Hagedorn temperature in holography: world-sheet and effective approaches

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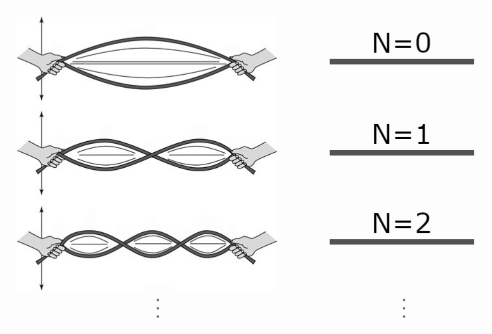
Based on:

[arXiv:2210.09893 [hep-th]], [arXiv:2306.00588 [hep-th]], [arXiv:2306.17126 [hep-th]]
[arXiv:2406.08405 [hep-th]], [arXiv:2407.07943 [hep-th]], [arXiv:2407.00375 [hep-th]]

with F. Bigazzi, F. Castellani, A. L. Cotrone, W. Mück and J. M. Penín.

String theory

- Building block of nature: one-dimensional string (length scale $\sqrt{\alpha'}$)
- String spectrum in flat space:



$\alpha' M^2 \approx N$ (integer)

(mass-shell condition)

$\Downarrow$

states belonging to the same level N have the same mass M

Asymptotic density of states

$$d(N) \approx e^{\#\sqrt{N}}, \quad N \to \infty$$

[Huang et al. '70, Sundborg '85]

The Hagedorn temperature [Hagedorn '68]

Thermal partition function (II superstring on ten-dim flat space):

$$Z \sim \int \mathrm{d}E\,\omega(E)\,e^{-\beta E}$$

- $\beta = 1/T$ inverse temperature, E energy.
- $E^2 - \vec{p}^2 = M^2 \approx N/\alpha'$, $\vec{p}$ center-of-mass momentum,
- with $\omega(E)\mathrm{d}E \approx E\,\mathrm{d}E \int \mathrm{d}^9\vec{p}\, d(N)d(N)$,

$$\omega(E) \approx E^{-11/2}\,e^{\beta_H E}\,, \quad E \to \infty,$$ [Bowick '89]

where $\beta_H = 2\pi\sqrt{2\alpha'}$.

Definition

The Hagedorn temperature $T_H = 1/\beta_H$ is defined as the temperature above which Z does not exist (divergent!)

Solvable models of superstrings on curved space [Canneti '24]

At fixed $b_i\mu \neq 0$ (world-sheet boson masses), $i = 1, ..., \mathsf{N}_b$,

$$d(\mathcal{E};\mu) \approx \mathcal{C}\left(\mathcal{E},\mu\right)\,\mathcal{E}^{-\frac{11-\mathsf{N}_b}{2}}\,e^{4\pi\sqrt{\mathcal{E}}}\,, \quad \mathcal{E}\to+\infty\,,$$

$$\mathcal{C}\left(\mathcal{E},\mu\right) = e^{-\pi\sum_i b_i\mu + 2\pi\log 2\sum_i b_i^2\mu^2/\sqrt{\mathcal{E}} + \mathcal{O}\left(\mu^4/\mathcal{E}^{3/2}\right)}\,,$$

$\mathcal{E}$ eigenvalue of the quadratic world-sheet Hamiltonian (oscillatory part).

N. B. - In general, μ can depend on the momentum of the string.

- $\mathcal{E}$ is non-integer, mass-shell condition $\mathcal{E} = \alpha' M^2/2$.

