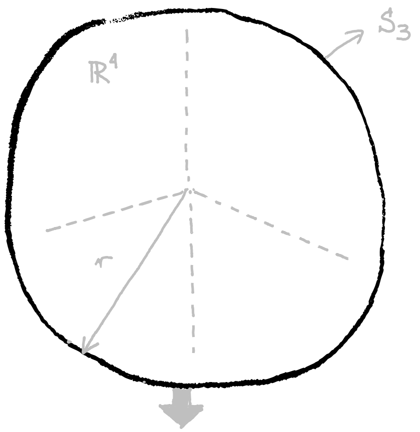
- Later we will use **canonical quantization** to determine the θ dependence of the **wave functional**

3. Results with the infinite T method

Finite action constraints and topology

According to **Picard-Lefschetz theory,** Euclidean path integral can be formulated in terms of a sum of integrations over **steepest descent flows** that start from **finite action saddles** [Witten]

In **infinite spacetime, gauge fields at saddles** must be **pure gauge transf. at ∞**



$$S_E \longrightarrow 0$$

$$A_\mu(r,\theta,\varphi,\delta) = \frac{i}{g} U_r(\theta,\varphi,\delta)\partial_\mu U_r^\dagger(\theta,\varphi,\delta), \quad U_r(\theta,\varphi,\delta) \in SU(3)$$

Finite action constraints and topology

This leads to maps S_3 ⟶ SU(3) that fall into equivalence or **homotopy classes**

"wrappings" of SU(3) over S_3 that cannot be connected by continuous deformations

The **steepest descent flows** are **continuous**

▶ the full flow from a saddle point falls into the same homotopy class

Homotopy classes characterized by **integer topological charge** Δn

▶ In an **infinite spacetime** $Z = \sum_{\Delta n} Z_{\Delta n}$

The θ term and the topological charge

The θ **term** turns out to be **proportional to the topological charge**

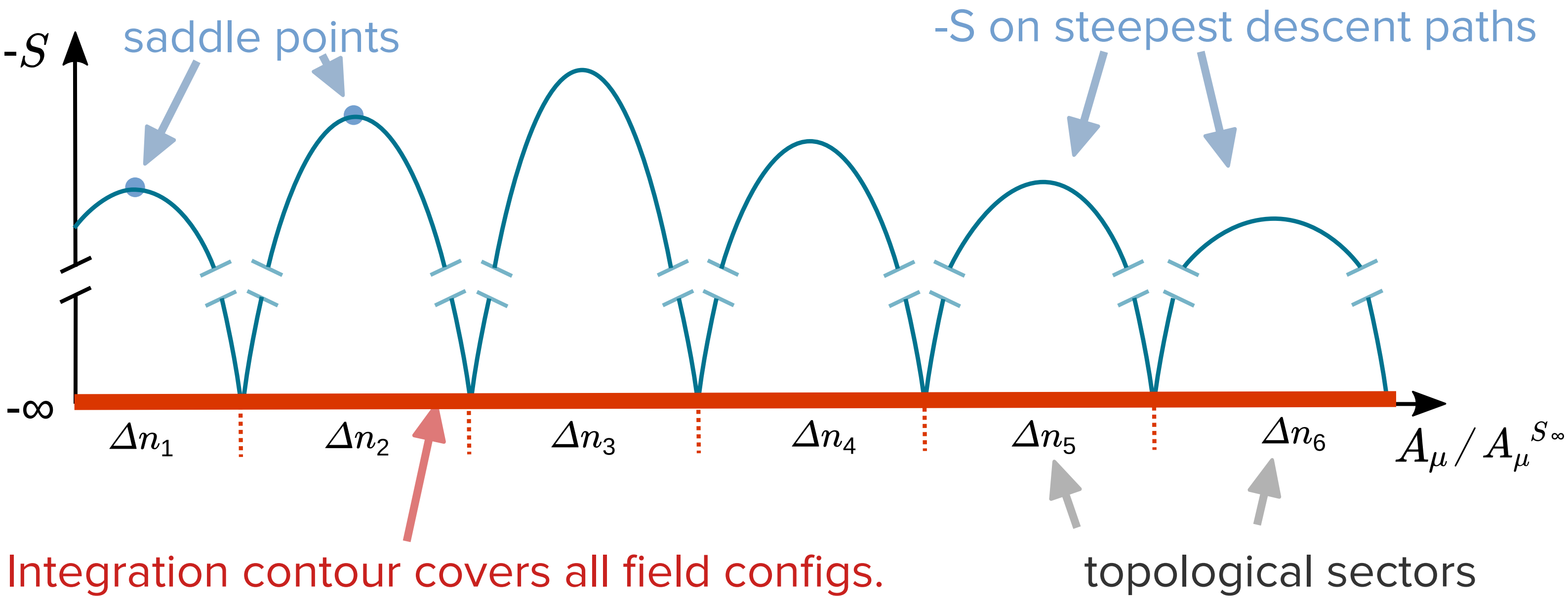
$$-S_\theta^E = i\theta \int d^4x \frac{g^2}{64\pi^2} \epsilon_{mnrs} F^a_{mn} F^a_{rs} = i\theta \Delta n$$

- In an **infinite spacetime** $Z = \sum_{\Delta n} e^{i\Delta n \theta} \tilde{Z}_{\Delta n}$

