Nonlinear Systems: Theory and Applications LE41 2011/'12

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Nonlinear Systems: Theory and Applications

Sezioni

Lecce, Perugia

Componenti

- Lecce $\begin{cases} M.\ Boiti \\ M.\ Gianfreda \\ B.\ Konopelchenko \quad (rappr.naz.) \\ G.\ Landolfi \\ L.\ Martina \\ F.\ Pempinelli \\ B.Prinari \\ L.\ Renna \end{cases}$
- Perugia $\begin{cases} S.\ De\ Lillo \\ G.\ Lupo \end{cases}$

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7. Dip. Fisica, Universita' del Salento, Lecce Italy (S. Zykov)
8. Dip. Matematica, Universita' del Salento, Lecce Italy (R. Vitolo)

Extended resolvent and applications

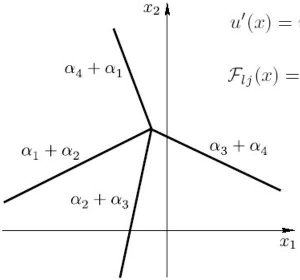
Boiti, Pempinelli

1. Extended resolvent generalizes the classical resolvent of differential operators.
2. It can be used to study the nonlinear integrable evolution equations, as the Kadomtsev-Petviashvili I and II equations
3. N solitons with N incoming rays and one outgoing ray
4. Complete description of the Jost solutions
5. Solution of the IVP for KPII

M. Boiti *et al*, Theor. Math. Phys., 159: 721733 (2009)

M. Boiti *et al*, Theor. Math. Phys., 165: 1237-1255 (2010)

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2},$$



$$u'(x) = u(x) - 2\partial_{x_1}^2 \log\det(E_{N_b} + \mathcal{F}c$$

$$\mathcal{F}_{lj}(x) = \frac{e^{(a_j - b_l)(x_1 + (a_j + b_l)x_2)}}{a_j - b_l}$$

1 Study of the Davey-Stewartson Equation

Singular Sector of Hydrodynamical type Systems

Konopelchenko

1. Singular sector of the classical one-layer Benney system
2. dispersionless Toda equation and large N limit Hermitian Random Matrix Model
3. dispersionless KdV and Hermitian Random Matrix Model
4. Hermitian Random Matrix Model and Euler-Poisson-Darboux equation
5. Gradient Catastrophe and Thom's Catastrophe
6. Instability of vortex filament by dispersionless da Rios system

Algebro-Geometric structure in Sato-Grassmannians

Konopelchenko

1. Algebraic varieties and curves in Birkhoff strata of Sato Grassmannian
2. Isomorphism among ∞-dim associative algebras and algebraic curves in Birkhoff strata
3. Regularization of degenerate algebraic curves.
4. Harrison's cohomology of algebraic varieties.
5. Deformations of hyperelliptic curves and the dispersionless KP hyerarchy.
6. The Yano-Ako system and the Frobenius manifold theory.

- B G Konopelchenko and G Ortenzi, J. Phys. A: Math. Theor. 42 (2009) 415207
- B. G. Konopelchenko, Theoretical and Mathematical Physics, 159(3): 842852 (2009)
- B G Konopelchenko, J. Phys. A: Math. Theor. 42 (2009) 454003
- B Konopelchenko,J. Phys. A: Math. Theor. 42 (2009) 095201
- B Konopelchenko, L Martinez Alonso and E Medina, J. Phys. A: Math. Theor. 43 (2010) 434020
- B G Konopelchenko and G Ortenzi, J. Phys. A: Math. Theor. 43 (2010) 195204
- B Konopelchenko, L Martinez Alonso and E Medina, Physics Letters A 375 (2011) 867872

Simmetries and Entanglement in Quantum Systems

Landolfi

1. Entanglement in continuous solvable models. Witness observables.
2. Darboux transformations to quadratic Hamiltonians
3. Spectral properties of the Weyl-ordered operators involving powers of position and momentum and their eigenfunctions
4. Observables canonically conjugated to the Hamiltonians.
5. Stationary position-momentum correlated states of time-dependent hamiltonians.
6. Generalized heterodyne detection for linear multimode fields.
7. Decoherence phenomena for non-autonomous quantum systems

- M. Gianfreda,. G. Landolfi and M. G. A. Paris, Theor. Math. Phys., 160(1): 925932 (2009)
- M. Gianfreda,. G. Landolfi, Theor. Math. Phys. (in press)
- M. Gianfreda,. G. Landolfi:Spectral problem for Weyl-ordered form of operators, preprint 2011
- M. Gianfreda,. G. Landolfi:On the feasibility and robustness of steady position-momentum correlations for time-dependent quadratic systems, preprint 2011
- L. Martina , G. Ruggeri, G. Soliani : Correlation Energy and Entanglement Gap in Continuous Models, Int. J. Quant. Inf. **6**, n. 3 (2011), 766

Inverse Scattering Transform: extensions and Applications

Prinari

1. IST for defocusing V-NLS equation with nonvanishing boundary conditions.
2. Dark-dark and dark-bright soliton interaction for 2-NLS.
3. Asymptotic states for solitons of the 2-NLS equation, to be generalized to N-components case.
4. NLS in non euclidean spaces
5. IST for discretized NLS
6. Algebraic methods for NLS with nontrivial boundary conditions.
7. Dispersive shock waves and NLS with discontinuous initial data.
8. IST for coupled Maxwell - Bloch systems
9. Analysis of a nonlinear nonlocal ODE system modeling the performance and clinical outcome of an existing medical ward

- B. Prinari, G. Biondini, and A. D. Trubatch:*Inverse Scattering Transform for the Multi-Component Nonlinear Schr¨odinger Equation with Nonzero Boundary Conditions*, Studies in Applied Math. **126** (2011) 245-302.
- M. Lo Schiavo, B. Prinari, A.V. Serio,*Mathematical modeling of quality in a medical structure: a case study*, Math. Comp. Mod. 2011 (in press)
- G. Dean, T. Klotz, B. Prinari, F. Vitale: Dark-dark and dark-bright soliton interactions in the two-component defocusing nonlinear Schrödinger equation, preprint 2011.

$$i\mathbf{q}_t = \mathbf{q}_{xx} - 2\sigma \|\mathbf{q}\|^2 \mathbf{q}$$

Chaotic systems and applications

Renna

1. Qualitative behavior of a periodically kicked mechanical oscillator, with damping.
2. Numerical analysis with (i) sinusoidal and (ii) Gaussian pulses
3. Forcing symmetry and resonance symmetry dominance
4. The mechanisms of diseases spread by a SIRS model with a variable population size
5. The mechanisms of diseases spread by a SIRS model with seasonal variability
6. Climate change detection by use of bayesian approaches.