Example [Canneti '24]

Ramond-Ramond (RR) supported pp-wave backgrounds [Metsaev '01]

$$\mathrm{d}s^2 = -2\,\mathrm{d}x^+\,\mathrm{d}x^- - f^2 \sum_{i=1}^{\mathsf{N}_b} b_i^2\, x_i^2\,(\,\mathrm{d}x^+)^2 + \sum_{j=1}^{8} dx_j^2\,, \quad \mu = f\alpha' p^+\,,$$

Once integrating over all the possible momenta,

$$\omega(E) \approx E^{-\frac{11-N_b}{2}} e^{\beta_H E}\;, \quad E \to +\infty\,,$$

with

$$\begin{aligned}\beta_H =& 2\pi\sqrt{2\alpha'} - \frac{\pi}{\sqrt{2}}\, f\alpha' \sum_i b_i + \\ &+ f^2\alpha'^{3/2}\left[\pi\sqrt{2}\log 2 \sum_i b_i^2 + \frac{\pi}{8\sqrt{2}}\left(\sum_i b_i\right)^2\right] + \mathcal{O}(f^3\alpha'^2)\,.\end{aligned}$$

N. B. - the above applies in the small curvature regime $f\sqrt{\alpha'} \ll 1$

- from flat to curved space: $10 \mapsto 10 - \mathsf{N}_b$

Motivations and goal

- Holographic correspondence [Maldacena '97, Witten '98, Gubser et al. '98]

Large N confining gauge theories at strong coupling	$\simeq$	Weakly coupled string theories on curved gravity backgrounds with fluxes

$\Rightarrow T_H^{\text{gauge theory}} \equiv T_H^{\text{string theory}}$

- Lack of first principle computations of T_H in Yang Mills (YM) theories, but evidences on Hagedorn behavior from lattice [Bringoltz et al. '05]

- Can we compute T_H of strongly coupled large N confining gauge theories from string theory?

  - in general we do not know how to compute Z nor ω

  - need to rephrase the divergence of Z from a spectrum perspective

Outline

Alternative methods [Sathiapalan '87, Kogan '87, O'Brien et al. '87, Atick et al '88]

World-sheet approach

The lightest state of the spectrum of a string winding once the thermal (compact) direction is tachyonic above T_H

$$m_W^2 \equiv M^2(|0\rangle; \beta \leq \beta_H) \leq 0$$

$\Rightarrow$ [Atick&Witten '88] flat space computation

Effective approach

Thermal scalar χ in flat space

$$S_\chi = \beta \int d^9x \left(\partial_i \chi^* \partial^i \chi + m_W^2\, \chi^* \chi\right) , \quad m_W^2 = \frac{\beta^2 - \beta_H^2}{4\pi^2 \alpha'^2} \, .$$

Can we extend these methods to curved target space?

Interplay [Bigazzi et al. '22, '23, '24] vs [Horowitz et al. '97, Urbach '22; Ekhammar et al. '23, Harmark '24]

World-sheet approach

Quadratic fluctuations around the temporal winding configuration, fixing the mass of the physical ground state to zero

Pros: very powerful
Cons: very demanding

$\simeq$

Effective approach

Quantum corrected normalizability condition for the solution to the eom from the effective action of χ in curved space

Pros: very convenient
Cons: arbitrary coefficients

$\Downarrow$

Strategy

Neglect higher derivative corrections to the thermal scalar effective action, introducing the missing terms through the comparison with the world-sheet sigma model

General confining backgrounds

Confining phase of $(q+1)$-dim finite T QFTs at low energy dual to type II superstring models on

$$\mathrm{d}s^2 \approx 2\pi\alpha' T_s \left(1+\frac{r^2}{l^2}\right)\left(dt^2+\eta_{ij}dx^i dx^j\right) + dr^2 + r^2 d\Omega_{d-1}^2 + ds_{\mathcal{M}}^2\,,$$

$r \ll l, \quad i,j = 1,...,q, \quad t \sim t+\beta,$

r radial coordinate $\sim$ QFT energy scale, $\quad l$ curvature radius,

T_s confining string tension,

$\mathcal{M}$ transverse (typically compact) $(9-d-q)$-dimensional space

N. B. - In general: running dilaton, non-trivial RR and Kalb-Ramond (KR) fields

- Simplifying assumption: the KR filed has no legs along time (for the general case, see [Bigazzi '24])

