Aspects of Generalized Symmetries

A high-energy physicist's perspective

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NP-QFT

Generalized symmetries

New paradigm: symmetries in (Euclidean) QFT = topological defect operators, of any dimension

[Gaiotto, Kapustin, Seiberg, Willett 14]

⋆ This leads to a substantial widening of the concept of symmetry, as well as of all related constructions and consequences.

⋆ $U(1)$ symmetry:

conserved current	$\partial^\mu j_\mu(x) = 0$
conserved charge	$Q = \int_\Sigma d^{d-1}x\, j_0$
unitary operators	$U_\alpha = e^{i\alpha Q}$ with $\alpha \in U(1)$
operators in reps:	$U_\alpha\, \mathcal{O}_q = e^{i\alpha q}\, \mathcal{O}_q\, U_\alpha$

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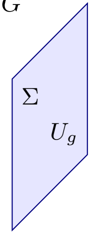
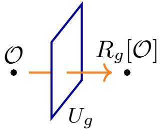
⋆ Standard 0-form symmetry G: codimension-1 defects $U_g[\Sigma]$, $g \in G$ along submanifolds Σ



that fuse according to group structure of G:

$$U_g \times U_h = U_{gh}$$

and act on local operators through representations:

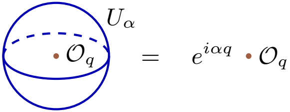


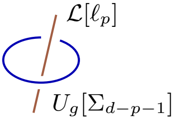
Charge conservation = topological character of defects

⋆ Symmetry defects allow us to treat
finite symmetries (*e.g.* charge conjugation) on equal footing

Various new structures:

- Defects of higher codimension: p-form symmetries (necessarily Abelian)
  Charges carried by p-dimensional extended operators





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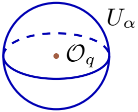
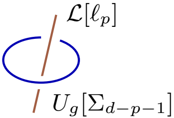
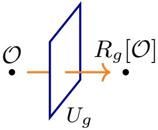
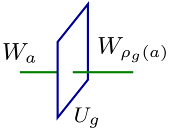


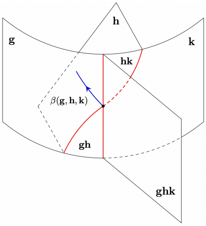
$$= \quad e^{i\alpha q} \cdot \mathcal{O}_q$$



- Symmetries that act on other symmetries (*e.g.*, n-groups): [Baez, Lauda 03]



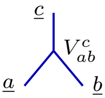




from [FB, Cordova, Hsin 18]

- Fusion algebras instead of groups

$$U_a \times U_b = \sum_c N_{ab}^c \;\; U_c$$



Familiar from Verlinde lines in 2d RCFT's

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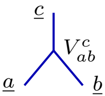


Familiar from Verlinde lines in 2d RCFT's

- TQFT coefficients: $N_{ab}^c \ \rightarrow \ Z_{\mathsf{TQFT}}[\Sigma_{d-p-1}]$ [Roumpedakis, Seifnashri, Shao 22]

- Symmetries obtained by "condensing" other symmetries on submanifolds
(gauging)
[Roumpedakis, Seifnashri, Shao 22]

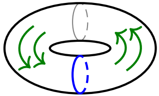
⋆ Symmetries no longer characterized by groups

$\longrightarrow$ "Categorical" or "Non-invertible" symmetries

"Background fields"

In QFT many physical quantities become manifest by turning on background fields

⋆ Insertions of networks of symmetry defects
play the role of (flat) background fields



E.g.: flat connections *vs* symmetry defects on T^2

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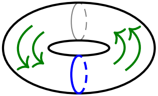

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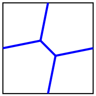
⋆ Gauging (condensation, generalized orbifolding) [Fuchs, Runkel, Schweigert '01]
represented as a sum over insertions on a mesh



Higher gauging: on a submanifold [Roumpedakis, Seifnashri, Shao 22]

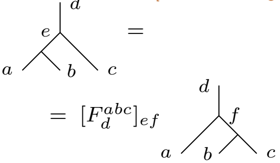
Mathematical Language: Category Theory

In 2 dimensions, (internal, finite) symmetries are described by fusion categories.

