ONE-LOOP INTEGRABLE S-MATRICES FROM TREE-LEVEL

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BASED ON 2302.04709

AND 2402.12087 WITH MATHEUS FABRI

INTRODUCTION

INTEGRABLE THEORIES IN 1+1 DIMENSIONS ARE CHARACTERISED BY THE PRESENCE OF HIGHER SPIN CONSERVED CHARGES

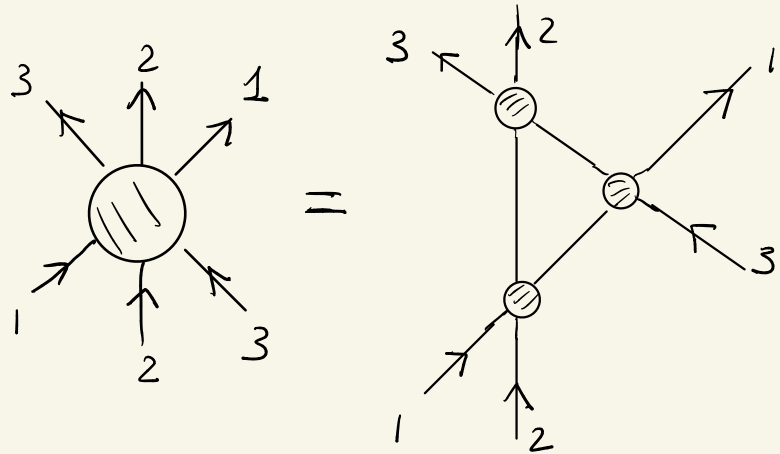
$\downarrow$

SCATTERING DIAGONAL AND FACTORISED

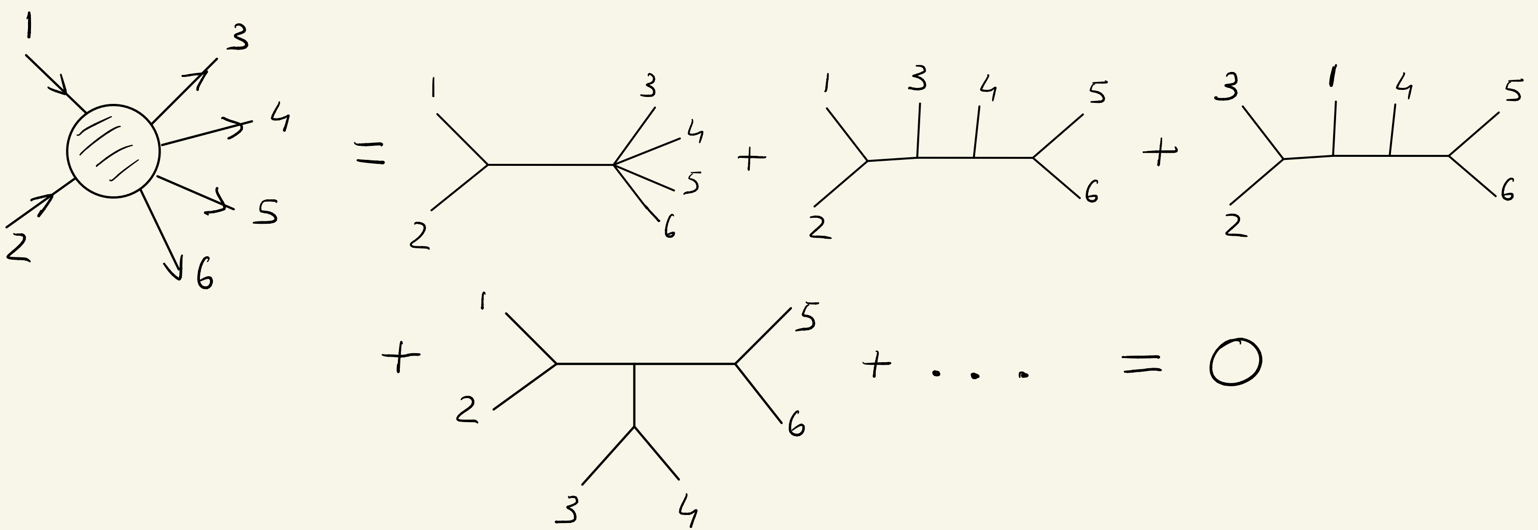
[ZAMOLODCHIKOV & ZAMOLODCHIKOV '78]

S-MATRIX BOOTSTRAP: USE AXIOMS TO CONSTRUCT $2\to 2$ S-MATRICES.

$$S_{3\to 3} = S_{2\to 2}\, S_{2\to 2}\, S_{2\to 2}$$



Perturbation Theory: Integrability manifest itself through surprising cancellations between Feynman diagrams contributing to non-elastic processes (incoming state $\neq$ outgoing state)



WHAT...