Our proposal

β_H is the solution of the implicit equation

$$\frac{T_s}{2}\,\beta_H^2 = 2\pi\left[\,\Delta(\mu) + \Delta\epsilon\,\right]\,, \quad \mu = \frac{\beta_H}{2\pi}\frac{\sqrt{2\pi\alpha' T_s}}{l}\,,$$

- μ mass of the d world-sheet bosons y^I at $\beta = \beta_H$, where $r^2 = y^I y^I$,
- $\Delta(\mu)$ zero-point energy of the world-sheet sigma model

$$\Delta(\mu) = 1 - \frac{d}{2}\mu + d\log 2\,\mu^2 + \mathcal{O}(\mu^4)\,,$$

- $\Delta\epsilon$ quartic and higher order contributions in the bosonic zero modes, captured by the effective approach

Our proposal

Solving the implicit equation order by order in $\sqrt{\alpha'}/l$

- leading order [Bigazzi et al. '22]

$$T_H = \sqrt{\frac{T_s}{4\pi}}$$

(trivial generalization of the flat space result)

- next-to-leading order (NLO) [Bigazzi et al. '23, Urbach '23]

$$T_H = \sqrt{\frac{T_s}{4\pi}} \left[1 + \frac{d}{2\sqrt{2}} \frac{\sqrt{\alpha'}}{l} \right]$$

- NNLO: $\Delta\epsilon$ comes into play, let us see an example

Example [Urbach '22, Ekhammar et al. '23, Bigazzi et al. '23]

String sitting at the center $(r = 0)$ of $(d+1)$-dim global-AdS space

$$\mathrm{d}s^2 = R_{AdS}^2 \left(\cosh^2 r \, \mathrm{d}t^2 + \mathrm{d}r^2 + \sinh^2 r \, \mathrm{d}\Omega_{d-1}\right) ,$$

dual to d-dim CFT on $S^{(d-1)}$ ("confining" as in [Witten '98]).

From the interplay of the methods ($q = 0$, $T_s = 1/2\pi\alpha'$, $l = R_{AdS}$),

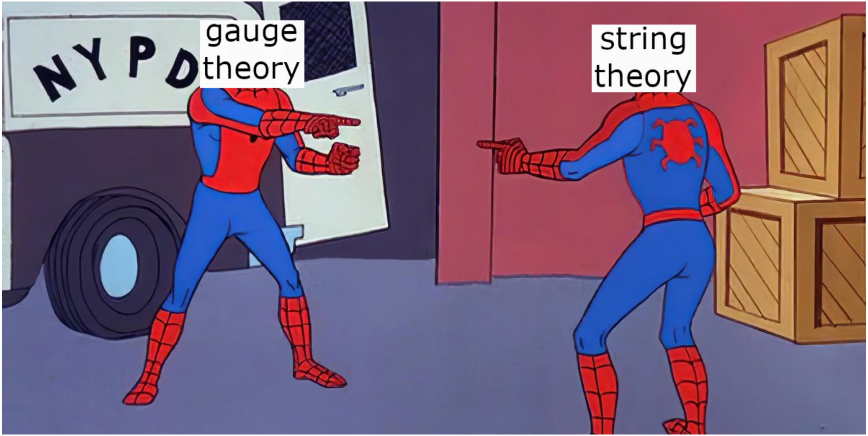
$$T_H = \sqrt{\frac{g}{2\pi}} + \frac{d}{8\pi} + \frac{d^2 + d - 8d\log 2}{32\sqrt{2}\pi^{3/2}\sqrt{g}} + \frac{4d^3 + 7d^2 - 2d}{1024\pi^2 g} + \mathcal{O}(g^{-3/2}) ,$$

where $g = 1/4\pi\alpha'$.

