Modular transformation of c=1 toric conformal block from isomonodromic deformations

Pavlo Gavrylenko (SISSA)

Based on joint works with Giulio Bonelli, Fabrizio Del Monte, Harini Desiraju, and Alessandro Tanzini

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Isomonodromy-CFT-gauge theory correspondence

The following objects are related:

- $\mathcal{N} = 2^*$ $SU(2)$ gauge theory
- 1-point c=1 Virasoro conformal blocks on a torus (AGT relation)
- Classical isomonodomric tau functions on a torus (spherical case: Gamayun, Iorgov, Lisovyy; toric case: Bonelli, Del Monte, PG, Tanzini)

Extended relation includes $c = \infty$ conformal blocks/quantum Lamé potential, and probably $c = -2$ tau functions.

Isomonodromic deformations: elliptic Calogero-Moser

- Autonomous Calogero-Moser system:

$$(2\pi i)^2 \frac{d^2 Q(t)}{dt^2} = m_0^2 \wp'(2Q(t)|\tau_0)$$

$\wp$ is a Weierstrass $\wp$-function:

$$\wp(z|\tau) = \frac{1}{z^2} + \sum_{(n,m)\in\mathbb{Z}^2\setminus(0,0)} \left(\frac{1}{(z-n-m\tau)^2} - \frac{1}{(n+m\tau)^2} \right)$$

- Non-autonomous Calogero-Moser system (special case of Painlevé VI equation):

$$(2\pi i)^2 \frac{d^2 Q(\tau)}{d\tau^2} = m^2 \wp'(2Q(\tau)|\tau)$$

- There is an autonomous limit $m = \frac{m_0}{\hbar}$, $\tau = \tau_0 + \hbar t$, $\hbar \to 0$, $\tau_0 = const$

Elliptic Calogero-Moser system, solutions

- Autonomous system is integrable = exactly solvable:

$$\frac{\theta_2(2Q(t)|2\tau_0)}{\theta_3(2Q(t)|2\tau_0)} = \frac{\theta_2(2\omega t + \phi_0)|2\tau_{SW})}{\theta_3(2\omega t + \phi_0)|2\tau_{SW})}$$

Parameters τ_{SW}, ω, ϕ_0 are functions of m and of initial conditions.

$$\theta_{3-\epsilon}(z|\tau) = \sum_{n\in\mathbb{Z}+\frac{\epsilon}{2}} q^{\frac{1}{2}n^2} e^{2\pi i n z}, \qquad q = e^{2\pi i \tau}$$

- Non-autonomous system is still exactly solvable:

$$\frac{\theta_2(2Q(\tau)|2\tau)}{\theta_3(2Q(\tau)|2\tau)} = \frac{\sum_{n\in\mathbb{Z}+\frac{1}{2}} C(a+n) e^{4\pi i n\eta} q^{(a+n)^2} \mathcal{B}(a+n,m,q)}{\sum_{n\in\mathbb{Z}} C(a+n) e^{4\pi i n\eta} q^{(a+n)^2} \mathcal{B}(a+n,m,q)}$$

- Solution is highly transcendental, C and $\mathcal{B}$ are explained later.

Isomonodromy-CFT-gauge theory correspondence

- $C(a) = \frac{G(1-m+2a)G(1-m-2a)}{G(1+2a)G(1-2a)}$, G is a Barnes G-function, $G(x+1) = \Gamma(x)G(x)$
- $\mathcal{B}(a, m, q)$ is a toric 1-point Virasoro conformal block in $c = 1$ conformal field theory with external dimension m^2 and internal dimension a^2. Namely, $\mathcal{B}(a, m, q) = \mathrm{tr}_{\mathcal{H}_{a^2}} \left(V_{m^2}(0) q^{L_0 - 1/24} \right)$.
- Equivalently, by AGT, $\mathcal{B}(a, m, q)$ is an instanton partition function of $\mathcal{N} = 2$ $SU(2)$ supersymmetric gauge theory with scalar VEV a and with adjoint multiplet of mass m. It is also called Nekrasov function.
- Such Fourier-like series are called "Kyiv formulas" and were first found by Gamayun, Iorgov, Lisovyy for Painlevé VI in 1207.0787.