L. Renna, F. Paladini, Theor. Math. Phys. 168 (2011) 1010-1019

Symmetries in Nonlinear models

Martina

1. Symmetries and solutions for the infrared limit of the pure Yang-Mills theory and the generalized 2-components Ginzburg-Landau Model
2. Symmetries for Dynamics in Non-Commutative Spaces and Generalizations
3. Symmetries of continuous and discrete Surfaces in Lie Algebras

- **L. M.**, A. Protogenov, V. Verbus, Theor. Math Phys. (2008)
- **L. M.**, A. Protogenov, V. Verbus, J. Nonlinear Math. Phys. **15**, 343-351 (2008)
- **L. M.**, A. Protogenov, V. Verbus, Theor. Math. Phys. **160**, n. (2009), 1058
- L. M., A. Protogenov, V. Verbus, Theor. Math. Phys. 167(3)(2011), 843855
- L.M. G. Martone, S. Zykov: Studies on the pure Yang-Mills model, in preparation
- P. A. Horváthy, L. M., P. C. Stichel, *SIGMA* **6** *(2010) 060, P. Aschieri et al. ed.s Noncommutative Spaces and Fields*
- L. M. , Theor. Math. Phys. **167** (3) (2011), 816825, arXiv:1011.3545
- A.M. Grundland, L.M.:Symmetries of the $\mathbb{CP}^{N-1}$ model and the continuous deformations of their associated Surfaces, in preparation

The pure Yang-Mills theory

Pure SU(2) Yang-Mills - No Matter

$$S = -\int F \wedge \star F,$$

$$F = A + A \wedge A, \qquad A = -T^a A^a_\mu(x)\, x^\mu, \quad T^a \in su(2)$$

$$\star F + A \wedge \star F - \star F \wedge A = 0, \quad F + A \wedge F - F \wedge A = 0.$$

$$\text{local gauge invariance} A \to VAV^{-1} + VV^{-1}, \quad V \in SU(2)$$

$$F = \frac{1}{2}\, T^a \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu\right) x^\mu \wedge x^\nu = \frac{1}{2}\, T^a F^a_{\mu\nu}\, x^\mu \wedge x^\nu.$$

Spin-Charge Separation

$$\mathrm{U_C}(1) \qquad A_a^i \to \left(A_a, X_a^{\pm}\right), \quad X_a^{\pm} = A_a^1 \pm A_a^2, \qquad A_a = A_a^3.$$

$$X_a^+ = \psi_1 e_a + \psi_2 \bar{e}_a,\; X_a^- = \left(X_a^+\right)^* = \psi_1^* \bar{e}_a + \psi_2^* e_a \qquad e_a e_a = 0,\; e_a \bar{e}_a = 1.$$

$$P_{ab} = \frac{}{2}\left(|\psi_1|^2 - |\psi_2|^2\right)\left(e_a \bar{e}_b - e_b \bar{e}_a\right).$$

$$\mathrm{U_I}(1) - \text{inner symmetry} e_a \to e^{-\lambda} e_a, \qquad \psi_1 \to e^{\lambda}\psi_1, \qquad \psi_2 \to e^{-\lambda}\psi_2.$$

$$p_i = \frac{}{2}\left(e_4 \bar{e}_i - e_i \bar{e}_4\right), \qquad q_i = \frac{}{2}\epsilon_{ijk} e_j \bar{e}_k, \qquad \vec{p}\cdot\vec{q} = 0, \quad |\vec{p}|^2 + |\vec{q}|^2 = \frac{1}{4}$$

$$n_+ =^{-2\eta} \frac{1}{\rho^2}\psi_1^*\psi_2, \qquad n_- =^{2\eta} \frac{1}{\rho^2}\psi_1\psi_2^*, \qquad n_3 = \frac{|\psi_1|^2 - |\psi_2|^2}{\rho^2}, \Rightarrow \vec{n} \xrightarrow{r\to\infty} \pm\hat{z}$$

$$\hat{C}_a = C_a + \partial_a \eta = \hat{\bar{e}}_b \partial_a \hat{e}_b = -2\,|\vec{q}|\left(\vec{k} \times \vec{l} \cdot \partial_a \vec{k}\right) = \frac{2\vec{p}\cdot\partial_a \vec{s}}{|\vec{p}|^2}$$

$$\vec{k} = \frac{\vec{p}}{|\vec{p}|}, \qquad \vec{l} = \frac{\vec{q}}{|\vec{q}|}, \qquad \vec{s} = \vec{p} \times \vec{q}$$

Quantizing in background

Path-Integral Quantization   Faddeev-Popov gauge fixing

$$\int \mathcal{D}A^{\,S[A]} = \left(\int \mathcal{D}\alpha\right) \int \mathcal{D}A^{\,S[A]} \delta\left[G\left[A\right]\right] \det\left(\frac{\delta G\left[A^{\alpha}\right]}{\delta\alpha}\right).$$

$$\text{InfraRed limit : Classical background + Quantum fluctuations}$$

$$X_a^{\pm} \to X_a^{\pm} + \hat{X}_a^{\pm}, \qquad A_a \to A_a + \hat{A}_a.$$

$$g.-f. \perp U_C(1) \quad G^{\pm}\left[A\right] = \mathrm{D}^{\pm}_{Aa}\left(X_a^{\pm} + \hat{X}_a^{\pm}\right) - \zeta^{\pm},$$

$$\begin{aligned}
\mathcal{L}_{\mathrm{YM}} &= \frac{1}{4}\mathcal{F}_{ab}^2 + \frac{1}{2}\left(\partial_a\rho\right)^2 + \frac{1}{8}\rho^2\left(\mathrm{D}_a^{\hat{C}}\vec{n}\right)^2 + \rho^2\left[\left(\partial_a\vec{p}\right)^2 + \left(\partial_a\vec{q}\right)^2\right] \\
&+ \frac{\rho^2}{2}\left(n_+\left(\partial_a\hat{\bar{e}}_b\right)^2 + n_-\left(\partial_a\hat{e}_b\right)^2\right) + \frac{1}{2}\rho^2 J_a^2 + \frac{3}{8}\left(1-n_3^2\right)\rho^4 - \frac{3}{8}\rho^4, \\
\mathcal{F}_{ab} &= \left(\partial_a J_b - \partial_b J_a\right) + \frac{1}{2}\,\vec{n}\cdot\mathrm{D}_a^{\hat{C}}\vec{n}\times\mathrm{D}_b^{\hat{C}}\vec{n} - n_3\left(\partial_a\hat{C}_b - \partial_b\hat{C}_a\right) - 2\rho^2 n_3 H_{ab} \\
J_a &= \frac{}{2\rho^2}\left(\psi_1^*\mathrm{D}_{Aa}^{C}\psi_1 - \psi_1\bar{\mathrm{D}}_{Aa}^{C}\psi_1^* + \psi_2^*\mathrm{D}_{Aa}^{C}\psi_2 - \psi_2\bar{\mathrm{D}}_{Aa}^{C}\psi_2^*\right)
\end{aligned}$$