Here, $R_{AdS} = 1$: to get physical result $T_H \mapsto T_H \, R_{AdS}$ and $\alpha' \mapsto \alpha'/R_{AdS}^2$.

Holography [Maldacena '97, Witten '98, Gubser et al. '98]

$$T_H^{\text{gauge theory}} \equiv T_H^{\text{string theory}}$$



Comparison with numerical results (from quantum spectral curve)
$d=4$: ($\mathcal{N}=4$) Super Yang-Mills theory on $S^1 \times S^3$

$$T_H = \sqrt{\frac{g}{2\pi}} + \frac{1}{2\pi} + \frac{5 - 8\log 2}{8\pi\sqrt{2\pi}\sqrt{g}} + \frac{45}{128\pi^2 g} + \mathcal{O}(g^{-3/2})$$
$$\approx 0.39894...\sqrt{g} + 0.15916... - \frac{0.00865...}{\sqrt{g}} + \frac{0.0356...}{g} + \mathcal{O}(g^{-3/2})$$

vs

$$T_H^{\text{[Ekhammar et al.'23]}} \approx (0.39894 \pm 0.00001)\sqrt{g} + (0.15916 \pm 0.00001) - \frac{(0.00865 \pm 0.00001)}{\sqrt{g}} + \frac{(0.0356 \pm 0.0001)}{g} + ...$$

That's an outstanding and non-trivial test of holography!

Conclusions

- T_H for strongly coupled large N confining gauge theories up to a certain order in the string length from the holographic dual description
- Agreement with numerical analysis (if available)
- Some ambiguities fixed in an analytical way from the string

Further extensions [Bigazzi et al. '24]

- non-trivial Yang-Mills θ-angle: $\frac{T_H}{\sqrt{T_s}}$ increases as θ increases
  (scheme: fixed 't Hooft coupling)
- effect of N_f quenched flavors: $\frac{T_H}{\sqrt{T_s}}$ decreases as N_f increases
  (scheme: fixed 't Hooft coupling)
- non-perturbative corrections to T_H at strong string coupling

Thanks a lot for your attention!

Some backup slides

Quadratic world-sheet Hamiltonian

For solvable (quadratic) models,

$$\mathcal{H} = \sum_{k\in\mathbb{Z}} \sum_{i=1}^{8} \left(|\omega^B_{ki}| N^B_{ki} + |\omega^F_{ki}| N^F_{ki}\right) + c(\mu)\,\mathbb{1}\,,$$

where

$$\omega^B_{ki} = \begin{cases} +\sqrt{k^2 + b_i^2\mu^2}\,, & k \geq 0 \\ -\sqrt{k^2 + b_i^2\mu^2}\,, & k < 0 \end{cases}\,, \quad \omega^F_{ki} = \begin{cases} +\sqrt{k^2 + f_i^2\mu^2}\,, & k \geq 0 \\ -\sqrt{k^2 + f_i^2\mu^2}\,, & k < 0 \end{cases}\,,$$

and

$$N^B_{ki} = \begin{cases} \frac{1}{\omega^B_{ki}} \alpha^i_{-k}\alpha^i_k\,, & k > 0 \\ {a^i}^\dagger a^i\,, & k = 0 \\ \frac{1}{\omega^B_{-ki}} \widetilde{\alpha}^i_k \widetilde{\alpha}^i_{-k}\,, & k < 0 \end{cases}\,, \quad N^F_{ki} = \begin{cases} S^i_{-k} S^i_k\,, & k > 0 \\ {s^i}^\dagger s^i\,, & k = 0 \\ \widetilde{S}^i_k \widetilde{S}^i_{-k}\,, & k < 0 \end{cases}\,.$$

Finally, $c(\mu)$ denotes the normal order constant $c(\mu) = \frac{1}{2} \sum_{k\in\mathbb{Z}} \sum_{i=1}^{8} \left(|\omega^B_{ki}| - |\omega^F_{ki}|\right)$.