- **Remember: Integer topological charge only enforced for infinite volume**

Path integral a la Picard-Lefschetz

$$Z = \sum_{\Delta n} Z_{\Delta n}$$

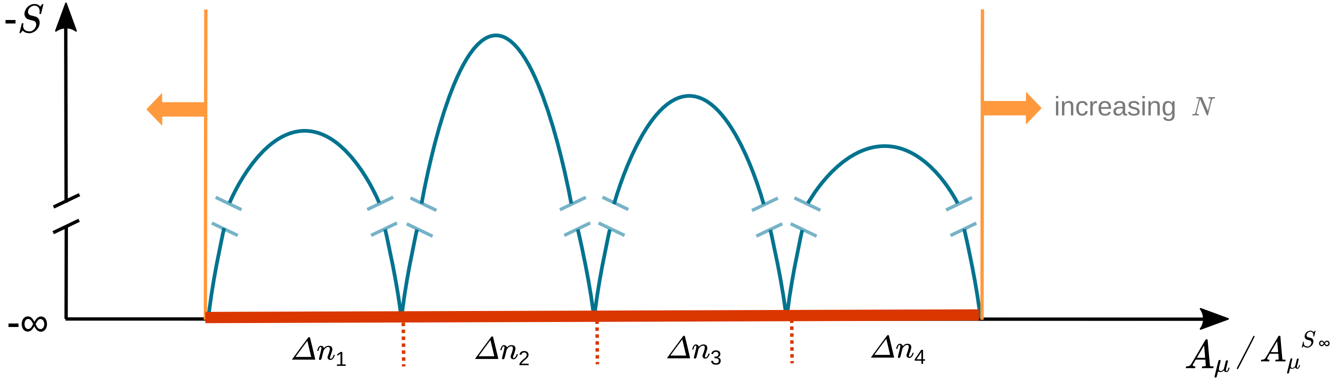


Ordering of limits

Need infinite spacetime volume to project into vacuum

Δn **required to be integer only in infinite volume** → **take infinite volume first**

$$Z = \lim_{N\to\infty} \sum_{|\Delta n|<N} \lim_{\Omega\to\infty} Z_{\Delta n}(\Omega)$$

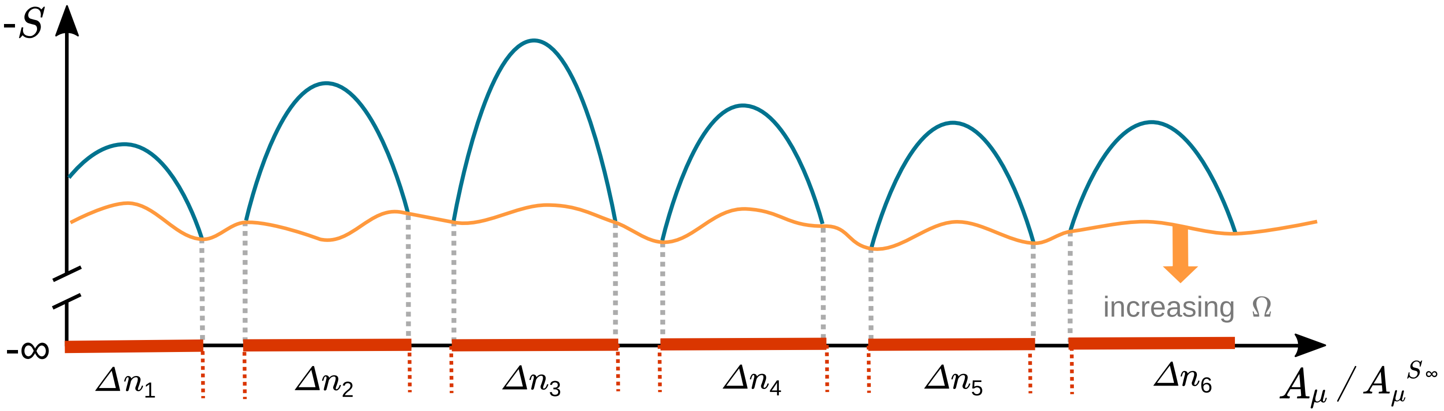


Integration contour continuous, exponential suppression of large N contributions

Alternative ordering of limits

$$Z = \lim_{\Omega\to\infty} \lim_{N\to\infty} \sum_{|\Delta n|<N} Z_{\Delta n}(\Omega)$$