- Fusion category:

[*cfr.* Moore, Seiberg 89]

Objects:	top. line defects
Tensor product:	stacking of lines
Morphisms:	fusion algebra $U_a \times U_b = \sum_c N_{ab}^c \, U_c$
Associator or F-symbol:	



Language familiar from 2d RCFTs [Moore, Seiberg 89]

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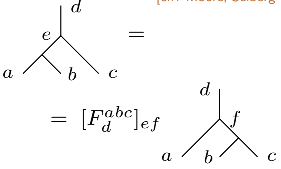

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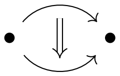
Includes standard 0-form symmetry G with 't Hooft anomaly: $F \in H^3\big(BG, U(1)\big)$

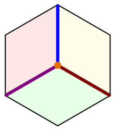
Higher categories

In d dimensions: symmetries form a $(d-1)$-category



- n-category:

Objects	0-form symmetry defects
1-morphisms between objects	junctions of 0-form defects, and 1-form defects
2-morphisms between 1-morphisms	junctions of junctions, ...
...	
n-morphisms	



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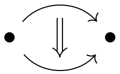

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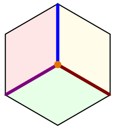

⋆ For 3d theories, Douglas and Reutter gave a definition of spherical (semi-simple) fusion 2-category [Douglas, Reutter 18]

Similar definitions exist in higher dimensions. Topic of active research.

Symmetry TFT

- ⋆ The rigid structure of the symmetry is captured by a Topological Quantum Field Theory (TQFT) in one higher dimension: SymTFT

[Gaiotto, Kapustin, Seiberg, Willett 14; Gaiotto, Kulp 20]
[Apruzzi, Bonetti, García-Etxebarria, Hosseini, Schafer-Nameki 21; Freed, Moore, Teleman 22]

Builds on ideas dating back to Wess and Zumino: anomaly inflow [Wess, Zumino 71]

$d+2$: anomaly polynomial $\rightarrow$ $d+1$: Chern–Simons TFT $\rightarrow$ d: QFT with anomaly

SymTFT is a fully-dynamical TQFT

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$d+2$: anomaly polynomial $\to$ $d+1$: Chern–Simons TFT $\to$ d: QFT with anomaly

SymTFT is a fully-dynamical TQFT

- It appears to capture all aspects of the symmetry: structure, anomalies, global forms, representations, spontaneous breaking, boundary conditions, ...

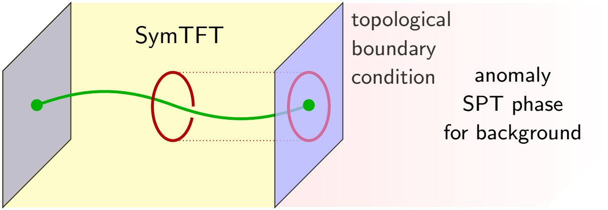
E.g.: 0-form symmetry G (finite group) with anomaly $F \in H^{d+1}\big(BG, U(1)\big)$
$\to$ $(d+1)$-dimensional G gauge theory with Dijkgraaf–Witten twist F

$$S_{\text{TQFT}} = 2\pi i \int_{X_{d+1}} F(\mathfrak{b}) \qquad \mathfrak{b} : G\text{-cocycle}$$

[Dijkgraaf, Witten, 89]

(Not always there is a simple state-sum or path-integral description)

"Slab" construction of the Symmetry TFT:



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  Those operators are trivialized at the boundary

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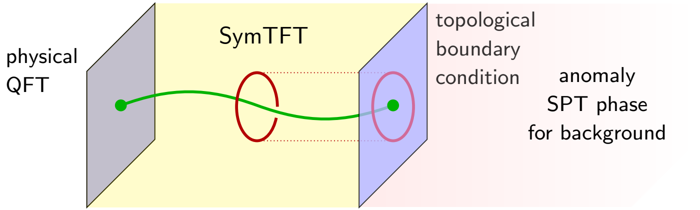

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- Bulk operators modulo Lagrangian algebra