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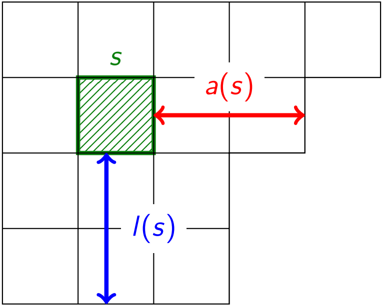
Explicit formulas

$$\mathcal{B}(a,m,q)=\prod_{n=1}^{\infty}(1-q^n)^{1-2m^2}\sum_{Y_+,Y_-}q^{|Y_+|+|Y_-|}\prod_{\epsilon,\epsilon'=\pm}\frac{N_{Y_\epsilon,Y_{\epsilon'}}(m+(\epsilon-\epsilon')a)}{N_{Y_\epsilon,Y_{\epsilon'}}((\epsilon-\epsilon')a)},$$

$$N_{\lambda,\mu}(x)=\prod_{s\in\lambda}(x+a_\lambda(s)+l_\mu(s)+1)\prod_{t\in\mu}(x-l_\lambda(t)-a_\mu(t)-1).$$

$Y_\pm$ are Youg diagrams. l and a are leg and arm lengths:



Several terms of expansion

$$\mathcal{B}(a,m,q) = 1 + q\left(\frac{(m^2-1)m^2}{2a^2}+1\right) + q^2\left(\frac{3(m^2-4)(m^2-1)^2m^2}{16(a^2-\frac{1}{4})^2} - \right.$$

$$\left. - \frac{(m^2-3)(m^2-1)(m^2+2)m^2}{4(a^2-\frac{1}{4})} + \frac{(m^2-1)(m^4-m^2+2)m^2}{4a^2}+2\right) +$$

$$+q^3\left(\frac{(m^2-9)(m^2-4)^2(m^2-1)^2m^2}{36(a^2-1)^2} + \frac{(m^2-4)(m^2-1)^2(2m^4-2m^2+9)m^2}{48(a^2-\frac{1}{4})^2} - \right.$$

$$- \frac{(m^2-4)(m^2-1)(11m^6-106m^4+131m^2+108)m^2}{216(a^2-1)} +$$

$$+ \frac{(m^2-1)(3m^8-22m^6+65m^4-46m^2+24)m^2}{24a^2} -$$

$$\left. - \frac{(m^2-1)(8m^8-24m^6+15m^4+m^2-162)m^2}{108(a^2-\frac{1}{4})} + 3\right) + \ldots$$

Trivial example: $\mathcal{N} = 4$ SYM

- $m = 0$
- $\mathcal{B}(a, 0, q) = \frac{1}{\prod_{i=1}^{\infty}(1-q^n)} = \frac{q^{1/24}}{\eta(q)}$
- $C(a) = 1$
- Defining equation: $\frac{\theta_2(2Q(\tau)|2\tau)}{\theta_3(2Q(\tau)|2\tau)} = \frac{\theta_2(2a\tau+2\eta|2\tau)}{\theta_3(2a\tau+2\eta|2\tau)}$
- $Q = a\tau + \eta$
- $P = 2\pi i a$
- $H = (2\pi i a)^2$

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Isomonodromic Lax pair

- Non-autonomous Calogero-Moser system is described by compatibility condition of two equations (Takasaki, or Korotkin for different form):

$$\mathcal{Y}(z)L_z(z,\tau) = \partial_z\mathcal{Y}(z), \qquad 2\pi i\partial_\tau\mathcal{Y}(z) = \mathcal{Y}(z)L_\tau(z,\tau),$$

which gives

$$2\pi i\partial_\tau L_z(z,\tau) = [L_z(z,\tau), L_\tau(z,\tau)] + \partial_z L_\tau(z,\tau)$$