SUPPOSE TO HAVE A THEORY WHICH IS INTEGRABLE AT THE TREE LEVEL.

1. IS THE THEORY INTEGRABLE AT ONE LOOP?

2. GIVEN THE TREE-LEVEL S-MATRIX OF AN INTEGRABLE THEORY CAN WE WRITE DOWN ITS ALL-LOOP S-MATRIX?

PLAN

1. REVIEW TREE-LEVEL INTEGRABILITY FOR SIMPLE MODELS

2. DERIVATION OF 1-LOOP S-MATRICES AS FUNCTIONS OF TREE-LEVEL S-MATRICES FOR THESE SIMPLE MODELS.

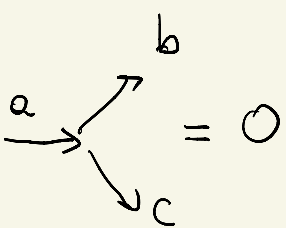
3. OPEN PROBLEMS AND DIRECTIONS.

WHAT CLASS OF THEORIES?

$$\mathcal{L} = \sum_{a=1}^{r} \left[\frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{m_a^2}{2} \phi_a^2 \right] - \sum_{n=3}^{+\infty} \frac{1}{n!} C^{(n)}_{a_1 \dots a_n} \phi_{a_1} \cdots \phi_{a_n}$$

$a = 1, 2, \dots, r$ (PARTICLE LABEL) $\qquad \mu = 0, 1$

<u>PURELY ELASTICITY CONDITION:</u> ALL TREE-LEVEL INELASTIC AMPLITUDES ARE ZERO.



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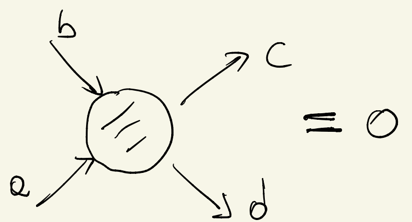
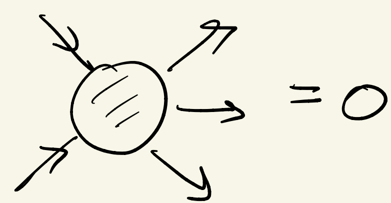


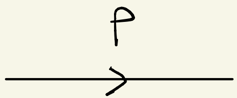
$\{a, b\} \neq \{c, d\}$

,

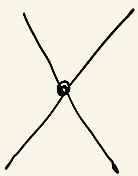


HOW TO CONSTRUCT THESE THEORIES?

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$$



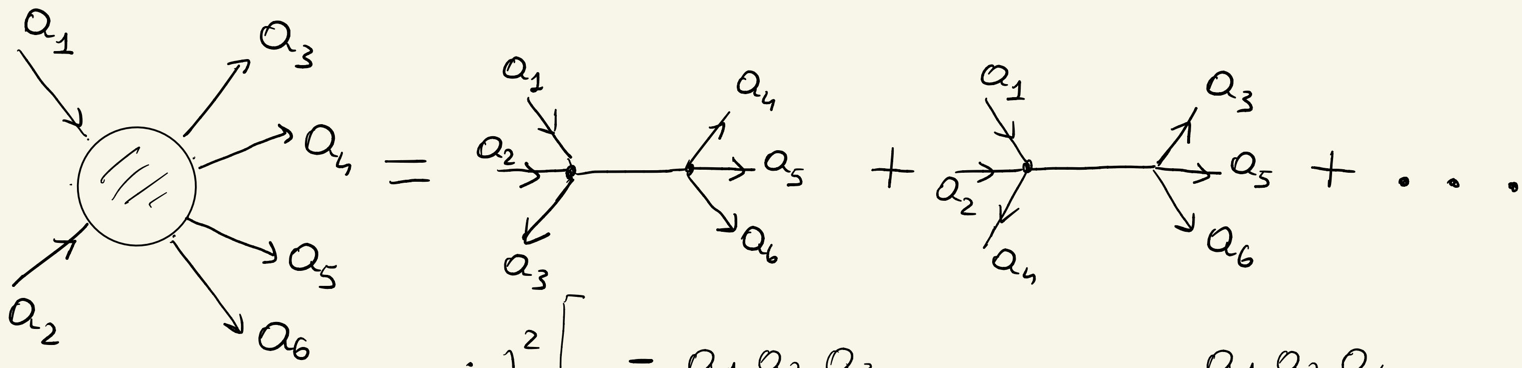
$$= \frac{i}{p^2 - m^2 + i\varepsilon}$$



$$= -i\lambda$$

$$P^\mu = (\underbrace{p}_{p_0+p_1}, \underbrace{\bar{p}}_{p_0-p_1}) = m\left(a, \frac{1}{a}\right) \qquad p\bar{p} = m^2$$