- $\mathrm{U_C}(1) \times \mathrm{U_I}(1)$ -Invariant Fields
- $\vec{n} \to \mathrm{O}(3)$ - nonlinear σ model
- $(\vec{p}, \vec{q}) \to G\,(4,\,2)$-nonlinear σ model
- Interaction terms: $T^{*1,0}\mathbb{S}^2_+ \times T^{*0,1}\mathbb{S}^2_- \to \mathbb{R}$
- Static Limit

$$\begin{aligned}\mathcal{H}_{\mathrm{statica}} &= \frac{1}{4}\mathcal{F}_{ij}^2 + \frac{1}{2}(\partial_i\rho)^2 + \frac{1}{8}\rho^2\left(\mathrm{D}_i^{\hat{C}}\vec{n}\right)^2 + \frac{1}{4}\rho^2\left(\partial_i\vec{l}\right)^2 \\ &+ \frac{\rho^2}{4}\left\{n_+\left[\left(\vec{m}-\vec{k}\right)\cdot\partial_i\vec{l}\right]^2 + n_-\left[\left(\vec{m}+\vec{k}\right)\cdot\partial_i\vec{l}\right]^2\right\} \\ &+ \frac{1}{2}\rho^2 J_i^2 + \frac{3}{8}\left(1-n_3^2\right)\rho^4 - \frac{3}{8}\rho^4,\end{aligned}$$

$$\mathcal{F}_{ij} = \partial_i J_j - \partial_j J_i + \frac{1}{2}\,\vec{n}\cdot\mathrm{D}_i^{\hat{C}}\vec{n}\times\mathrm{D}_j^{\hat{C}}\vec{n} + n_3\left(\vec{l}\cdot\partial_i\vec{l}\times\partial_j\vec{l} - \rho^2\epsilon_{ijk}l_k\right)$$

$\rho \in \mathbb{R}$; 4 - v.- f. $J_a \in \mathbb{R}$; 2 independent comp.s $\vec{n} \in \mathbb{R}$, $\vec{n}^2 = 1$, 4 comp.s $\hat{e}_a$

- London Limit $\rho \to \Delta$,

$$\begin{aligned}\mathcal{L} &= \frac{\Delta^2}{8}\left(\mathrm{D}_a^{\hat{C}}\vec{n}\right)^2 + \frac{3}{8}\Delta^4\left(1-n_3^2\right) \\ &+ \frac{1}{16}\left[\vec{n}\cdot\mathrm{D}_a^{\hat{C}}\vec{n}\times\mathrm{D}_b^{\hat{C}}\vec{n} - 2n_3\left(\partial_a\hat{C}_b - \partial_b\hat{C}_a\right)\right]^2.\end{aligned} \tag{1}$$

Reductions of the static pure Yang-Mills Model

L.M., G. Martone (2011)

- $\vec{n} \neq \text{cost}$, $\rho = \Delta = \text{cost}$ e $J_a = 0$,

$$\mathcal{L} = \frac{\Delta^2}{8}\left(\partial_a \vec{n}\right)^2 + \frac{1}{16}\left(\vec{n}\cdot\partial_a\vec{n}\times\partial_b\vec{n} - 4\Delta^2 n_3 H_{ab}\right)^2 - \frac{3}{8}\Delta^4 n_3^2; \tag{2a}$$

- $\vec{n} = \hat{\vec{z}} = \text{cost}$, $\rho \neq \text{cost}$ e $J_a \neq 0$,

$$\mathcal{L} = \frac{1}{4}\left(\partial_a J_b - \partial_b J_a - 2\rho^2 H_{ab}\right)^2 + \frac{1}{2}\left(\partial_a\rho\right)^2 + \frac{1}{2}\rho^2 J_a^2 - \frac{3}{8}\rho^4; \tag{2b}$$

- $\vec{n} \neq \text{cost}$, $\rho = \Delta = \text{cost}$ e $J_a \neq 0$, (Current States)

$$\begin{aligned}\mathcal{L} = \frac{1}{4}&\left[\left(\partial_a J_b - \partial_b J_a\right) + \frac{1}{2}\left(\vec{n}\cdot\partial_a\vec{n}\times\partial_b\vec{n}\right) - 2\Delta^2 n_3 H_{ab}\right]^2 \\ &+ \frac{\Delta^2}{2}J_a^2 + \frac{\Delta^2}{8}\left(\partial_a\vec{n}\right)^2 - \frac{3}{8}\Delta^4 n_3^2.\end{aligned} \tag{2c}$$

The Skyrme Faddeev- Model

- T. H. R. Skyrme, *Proc. R. Soc. Lond. A* **260** (1961), 127; *Nucl. Phys.* **31** (1962), 556.
- L. Faddeev, *Quantisation of Solitons*, preprint IAS Print-75-QS70, 1975

$$E\left[\vec{n}\right] = \int_{\mathbb{R}^3} \left\{ \left(\partial_a \vec{n}\right)^2 + \left(\frac{1}{2} \left(\vec{n} \cdot \partial_a \vec{n} \times \partial_b \vec{n}\right) \right)^2 \right\}^3 x,$$

$$x \to \Lambda x \Rightarrow \int_{\mathbb{R}^3} \left(\partial_a \vec{n}\right)^{2\;3} x \quad \to \quad \Lambda \int_{\mathbb{R}^3} \left(\partial_a \vec{n}\right)^{2\;3} x$$

$$\int_{\mathbb{R}^3} \left(\vec{n} \cdot \partial_a \vec{n} \times \partial_b \vec{n}\right)^{2\;3} x \quad \to \quad \Lambda^{-1} \int_{\mathbb{R}^3} \left(\vec{n} \cdot \partial_a \vec{n} \times \partial_b \vec{n}\right)^{2\;3} x \Rightarrow \Lambda = 1$$

$$\partial_a^2 \vec{n} - \left(\partial_a \mathcal{F}_{ab}\right) \left(\vec{n} \times \partial_b \vec{n}\right) = \left(\vec{n} \cdot \partial_a^2 \vec{n}\right) \vec{n}.$$

The Hopf Charge: hopfions

$$\lim_{|\vec{x}|\to\infty} \vec{n}\left(\vec{x}\right) = \vec{n}_\infty = (0,0,1) \Leftrightarrow \vec{n} : \mathbb{S}^3 \to \mathbb{S}^2$$

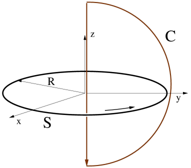
The Hopf Invariant $\quad N\left[\vec{n}\right] \in \pi_3\left(\mathbb{S}^2\right) = \mathbb{Z}$

$\mathcal{H} = \frac{1}{2}\left(\vec{n}\cdot\partial_a\vec{n}\times\partial_b\vec{n}\right) x_a \wedge x_b$ is closed $\mathcal{H} = 0$

$H^2\left(\mathbb{S}^3\right) = \{0\} \Rightarrow \mathcal{A} = \mathcal{A}_a x_a \;:\; \mathcal{H} = \mathcal{A}$

$$N\left[\vec{n}\right] = \frac{1}{4\pi^2}\int_{\mathbb{S}^3} \mathcal{H}\wedge\mathcal{A}.$$

$n_1 + n_2 =^{(m\phi - n\psi)} \sin\Theta,\; n_3 = \cos\Theta \qquad C : \vec{n} = \vec{n}_\infty,\; S : \vec{n} = -\vec{n}_\infty$



The energy bound

$$E[\vec{n}] \geq c\,|N[\vec{n}]|^{3/4}, \quad c \approx (3/16)^{3/8}$$

- A. F. Vakulenko, L. V. Kapitansky, *Sov. Phys. Dokl.* **24** (1979), 433
- A. Kundu e Y. P. Rybakov, *J. Phys. A* **15** (1982), 269
- R. S. Ward, *Nonlinearity* **12** (1999), 241
- M. F. Atiyah e N. S. Manton, *Phys. Lett. A* **222** (1989), 438
- L. D. Faddeev e A. J. Niemi, *Nature* **387** (1997), 58.
- R. Battye e P. Sutcliffe, *Proc. Roy. Soc. London A* **455** (1999), 4305;*Phys. Rev. Lett.* **81** (1998), 4798; J. Hietarinta e P. Salo, *Phys. Lett. B* **451** (1999), 60.
- P. Sutcliffe, *Proc. R. Soc. A* **463** (2007), 3001

vortices of higher topological charge are metastable configurations
$N = 7$ Trefoil Knot configurations