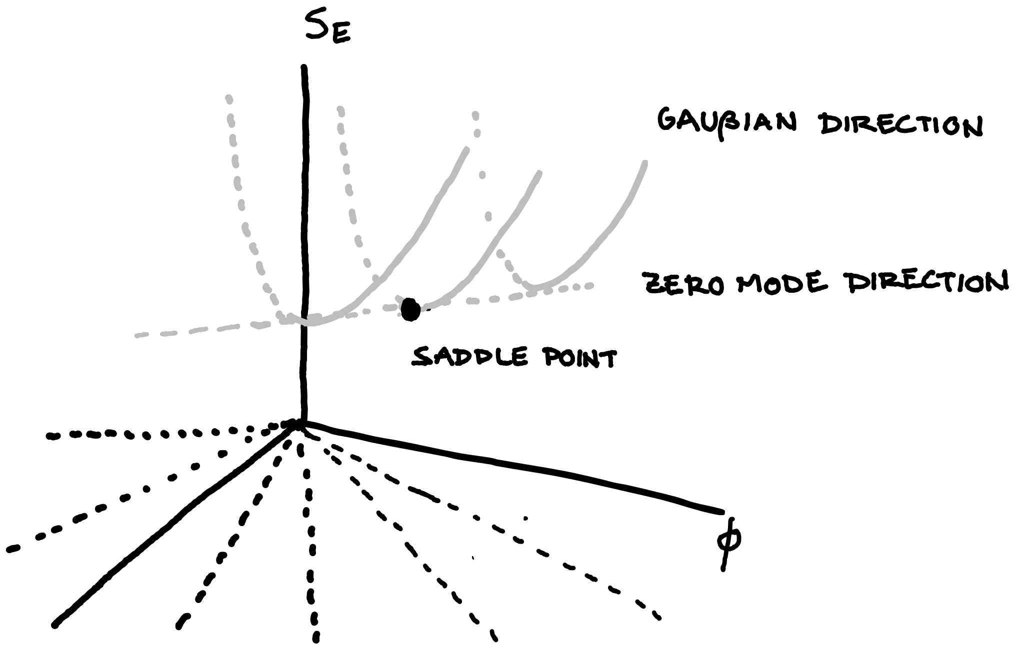
For finite spacetime volume, **topological charge is not necessarily quantized**
Insisting on integer charge means that one **misses configurations**



Integration contour **not connected, singular deformation of original contour**

→ Does not capture full path integral

Strategy to compute correlators



Integration contour through **saddle points of finite action** for each Δn

Can do **Gaußian integration** over fluctuations, including those with infinite action

Strategy to compute correlators

Local fermionic Green functions are obtained by:

Determining **fluctuation determinants**

Calculating **fermion propagator**

Summing over all saddle points in **all** Δn sectors

$$Z\langle\psi(x)\bar{\psi}(y)\rangle \approx \sum_{\Delta n} \sum\!\!\!\!\!\!\int_{\substack{\text{saddle d.o.f} \\ \Delta n \text{ fixed}}} e^{i\Delta n\theta} e^{-S_{E,\text{saddle}}} \Pi_f (\text{Fermionic det})|_{\text{saddle}} (\text{Gauge det})^{-1/2}|_{\text{saddle}} (\text{Propagator(x,y)})$$

Dilute instanton gas

The saddles for arbitrary Δn are approximated as **superpositions** of $\Delta n = \pm 1$ saddles called **“instantons”**

We use **standard** results for **propagators in instanton backgrounds** [Diakonov]

We use **standard** results based on the **index theorem** for determining the complex **phases of fluctuation determinants**

$$\Delta n = \#(\text{Right-handed zero modes of } \not{D}) - \#(\text{Left-handed zero modes of } \not{D})$$

$$\not{D}\varphi_{0R} = 0 \qquad \not{D}\varphi_{0L} = 0$$

Results

$$\langle\psi(x)\bar\psi(x')\rangle = \lim_{\substack{N\to\infty\\ N\in N}} \lim_{VT\to\infty} \frac{\sum_{\Delta n=-N}^{N} e^{i\theta\Delta n}\langle\langle\psi(x)\bar\psi(x')\rangle\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} e^{i\theta\Delta n}\tilde Z_{\Delta n}} = S^E_{0\mathrm{inst}}(x,x') + \kappa\bar h(x,x')m^{-1}\mathrm{e}^{-\mathrm{i}\alpha\gamma^5}.$$

$$h(x,x')P_{\mathrm{L/R}} = \frac{1}{\int d\Omega}\int d\Omega \int_{VT} \mathrm{d}^4x_{0,\bar\nu}\,\varphi_{0\mathrm{L/R}}(x-x_{0,\bar\nu})\varphi^\dagger_{0\mathrm{L/R}}(x'-x_{0,\bar\nu})$$

Topological classification only **enforced in infinite volume**, which **fixes ordering**

$$S^E_{0\mathrm{inst}}(x,x') = (-\not{D} + m e^{-\mathrm{i}\alpha\gamma_5})\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip(x-x')}}{p^2+m^2}$$

Alignment between perturbative and non-perturbative phases: **No CP violation**

Results with the usual order of limits

The alternative order of limits gives standard results:

$$\lim_{VT\to\infty} \lim_{\substack{N\to\infty \\ N\in N}} \frac{\sum_{\Delta n=-N}^{N} e^{i\theta\Delta n}\langle\langle\psi(x)\bar{\psi}(x')\rangle\rangle_{\Delta n}}{\sum_{\Delta n=-N}^{N} e^{i\theta\Delta n}\tilde{Z}_{\Delta n}} = S^{E}_{0\text{inst}}(x,x') - \kappa\bar{h}(x,x')m^{-1}\mathrm{e}^{\mathrm{i}\theta\gamma_5}$$

Misaligned phases: CPV

General correlators

In a similar way one can derive more general local correlators

$$\langle \left(\prod_{j=1}^{N_f} (\bar{\psi}_j(x_j) P_L \psi_j(y_j)) \right) \rangle = e^{\mathrm{i} \sum_j \alpha_j} f(x_k, y_k)$$