  = topological symmetry defects of boundary theory

  Boundary condition hosts higher category of the symmetry

  Fusion in the bulk $\Rightarrow$ fusion on the boundary

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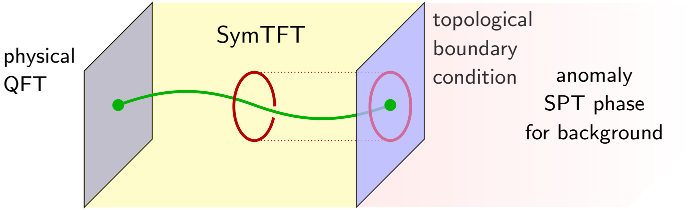

- Operators that end (Lagrangian algebra): charges of physical operators
  Phases from braiding between Lag. algebra and symmetry defects

[Bhardwaj, Schafer-Nameki 23]

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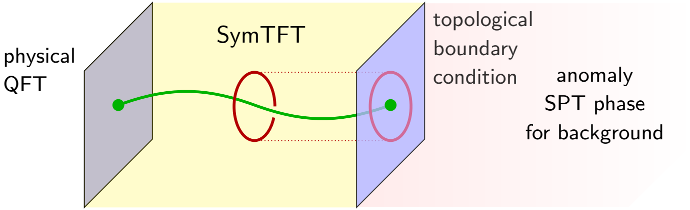

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- Different choices of boundary conditions: global forms of the QFT
  Related by gauging discrete symmetries in QFT $\Rightarrow$ topological operations

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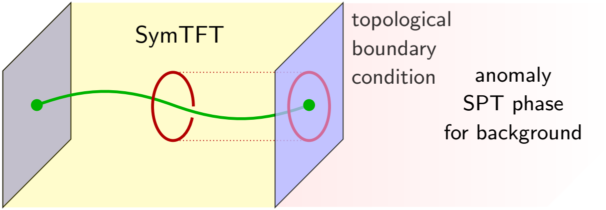

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- Anomalies: appear as bulk phases produced under moves
  Also appears as lack of boundary conditions [Kaidi, Ohmori, Zheng 22]

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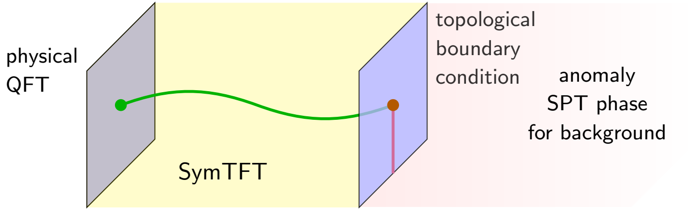

- Operators that cannot end on boundary give twisted sectors
  Representations of both untwisted and twisted sectors

[Lin, Okada, Seifnashri, Tachikawa 22]

Examples of SymTFT's

- $\mathbb{Z}_N$ 0-form symmetry [Maldacena, Moore, Seiberg 01; Banks, Seiberg 10]

  SymTFT is $(d+1)$-dim $\mathbb{Z}_N$ gauge theory. Path integral description as BF theory:

$$S_{\text{SymTFT}} = \frac{i}{2\pi}\int A_1 \wedge dB_{d-1} \qquad A, B : U(1)\ (p\text{-form})\ \text{gauge fields}$$

Anomalies: $H^{d+1}\big(B\mathbb{Z}_N, U(1)\big) = \mathbb{Z}_N$ for d even. $\qquad S_{\text{anom}} \sim k \int A_1\, (dA_1)^d$

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- G simple Lie group [Brennan, Sun 24; Antinucci, FB 24; Bonetti, Del Zotto, Minasian 24]

  SymTFT is non-Abelian BF theory studied in [Horowitz 89] :

$$S_{\text{SymTFT}} = \frac{i}{2\pi}\int \operatorname{Tr}\big(b_{d-1} \wedge F_2\big) \qquad F \text{ is field strength of G-connection } A$$

In both cases, chiral anomalies described by Chern–Simons terms.

Constructions of non-invertible symmetries

⋆ Gauge a 0-form symmetry that acts on a higher-form symmetry (n-group).