WE CONSIDER THE PROCESS : $1+2 \longrightarrow 3+4+5+6$



$$= \frac{-i\lambda^2}{m^2} \underbrace{\left[\frac{-a_1 a_2 a_3}{(a_1+a_2)(a_2-a_3)(-a_3+a_1)} - \frac{a_1 a_2 a_4}{(a_1+a_2)(a_2-a_4)(-a_4+a_1)} + \dots \right]}_{H(a_1, a_2, a_3, a_4, a_5, a_6)}$$

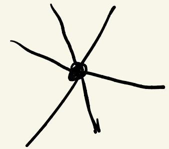
IF MOMENTUM IS CONSERVED

$$a_1 + a_2 = a_3 + a_4 + a_5 + a_6$$

$$\frac{1}{a_1} + \frac{1}{a_2} = \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} + \frac{1}{a_6}$$

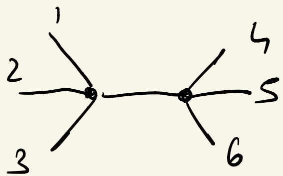
$$\Longrightarrow H(a_1, a_2, a_3, a_4, a_5, a_6) = -1$$

WE CAN THEN INTRODUCE A PROPERLY TUNED 6-POINT COUPLING



$$= -\frac{i\lambda^2}{m^2}$$

SUCH THAT



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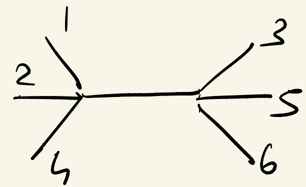
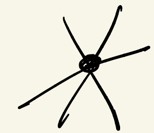


$+ \dots +$



$= 0$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 - \frac{\lambda^2}{6!\,m^2}\varphi^6$$

IMPOSING SIMILARLY THE CANCELLATION OF $2 \to 6$, $2 \to 8$, ... PROCESSES WE OBTAIN THE EXPANSION OF THE FOLLOWING LAGRANGIAN [P. DOREY '98]

$$\boxed{\mathcal{L}_{ShG} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{m^4}{\lambda}\left[ch\left(\frac{\sqrt{\lambda}}{m}\varphi\right) - 1\right]}$$

ALL THIS CONSTRUCTION RELIES ON THE FACT THAT H IS A CONSTANT (SEEDS OF INTEGRABILITY).

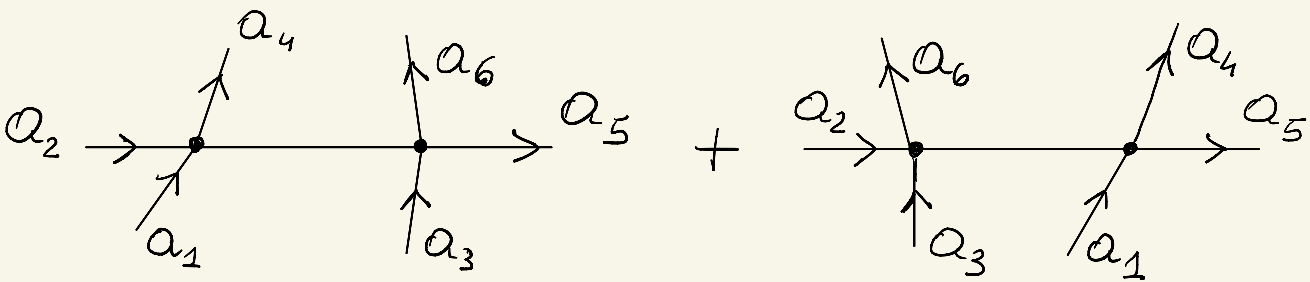
[GABAI, MAZAC, SHIEBER, VIEIRA, ZHOU '18] [P. DOREY, DP '21]

[BERCINI, TRANCANELLI '18]

ELASTIC PROCESSES FACTORISE

$$1+2+3 \longrightarrow 4+5+6$$

IN THE KINEMATICAL REGION $\{a_1 \simeq a_4,\ a_2 \simeq a_5,\ a_3 \simeq a_6\}$



$$\propto \frac{1}{(a_1-a_4)+i\varepsilon} + \frac{1}{-(a_1-a_4)+i\varepsilon} = (-2\pi i)\,\delta(a_1-a_4)$$