Examples of confining backgrounds

- $q = 3$, $d = 2$: Witten background [Witten '98]

  sourced by N $D4$-branes wrapped on S^1
  supported by RR four-form field strength
  running dilaton

- $q = 3$, $d = 3$: Maldacena-Núñez background [Maldacena et al. '00]

  sourced by N $D5$-branes wrapped on S^2
  supported by RR three-form field strength
  running dilaton

- global-AdS_{d+1} $\times \mathcal{M}_{9-d}$: $q = 0$, $T_s = 1/2\pi\alpha'$, $l = R_{AdS}$

  dual to IR regime of d-dimensional CFTs on $S^{(d-1)}$
  "confining" in the sense described in [Witten '98]
  critical temperature set by $S^{(d-1)}$ (inverse) radius

Computation of $\Delta\mathcal{E}$ [Ekhammar et al. '23, Bigazzi et al. '23]

1 global-AdS_{d+1}:

$$ds^2 = (1+R^2)d\tau^2 + \frac{dR^2}{1+R^2} + R^2 d\Omega_{d-1}^2\,, \quad R_{AdS} \equiv 1$$

2 Thermal scalar action:

$$S_\chi \approx \beta \int dR\, R^{d-1} \left\{ g^{pq}\, \partial_p \chi^*\, \partial_q \chi + m_{eff}^2(R)\, \chi^* \chi \right\}\,,$$

$$\chi = \chi(R)\,, \quad m_{eff}^2(R) = (1+R^2)\left(\tfrac{\beta}{2\pi\alpha'}\right)^2 - \tfrac{2}{\alpha'}$$

3 eom:

$$-\frac{1}{R^{d-1}}\partial_R \left(R^{d-1}(1+R^2)\partial_R \right) \chi(R) + m_{eff}^2(R)\, \chi(R) = 0$$

Computation of $\Delta\mathcal{E}$ [Ekhammar et al. '23, Bigazzi et al. '23]

4 LO, with $\omega = \frac{\beta}{2\pi\alpha'}$:

$$-\frac{1}{2}\chi''(R) - \frac{1}{2}(d-1)\frac{1}{R}\chi'(R) + \frac{1}{2}\omega^2 R^2 \chi(R) = \omega\left(n + \frac{d}{2}\right)\chi(R)\,,$$

given the normalizability condition

$$E_n = \omega\left(n + \frac{d}{2}\right) = \frac{1}{2}\left[\frac{2}{\alpha'} - \omega^2\right]$$

$\Rightarrow$ d-dimensional harmonic oscillator ("unperturbed" problem)

$$\chi_n(R) = \alpha_n e^{-\frac{\omega}{2}R^2} L_{n/2}^{d/2-1}(\omega R^2)\,, \quad \langle \chi_n | \chi_m \rangle = \int_0^\infty dR\, R^{d-1}\, \chi_n^*\, \chi_m\,.$$

Computation of $\Delta\mathcal{E}$ [Ekhammar et al. '23, Bigazzi et al. '23]

5 Perturbation theory:

$$\Delta H = -\frac{1}{2}\left[(d+1)R\frac{\partial}{\partial R} + R^2\frac{\partial^2}{\partial R^2}\right]$$

6 Corrected normalizability condition
(for the ground state, i. e. the winding mode)

$$\frac{d}{2}\frac{\beta}{2\pi\alpha'} + \Delta E^{(1)} + \Delta E^{(2)} = \frac{1}{2}\left[\frac{2}{\alpha'} - \frac{\beta^2}{2\pi^2\alpha'}\Delta c - \left(\frac{\beta}{2\pi\alpha'}\right)^2\right],$$

where

$$\Delta E^{(1)} = \langle\chi_0|\Delta H|\chi_0\rangle = \frac{d\,(d+2)}{8},$$

$$\Delta E^{(2)} = \frac{|\langle\chi_0|\Delta H|\chi_4\rangle|^2}{-4\,\omega} = -\frac{d(d+2)\pi\alpha'}{16\,\beta}.$$

Computation of $\Delta\mathcal{E}$ [Ekhammar et al. '23, Bigazzi et al. '23]

7 World-sheet data:

- $\Delta c = -d\log 2$ [Bigazzi et al. '23],
- $\Delta E^{(1)} = \langle 0|\Delta H^{\text{WL}}_{(4)}|0\rangle$ [Bigazzi et al. '24].