E.g.: 4d $SU(N)$ Yang-Mills with $\mathbb{Z}_N$ 1-form symmetry, $U_a \times U_b = U_{a+b}$

gauge charge conjugation $C : U_a \to U_{-a}$.

For $a \neq -a$: $\widetilde{U}_a = U_a \oplus U_{-a} \quad \Rightarrow \quad \widetilde{U}_a \times \widetilde{U}_b = \widetilde{U}_{a+b} \oplus \widetilde{U}_{a-b}$

[Bhardwaj, Bottini, Schafer-Nameki, Tiwari 22; Antinucci, Galati, Rizi 22]

⋆ In 4d QED: Abelian symmetry with ABJ anomaly.

[Choi, Lam, Shao 22]
[Cordova, Ohmori 22]

Conserved current is spoiled, but $\mathbb{Q}/\mathbb{Z} \subset U(1)$ survives as non-invertible.

Topological defects constructed via quantum Hall state coupled to photon:

$$d * j = \frac{1}{8\pi^2} F \wedge F \qquad \Rightarrow \qquad U_{\theta = \frac{p}{q}} = \exp\Big[2\pi i\,\theta \int_{\Sigma_3} * j\Big] \underbrace{Z\big[\mathcal{A}^{q,p}, F\big]}_{\supset \exp\left[\frac{i\,p/q}{4\pi} \int_{\mathcal{M}_4} F \wedge F\right]}$$

For $\theta = \frac{1}{q}$: $\quad U(1)_1$ CS theory, $\quad Z = \int \mathcal{D}C \; e^{\frac{i}{4\pi} \int q\, C dC + 2\, C dA}$

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Part of the algebra:
$$\begin{cases} U_{\frac{p}{q}} \times U_{-\frac{p}{q}} = \mathcal{C}[\mathbb{Z}_q] \\ U_{\frac{p}{q}} \times U_{\frac{\ell}{q}} = \mathcal{A}^{q,(p^{-1}+\ell^{-1})^{-1}}\, U_{\frac{p+\ell}{q}} \qquad \text{if } \gcd(p+\ell, q) = 1 \end{cases}$$

⋆ Non-Abelian examples with finite symmetry: [Kaidi, Ohmori, Zheng 21]

- $\mathcal{N}=1$ $PSU(N)$ SYM: $\mathbb{Z}_N$ non-invertible chiral symmetry (R-symmetry)

⋆ Self-duality symmetries

E.g.: 2d Ising model has $\mathbb{Z}_2$ symmetry (spin flip)

and Kramers–Wannier symmetry (self-duality under $\mathbb{Z}_2$ gauging)

Symmetry elements: $\mathbb{1},\ \eta,\ \mathcal{N}$ [Tambara, Yamagami 98]

s.t. $\eta \times \eta = \mathbb{1}\ , \qquad \eta \times \mathcal{N} = \mathcal{N} \times \eta = \mathcal{N}\ , \qquad \mathcal{N} \times \mathcal{N} = \mathbb{1} \oplus \mathcal{N}$

Tambara-Yamagami symmetry

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Tambara-Yamagami symmetry

- Similar structure in some 4d gauge theories with a conformal manifold

E.g.: $\mathcal{N} = 4 \; \mathfrak{su}(N)$ SYM has $SL(2,\mathbb{Z})$ duality, $S : \tau \to -\frac{1}{\tau}$

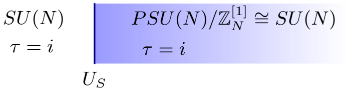
At $\tau = i$ is almost self dual, but $SU(N) \leftrightarrow PSU(N)$

Combine with topological gauging of $\mathbb{Z}_N$ 1-form symmetry

Non-invertible 0-form self-duality symmetry:

$$U_S \times \overline{U}_S = \mathcal{C}[\mathbb{Z}_N]$$

$$U_S \times U_S = \mathcal{C}[\mathbb{Z}_N] \times U_C$$



[Kaidi, Ohmori, Zheng 21; Choi, Cordova, Hsin, Lam, Shao 21 & 22]