PARAMETERISING $p_i = m(\mathrm{ch}\theta_i, \mathrm{sh}\theta_i)$

AND DEFINING $\theta_{ij} \equiv \theta_i - \theta_j$

WE HAVE

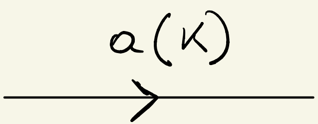
$$S^{(0)}_{3\to 3}(\theta_1, \theta_2, \theta_3) = \Big[S^{(0)}_{2\to 2}(\theta_{13}) S^{(0)}_{2\to 2}(\theta_{23}) + S^{(0)}_{2\to 2}(\theta_{12}) S^{(0)}_{2\to 2}(\theta_{23})$$
$$+ S^{(0)}_{2\to 2}(\theta_{12}) S^{(0)}_{2\to 2}(\theta_{13}) \Big] \delta(\theta_{14}) \delta(\theta_{25}) \delta(\theta_{36})$$

1-LOOP

S-MATRICES

One-loop S-matrices from the tree-level

$$\Pi_a^{(F)}(K) = \Pi_a^{(R)}(K) + \frac{\pi}{\omega_a(K_1)}\,\delta\big(K_0 - \overset{\sqrt{K_1^2+m_a^2}}{\overbrace{\omega_a(K_1)}}\big)$$



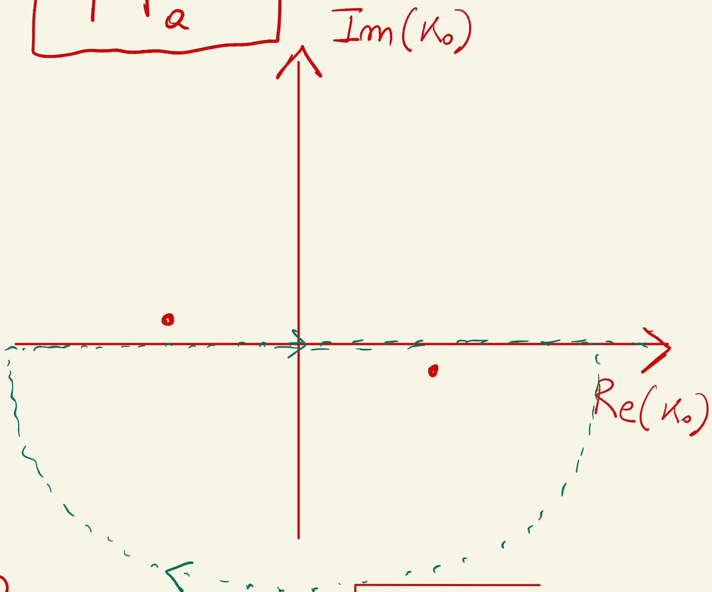
$$\Pi_a^{(F)}(K) = \frac{i}{K^2 - m_a^2 + i\varepsilon}$$

Feynman propagator

$$\Pi_a^{(R)}(K) = \frac{i}{K^2 - m_a^2 - iK_0\varepsilon}$$

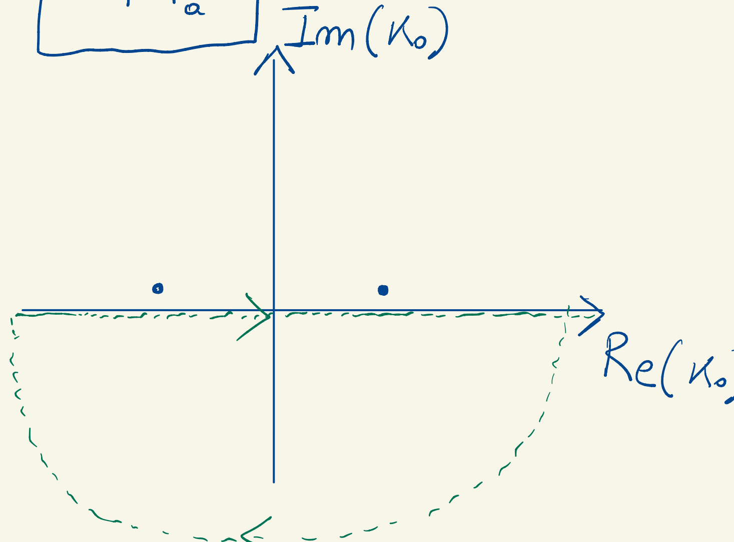
Retarded propagator

$$\boxed{\Pi_a^{(F)}}$$



Poles: $k_0 = \pm\sqrt{k_1^2 + m_a^2} \mp i\varepsilon$

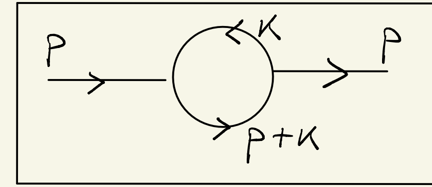
$$\boxed{\Pi_a^{(R)}}$$



Poles: $k_0 = \pm\sqrt{k_1^2 + m_a^2} + i\varepsilon$

LOOP DIAGRAMS CONTAINING ONLY PROPAGATORS OF RETARDED TYPE ARE ZERO. WE CAN CLOSE A PATH WHICH DOES NOT CONTAIN POLES.