Stereographic form of the Skyrme-Faddeev model

$$S^2 \leftrightarrow \mathbb{C} \; \vec{n} = \left(\frac{w+\bar{w}}{w\bar{w}+1}, -\frac{i(w-\bar{w})}{w\bar{w}+1}, \frac{1-w\bar{w}}{w\bar{w}+1} \right) \quad w = \frac{n_1 + \imath n_2}{1-n_3}$$

$$\mathcal{L}_w = \frac{\sum_{i=0}^{3} g_i \partial_i w \; \partial_i}{8\pi^2 (1+w)^2} + \lambda \frac{\sum_{i,j=0,i<j}^{3} g_i \; g_j \left(\partial_i w \; \partial_j - \partial_j w \; \partial_i\right)^2}{16\pi^2 (1+w)^4}.$$

$$U = (w,) \quad U_i = \partial_i U, \quad U_{i,j} = \partial_i \partial_j U.$$

$$\sum_{0 \le i \le j \le 3} K_{ij} [U, U_0, \ldots, U_3] \, U_{ij} - K_0 [U, U_0, \ldots, U_3] = 0$$

$$K_{ij} = \quad g_i \left\{ \delta_{ij} \left[\left(1 + \tfrac{1}{2} U^\dagger U\right)^2 \sigma_1 + \tfrac{\lambda}{2} \mathbf{A} \textstyle\sum_l (1-\delta_{il}) \, g_l U_l \otimes U_l \right] - \lambda (1-\delta_{ij}) \, \mathbf{A} g_j U_i \otimes U_j \right\},$$

$$K_0 = \quad \left\{ \left(1 + \tfrac{1}{2} U^\dagger U\right) \mathbf{AB} \textstyle\sum_{0 \le l \le 3} g_l U_l \otimes U_l - \frac{2\lambda}{1+\frac{1}{2}U^\dagger U} \textstyle\sum_{0 \le l < m \le 3} g_l g_m \left[\mathbf{AC} U_l \otimes U_m \right]^2 \right\} U$$

Lie-point Symmetry Group $\mathbb{R}^4 \otimes SO(3,1) \odot SO(3)$

$\rightarrow$ Lagrangian Symmetries

Hedgehog Solutions

symm. 1D s.alg. $\vec{v} = \imath\left(x\partial_y - y\partial_x\right) + \alpha\left(w\partial_w - \partial\right) \Rightarrow$

$$w = e^{\imath\alpha\varphi}\left(\cot[\theta] + \imath\cot[\chi(r)]\csc[\theta]\right)$$

$$\begin{aligned}\vec{n}\cdot\vec{\sigma} &= U\left(\vec{n}_\infty\cdot\vec{\sigma}\right)U^\dagger \\ U = \exp\left[\chi(r)\,\vec{\nu}(\vartheta,\varphi)\cdot\vec{\sigma}\right] &= \cos\chi(r)\,I + \sin\chi(r)\,\vec{\nu}(\vartheta,\varphi)\cdot\vec{\sigma}\end{aligned}$$

$$\vec{\nu}(\vartheta,\varphi) = \left(\sin(m\vartheta)\cos(n\varphi), \sin(m\vartheta)\sin(n\varphi), \cos(m\vartheta)\right)$$

$$E[\chi]_{n=m=1} = \frac{16\pi}{3}\Delta\int_{\mathbb{R}^+}\left\{\left(\tilde{r}^2 + 2\sin^2\chi\right)\chi'^2 + 2\sin^2\chi + \frac{\sin^4\chi}{\tilde{r}^2}\right\}\tilde{r}$$

$$\left(\tilde{r}^2 + 2\sin^2\chi\right)\chi'' + \sin 2\chi\,\chi'^2 + 2\tilde{r}\chi' - \sin 2\chi\left(1 + \frac{\sin^2\chi}{\tilde{r}^2}\right) = 0$$

$$\tilde{r} = (1/2)\Delta\, r \qquad \chi(0) = \pi \text{ and } \chi(\infty) = 0$$

Hedgehog Solutions

$$g\left(\tilde{r}\right)=\sin\frac{\chi\left(\tilde{r}\right)}{2}.$$

$$\left(8g^4-8g^2-\tilde{r}^2\right)\left(g^2-1\right)g''+g\left[8g^2\left(g^2-2\right)+\tilde{r}^2+8\right]g'^2$$
$$-2\tilde{r}\left(g^2-1\right)g'-\frac{2g\left(2g^2-1\right)\left(g^2-1\right)^2\left(4g^4-4g^2-\tilde{r}^2\right)}{\tilde{r}^2}=0,$$

NO Painlevé

Approximated solutions by rational f.

$$g_{rat}\left(r\right)=\frac{1+a_1\tilde{r}+a_2\tilde{r}^2}{1+a_1\tilde{r}+b_2\tilde{r}^2+b_3\tilde{r}^3+b_4\tilde{r}^4},$$

$$a_1=0.216,\quad a_2=0.230,\quad b_2=0.752,\quad b_3=-0.018,\quad b_4=0.302,$$

Hedgehog Profile

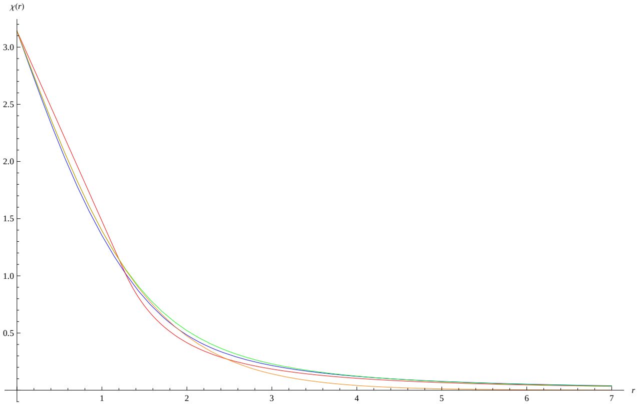


Figura: Blu : numerical solution. Green: $\chi_{rat} = 2 \arcsin g_{rat}$. Red: test χ_p-function. Orange: Atiyah - Manton test function. Length unity $2\Delta^{-1}$

Hedgehog Shape

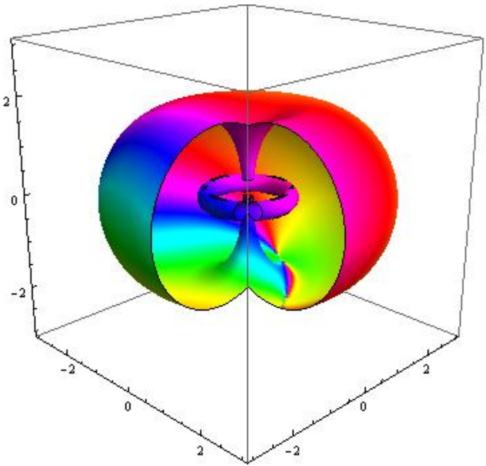


Figura: Hedgehog N=1 level surfaces $n_3 = 0.9$ e $n_3 = -0.9$. Color $= Hue\left[\arctan\left(n_2/n_1\right)\right]$