⋆ Non-Invertible Symmetries and String Theory

For QFTs with a realization in string theory,
geometric tools might be used to identify the non-invertible symmetry
or uncover underlying general structures (*e.g.* SymTFT)

- Holography
- Geometry engineering

Symmetry TFT from Holography

- Relevance of topological sectors in holography was noticed long ago: [Witten 98]

AdS/CFT: 4d $\mathfrak{su}(N)$ $\mathcal{N}=4$ SYM $\longleftrightarrow$ IIB string theory on $\text{AdS}_5 \times S^5$

SUGRA: at low momenta, drop kinetic terms and be left with a topological theory:

[Aharony, Witten 98; Witten 98; Belov, Moore 04; Kravec, McGreevy, Swingle 14]

$$\int_{X_{10}} B_2 \wedge F_3 \wedge F_5 \qquad \xrightarrow{S^5} \qquad \frac{N}{2\pi} \int_{\text{AdS}_5} B_2 \wedge dC_2$$

Chern-Simons-like TQFT, equivalent to 5d 2-form $\mathbb{Z}_N$ gauge theory

Top. sector is SymTFT for $\mathbb{Z}_N$ 1-form symmetry

Symmetry TFT from Holography

- Relevance of topological sectors in holography was noticed long ago: [Witten 98]

AdS/CFT: 4d $\mathfrak{su}(N)$ $\mathcal{N}=4$ SYM $\longleftrightarrow$ IIB string theory on $\mathrm{AdS}_5 \times S^5$

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Top. sector is SymTFT for $\mathbb{Z}_N$ 1-form symmetry

- Global variants are described by boundary conditions for the topological sector:

electric top. b.c. $\quad B_2\big|_{\partial \mathrm{AdS}_5} = 0 \quad SU(N)$

magnetic top. b.c. $\quad C_2\big|_{\partial \mathrm{AdS}_5} = 0 \quad PSU(N)_0 \cong \big[SU(N)/\mathbb{Z}_N\big]_0$

★ SymTFT determined from string theory [Apruzzi, Bah, Bonetti, Schafer-Nameki 22]

[Apruzzi, Bonetti, Garcia Etxebarria, Hosseini, Schafer-Nameki 21]

⋆ Non-invertible self-duality symmetry of $\mathcal{N}=4$ SYM

In IIB String Theory, $SL(2,\mathbb{Z})$ is a gauge symmetry
spontaneously broken by axio-dilaton VEV $\tau = C_0 + i\, e^{-\phi} \;\to\; \dfrac{a\tau+b}{c\tau+d}$

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$\Rightarrow$ SymTFT is 5d 2-form $\mathbb{Z}_N$ gauge theory with $S : \left(\begin{smallmatrix} B_2 \\ C_2 \end{smallmatrix}\right) \to \left(\begin{smallmatrix} -C_2 \\ B_2 \end{smallmatrix}\right)$ gauged

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⋆ In holography, symmetry defect become dynamical objects (swapland)

E.g. for $U(1)$: defect = background field = b.c. for dynamical A_μ in the bulk

In other cases, defects $\leftrightarrow$ branes in the bulk [Apruzzi, Bah, Bonetti, Schafer-Nameki 22]
[Garcia Etxebarria 22; Heckman, Hübner, Torres, Zhang 22]

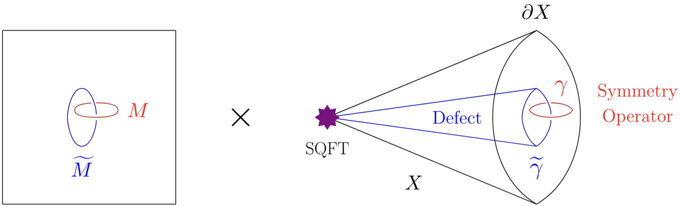
Topological only within IR topological sector, or equivalently at infinity

[*cfr.* Heckman, Hübner, Murdia 24]

Many other cases discussed *e.g.* in [van Beest, Gould, Schafer-Nameki, Wang 22; Bashmakov, Del Zotto, Hasan, Kaidi 22; Antinucci, Copetti, Galati, Rizi 22; Heckman, Hübner, Torres, Yu, Zhang 22]