8 Solving for T_H up to NNNLO,

$$\boxed{T_H = \sqrt{\frac{g}{2\pi}} + \frac{d}{8\pi} + \frac{d^2 + d - 8d\log 2}{32\sqrt{2}\pi^{3/2}\sqrt{g}} + \frac{4d^3 + 7d^2 - 2d}{1024\pi^2 g} + \mathcal{O}(g^{-3/2})\,,}$$

where $g = 1/4\pi\alpha'$.

N. B . To get physical result $T_H \mapsto T_H R_{AdS}$ and $\alpha' \mapsto \alpha'/R^2_{AdS}$

Witten YM theory and finite θ-angle effects [Bigazzi et al. '24]

Dual θ-backreacted background [Witten '98, Barbón et al. '99, Dubovsky et al. '11]

$$\mathrm{d}s^2 = \left(\frac{u}{R}\right)^{3/2}\left[\sqrt{H_0}\,\mathrm{d}x_\mu\,\mathrm{d}x^\mu + \frac{f}{\sqrt{H_0}}\,\mathrm{d}x_4^2\right] + \left(\frac{R}{u}\right)^{3/2}\sqrt{H_0}\left[\frac{\mathrm{d}u^2}{f} + u^2\,\mathrm{d}\Omega_4^2\right],$$

where

$$x_4 \sim x_4 + 2\pi M_{\mathsf{KK}}^{-1}\,, \quad f = 1 - \frac{u_0^3}{u^3}\,, \quad H_0 = 1 - \frac{u_0^3}{u^3}\frac{\Theta^2}{1+\Theta^2}\,,$$

$$u_0 = \frac{4R^3}{9}M_{\mathsf{KK}}^2\frac{1}{1+\Theta^2}\,, \quad \Theta = \frac{\lambda}{4\pi^2}\left(\frac{\theta + 2\pi k}{N}\right)\,, \quad k\in\mathbb{Z}\,,$$

$\lambda \gg 1$ ('t Hooft coupling), $\quad N \gg 1$ (number of colors),

$$u - u_0 = \frac{1}{2}M_{\mathsf{KK}}r^2\sqrt{1+\Theta^2}\left(1 - \frac{r^2\sqrt{1+\Theta^2}(1+3\Theta^2)}{8\sqrt{u_0R^3}}\right).$$

$$\frac{T_H}{\sqrt{T_s}} \sim \frac{1}{2\sqrt{\pi}} + \frac{3\sqrt{3}}{8\sqrt{\pi\lambda}}\left(1+\Theta^2\right)$$

Witten-Sakai-Sugimoto model [Witten '98, Sakai et al. '04]

Low energy limit: 4-dim $SU(N)$ non-susy YM theory coupled to N_f quarks

Dual gravity solution with backreacting (smeared) flavors [Bigazzi et al. '15]:

$$\mathrm{d}s^2 = e^{2\hat{\lambda}}\left(-\,\mathrm{d}t^2 + \mathrm{d}x_i\,\mathrm{d}x^i\right) + \alpha' e^{-2\varphi}\,\mathrm{d}\rho^2 + \ldots\,, \quad \epsilon_f = \frac{1}{12\pi^3}\lambda^2\frac{N_f}{N} \ll 1\,,$$