- Lax matrices are

$$L_z(z,\tau) = \begin{pmatrix} P(\tau) & mx(-2Q(\tau),z) \\ mx(2Q(\tau),z) & -P(\tau) \end{pmatrix},$$

$$L_\tau(z,\tau) = -m\begin{pmatrix} 0 & y(-2Q(\tau),z) \\ y(2Q(\tau),z) & 0 \end{pmatrix},$$

where $x(\xi,z) = \frac{\theta_1(z-\xi|\tau)\theta_1'(0|\tau)}{\theta_1(z|\tau)\theta_1(\xi|\tau)}$, $\quad y(\xi,z) = \partial_\xi x(\xi,z)$.

Isomonodromic Lax pair

- Transformation properties of the Lax matrix:

$$L_z(z+\tau,\tau) = e^{-2\pi i Q(\tau)\sigma_3} L_z(z,\tau) e^{2\pi i Q(\tau)\sigma_3}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$L_z(z+1,\tau) = L_z(z,\tau)$$

$Q(\tau)$ is a modulus of the vector bundle

Monodromy parameterization

- Monodromies are M_0, M_A, M_B:

$$\mathcal{Y}(z+1,\tau) = M_A \mathcal{Y}(z,\tau),$$
$$\mathcal{Y}(z+\tau,\tau) = M_B \mathcal{Y}(z,\tau) e^{2\pi i Q(\tau)\sigma_3},$$
$$\mathcal{Y}(e^{2\pi i} z,\tau) = M_0 \mathcal{Y}(z,\tau).$$

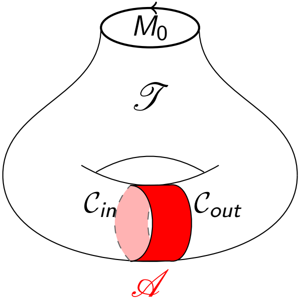
- Relation of monodromies to parameters in the Kyiv formula:

$$M_A = \begin{pmatrix} e^{2\pi i a} & 0 \\ 0 & e^{-2\pi i a} \end{pmatrix}, \quad M_B = \begin{pmatrix} \frac{\sin \pi(2a-m)}{\sin 2\pi a} e^{-2\pi i \eta} & -\frac{\sin \pi m}{\sin 2\pi a} e^{-2\pi i \eta} \\ \frac{\sin \pi m}{\sin 2\pi a} e^{2\pi i \eta} & \frac{\sin \pi(2a+m)}{\sin 2\pi a} e^{2\pi i \eta} \end{pmatrix}$$

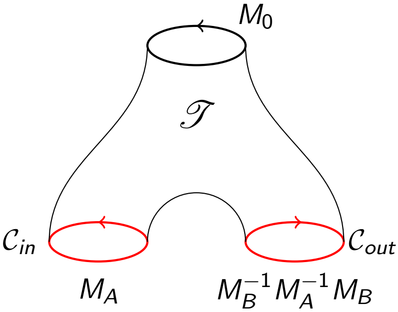
$$M_0 = M_A^{-1} M_B^{-1} M_A M_B.$$

- a and η are the complex Fenchel-Nielsen coordinates.