Rational Maps Ansatz

$$\vec{n} : S^3 \rightarrow S^2 \hookrightarrow \quad S^2 \times S \quad \rightarrow \quad S^2$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$CP^1 \leftrightarrow z = \tan\left[\theta/2\right] e^{\imath\varphi} \quad \rightarrow w\left(z, r\right)$$

$\omega \in SO\left(3\right)$

On the sphere $\omega_S\left(z\right) = \frac{\alpha z + \beta}{-\bar{\beta} z \bar{\alpha}}, \quad |\alpha|^2 + |\beta|^2 = 1$

In inner $\vec{n}$ or w-space $\Leftrightarrow \omega_T\left(w\right) = \frac{\gamma w + \delta}{-\bar{\delta} w \bar{\gamma}}, \quad |\gamma|^2 + |\delta|^2 = 1$

symmetric map $w\left(\omega_S\left(z\right)\right) = \omega_T\left(w\left(z\right)\right)$

$R\left(z\right) : CP^1 \rightarrow CP^1, \ \deg\left(R\right) = N$ 2-dim + 2-dim Irreducible representations of the $SO(3)$ subgroups (Platonic symm) $\rightarrow$ Klein Polynomials

F. Klein,Lectures on the Icosahedron,(London, Kegan Paul, 1913)

Rational Maps Ansatz

$$\vec{\nu}_R = \frac{1}{1+|R|^2}\left(R+\bar{R}, -\imath\left(R-\bar{R}\right), 1-|R|^2\right) \; U_R = \exp\left[\chi\left(r\right)\vec{\nu}_R\cdot\vec{\sigma}\right]$$

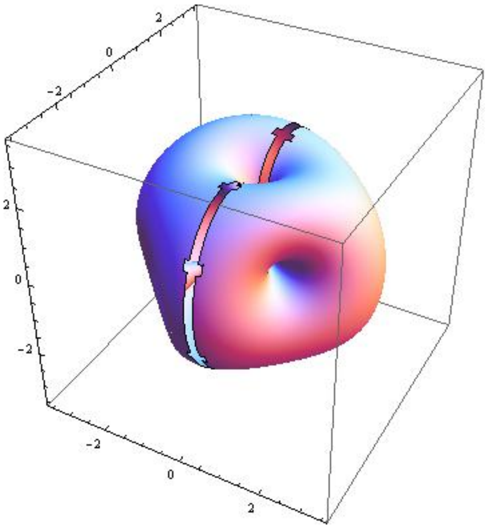
$$E\left[\chi\right]_R = \frac{16\pi}{3}\Delta\int_{\mathbb{R}^+}\left\{\left(\tilde{r}^2+2B_R\sin^2\chi\right)\chi'^2+2B_R\sin^2\chi+I_R\frac{\sin^4\chi}{\tilde{r}^2}\right\}\tilde{r}$$

$$B_R = -N\int_{\mathbb{C}}\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^2\frac{2\imath dz\, d\bar{z}}{\left(1+|z|^2\right)^2}$$

$$I_R = \int_{\mathbb{C}}\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^4\frac{2\imath dz\, d\bar{z}}{\left(1+|z|^2\right)^2}$$

$$R_D = z^2, R_T = \frac{z^3-\sqrt{3}\imath z}{\sqrt{3}\imath z^2-1}, R_O = \frac{z^4+2\sqrt{3}\imath z^2+1}{z^4-2\sqrt{3}\imath z^2+1}, R_Y = \frac{z^7-z^5-7z^2-1}{z^7+z^5-7z^2+1}$$

Rational Maps Ansatz



Conclusions and open problems

- 2c-GL model in Condensed Matter and Pure Yang- Mills in intermediate energies are relevant
- Reduce to similar equations: Skyrme-Faddeev model
- Localized perturbations are Knotted Vortices
- Knotted Vortices are stabilized by Hopf index
- Current states possess different energy bounds
- Approximate solutions can be found in the axisymmetric setting and/or in the rational map ansatz
- Higher symmetries (if any) are unknown
- Reduction/ modification to integrable systems is unknown (not even in 2D)
- Interaction among hopfions is under considerations by numericals and by lattice toroidal moment models (Protogenov,Verbus)

Dynamics in Non-Commutative Spaces and Generalizations

V. Bargmann
On Unitary ray representations of continuous groups
Ann. Math. **59** *(1954) 1.*

J.-M. Lévy-Leblond
$(2+1)\,D$ $[K_1, K_2] = i\kappa$
Group Theory and Applications, Loebl Ed. (1972)

The "Exotic" Galilean symmetry

Can Physics carry "exotic" structure ?

1. Kirillov - Konstant - Souriau method of the Group Coadjoint Orbits
2. Acceleration-dependent Lagrangian
3. A. Ballesteros *et al.*
   Moyal quantization of $2+1$ dimensional Galilean systems
   Journ. Math. Phys. **33**, *3379 (1992).*
4. D. R. Grigore
   Transitive symplectic manifolds in $1+2$ dimensions
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   Phys. Lett. **B 479**, *284 (2000).*
4. J. Lukierski *et al.*
   Galilean-invariant $(2+1)$-dimensional models with a Chern-Simons-like term and $d=2$ noncommutative geometry
   Annals of Physics (N. Y.) **260**, *224 (1997).*

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The Duval - Horvathy Model

$$\Omega = \Omega_0 + eB\, dq_1 \wedge dq_2, \qquad H = H_0 + eV$$
$$B = B(\vec{x}, t), \qquad V = V(\vec{x}, t)$$

DH-model

$$\begin{aligned} m^* \dot{x}_i &= p_i - em\theta\, \epsilon_{ij} E_j, \quad \textit{anomalous velocity} \\ \dot{p}_i &= eE_i + eB\, \epsilon_{ij} \dot{x}_j \quad \textit{Lorentz F.} \\ m^* &= m(1 - e\theta B) \quad \textit{effective mass} \end{aligned}$$

Poisson Structure

$$\{x_1, x_2\} = \frac{m}{m^*}\,\theta, \quad \{x_i, p_j\} = \frac{m}{m^*}\,\delta_{ij}, \quad \{p_1, p_2\} = \frac{m}{m^*}\, eB$$

Coupling to an external E.M. field

$$\begin{array}{llll} \text{Cartan 1-form} & \lambda & = & (p_i - A_i)dx_i - \frac{\vec{p}^{\,2}}{2m}dt + \frac{\theta}{2}\epsilon_{ij}p_i dp_j \\ \text{Lagrange } 2-\text{form} & \sigma & = & d\lambda \qquad \sigma(\tilde{\gamma},\cdot) = 0 \end{array}$$

$$\begin{aligned} \mathcal{A} &= \int_{\tilde{\gamma}} \lambda = \int_{\tilde{\gamma}} \left(f_i(\xi)\dot{\xi}^i - H(\xi)\right) dt \qquad \left[\xi = (\vec{x},\vec{p}),\ \tilde{\gamma} = (\gamma,\dot{\gamma},t) \subset T^*\mathbb{R}^2 \times \mathbb{R}\right] \\ &\neq \int_{\tilde{\gamma}} \frac{\partial L}{\partial v_i}dx^i + \left(L - \frac{\partial L}{\partial v_i}v_i\right)dt = \int_{t_1}^{t_2} L dt \end{aligned}$$