Reproduced by the following **effective interaction** (after factoring out ordinary props/)

$$\mathcal{L}_{\text{eff}} \supset e^{\mathrm{i} \sum_j \alpha_j} \Gamma \prod_j \bar{\psi}_j P_R \psi_j$$

To be **matched** to **chiral Lagrangian** with $U \sim \bar{\psi} P_R \psi$

$$\xi = -\sum_j \alpha_j$$

$$\mathcal{L}_{\text{pion}} \supset |b| e^{-i\xi} f_\pi^4 \det U$$

Consequences for d_n and CP violation

$$d_n \propto \xi + \alpha_u + \alpha_d + \alpha_s = 0$$

- All phases of all fermion correlators are fixed by the α_i
- θ effects cancel
- All phases can be eliminated with chiral field redefinitions

No CP violation in fermion correlators

Why CP conservation?

As Δn is CP-odd, and it is a spacetime integral, a **measure of local CP violation** is

$$\frac{\langle \Delta n \rangle}{VT} = \sum_{\Delta n} \frac{\Delta n P(\Delta n)}{VT}$$

With the sum being only over discrete numbers for $VT \to \infty$, each element in the series is zero resulting in **CP conservation.**

The infinite volume limit gives a **local behaviour** (insensitivity to b.c.s)

$$\langle \psi(x) \bar{\psi}(y) \rangle_{\Delta n}, Z_{\Delta n}$$

Δn-independent up to volume suppressed effects

Strong CPV in vacuum as a boundary effect

$$\langle\psi(x)\bar{\psi}(y)\rangle = \frac{\sum_{\Delta n} e^{i\Delta n\theta}\langle\langle\psi(x)\bar{\psi}(y)\rangle\rangle_{\Delta n}}{\sum_{\Delta m} e^{i\Delta m\theta}\tilde{Z}_{\Delta m}}$$

When boundary conditions at infinity do not matter, we have

$$\langle\langle\psi(x)\bar{\psi}(y)\rangle\rangle_{\Delta n} \to \langle\langle\psi(x)\bar{\psi}(y)\rangle\rangle_0, \quad Z_{\Delta n} \to Z_0$$

$$\langle\langle\psi(x)\bar{\psi}(y)\rangle\rangle = \frac{\sum_{\Delta n} e^{i\Delta n\theta}\langle\langle\psi(x)\bar{\psi}(y)\rangle\rangle_0}{\sum_{\Delta m} e^{i\Delta m\theta}\tilde{Z}_0} = \frac{\langle\langle\psi(x)\bar{\psi}(y)\rangle\rangle_0}{\tilde{Z}_0}$$ θ -independent!

CP violation in the vacuum is **only possible if physics is sensitive to b.c.s at infinity**

4. Results with the wave-functional method

Goals

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}A_f]_{T/2}[\mathcal{D}A_i]_{-T/2}\langle 0|A_f\rangle\langle A_i|0\rangle \int_{A_i,A_f,T} \left(\prod \mathcal{D}A\right) e^{iS}$$

Understand the θ-**dependence of wave functionals**

Fix θ-**dependence of correlators without infinite volume limit**

Show cancellation of θ-dependence to confirm $VT \to \infty$ results

For simplicity we focus on the case of a **pure gauge theory**

Chern-Simons Number

In **pure gauge theory**

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}A_f]_{T/2}[\mathcal{D}A_i]_{-T/2}\langle 0|A_f\rangle\langle A_i|0\rangle \int_{A_i,A_f,T} \left(\prod \mathcal{D}A\right) e^{iS}$$

$$S = \int d^4x \left(-\frac{1}{2g^2}\mathrm{tr}F_{\mu\nu}F^{\mu\nu} + \frac{1}{16\pi^2}\theta\,\mathrm{tr}F_{\mu\nu}\widetilde{F}^{\mu\nu}\right) = S_0 + S_\theta .$$

With **appropriate gauge-fixing**

$$S_\theta = \theta(W[\mathbf{A}_f] - W[\mathbf{A}_i]) \qquad W[\mathbf{A}] = \frac{1}{4\pi^2}\varepsilon_{ijk}\int_S \mathrm{d}^3x\,\mathrm{tr}\left[\frac{1}{2}A_i\partial_j A_k - \frac{\mathrm{i}}{3}A_iA_jA_k\right]$$

Chern-Simons number (CSN)

$W[\mathbf{A}]$ changes by integer numbers under spatial gauge transf with $\mathrm{U}(\mathbf{x}) \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 1$

We single out $U^{(1)}(\mathbf{x})$ with $W[\mathbf{A}_{U^{(1)}}] = W[\mathbf{A}] + 1$

The lore: θ vacua

Classical vacua: pure gauge configurations $\mathbf{A}_U = U\mathbf{A}U^{-1} + \mathrm{i}U\nabla U^{-1}$

For $\mathrm{U}(\mathbf{x}) \underset{|\mathbf{x}|\to\infty}{\longrightarrow} 1$ they have **integer CSN**, corresponding to states

$$|n\rangle \quad / \quad \hat{W}|n\rangle = n|n\rangle, \quad U^{(1)}|n\rangle = |n+1\rangle$$

$U^{(1)}$ are a **gauge redundancy** so they can only **rephase** physical states:

$$U^{(1)}|0\rangle = e^{i\hat{\theta}}|0\rangle \quad \Rightarrow \quad |0\rangle_{\hat{\theta}} = \sum_n e^{in\hat{\theta}}|n\rangle = e^{i\hat{\theta}\hat{W}} \sum_n |n\rangle \equiv |\hat{\theta}\rangle$$

Special case of wave-functionals of type

$$\Psi^{(a)}[\mathbf{A}] = e^{i\varphi^{(a)}[\mathbf{A}]}\Psi^{(a)}_{\mathrm{g.i.}}[\mathbf{A}]$$

gauge dep. gauge inv.