⋆ Similar constructions in geometric engineering [Del Zotto, Heckman, Park, Rudelius 15] [Heckman, Hübner, Torres, Zhang 22]

String theory / M-theory on $\mathbb{R}^{d-1,1} \times X$ [image taken from 2209.03343]



BPS m-dimensional operators from p-branes on “unscreen defect group”:

$$\mathbb{D} = \bigoplus_m \mathbb{D}^{(m)} \qquad \mathbb{D}^{(m)} = \bigoplus_{p-k=m-1} \frac{H_k(X,\partial X)}{H_k(X)}$$

- Topological operators from dual q-branes on ∂X at infinity

Anomalies

For invertible symmetries, 't Hooft anomalies are additive and described by cohomology classes (group cohomology or more generally cobordism).

Not additive in general.

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⋆ Symmetry is non-anomalous if:

- can be gauged
- exists a trivially gapped (SPT) phase supporting it
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2d: complicated algebraic conditions. Simplify for Tambara–Yamagami type.

In higher dimensions, not well understood.

⋆ In 4d, when SymTFT is gauging of DW theory (as for self-duality of $\mathcal{N} = 4$ SYM):

Symmetry non-anomalous if $\exists$ duality-invariant Lagrangian algebra of DW theory

E.g.: $N = 2, 5, 8, 10, \ldots$ for $\mathfrak{su}(N)$ $\mathcal{N} = 4$ SYM [Antinucci, FB, Copetti, Galati, Rizi 23] [Cordova, Hsin, Zhang 23; Sun, Zheng 23]

Spontaneous Breaking on Non-Invertible Symmetries

⋆ *E.g.*: 2d tricritical Ising model ($c = \frac{7}{10}$ minimal model)

Relevant deformation by $\alpha\,\mathcal{O}_{\Delta=6/5}$ that preserves Tambara-Yamagami symmetry:

$\alpha > 0$: flow to $c = 1/2$ Ising

$\alpha < 0$: spontaneous breaking of TY symmetry [Chang, Lin, Shao, Wang, Yin 18]

$\rightsquigarrow$ 3 degenerate gapped vacua with different physical properties [Huse 84]

⋆ *E.g.*: 4d $SU(2)$ $\mathcal{N}=4$ SYM, deformed by $W = m^2 \sum \Phi_i^2$ ($\mathcal{N}=1^*$ theory)

At $\tau = i$: non-invertible self-duality symmetry, spontaneously broken

[Aguilara-Damia, Argurio, FB, Benvenuti, Copetti, Tizzano 23]

3 gapped vacua: 1 Higgsed and 2 confined

H :	$D_{(1,0)}$ = Wilson condenses	$\mathbb{Z}_2$ gauge theory (TQFT)
$C^{(0)}$:	$D_{(0,1)}$ = non-genuine 't Hooft cond.	SPT_0
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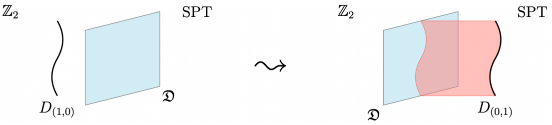
- S duality: $H \overset{S}{\longleftrightarrow} C^{(0)}$ while $C^{(1)}$ is a singlet [Dorey 99]

Order parameter $\mathcal{O} = \text{Tr}\,\Phi_i^2$: $\langle\mathcal{O}\rangle_H = -\langle\mathcal{O}\rangle_{C^{(0)}}$ $\langle\mathcal{O}\rangle_{C^{(1)}} = 0$

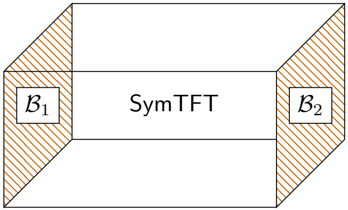
★ Spontaneous symmetry breaking of (discrete) non-invertible symmetry
$\rightarrow$ degenerate vacua with inequivalent physical properties

Non-invertible symmetry relates untwisted and twisted sectors:



⋆ Patterns of discrete non-invertible symmetry breaking in 2d are classified by topological boundary conditions of the SymTFT

[Bhardwaj, Bottini, Pajer, Schafer-Nameki 23]



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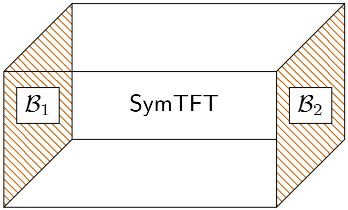

E.g.: the SymTFT of TY admits a unique top. b.c. that leads to 3 vacua

- Higher dimensions are more complicated

[Bhardwaj, Pajer, Schafer-Nameki, Tiwari, Warman, Wu 24]

Examples and Applications

Physics of 4d Yang–Mills theory at $\theta = \pi$

[Gaiotto, Kapustin, Komargodski, Seiberg 17]

4d $SU(N)$ gauge theory depends on theta angle θ

$$S \supset \frac{i\,\theta}{8\pi^2}\int \mathrm{Tr}\, F \wedge F$$

- 1-form (center) symmetry $\mathbb{Z}_N$

  At $\theta = 0, \pi$: CP symmetry (equivalently, time reversal) $\theta \to -\theta$

The angle θ is 2π periodic up to a counterterm:

$$\theta \to \theta + 2\pi \qquad \Rightarrow \qquad \Delta S = \frac{2\pi i\,(N-1)}{2N}\int \underbrace{\mathcal{P}(\mathfrak{b})}_{\simeq\, \mathfrak{b} \cup \mathfrak{b}}$$

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Slightly different physics at N even/odd. With regularization preserving CP at $\theta = 0$, at $\theta = \pi$ there is a mixed 't Hooft anomaly between $\mathbb{Z}_N$ and CP.

⋆ Assuming confinement for all values of θ, CP spontaneously broken at $\theta = \pi$.

(Other less probably scenarios are possible: TQFT, or massless)

Two-dimensional adjoint QCD [Komargodski, Ohmori, Roumpedakis, Seifnashri 20]

2d $SU(N)$ QCD with one massless adjoint Majorana fermion. Does it confine?

⋆ Bosonization: n Majorana fermsions $\simeq$ Spin$(n)_1$ WZW

Symmetries of QCD same as of Spin$\big(N^2-1\big)_1/SU(N)_N$ coset model

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- $\sim\ 2^{2N}$ non-invertible lines, charged under $\mathbb{Z}_N$ 1-form symmetry
  Lines charged under $\mathbb{Z}_N^{[1]}$ create strings, ground states of Wilson lines, degenerate with vacuum

Fundamental Wilson line has perimeter low $\Rightarrow$ deconfinement

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- Generalized naturalness

$$\mathcal{O} = \mathrm{Tr}(\psi_+\psi_-)\,\mathrm{Tr}(\psi_+\psi_-)$$

Invariant under ordinary symmetries, but breaks some non-invertible symmetries

$\Rightarrow$ not generated along RG flow

[*cfr.* Gorbenko, Zan 20; Jacobsen, Saleur 23]

2d Modular Bootstrap

[Lin, Shao 23]

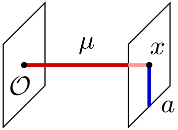
Conformal bootstrap determines rigorous bounds on unitary CFTs

[Rattazzi, Rychkov, Vichi, Tonni 08]

2d CFTs: modular bootstrap exploits modular invariance on T^2

$$\mathcal{H}_a = \bigoplus_\mu W_a^\mu \times \mathcal{V}_\mu$$

$$Z_\mu^{\text{3d}}(-1/\tau) = \sum_{\nu \in \text{SymTFT}} S_{\mu\nu}\, Z_\nu^{\text{3d}}(\tau)$$



Expansion in Virasoro characters: $Z_\mu^{\text{3d}} = \sum_{(h,\bar{h})\in\mathcal{H}_\mu} n_{\mu;h,\bar{h}}\, \chi_h(\tau)\, \chi_{\bar{h}}(\bar{\tau})$

Positive-definite functionals on ranges of spectra rule them out.