Approximate solution of the linear system



Pants decomposition of $C_{1,1}$



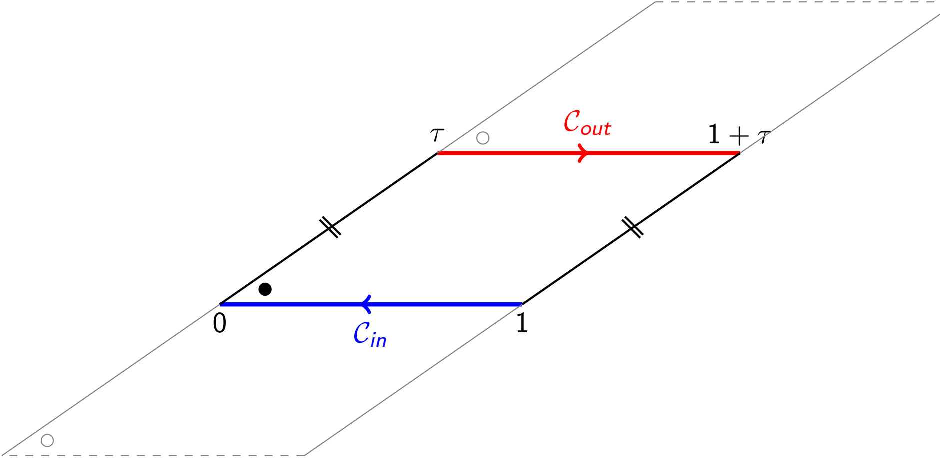
Trinion

- Linear system on the trinion:

$$\partial_z\widetilde{\mathcal{Y}}(z) = \widetilde{\mathcal{Y}}(z)L_{3pt}(z), \quad L_{3pt}(z) = -2\pi i A_0 - 2\pi i\frac{A_1}{1-e^{2\pi i z}}.$$

- Monodromies of $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ should match, so function $\widetilde{\mathcal{Y}}(z)^{-1}\mathcal{Y}(z)$ is single-valued on the trinion.

Another way to draw a torus



Riemann-Hilbert problem on a torus

- Consider the ratio $\Psi(z) = \widetilde{\mathcal{Y}}(z)^{-1}\mathcal{Y}(z)$
- It is single-valued and invertible on a trinion (or in the fundamental domain) $\Im z \in (-\Im\tau/2, \Im\tau/2)$
- $\Psi(z+1) = \Psi(z)$
- Jump on the B-cycle: $\Psi(z+\tau) = J(z)\Psi(z)e^{2\pi i \boldsymbol{Q}[J]}$

  Formula for the jump: $J(z) = \widetilde{\mathcal{Y}}(z+\tau)^{-1} M_B \widetilde{\mathcal{Y}}(z)$
- Solution of the Riemann-Hilbert problem gives $\mathcal{Y}(z)$ from known $\widetilde{\mathcal{Y}}(z)$
- Ordinary Riemann-Hilbert problem on a complex sphere: find holomorphic invertible matrix-valued function $\Psi(z)$ with jump $J(z)$ on some contour Γ, such that $\Psi_+(z)|_\Gamma = J(z)\Psi_-(z)|_\Gamma$

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Fredholm determinant from the 3-pt problem

- Formula for solution on the 3-punctured sphere:

$$\widetilde{\mathcal{Y}}(z) = (1-e^{-2\pi i z})^m \times \mathrm{diag}(e^{2\pi i a z}, e^{-2\pi i a z}) \times$$
$$\times \begin{pmatrix} {}_2F_1(m, m-2a, -2a, e^{-2\pi i z}) & -\frac{m}{2a}\,{}_2F_1(1+m, m-2a, 1-2a, e^{-2\pi i z}) \\ \frac{m e^{-2\pi i z}}{2a+1}\,{}_2F_1(1+m, 1+m+2a, 2+2a, e^{-2\pi i z}) & {}_2F_1(m, 1+m+2a, 1+2a, e^{-2\pi i z}) \end{pmatrix}$$

- Determinant:

$$\mathcal{T}^{(1,1)} = \det_{\mathcal{H}_-\oplus\mathcal{H}_+} \begin{pmatrix} \mathbb{I} - e^{-2\pi i\rho+\tau\partial_z}\Pi_- J^{-1} & \Pi_- J \\ \Pi_+ J^{-1} & \mathbb{I} - e^{2\pi i\rho-\tau\partial_z}\Pi_+ J \end{pmatrix}.$$

Π_+ is a projector onto modes $e^{-2\pi i n}$, $n \geq 0$, $\qquad \Pi_- = \mathbb{I} - \Pi_+$