The lore: θ vacua lead to CP violation

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}\mathbf{A}_f]_{T/2}[\mathcal{D}\mathbf{A}_i]_{-T/2}\langle 0|\mathbf{A}_f\rangle\langle \mathbf{A}_i|0\rangle \int_{\mathbf{A}_i,\mathbf{A}_f,T}\left(\prod \mathcal{D}\mathbf{A}\right) e^{\mathrm{i}(S+S_\theta)}$$

$$e^{-i\hat{\theta}W[\mathbf{A}_f]}\,e^{i\hat{\theta}W[\mathbf{A}_i]}$$

$$e^{i\theta(W[\mathbf{A}_f]-W[\mathbf{A}_i])}$$

$$e^{i(\theta-\hat{\theta})(W[\mathbf{A}_f]-W[\mathbf{A}_i])}$$

No cancellation: CPV

The trouble with θ vacua

$\hat{\theta}$ thought to be **arbitrary**

Hilbert space decomposes in **sectors** that **don't talk to each other**:

$$\langle\hat{\theta}|\hat{\theta}'\rangle = \sum_{mn} e^{-i\hat{\theta}m + i\hat{\theta}'n}\langle m|n\rangle = \sum_{n} e^{-in(\hat{\theta}-\hat{\theta}')} = \delta(\hat{\theta}-\hat{\theta}')$$

$|\hat{\theta}\rangle$ are **not normalizable, contradicting basic postulates of quantum mechanics**
[Okubo & Marshak]

It is also **naive** to consider wave functional **only having support on classical vacua**

How to define normalizable states

$U^{(1)}$: **Gauge redundancy changes W by 1**

All **physical configs**. covered with $0 \leq W[\mathbf{A}] < 1$

Can define **inner product** within **gauge-fixed hypersurface** $\mathcal{A}$ with $0 \leq W[\mathbf{A}] < 1$

- Under this product $|\hat{\theta}\rangle$ are not orthogonal!

- For general states $\Psi^{(a)}[\mathbf{A}] = e^{i\varphi^{(a)}[\mathbf{A}]}\Psi^{(a)}_{\text{g.i.}}[\mathbf{A}]$

  hermiticity of H demands a physical space with **unique** $\varphi^{(a)} = \varphi$

Traditional vs physical picture

Traditional picture

Many mutually orthogonal spaces of states with:

unphysical, infinite norm states

different $\hat{\theta}$ → **different rephasings** under $U^{(1)}$

Physical picture

Single space of states with:

physical normalizable states

unique φ → **common rephasing** under $U^{(1)}$

Fixing phase φ: canonical quantization

Wave functionals $\Psi^{(a)}[A_i]$ satisfy Schrödinger equation

$$H\Psi^{(a)}[A] = E^{(a)}\Psi^{(a)}[A]$$

We quantize in the gauge $A_0 = 0$, which still allows gauge transformations $U(\mathbf{x})$

$$[A^{i,a}(\mathbf{x}), \Pi^{j,b}(\mathbf{x})] = \mathrm{i}\delta^{ij}\delta^{ab}\delta^3(\mathbf{x}-\mathbf{y}) \Rightarrow \mathbf{\Pi}^a = \frac{\delta}{\mathrm{i}\delta\mathbf{A}^a} \,.$$

$$\mathcal{H} = \frac{1}{2}\left[(\mathbf{E}^a)^2 + (\mathbf{B}^a)^2\right] = \frac{1}{2}\left[\left(g\frac{\delta}{\mathrm{i}\delta\mathbf{A}^a} - \frac{g^2}{8\pi^2}\theta\mathbf{B}^a\right)^2 + (\mathbf{B}^a)^2\right] \,.$$

Fixing phase φ: change of basis

Under a **change of basis**

$$\Psi[\mathbf{A}] = \mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}]}\Psi'[\mathbf{A}]$$

The **Hamiltonian** becomes θ **- independent**

$$\mathcal{H}' = \frac{1}{2}\left[\left(g\frac{\delta}{\mathrm{i}\delta\mathbf{A}^a}\right)^2 + (\mathbf{B}^a)^2\right] .$$

Basis allows **separation of variables** $\mathbf{A} = \mathbf{A}_{\mathrm{gauge}} + \mathbf{A}_{||}$

$$\Psi'[\mathbf{A}] = e^{i\varphi[\mathbf{A}_{\mathrm{gauge}}]}\tilde{\Psi}_{\mathrm{g.i.}}[\mathbf{A}_{||}]$$

Physical states

$\varphi^{(a)}[\mathbf{A}_{\text{gauge}}]$ forced to be **linear** in $\mathbf{A}_{\text{gauge}}$, while gauge transformations **shift** $\mathbf{A}_{\text{gauge}}$