where λ is the 't Hooft coupling and

$$\hat{\lambda} \approx f_0(\tilde{r}) + \frac{3}{4}\log\frac{u_0}{R} + \frac{\epsilon_f}{4}\left(3A_1 - 4k\right)\,, \quad \tilde{\lambda} \approx f_0(\tilde{r}) + \frac{3}{4}\log\frac{u_0}{R} - \frac{3}{2}\tilde{r} - \frac{\epsilon_f}{4}\left(A_1 + 4k\right)\,,$$

$$\phi \approx f_0(\tilde{r}) + \frac{3}{4}\log\frac{u_0}{R} + \log g_s + \frac{\epsilon_f}{4}\left(11A_1 - 4k\right)\,, \quad \nu \approx \frac{1}{3}f_0(\tilde{r}) + \frac{1}{4}\log\frac{u_0}{R} + \log\frac{R}{l_s} + \frac{\epsilon_f}{12}\left(11A_1 - 20k\right)\,,$$

$$f_0(\tilde{r}) = -\frac{1}{4}\log\left(1 - e^{-3\tilde{r}}\right)\,, \quad \tilde{r} = \frac{u_0^3}{l_s^3 g_s^2}\rho\,, \quad A_1 = \frac{81\sqrt{3}\pi^2(-9 + \sqrt{3}\pi - 12\log 2 + 9\log 3)}{43120 \times 2^{2/3}\Gamma[-14/3]\Gamma[-2/3]^2}\,,$$

$$k = \frac{\pi^{3/2}(3 + \sqrt{3}\pi - 12\log 2 + 9\log 3)}{78\Gamma[1/6]\Gamma[-2/3]}\,, \quad r = \frac{2}{3}R^{3/4}u_0^{1/4}e^{-\frac{3u_0^3}{2g_s^2 l_s^3}\rho}\left[1 + \epsilon_f\left(\frac{11}{12}A_1 - \frac{29}{3}k\right)\right]\,.$$

$$\frac{T_H}{\sqrt{T_s}} \sim \frac{1}{2\sqrt{\pi}} + \frac{3\sqrt{3}}{8\sqrt{\pi\,\lambda}}\left(1 - 0.35\,\epsilon_f\right)$$

Non-perturbative corrections at strong string coupling

- in 11 dim: two-dim membrane wrapped around the temporal circle and the eleventh dimension.

- the background:

$$\begin{aligned} \mathrm{d}s_{11}^2 &= G_{\tau\tau}\,\mathrm{d}\tau^2 + ... + G_{AA}(\,\mathrm{d}x_{11} + C_\mu\,\mathrm{d}x^\mu)^2 \\ &= e^{-2\phi/3}ds_{10}^2 + e^{4\phi/3}(dx_{11} + C_\mu dx^\mu)^2\,, \end{aligned}$$

$$g_{\tau\tau} = G_{\tau\tau}G_{AA}^{1/2}\,, \quad C_\tau = 0\,, \quad G_{AA} = e^{4\phi/3}$$

- Thermal scalar mass (generalization of [Russo '01]):

$$m_{eff}^2 = \frac{e^{2\phi/3}}{4\pi^2\alpha'^2}\left[\beta^2 g_{\tau\tau} - 8\pi^2\alpha'\mathcal{F}(g_{eff})\right]\,,$$

$$\mathcal{F}(g_{eff}) \approx 1 + \tfrac{16}{\pi}\tfrac{1}{\sqrt{g_{eff}}}e^{-\frac{2\pi}{g_{eff}}}\,, \quad g_{eff} = \tfrac{\sqrt{\alpha'}}{\beta}\tfrac{\sqrt{G_{AA}}}{\sqrt{G_{\tau\tau}}}\,2\pi g_s\,.$$

$$\frac{T_s}{2}\beta_H^2 \approx 2\pi\left[1 + \frac{2^{7/2}e^{-\frac{1}{\eta}}}{\pi^{3/2}\sqrt{\eta}}\right]\,, \quad \eta \equiv \frac{(G_{AA}(r=0))^{3/4}}{2\sqrt{2}\pi}g_s\,.$$