- $e^{2\pi i\rho}$ is a $U(1)$ modification of M_B

- Relation to the solution: $\#\mathcal{T}^{(1,1)} = e^{2\pi i\rho}\prod_{\epsilon=\pm}\frac{\theta_1(\rho-\epsilon Q)}{\eta(\tau)}\mathcal{T}_{CM}$. Value of Q is defined by zeros of $\mathcal{T}^{(1,1)}$ in ρ

- Tau function: $2\pi i\partial_\tau \log\mathcal{T}_{CM}(\tau) := H_{CM}(\tau)$

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Modular transformation $\widetilde{\tau} = -1/\tau$

$$(2\pi i)^2 \frac{d^2 Q(\tau)}{d\tau^2} = m^2 \wp'(2Q(\tau)|\tau)$$

- If $Q(a, \eta, m, \tau)$ is a solution, then $-\tau Q(\tilde{a}, \tilde{\eta}, m, -1/\tau)$ is another solution.
- Monodromy parameterization in different channels:

$$M_A = \begin{pmatrix} e^{2\pi i a} & 0 \\ 0 & e^{-2\pi i a} \end{pmatrix}, \quad M_B = \begin{pmatrix} \frac{\sin\pi(2a-m)}{\sin 2\pi a} e^{-2\pi i \eta} & -\frac{\sin\pi m}{\sin 2\pi a} e^{-2\pi i \eta} \\ \frac{\sin\pi m}{\sin 2\pi a} e^{2\pi i \eta} & \frac{\sin\pi(2a+m)}{\sin 2\pi a} e^{2\pi i \eta} \end{pmatrix}$$

up to overall conjugation:

$$M_A = \begin{pmatrix} \frac{\sin\pi(2\tilde{a}-m)}{\sin 2\pi\tilde{a}} e^{-2\pi i \tilde{\eta}} & -\frac{\sin\pi m}{\sin 2\pi\tilde{a}} e^{-2\pi i \tilde{\eta}} \\ \frac{\sin\pi m}{\sin 2\pi\tilde{a}} e^{2\pi i \tilde{\eta}} & \frac{\sin\pi(2\tilde{a}+m)}{\sin 2\pi\tilde{a}} e^{2\pi i \tilde{\eta}} \end{pmatrix}, \quad M_B = \begin{pmatrix} e^{-2\pi i \tilde{a}} & 0 \\ 0 & e^{2\pi i \tilde{a}} \end{pmatrix}$$

Explicit monodromy mapping

We should have $\operatorname{tr} M_A = \operatorname{tr} M_A$, $\operatorname{tr} M_A M_B = \operatorname{tr} M_A M_B$, $\operatorname{tr} M_B = \operatorname{tr} M_B$. Explicit expressions are

$$a = \frac{1}{4\pi i} \log \frac{\sin(\pi\widetilde{a} - \pi m/2 + \pi\eta)\sin(\pi\widetilde{a} + \pi m/2 - \pi\eta)}{\sin(\pi\widetilde{a} - \pi m/2 - \pi\eta)\sin(\pi\widetilde{a} + \pi m/2 + \pi\eta)},$$

$$\widetilde{\eta} = a + \frac{1}{2\pi i} \log \frac{\sin(\pi\widetilde{a} - \pi m/2 - \pi\eta)}{\sin(\pi\widetilde{a} + \pi m/2 - \pi\epsilon)},$$

and the inverse formulas

$$\widetilde{a} = \frac{1}{4\pi i} \log \frac{\sin(\pi a - \pi m/2 + \pi\widetilde{\eta})\sin(\pi a + \pi m/2 - \pi\widetilde{\eta})}{\sin(\pi a - \pi m/2 - \pi\widetilde{\eta})\sin(\pi a + \pi m/2 + \pi\widetilde{\eta})},$$

$$\eta = \widetilde{a} + \frac{1}{2\pi i} \log \frac{\sin(\pi a - \pi m/2 - \pi\widetilde{\eta})}{\sin(\pi a + \pi m/2 - \pi\widetilde{\eta})}.$$