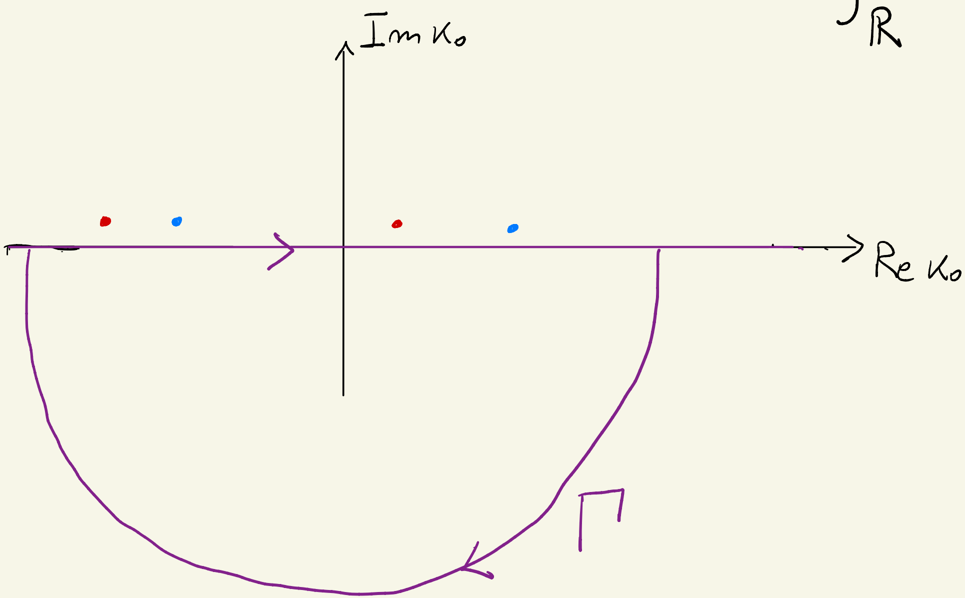
EXAMPLE: BUBBLE DIAGRAM



$$\int_{\mathbb{R}} dk_1 \int_{\mathbb{R}} dk_0 \, \Pi^{(R)}(k) \, \Pi^{(R)}(p+k)$$

$$= \int_{\mathbb{R}} dk_1 \int_{\Gamma} dk_0 \, \Pi^{(R)}(k) \, \Pi^{(R)}(p+k)$$

$$= 0$$



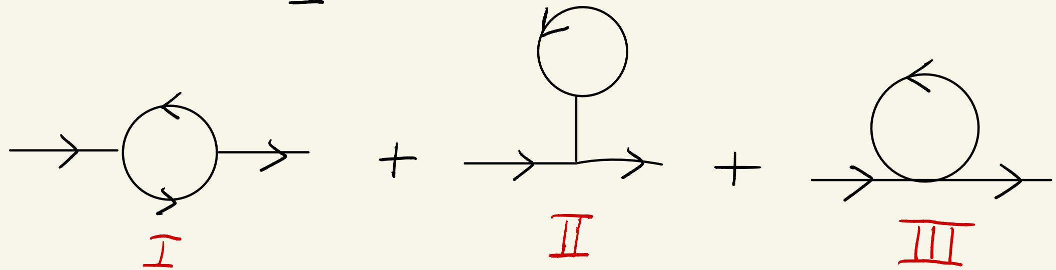
POLES

$$\Pi^{(R)}(k) \longrightarrow k_0 = \pm\sqrt{k_1^2 + m^2} + i\varepsilon$$

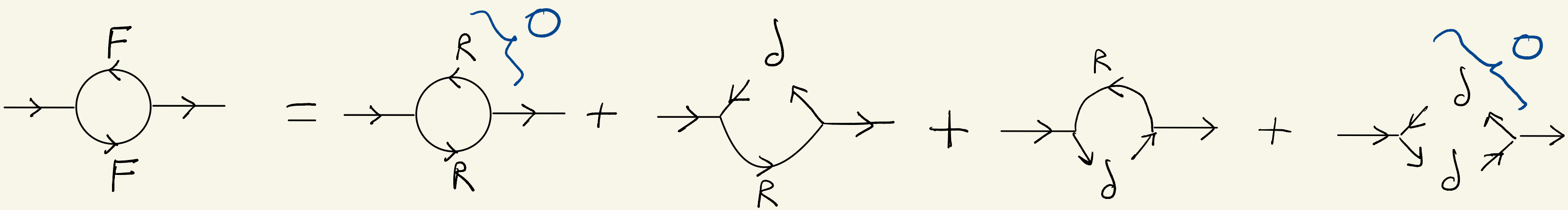
$$\Pi^{(R)}(p+k) \longrightarrow k_0 = -p_0 \pm \sqrt{(p_1+k_1)^2 + m^2} + i\varepsilon$$

CONSIDER A $1 \to 1$ ONE-LOOP AMPLITUDE.