Hamiltonian EOM

$$\dot{\xi}^i = \{\xi^i, H\}, \quad \{\xi^i, \xi^j\} = \left[\left(\partial_{\xi^a} f_b - \partial_{\xi^b} f_a\right)^{-1}\right]_{ij}$$

$$m^* \to 0 \Leftrightarrow \frac{1}{eB_{cr}} = \theta \Rightarrow \text{Constrained System} \Rightarrow \text{Symplectic reduction}$$

The Symplectic Reduction

$$Q_i = x_i + \frac{1 - \sqrt{\frac{m^*}{m}}}{B} \varepsilon_{ij} p_j, \; P_i = \sqrt{\frac{m^*}{m}} p_i - \frac{B}{2} \varepsilon_{ij} Q_j$$

$$(m^* \to 0) \qquad \lambda = f_i \left(\vec{Q} \right) dQ^i - H \left(\vec{Q}, \vec{p} \right)$$

$$\frac{\partial H}{\partial p^i} = 0 \Rightarrow \frac{p_i}{m} = \varepsilon_{ij} \frac{E_j}{B_{cr}} \qquad \text{Hall's motions}$$

$$\frac{\partial H}{\partial Q^i} = -E_i$$

$$\{Q_1, Q_2\} = \frac{1}{eB_{cr}} = \theta, \quad H = V \left(\vec{Q} \right) \qquad \text{(Peierl's subst.)}$$

Quantization and Anyons

$$\begin{array}{l} z = \frac{\sqrt{B}}{2}\left(Q_1 + iQ_2\right) - i\frac{P_1+iP_2}{\sqrt{B}} \\ w = \frac{\sqrt{B}}{2}\left(Q_1 - iQ_2\right) - i\frac{P_1-iP_2}{\sqrt{B}} \end{array}, \Omega_K = \frac{dz \wedge d\overline{z} + dw \wedge d\overline{w}}{2\imath}$$

Bargmann - Fock w.f. $\quad \psi = f\left(z, w\right) \exp\left[-\frac{z\overline{z}+w\overline{w}}{4}\right]$

$$\left[\hat{\overline{z}}, \hat{z}\right] = \left[\hat{\overline{w}}, \hat{w}\right] = 2, \quad \left[\hat{w}, \hat{z}\right] = \left[\hat{\overline{w}}, \hat{\overline{z}}\right] = 0$$

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad \hat{H}_0 = \frac{B}{2m^*}\left(\hat{w}\hat{\overline{w}} + 1\right)$$

$$m^* \rightsquigarrow 0 \quad \text{and} \quad \hat{\overline{w}} f = 0 \Rightarrow \Psi = f\left(z\right) e^{-\frac{z\overline{z}}{4}}$$

ANYONS at the Lowest Landau Level

R. B. Laughlin
Phys.Rev.Lett. **50** ,*1395(1983)*

General noncommutative mechanics

$$\mathcal{L} = p_i \dot{x}_i + \tilde{A}_i(\vec{x},\vec{p})\dot{p}_i - H(\vec{p},\vec{x})$$

$$\{x_i, x_j\} = \epsilon_{ij}\tilde{B} \ \left(\tilde{B} = \epsilon_{k\ell}\partial_{p_k}\tilde{A}_\ell(\vec{x},\vec{p})\right), \ \{x_i, p_j\} = \delta_{ij}, \ \{p_i, p_j\} = 0$$

$$x_i \to q_i = x_i - \tilde{A}_i(\vec{x},\vec{p}) \qquad \text{Commutative Coordinates}$$

$$p_i \dot{x}_i + \tilde{A}_i \dot{p}_i = p_i \dot{q}_i + \frac{d}{dt}(\tilde{A}_i p_i)$$

Examples

- $\tilde{A}_i = \tilde{A}_i(\vec{p})$,

  $\{x_i, x_j\} = \epsilon_{ij}\tilde{B}(\vec{p}) \quad \{p_i, p_j\} = 0$ — DH model

  $\{x_i, p_j\} = \delta_{ij}$ — Berry phase in momentum space

- $\tilde{A}_i = f(p^2)(\vec{x}\cdot\vec{p})p_i$, $\quad \{x_i, x_j\} = \frac{f(p^2)\epsilon_{ij}}{1-p^2 f(p^2)}\,\epsilon_{k\ell}x_k p_\ell$, $\quad \{x_i, p_j\} = \delta_{ij} + \frac{f(p^2)}{1-p^2 f(p^2)}p_i p_j$

1) $f = \dfrac{\theta}{1+p^2\theta}$

   H.S. Snyder, *Phys. Rev.* **71**, 38 (1947)

2) $f \to \infty, \; H = \kappa \ln(p^2/2)$

   Conserved q. $\quad G_i = p_i t + \dfrac{p^2}{2\kappa}x_i \qquad \{G_i, p_j\} = \dfrac{\delta_{ij}p^2 - 2p_i p_j}{2\kappa},$

   $\{H, G_i\} = p_i \;, \quad \{G_i, G_j\} = 0$

κ-deformed Galilei algebra $\{H, p_i, J, G_i\}$

de Azcarraga *et al. J. Math. Phys.* **36**, 6879 (1995)

Group Coadjoint Orbit $SO(2,1)$

$$
\begin{aligned}
\Omega_r &= dp_\alpha \wedge dx^\alpha + \frac{s}{2}\epsilon^{\alpha\beta\gamma}\frac{p_\alpha dp_\beta \wedge dp_\gamma}{(p^2)^{3/2}}, \\
H_r &= \frac{1}{2m}\left(p^2 - m^2c^2\right).
\end{aligned}
$$

L. Fehér, Ph. D. Thesis (1987);

Skagerstam , Stern, *Int. Journ. Mod. Phys.* **A 5**, 1575 (1990)

Lorentz alg. $J_\mu = \epsilon_{\mu\nu\rho}x^\nu p^\rho + s\frac{p_\mu}{\sqrt{p^2}}, \quad \{J^\alpha, J^\beta\} = \epsilon^{\alpha\beta\gamma}J_\gamma,$

Jackiw-Nair limit $\quad s/c^2 \to m^2\theta \quad \Rightarrow \quad \Omega_r\Big|_{H_r=0} \to \Omega_0$

$c \to \infty \quad H_r \to 0 \quad \frac{\epsilon_{ij}J^j}{c} \to K_i = mx_i - p_i t + m\theta\epsilon_{ij}$

Minimal substitution/addition

minimal addition for NC variables - Souriau method

DH - model $\mathcal{L}_{DH-em} = \mathcal{L}_{DH} + e\left(A_i\dot{x}_i + A_0\right)$

local gauge T. $A_\mu(\vec{x},t) \to A_\mu(\vec{x},t) + \partial_\mu\Lambda(\vec{x},t)$