$\Psi'^{(a)}[\mathbf{A}]$ **transforms** with a **phase** under **gauge transformations** (as expected!)

$$\Psi'[\mathbf{A}_{U(\mathbf{x})}] = e^{\mathrm{i}\theta_{\Psi,U}}\Psi'[\mathbf{A}]$$

The $\theta_{\Psi,U}$ are a **1D representation** of the gauge group

$$\theta_{\Phi,U_1U_2} = \theta_{\Phi,U_1} + \theta_{\Phi,U_2}, \quad \theta_{\Phi,U_1^{-1}} = -\theta_{\Phi,U_1}$$

In a **simple Lie Group**, any element can be written as

$$U = U_1U_2U_1^{-1}U_2^{-1}$$

$$\theta_{\Psi,U} = \theta_{\Psi,U_1} + \theta_{\Psi,U_2} - \theta_{\Psi,U_1} - \theta_{\Psi,U_2} = 0$$

Cancellation of θ

Back to **original basis**

$$\Psi[\mathbf{A}] = \mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}]}\Psi'[\mathbf{A}] = \mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}]}\tilde{\Psi}_{\mathrm{g.i}}[\mathbf{A}]$$

Evolve with CP-even H'
No ~~CP~~ phase in constraints

$\mathrm{e}^{-\mathrm{i}\theta W[\mathbf{A}_f]}\tilde{\Psi}_{0,\mathrm{g.i.}}[\mathbf{A}_f]$ $\quad$ $\mathrm{e}^{\mathrm{i}\theta W[\mathbf{A}_i]}\Psi_{0,\mathrm{g.i.}}[\mathbf{A}_i]$ $\quad$ $\mathrm{e}^{\mathrm{i}\theta(W[\mathbf{A}_i]-W[\mathbf{A}_f])}$

$$\langle 0|e^{-iHT}|0\rangle = \int[\mathcal{D}\mathbf{A}_f]_{T/2}[\mathcal{D}\mathbf{A}_i]_{-T/2}\Psi_0[\mathbf{A}_f]^*\Psi_0[\mathbf{A}_i]\int_{\mathbf{A}_i,\mathbf{A}_f,T}\left(\prod\mathcal{D}\phi\right)e^{\mathrm{i}S+\mathrm{i}S_\theta}$$

$$= \int[\mathcal{D}\mathbf{A}_f]_{T/2}[\mathcal{D}\mathbf{A}_i]_{-T/2}\tilde{\Psi}_{0,\mathrm{g.i.}}[\mathbf{A}_f]^*\tilde{\Psi}_{0,\mathrm{g.i.}}[\mathbf{A}_i]\int_{\mathbf{A}_i,\mathbf{A}_f,T}\left(\prod\mathcal{D}\phi\right)e^{\mathrm{i}S}$$

θ **disappears** from the **partition function** → **CP conservation**

Is there a symmetry related to parity?

Even in the presence of θ, the Hamiltonian has a **discrete symmetry**, which can be seen to enforce

$$\Psi[\mathbf{A}^P] = \pm \mathrm{e}^{-2\mathrm{i}\theta W[\mathbf{A}]}\Psi[\mathbf{A}]$$

parity = rephasing

$$\mathrm{i}S \underset{P}{\longrightarrow} \mathrm{i}S - 2\mathrm{i}\theta(W[\mathbf{A}_f] - W[\mathbf{A}_i])$$

$$+2\mathrm{i}\theta W[\mathbf{A}_f] \quad -2\mathrm{i}\theta W[\mathbf{A}_i] \quad -2\mathrm{i}\theta W[\mathbf{A}_f] + 2\mathrm{i}\theta W[\mathbf{A}_i]$$

$$\langle 0|e^{-iHT}|0\rangle = \int [\mathcal{D}\phi_f]_{T/2}[\mathcal{D}\phi_i]_{-T/2}\Psi_0[\phi_f]^*\Psi_0[\phi_i]\int_{\phi_i,\phi_f,T}\left(\prod \mathcal{D}\phi\right) e^{\mathrm{i}S}$$

The **partition function** is **parity invariant!**

Conclusions

- **In the path integral formalism in infinite volume** (ensuring projection into vacuum):

  QCD with an arbitrary θ **does not predict CP violation**, as long as the sum over topological sectors is performed at **infinite volume**

  This **ordering of limits** is the correct one because the topological classification is only enforced for an infinite volume

- **In canonical quantization** in **pure gauge** theory:

  The **phase** of the **wave functionals** of stationary states is **correlated with** θ

  This leads to θ**-independent path integral and** a **conserved parity symmetry**

Concrete challenges to the usual lore

- In an infinite volume path integral, why is it OK to sum over all topological sectors before taking $VT \to \infty$, when this is a singular deformation of the original path integral contour?

- Why is it OK to consider $|\theta\rangle$ vacua which:

  are non-normalizable and contradict postulates of quantum mechanics?

  lead to wave-functionals which only have support on classical minima?

Thank you!