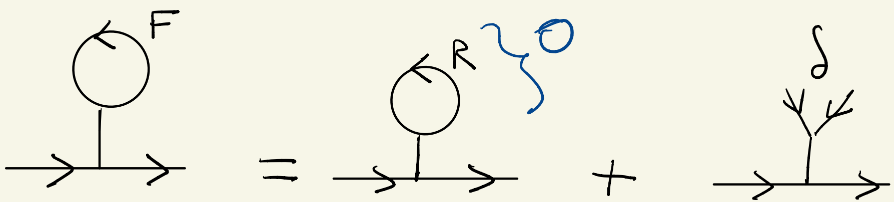
$\mathcal{M}^{(1)}_{1\to 1} =$



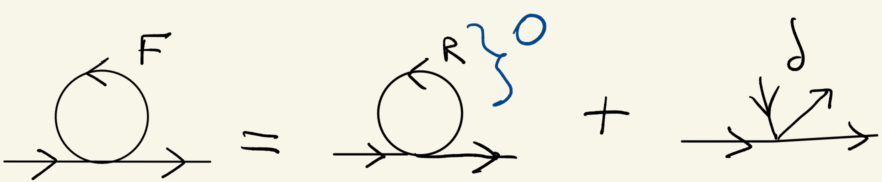
I.



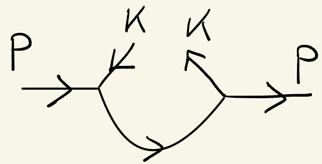
II.



III.



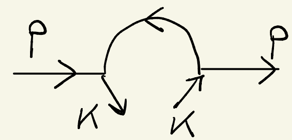
$M^{(1)}_{1\to 1} =$



S-CHANNEL

I

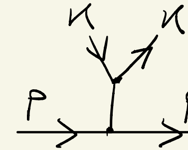
$+$



T-CHANNEL

I

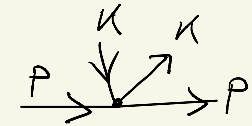
$+$



U-CHANNEL

II

$+$



4-POINT COUPLING

III

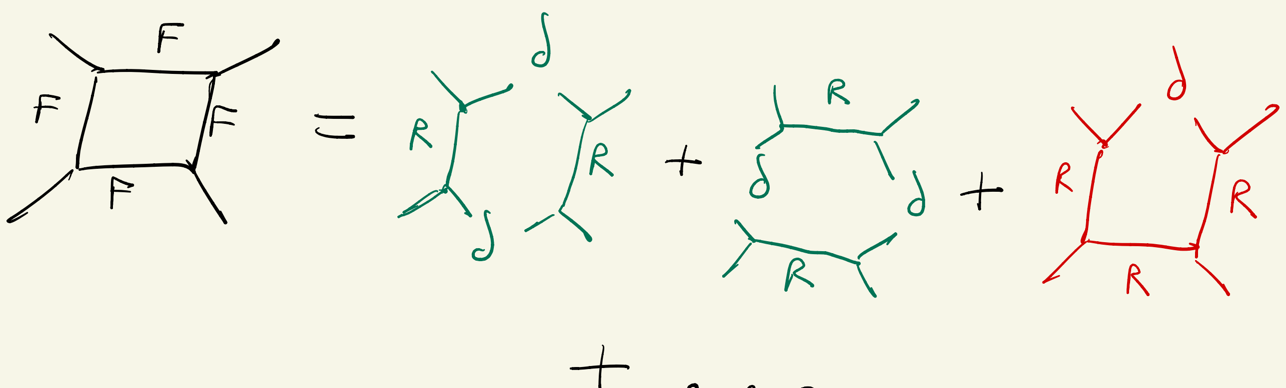
$$M^{(1)}_{1\to 1} = \frac{1}{8\pi}\int_{-\infty}^{+\infty} d\theta_k \; M^{(0)}_{2\to 2}(\theta_k - \theta_p)$$

2-TO-2 PROCESSES: $a(p)+b(p') \to c(q)+d(q')$

- IN 1-TO-1 PROCESSES ONLY DIAGRAMS WITH A SINGLE DIRAC DELTA FUNCTION SURVIVE (SINGLE-CUT CONTRIBUTIONS).

- IN 2-TO-2 PROCESSES WE ALSO HAVE DIAGRAMS CONTAINING PAIRS OF DIRAC DELTA FUNCTIONS (DOUBLE CUT CONTRIBUTIONS)

Ex.