Gauge Invariance $\mathcal{L}_{DH-em} \to \mathcal{L}_{DH-em} + \frac{d}{dt}\Lambda$

minimal substitution for NC variables

$$\widetilde{\mathcal{L}}_{ext} = P_i\dot{X}_i + \frac{\theta}{2}\varepsilon_{ij}P_i\dot{P}_j - \frac{1}{2}(P_i - e\hat{A}_i)^2 + e\hat{A}_0$$

$$\delta\hat{A}_\mu(\vec{X},t) = \hat{A}'_\mu(\vec{X}+\delta\vec{X},t) - \hat{A}_\mu(\vec{X},t) = \partial_\mu\Lambda(\vec{X},t)$$

$$\delta X_i = -e\theta\epsilon_{ij}\partial_j\Lambda, \quad \delta P_i = e\partial_i\Lambda$$

$$\Rightarrow \delta\widetilde{\mathcal{L}}_{ext} = e\frac{d}{dt}\left(\Lambda + \frac{\theta}{2}\varepsilon_{ij}\partial_i\Lambda\, p_j\right)$$

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minimal addition for NC variables - Souriau method

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$$\delta\hat{A}_\mu(\vec{X}, t) = \hat{A}'_\mu(\vec{X} + \delta\vec{X}, t) - \hat{A}_\mu(\vec{X}, t) = \partial_\mu \Lambda(\vec{X}, t)$$

$$\delta X_i = -e\theta\epsilon_{ij}\partial_j\Lambda, \quad \delta P_i = e\partial_i\Lambda$$

$$\Rightarrow \delta\widetilde{\mathcal{L}}_{ext} = e\frac{d}{dt}\left(\Lambda + \frac{\theta}{2}\varepsilon_{ij}\partial_i\Lambda\, p_j\right)$$

Generalized Gauge Transf.

$$\delta_0 \hat{A}_\mu(\vec{X},t) \;:=\; \hat{A}'_\mu(\vec{X},t) - \hat{A}_\mu(\vec{X},t) = \partial_\mu \Lambda(\vec{X},t) + e\{\hat{A}_\mu(\vec{X},t), \Lambda(\vec{X},t)\}$$

$$\hat{F}_{\mu\nu} \;=\; \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + e\left\{\hat{A}_\mu, \hat{A}_\nu\right\} \qquad (\{X_i, X_j\}_{LSZ} = \theta\delta_{ij})$$

R. Jackiw, *Phys.Rev.Lett.* **41** (1978) 1635

LSZ - DH correspondence

$$(X_i, P_i)_{(LSZ)} \leftrightarrow (x_i, p_i)_{(DH)}, \qquad \widehat{A}_\mu(\overrightarrow{X}, t) \to A_\mu(\overrightarrow{x}, t)$$

$$x_i = X_i + e\theta\varepsilon_{ij}\widehat{A}_j(\overrightarrow{X}, t), \qquad p_i = P_i - e\widehat{A}_j(\overrightarrow{X}, t)$$

$$\delta x_i = 0 \quad \Rightarrow \delta_0 x_i \;=\; e\{x_i, \Lambda\}$$

$$\{x_i, x_j\}_{DH} = \frac{\theta}{1 - e\theta B\left(\vec{x}, t\right)}, \qquad \Leftrightarrow B\left(\vec{x}, t\right) = \frac{\hat{B}\left(\vec{X}, t\right)}{1 + e\theta\hat{B}\left(\vec{X}, t\right)}$$

$$F_{\mu\nu}(\vec{x}, t) = \frac{\hat{F}_{\mu\nu}(\vec{X}, t)}{1 + e\theta\hat{B}(\vec{X}, t)}$$

classical $(*_{Moyal} \to \cdot)$ Seiberg-Witten transformation

Non canonical systems in 3D

$$\dot{\mathbf{r}} = \frac{\partial \epsilon_n(\mathbf{k})}{\partial \mathbf{k}} - \dot{\mathbf{k}} \times \vec{\Theta}(\mathbf{k}), \quad \dot{\mathbf{k}} = -e\mathbf{E} - e\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}) \qquad \text{(Bloch electron)}$$

$$\dot{\mathbf{r}} = \frac{\partial E_s(\mathbf{k})}{\partial \mathbf{k}} + \dot{\mathbf{k}} \times \vec{\Theta}_s, \qquad \dot{\mathbf{k}} = -e\vec{E}, \qquad \text{(Spin-Hall effect)}$$

$$E_s(\mathbf{k}) = \frac{\hbar^2}{2m}\left(A - Bs^2\right)k^2, \ \vec{\Theta}_s = s\left(2s^2 - \frac{7}{2}\right)\frac{\vec{k}}{k^3}, \qquad s = \pm\frac{1}{2}, \pm\frac{3}{2}$$

$$\dot{\vec{r}} = \vec{p} - \frac{s}{\omega}\,\mathrm{grad}(\frac{1}{n}) \times \vec{p}, \qquad \dot{\vec{p}} = -n^3\omega^2\mathrm{grad}(\frac{1}{n}), \quad \text{(Optical Magnus)}$$

$$M\left(\frac{\partial A_j}{\partial q^i}\right)\dot{\vec{q}} + \vec{F} \times \vec{r} = -\frac{\partial h}{\partial \vec{r}}, \qquad M\dot{\vec{r}} = \frac{\partial h}{\partial \vec{q}}, \qquad \text{(Bogoliubov q-particle)}$$

$$\dot{x}_i = \frac{p_i}{m} + \Theta_{ij}\frac{\partial V}{\partial x_j}, \qquad \dot{p}_i = -m\frac{\partial V}{\partial x_j} + m\Theta_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j} \quad \text{(NC Kepler problem)}$$

Lagrange-Souriau 2-form

$$2D)\ \sigma_{DH} = dp_i \wedge dx_i + \frac{1}{2}\theta\, \epsilon_{ij}\, dp_i \wedge dp_j + eB\, dx_1 \wedge dx_2 + d\left(\frac{\vec{p}^{\,2}}{2m} + eV\right) \wedge dt$$

$$3D)\ \sigma \;=\; [(1-\mu_i)\, dp_i - e\, E_i\, dt] \wedge (dr_i - g_i\, dt) + \tfrac{1}{2}\, e\, B_k\, \epsilon_{kij}\, dr_i \wedge dr_j + \tfrac{1}{2}\kappa_k\, \epsilon_{kij}\, dp_i \wedge dp_j + q_k\, \epsilon_{kij}\, dr_i \wedge dp_j$$

only gauge invariant quantities

closure condition $d\sigma = 0$ (Maxwell's principle)

Kernel condition $\sigma\,(\delta y, \cdot) = 0,\ \delta y = (\delta\vec{r}, \delta\vec{p}, \delta t)$