Additional material

Finite volumes from an infinite spacetime

- We aim to derive an **effective finite-volume description starting from an infinite-volume path integral** guaranteed to capture the vacuum state

- The finite volume description can can help make **contact with results from canonical quantization and with lattice computations**

Finite volumes in an infinite spacetime

Assume **local operator** $\mathcal{O}_1$ with **support** in finite spacetime volume Ω_1

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum_{\Delta n=-\infty}^{\infty} e^{i\Delta n \theta} \int_{\Delta n} \mathcal{D}\phi \, \mathcal{O}_1 \, \mathrm{e}^{-S_\Omega[\phi]}}{\sum_{\Delta n=-\infty}^{\infty} e^{i\Delta n \theta} \int_{\Delta n} \mathcal{D}\phi \, \mathrm{e}^{-S_\Omega[\phi]}}$$

$$= \frac{\sum_{\Delta n=-\infty}^{\infty} \sum_{\Delta n_1=-\infty}^{\infty} e^{i\Delta n \theta} \int_{\Delta n_1} \mathcal{D}\phi \, \mathcal{O}_1 \, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2 = \Delta n - \Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_2}[\phi]}}{\sum_{\Delta n=-\infty}^{\infty} \sum_{\Delta n_1=-\infty}^{\infty} e^{i\Delta n \theta} \int_{\Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_1}[\phi]} \int_{\Delta n_2 = \Delta n - \Delta n_1} \mathcal{D}\phi \, \mathrm{e}^{-S_{\Omega_2}[\phi]}} \, .$$

[Note: Integer Δn_1 is only an approximation, carried out in a surface kept finite, with reduced impact in full path integral.]

Finite volumes in an infinite spacetime

Path integrations over Ω_2 give just the **partition functions** we calculated before

In the **infinite volume** limit the **Bessel functions tend to common value** and dependence on Δn factorizes out and cancels:

$$\langle \mathcal{O}_1 \rangle_\Omega = \frac{\sum_{\Delta n_1 = -\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi \, (-1)^{-N_f \Delta n_1} e^{-i\,\alpha \Delta n_1} \mathcal{O}_1 \, e^{-S_{\Omega_1}[\phi]}}{\sum_{\Delta n_1 = -\infty}^{\infty} \int_{\Delta n_1} \mathcal{D}\phi \, (-1)^{-N_f \Delta n_1} e^{-i\,\alpha \Delta n_1} e^{-S_{\Omega_1}[\phi]}} \, .$$

We recover a **path integration** over a **finite volume, without θ dependence as in canonical quantization: CP is conserved**

Extra phases precisely **cancel those from fermion determinants in Ω_1**

Baluni’s CP-violating effective Lagrangian

Baluni’s CP-violating Lagrangian (used by [Crewther et al]) is based on searching for field redefinitions that minimize the QCD mass term

$$\mathcal{L}_M(U_{R,L}) = \bar{\psi} U_R^\dagger M U_L \psi_L + \text{h.c.}, \quad U_{R,L} \in SU_{R,L}(3)$$

$$\langle 0|\delta\mathcal{L}|0\rangle = \min_{U_{R,L}} \langle 0|\mathcal{L}_M(U_{R,L})|0\rangle$$

However, there is an **extra assumption**: that the **phase of the fermion condensate is aligned with**

$$\langle \bar{\psi}_R \psi_L \rangle = \Delta e^{\mathrm{i}c\theta} \mathbb{I}$$

This assumption **does not hold** for the chiral Lagrangian with $\xi = -\alpha$, but is valid for $\xi = \theta$

Crewther et al's calculations

Using [Baluni]'s CP-violating Lagrangian and current algebra [Crewther et al] get

$$\langle 0|\delta\mathcal{L}|\eta'\pi^0\pi^0\rangle = \frac{m_u m_d}{(m_u+m_d)^2}\frac{m_\pi^2}{f_\pi}\bar{\theta}\,.$$

From our general Chiral Lagrangian we get

$$\mathcal{L}_{\mathrm{M}}^{\mathrm{EFT}} \supset \frac{B_0\sin(\xi+\alpha_u+\alpha_d)}{f_\pi\sqrt{\frac{1}{m_u^2}+\frac{1}{m_d^2}+\frac{2\cos(\xi+\alpha_u+\alpha_d)}{m_u m_d}}}\left[(\pi^0)^2+2\pi^+\pi^-\right]\eta'$$

$$\langle 0|\delta\mathcal{L}|\eta'\pi^0\pi^0\rangle = \frac{m_u m_d}{(m_u+m_d)^2}\frac{m_\pi^2}{f_\pi}(\xi+\bar{\alpha}),\,.$$

Match for $\xi = \theta$

So once more, traditional results are built on **(hidden) assumption** $\xi = \theta$

The η’ mass

Chiral Lagrangian with alignment in the phases of mass terms and anomalous terms still predicts a **nonzero value of the ’ mass**

$$\mathcal{L} = f_\pi^2 \mathrm{Tr} \partial_\mu U \partial^\mu U^\dagger + a f_\pi^3 \mathrm{Tr} M U + |b| e^{i \arg \det M} f_\pi^4 \det U + \mathrm{h.c.}$$

$$m_{\eta'}^2 = 8|b| f_\pi^2$$

Can be seen to be **proportional** to the **topological susceptibility** over **finite volumes** of the **pure gauge theory**, in line with [Witten, Di Vecchia & Veneziano]

Classic arguments linking topological susceptibility to CP violation ([Shifman et al]) rely on analytic expansions in which **don’t apply** with our limiting procedure

***Z* from infinite-volume partition function becomes non-analytic in** θ**.**
This possibility has been mentioned by [Witten]

[Witten, Nucl. Phys. B 156 (1979)]

the physics is of order e^{-N}, contrary to the basic assumptions of this paper, or else the physics is non-analytic as a function of θ, In the latter case, which is quite plausible, the singularities would probably be at $\theta = \pm\pi$, as Coleman found for the massive Schwinger model [10]. It is also quite plausible that θ is not really an angular variable.)