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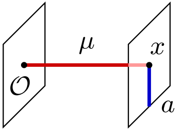

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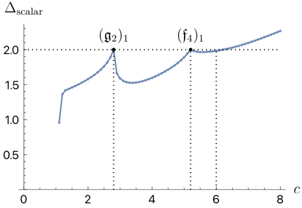
Positive-definite functionals on ranges of spectra rule them out.

- ⋆ *E.g.*: upper bounds on dimension of lightest symmetry-preserving scalar for Ising (TY) symmetry.



[image from 2302.13900]

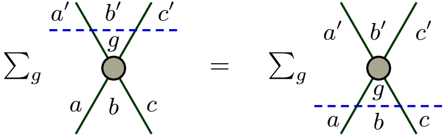
$\Delta < 2 \Rightarrow$ no stable CFT

For $1 < c < 6.7$: no stable Ising-preserving CFT

S-Matrix Modified Crossing Simmetry in 2d

S-matrices of 2d massive solitons are found to satisfy modified crossing relations.

If solitons are related by spontaneously broken non-invertible symmetry, modified crossing relations can be computed: [Copetti, Cordova, Komatsu 24]



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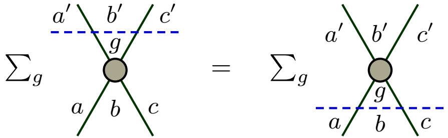

$$S^{ab}_{cd}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

⋆ Tested in tricritical Ising with massive deformation to 3 vacua.

Integrability + Unitarity + Yang–Baxter + Crossing fix the exact S-matrix:

$$S^{ab}_{cd}(\theta) = Z(\theta)\left[\sqrt{\frac{d_a d_c}{d_b d_d}}\sinh\!\left(\frac{\theta}{4}\right)\delta_{bd} + \sinh\!\left(\frac{i\pi-\theta}{4}\right)\delta_{ac}\right]$$

⋆ Modified crossing might play a role in 3d Chern–Simons-matter theories and in 4d scattering on monopoles [Mehta, Minwalla, Patel, Prakash, Sharma 22; Csaki, Hong, Shirman, Telem, Terning, Waterbury 20; van Beest, Boyle Smith, Delmastro, Komargodski, Tong 23]

Density of states and entanglement in 2d

⋆ Cardy's formula determines the density of states in CFTs at high temperatures.

With invertible finite symmetry G, density of states in a given rep μ:

$$\mathrm{Tr}_{\mathcal{H}^\mu}\, e^{-\beta H} \;\simeq\; \frac{(\dim \mu)^2}{|G|}\, e^{\pi c/6\beta} \qquad \text{for} \quad \beta \ll 1$$

Generalizes to fusion categories of 2d non-invertible symmetries:

[Lin, Okada, Seifnashri, Tachikawa 22]

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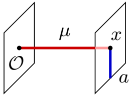

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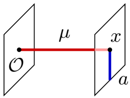

⋆ Inclusion of boundaries yields symmetry-resolved entanglement entropy:

[Choi, Rayhaun, Zheng 24; Heymann, Quella 24; Das, Molina-Vilaplana, Saura-Bastida 24]

$$S^\rho_{\mathsf{EE}} \simeq \frac{c}{3}\log\frac{L}{\epsilon} + \log g_1 + \log g_2 + \log \frac{d_\rho N_{\rho \underline{B}_2}^{\underline{B}_1}}{d_{\underline{B}_1} d_{\underline{B}_2}}$$

Here $g_i = \langle B_i | 0 \rangle$ are [Affleck, Ludwig 91] central charges

SymTFT: interfaces ρ between top. b.c.'s provide representations

Outlook

Non-invertible symmetries provide new rich rigid structures in QFTs
and powerful constraints on their RG flows.
Potential impacts of symmetries in all sort of fields.

- Mathematical structure is rather intricate: higher fusion categories

  Collaborative effort (high energy physics, condensed matter, mathematics) to develop the language

- Most new results to date are in 2d

  Development of higher categories allows us to go up in d

- Phenomenological applications are still limited

  Progess in 4d might lead to more applications