DOUBLE CUT CONTRIBUTIONS

- IF $\{c,d\} \neq \{a,b\}$ THEN $S^{(2\text{-CUT})}_{ab\to cd} = 0$

- IF $\{c,d\} = \{a,b\}$ THEN

$$S^{(2\text{-CUT})}_{ab\to ab}(\theta_p - \theta_{p'}) = \frac{1}{2}\left(S^{(0)}_{ab\to ab}(\theta_p - \theta_{p'})\right)^2$$

SINGLE-CUT CONTRIBUTIONS IN TWO-TO-TWO AMPLITUDES

$$M^{(1\text{-cut})}_{ab\to cd}(p,p';q,q') = \frac{1}{8\pi}\sum_{e=1}^{r}\int d\theta_k \; M^{(0)}_{abe\to cde}(p,p',k;q,q',k)$$

- IF $\{a,b\} \neq \{c,d\}$ THEN $M^{(0)}_{abe\to cde} = 0$
- IF $\{a,b\} = \{c,d\}$ THEN $M^{(0)}_{abe\to abe}$ FACTORISES INTO TREE-LEVEL AMPLITUDES.

$$M^{(0)}_{abe\to abe} \sim M^{(0)}_{ae\to ae}\cdot M^{(0)}_{be\to be}$$

WE OBTAIN THE FOLLOWING FINAL RESULT FOR THE ONE-LOOP

S-MATRIX

$$a(p)+b(p') \longrightarrow a(p)+b(p')$$

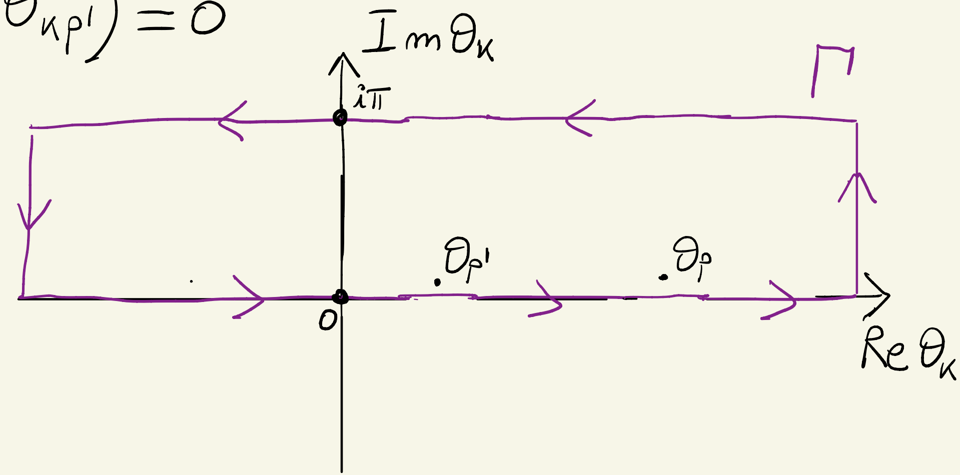
$$\boxed{\begin{aligned} S^{(1)}_{ab}(\theta_{pp'}) &= +\left(C_1+\theta_{pp'}C_2\right)\frac{\partial}{\partial\theta_{pp'}}S^{(0)}_{ab}(\theta_{pp'}) \\ &+ \frac{\left(S^{(0)}_{ab}(\theta_{pp'})\right)^2}{2} + \frac{i}{4\pi}\frac{\partial}{\partial\theta_{pp'}}\sum_{e=1}^{r}\int_{-\infty-i0}^{+\infty-i0} d\theta_k\, S^{(0)}_{ea}(\theta_{kp})\,S^{(0)}_{eb}(\theta_{kp'}) \end{aligned}}$$

INTEGRATED S-MATRICES

$$S^{(0)}_{ea}(\theta) = -S^{(0)}_{ea}(\theta \pm i\pi) \qquad \text{(CROSSING SYMMETRY)}$$

$$\Rightarrow \sum_{e=1}^{n} S^{(0)}_{ea}(\theta_{\kappa p})\, S^{(0)}_{eb}(\theta_{\kappa p'}) \quad \text{IS PERIODIC OF PERIOD } i\pi.$$

$$\Rightarrow \sum_{e=1}^{n} \oint_{\Gamma} S^{(0)}_{ea}(\theta_{\kappa p})\, S^{(0)}_{eb}(\theta_{\kappa p'}) = 0$$



WE INTRODUCE AN AUXILIARY PARAMETER β AND DEFINE

$$I_\beta \equiv \sum_{e=1}^{r} \int_{-\infty-i\varepsilon}^{+\infty-i\varepsilon} d\theta_\kappa \, e^{i\beta\theta_\kappa} \, S_{ea}^{(0)}(\theta_{\kappa p}) \, S_{eb}^{(0)}(\theta_{\kappa p'})$$