Dual Maxwell Laws

$$\partial_{p_i} E_j = \partial_{p_i} B_j = 0$$

$$\begin{aligned}
\partial_{r_j} B_j &= 0, & \varepsilon_{kij}\partial_{r_i} E_j &= -\partial_t B_k, \\
\partial_{p_j} \kappa_j &= 0, & \varepsilon_{kij}\partial_{p_i} \left[\left(1-\mu_j\right) g_j\right] &= \partial_t \kappa_k, \\
\partial_t \mu_i &= \partial_{r_i} \left[\left(1-\mu_i\right) g_i\right], & \tfrac{1}{2}\varepsilon_{kij}\, \partial_{r_i} \left[\left(1-\mu_j\right) g_j\right] &= \partial_t q_k, \\
\partial_{r_i} \mu_j &= \varepsilon_{ijk}\partial_{r_j} q_k, & \partial_{r_i}\kappa_j &= \varepsilon_{ijk}\partial_{p_k} \mu_i + \partial_{p_i} q_j - \delta_{ij}\partial_{p_k} q_k \\
\partial_{r_j} \left[\left(1-\mu_i\right) g_i\right] + \partial_{r_i} \left[\left(1-\mu_j\right) g_j\right] &= 0 & i \neq j &= 1,2,3
\end{aligned}$$

$$\partial_t \left(\sum_i \mu_i \right) + \partial_{r_i} \left[\left(1-\mu_i\right) g_i\right] = 0 \text{ mass conservation}$$

$$\kappa_i = \sum_{j\neq i} \left(r_j \partial_{p_j} q_i \left(\vec{p}\right) - r_i \partial_{p_j} q_j \left(\vec{p}\right)\right) + \chi_i \left(\vec{p}\right)$$

Equation of Motion

$$\sigma(\delta y, \cdot) = 0 \quad \delta y = (\delta r_i, \delta p_i, \delta t)$$

$$M^* \, \dot{\vec{r}} = \left((1 - \mathrm{diag}(\mu_i)) \, \vec{g} + e \, \mathcal{K} M^{-1} \cdot \vec{E} \right)$$

$$M^* \dot{\vec{p}} = \frac{e}{\det(M)} \left(R \, \vec{E} - \vec{g}^{\,T} N \vec{B} \right)$$

$$\begin{aligned} M^\star &= M + \left(2 \left(\epsilon_{ijk} \; q_k \right) - e \, \Theta M^{-1} \mathcal{B} \right) \; , M = \mathbf{1} - \mathrm{diag}(\mu_i) - \left(\epsilon_{ijk} \; q_k \right) \\ \Theta_{ij} &= \epsilon_{ijk} \kappa_k \; , \; \mathcal{B}_{ij} = \epsilon_{ijk} B_k , \qquad \det(M) \neq 0 \end{aligned}$$

$$g_i \equiv p_i$$

Hamiltonian Structure

$$\partial_t \vec{\mathcal{A}} = \partial_t \vec{\mathcal{R}} \equiv 0 \Rightarrow \sigma = \omega - dH \wedge dt \qquad d\omega = 0,$$

$$\mathcal{H} = \mathcal{E}\left(\vec{p}, t\right) + \varphi\left(\vec{r}, t\right)$$

$$\omega = \omega_{\alpha\beta}\, d\xi_\alpha \wedge d\xi_\beta = \left(\delta_{i,j} + \Xi_{ij}\right) dr_i \wedge dp_j + \frac{1}{2}\left[\mathcal{B}_{ij}\, dr_i \wedge dr_j - \Theta_{ij}\, dp_i \wedge dp_j\right]$$

$$\omega^{\alpha,\beta} = \left(1 - \tfrac{1}{2}\mathrm{Tr}\left(\Xi^2 + X\left(\mathbf{1} + 2\,\Xi\right)\Theta\right)\right)^{-1}$$

$$\left\{\begin{pmatrix} \Theta + [\Xi, \Theta] & 0 \\ 0 & -\mathcal{B} + [\Xi, \mathcal{B}] \end{pmatrix} + \left[1 - \tfrac{1}{6}\mathrm{Tr}\left(\Xi^2 + \mathcal{B}\,\Theta\right)\right]\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} -\right.$$

$$\left.\begin{pmatrix} 0 & \left(\Xi^2 + X\,\Theta\right)^T \\ -\left(\Xi^2 + \mathcal{B}\,\Theta\right) & 0 \end{pmatrix}\right\},$$

$$\sqrt{\det\left(\omega_{\alpha\beta}\right)} = 1 - \tfrac{1}{2}\mathrm{Tr}\left(\Xi^2 + \mathcal{B}\left(\mathbf{1} + 2\,\Xi\right)\Theta\right) \neq 0$$

Monopole in Momentum Space

$$\Theta = \theta \frac{\mathbf{k}}{k^3} \qquad (k \neq 0)$$

A. Bérard, H. Mohrbach *Phys. Rev.*(2004)

$B \equiv 0, \quad \vec{E} = E\hat{x}, \quad \epsilon_n(\mathbf{k}) = \mathbf{k}^2/2$

$\mathbf{r}(t) = x(t)\widehat{\mathbf{k}}_0 + y(t)\widehat{\mathbf{E}} + z(t)\widehat{\mathbf{n}}, \qquad \widehat{\mathbf{k}}_0 \perp \widehat{\mathbf{E}},\ \widehat{\mathbf{n}} = \frac{\mathbf{k}_0 \wedge \mathbf{E}}{k_0 E}$

$$z(t) = \frac{\theta}{k_0} \frac{eEt}{\sqrt{k_0^2 + e^2E^2t^2}} \Rightarrow \quad \Delta z = \frac{2\theta}{k_0}$$

Fang et al.*Science* **302**, 92 (2003)

Perovskite structure:$SrRuO_3$, AHE, Rashba-Dresselhaus spin-orbit Hamiltonian

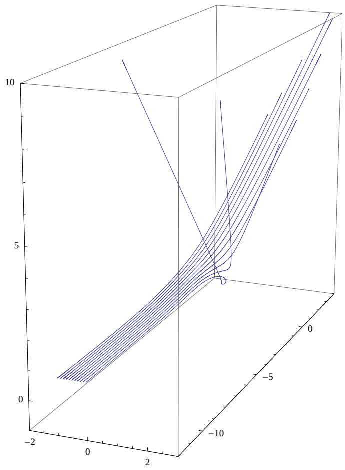
$$H = \sum_i f_i(\mathbf{k})\sigma_i$$

Charge in Magnetic and Dual Monopole

$$
\begin{aligned}
M^* \dot{r}_i &= \left(p_i - e\theta \frac{r_i}{|\vec{p}||\vec{r}|^3} \right) |\vec{r}|^3 |\vec{p}|^3 , \\
M^* \dot{p}_i &= e \varepsilon_{ijk} p_j r_k |\vec{p}|^3
\end{aligned}
$$

$$M^* = |\vec{r}|^3 |\vec{p}|^3 - e\theta \; \vec{r} \cdot \vec{p}$$

D. J. P. Morris et al. *Science* **326**, 411 (2009)


10
5
0
−2
0
2
0
−5
−10

Conclusions and Outlook

- Generalized models of noncommutative mechanics can be considered
- Quantization of the exotic models allows to identify the classical analogs of the Anyons
- The second central extension can be considered as a nonrelativistic shadow of the particle spin in relativistic models.
- In noncommutative models the Minimal Coupling and the Minimal Addition of a gauge field are not equivalent procedures (modulo total time derivatives). A local Seiberg-Witten transformations allows to map systems in different phase spaces (endowed with different symplectic structure) and fields acting on, in order to obtain the same physical results.
- Monopoles in momentum space can be conveniently described in the presented formalism.
- The general Hamiltonian Structure of systems described by non commutative configuration variables is described.