Modular transformation

- Trivial example: $Q(a,\eta,0,\tau) = a\tau + \eta$,
  $Q(a,\eta,0,\tau) = -\tau Q(\eta,-a,0,-1/\tau)$, $\tilde{a} = \eta$, therefore $\tilde{\eta} = -a$.
- Transformation of the tau function:

$$\mathcal{T}_{CM}(a,\eta,m,\tau) = \Upsilon \cdot \mathcal{T}_{CM}(\tilde{a},\tilde{\eta},m,-1/\tau) e^{2\pi i Q(a,\eta,m,\tau)^2/\tau} \tau^{-m^2}$$

- Trivial example: $\mathcal{T}_{CM}(a,\eta,0,\tau) = e^{2\pi i a^2 \tau}$,

$$\mathcal{T}_{CM}(\tilde{a},\tilde{\eta},0,-1/\tau) = e^{-2\pi i \eta^2/\tau} = \\ = e^{-4\pi i \tilde{a}\tilde{\eta}} \mathcal{T}_{CM}(a,\eta,0,\tau) e^{-2\pi i Q(a,\eta,0,\tau)^2/\tau}$$

- Tau function $e^{2\pi i \rho} \frac{\theta_1(\rho - Q|\tau)\theta_1(\rho+Q|\tau)}{\eta(\tau)^2} \mathcal{T}_{CM}$ has modular prefactor $e^{2\pi i \rho^2/\tau} \tau^{m^2}$.

Differential of the tau function

- Introduce generalized monodromies and isomonodromic deformations.

$$\mathcal{Y}(z) \sim C_k (z - z_k)^{\boldsymbol{m}_k} G_k \left(1 + O(z - z_k)\right)$$

(in our case $z_k = 0$). Instead of preserving M_k, now preserve C_k.

- Prove by explicit derivative computation that $\mathrm{d} \log \mathcal{T} = \omega - \omega_{3pt}$, where

$$\omega = \frac{1}{2\pi i} H_\tau \mathrm{d}\tau + P \mathrm{d}_{\mathcal{M}} Q + \mathrm{tr}\, \boldsymbol{m} \mathrm{d}_{\mathcal{M}} G_0 G_0^{-1}$$

is the analog of Bertola-Malgrange 1-form,

$$\omega^{3pt} = \mathrm{tr}\, \boldsymbol{m} \mathrm{d} G_0^{3pt} (G_0^{3pt})^{-1} + \mathrm{tr}\, \boldsymbol{a} \mathrm{d} G_+^{3pt} (G_+^{3pt})^{-1} - \mathrm{tr}\, \boldsymbol{a} \mathrm{d} G_-^{3pt} (G_-^{3pt})^{-1}$$

- Toric analog of Its, Lisovyy, Prokhorov approach.

Connection constant

- Differential of the connection constant is

$$\mathrm{d}\log\left(\Upsilon\frac{\widetilde{\Phi}}{\Phi}\right) = d\log\frac{\mathcal{T}}{\tilde{\mathcal{T}}} = \tilde{\omega}^{3pt} - \omega^{3pt}.$$

- Then it can be integrated:

$$\Upsilon = e^{4\pi i\tilde{a}\tilde{\eta}}\frac{\hat{G}\left(a-m/2+\tilde{\eta}\right)\hat{G}\left(a-m/2-\tilde{\eta}\right)}{\hat{G}\left(a+m/2+\tilde{\eta}\right)\hat{G}\left(a+m/2-\tilde{\eta}\right)}\frac{(2\pi)^m\hat{G}(m)}{e^{i\pi m^2/2}},$$

$$\frac{\widetilde{\Phi}}{\Phi} = \frac{G(1+2a)G(1-2a)}{G(1-m+2a)G(1-m-2a)}\frac{G(1-m+2\tilde{a})G(1-m-2\tilde{a})}{G(1+2\tilde{a})G(1-2\tilde{a})},$$