THEN IT HOLDS THAT

$$\sum_{e=1}^{r} \oint_{\Gamma} d\theta_\kappa \, e^{i\beta\theta_\kappa} \, S_{ea}^{(0)}(\theta_{\kappa p}) \, S_{eb}^{(0)}(\theta_{\kappa p'}) = I_\beta - e^{-\pi\beta} I_\beta$$

$$\Rightarrow \quad I_\beta = \frac{1}{1-e^{-\pi\beta}} \sum_{e=1}^{n} \oint_\Gamma d\theta_\kappa \, e^{i\beta\theta_\kappa} \, S^{(0)}_{ea}(\theta_{\kappa p}) \, S^{(0)}_{eb}(\theta_{\kappa p'})$$

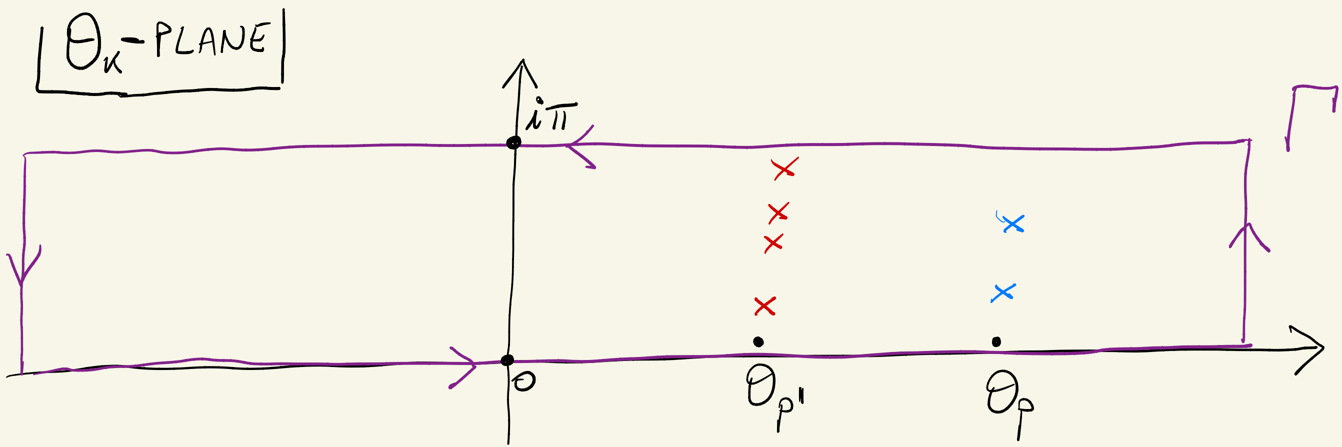
TAYLOR EXPANDING NUMERATOR AND DENOMINATOR IN β WE OBTAIN

$$I_0 = \lim_{\beta \to 0} I_\beta = \frac{i}{\pi} \sum_{e=1}^{n} \oint_\Gamma d\theta_\kappa \, \theta_\kappa \, S^{(0)}_{ea}(\theta_{\kappa p}) \, S^{(0)}_{eb}(\theta_{\kappa p'})$$

$$I_0 = \frac{i}{\pi} \sum_{e=1}^{n} \oint_{\Gamma} d\theta_k \; \theta_k \, S^{(0)}_{ea}(\theta_{kp}) \, S^{(0)}_{eb}(\theta_{kp'})$$

×: POLES OF S_{eb}

×: POLES OF S_{ea}



THE INTEGRAL CAN NOW BE COMPUTED BY CAUCHY THEOREM.

RESULTS

1. WE CHECKED OUR FORMULA ON THE FULL CLASS OF SIMPLY-LACED AFFINE TODA THEORIES FINDING EXACT AGREEMENT WITH RESULTS BOOTSTRAPPED IN THE PAST.
[H. BRADEN, E. CORRIGAN, P. DOREY, R. SASAKI '89]

2. OUR FORMULA PRODUCES THE CORRECT POLE STRUCTURE BOTH FOR BOUND STATES (SIMPLE POLES) AND LANDAU SINGULARITIES.
THESE POLES ARE NOT CAPTURED BY UNITARITY METHODS.
[L. BIANCHI, V. FORINI, B. HOARE '13] [L. BIANCHI, B. HOARE '14]

OPEN PROBLEMS

1. DERIVATIVE INTERACTIONS (GENERALIZED SINE-GORDON THEORIES, $T\bar{T}$-DEFORMATIONS, ...)

2. FERMIONS (THIRRING MODEL)

3. NON-RELATIVISTIC THEORIES & MASSLESS PARTICLES (SIGMA MODELS ON STRINGS)

4. CAN WE GO BEYOND 1-LOOP AND CONSTRUCT ITERATIVELY THE FULL ALL-LOOP S-MATRIX?

THANK you