To write a formal expression for $d^2E/d\theta^2$, let us think of the path integral formulation of the theory:

$$Z = \int dA_\mu \exp i \int \mathrm{Tr}\left[-\tfrac{1}{4}F_{\mu\nu} + \frac{g^2\theta}{16\pi^2 N} F_{\mu\nu}\tilde{F}_{\mu\nu}\right]. \tag{5}$$

Partition function and analiticity

Usual partition function is analytic in θ

$$Z_{\text{usual}} = \lim_{VT\to\infty} \lim_{\substack{N\to\infty \\ N\in N}} \sum_{\Delta n=-N}^{N} Z_{\Delta n} = \mathrm{e}^{2\mathrm{i}\kappa_{N_f} VT \cos(\bar{\alpha}+\theta+N_f\pi)}$$

θ-dependence of observables (giving CP violation) usually relies on expansion. e.g.

$$\frac{\langle \Delta n \rangle}{\Omega} = \mathrm{i}\,(\theta-\theta_0) \left.\frac{\langle \Delta n^2 \rangle}{\Omega}\right|_{\theta_0} + \mathcal{O}(\theta-\theta_0)^2$$

topological susceptibility

[Shifman et al]

In our limiting procedure the former is not valid, as Z becomes nonanalytic in θ

$$Z = \lim_{\substack{N\to\infty \\ N\in N}} \lim_{VT\to\infty} \sum_{\Delta n=-N}^{N} Z_{\Delta n} = I_0(2\mathrm{i}\kappa_{N_f} VT) \lim_{\substack{N\to\infty \\ N\in N}} \sum_{|\Delta n|\le N} \mathrm{e}^{\mathrm{i}\Delta n(\bar{\alpha}+\theta+N_f\pi)}$$

θ drops out from observables, there is no CP violation

Dvali's footnote

[2] The 3-form language of [14] clarifies the claim of [24] that by changing the order of limits in ordinary instanton calculation, one ends up with $\vartheta = 0$. In this approach one performs calculation in the finite volume and then takes it to infinity. In 3-form language the meaning of this is rather transparent. The finite volume is equivalent of introducing an infrared cutoff in form of a shift of the massless pole in (28) away from zero. This effectively gives a small mass to the 3-form. For any non-zero value of the cutoff, the unique vacuum is $E_0 = 0$ which is equivalent to $\vartheta = 0$. Other states $E \neq 0$ (corresponding to $\vartheta \neq 0$) have finite lifetimes which tend to infinity when cutoff is taken to zero. In this way the $\vartheta \neq 0$ vacua are of course present but one is constrained to $\vartheta = 0$ by the prescription of the calculation. Thus, changing the order of limits by no means eliminates the ϑ-vacua. As usual, when taking the limit properly, one must keep track of states that become stable in that limit. These are the states with $\vartheta \neq 0$ ($E \neq 0$), which become the valid vacua in the infinite volume limit. The effect is in certain sense equivalent to introducing an auxiliary axion and then decoupling it.

Dvali's 3-form formalism

[Dvali] has the following line of reasoning from which he concludes that QCD violates CP

Nonzero topological susceptibility at zero momentum / full volume → Massless pole in CS current-current correlators → EFT for massless 3-form with CP violating vacua

With **our ordering of limits**, we have that the **topological susceptibility** is:

zero at zero momentum/full volume

nonzero at finite volume/nonzero momentum (matching lattice)

Dvali's first premise is violated and his argument does not apply

Dvali's criticism

[Dvali] argues that in a calculation at finite volume which is then sent to infinity, CP violation can't be captured because the infrared regulation gives a mass to the 3 form.

We make the following observations:

- [`t Hooft]'s original calculations (at finite volume, taken to ∞ in the end) lead to CP violation for arbitrary θ, in conflict with Dvali's argument
- If finite volume is problematic, more reason to take the infinite volume limit as soon as possible, as we do, leading to no CP violation for arbitrary
- Dvali's formalism has no explicit/direct link to UV parameter
- Dvali's critique of finite volumes can be turned against his own construction, as it is based on assuming nonzero topological susceptibility, while the only nonperturbative evidence for it comes from lattice results at finite volume

Dvali's criticism

[Dvali]'s construction can be seen to imply boundary conditions that do not correspond to vanishing physical fields at the boundary, and so does not capture the standard partition functions

$$\tilde{F}F \propto \partial_\mu K^\mu$$

[Dvali] argues

$$\partial_\mu K^\mu = \sqrt{\chi}\,\theta_L, \quad \text{const.}$$

- This implies a single frozen topological sector as $\Delta n \propto \int d^4x\,\partial_\mu K^\mu = \text{const}$
- Constant, gauge-invariant $\partial_\mu K^\mu$ does not vanish at the boundary
- No reason for periodicity in θ_L so no clear relation to usual θ angle
- Does not correspond to QCD partition function