where $\hat{G}(x) = \frac{G(1+x)}{G(1-x)}$

Tau functions vs. conformal blocks

- Tau function better related to conformal filed theory: $Z_D \sim \#\Phi\mathcal{T}^{(1,1)}$
- Modular transformation of the tau function:

$$Z_D = -\Upsilon \tau^{-m^2} e^{-2\pi i \rho^2/\tau} \widetilde{Z}_D$$

- Modular transformation of conformal block:

$$\mathcal{B}(a, m, \tau) = \tau^{-m^2} \int_{-\infty+i\Lambda}^{\infty+i\Lambda} \mathrm{d}\widetilde{a} S(a, \widetilde{a}) \mathcal{B}(\widetilde{a}, m, \tilde{\tau})$$

- Tau function is an element in the space of conformal blocks that diagonalizes Verlinde loop operators. It also diagonalizes the action of the modular transformations.

Modular transformation of conformal blocks

- $Z_D(a,\eta,m,\tau,\rho) =$
$\frac{1}{\eta(\tau)}\sum_{n,k\in\mathbb{Z}} e^{2\pi i\eta n}e^{2\pi i\tau\left(k+\frac{n}{2}+\frac{1}{2}\right)^2}e^{4\pi i\left(k+\frac{n}{2}+\frac{1}{2}\right)\left(\rho+\frac{1}{2}\right)}\mathcal{B}\left(a+\frac{n}{2},m,\tau\right)$

- $Z_D(\widetilde{a},\widetilde{\eta},m,\tilde{\tau},\widetilde{\rho}) =$
$\frac{1}{\eta(\tilde{\tau})}\sum_{n,k\in\mathbb{Z}} e^{2\pi i\widetilde{\eta} n}e^{2\pi i\tilde{\tau}\left(k+\frac{n}{2}+\frac{1}{2}\right)^2}e^{4\pi i\left(k+\frac{n}{2}+\frac{1}{2}\right)\left(\widetilde{\rho}+\frac{1}{2}\right)}\mathcal{B}\left(\widetilde{a}+\frac{n}{2},m,\tilde{\tau}\right)$

- $\mathcal{B}(a,m,\tau) = \int_0^1 \mathrm{d}\rho \int_0^1 \mathrm{d}\eta\, \eta(\tau)e^{-i\pi\tau/2}e^{-2\pi i(\rho+1/2)}Z_D(a,\eta,m,\tau,\rho)$

- The answer for the fusion kernel:

$$S(a,\widetilde{a}) = \Upsilon(a,\widetilde{\eta})\left.\frac{d\widetilde{\eta}}{d\widetilde{a}}\right|_a$$

- Explicit expression, simpler than Ponsot-Teschner integral.

Relation to canonical transformation

- Generating function of the canonical transformation: $4\pi i F(a,\widetilde{\eta}) = \Lambda(a - m/2 - \widetilde{\eta}) - \Lambda(a + m/2 + \widetilde{\eta}) - \Lambda(a + m/2 - \widetilde{\eta}) + \Lambda(a - m/2 + \widetilde{\eta}) + \Lambda(m) + i\pi m^2/2$
- $\Lambda(x) = \log \hat{G}(x) + x \log \sin \pi x$, $\Lambda'(x) = \log(2\pi \sin \pi x)$.
- Canonical transformation: $\widetilde{a} = \frac{\partial F(a,\widetilde{\eta})}{\partial \tilde{\eta}}$, $\eta = \frac{\partial F(a,\widetilde{\eta})}{\partial a}$.
- Fusion kernel: $S = e^{4\pi i\left(F - a\frac{\partial F}{\partial a} - m\frac{\partial F}{\partial m}\right)} \left(\frac{\partial^2 F}{\partial \tilde{\eta}^2}\right)^{-1}$

Thank you for your attention!