Entanglement spectra from holography

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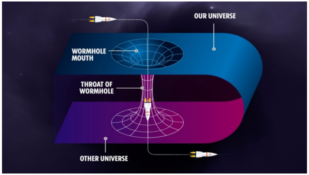
Motivations

Entanglement is a key feature distinguishing the classical from the quantum realm

- Many-body quantum systems and phase transitions [Cardy, Calabrese, 2004]
- Counting of the degrees of freedom (c-theorem) [Casini, Huerta, 2011]
- Connection with the Einstein-Rosen bridge (ER=EPR) [Maldacena, Susskind, 2013]



- Rényi entropies measured in cold atomic systems [Islam et al., 2015]

Entanglement spectrum

Advantages:

- Encodes more information than entanglement entropy [Li, Haldane, 2008]
- Theoretical understanding of entanglement [Cardy, Tonni, 2016] [Tonni, Rodrigues-Laguna, Sierra, 2017] [Alba, Calabrese, Tonni, 2017]

Entanglement spectrum characterized by Schmidt coefficients

$$D(\lambda) = \sum_i \delta(\lambda - \lambda_i) \tag{1}$$

Reduced density matrix can be written in terms of a modular Hamiltonian K_A [Bisognano, Wichmann, 1975-76]

$$\rho_A = e^{-\beta K_A}\,, \qquad E_i = -\frac{1}{\beta}\log \lambda_i \tag{2}$$

Density of states $D(E)$ characterizes the entanglement spectrum!

CFT spectra: Cardy formula

Universal formula for the density of states at high-energy of 2d CFT [Cardy, 1986]

$$S(E) = \log D(E) \approx 2\pi \sqrt{\frac{c}{6}\left(E - \frac{c}{24}\right)} \tag{3}$$

- Microcanonical entropy valid in the regime

$$c \text{ fixed}\,, \qquad E \gg \frac{c}{24} \tag{4}$$

- Coincides with the Bekenstein-Hawking entropy of a BTZ black hole [Strominger, 1998]. Holography extends the regime of validity to

$$c \gg 1\,, \qquad E \gtrsim c \tag{5}$$

Goal of this talk

Goals:

- Compute the **density of states** associated with the modular Hamiltonian of a **holographic CFT**
- Generalize Cardy formula to entanglement spectra in **higher dimensions**

1. General strategy
2. Results
3. Extensions and conclusions

General strategy

Properties of the holographic Rényi entropies

[Hung, Myers, Smolkin, Yale, 2011]

$$S_n = \pi V_\Sigma \left(\frac{L}{\ell_\mathrm{P}}\right)^{d-1} \frac{n}{n-1}\left[2 - x_n^{d-2}\left(1+x_n^2\right)\right] \tag{6}$$

$$x_n \equiv \frac{1}{d\,n} + \sqrt{\left(\frac{1}{d\,n}\right)^2 + \frac{d-2}{d}} \tag{7}$$

- $\ell_P^{d-1} = 8\pi G_N$ Planck length
- V_Σ regularized volume of hyperbolic space
- Limit $n \to \infty$ defines the minimal eigenenergy

$$E_0 = -\log \lambda_{\max} = \lim_{n\to\infty} S_n \tag{8}$$

Density of states from the Rényi entropies

Dual interpretation of partition function

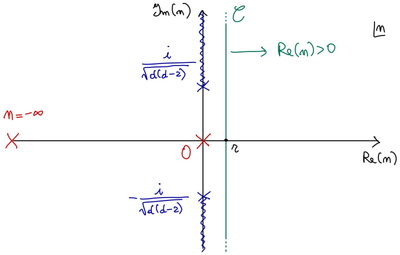
$$Z(n) = \mathrm{Tr}\left(\rho_A^n\right) = e^{(1-n)S_n} = \\ = \int_0^{\infty} D(E) e^{-nE} dE \tag{9}$$

The density of states is computed as an inverse Laplace transform

$$D(E) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{nE} e^{(1-n)S_n} dn \tag{10}$$

The contour $\mathcal{C}$ runs on the right of all the singularities of $e^{(1-n)S_n}$

Structure of singularities of $e^{(1-n)S_n}$



Denote with $r > 0$ the real value along which the contour runs

$$D(E) = \lim_{K\to\infty} \int_{r-iK}^{r+iK} e^{nE} e^{(1-n)S_n} \frac{dn}{2\pi i} \equiv \lim_{K\to\infty} \int_{r-iK}^{r+iK} e^{f(n)} \frac{dn}{2\pi i} \tag{11}$$

Saddle point approximation

[Bao, Penington, Sorce, Wall, 2019]

S_n proportional to $1/G_N$ $\Rightarrow$ **Perform a saddle point approximation!**

1. Taylor-expand the function $f(n)$ around the locus n_*

$$f(n) \equiv nE + (1-n)S_n = f(n_*) + f'(n_*)(n-n_*) + \frac{1}{2}f''(n_*)(n-n_*)^2 + \dots \tag{12}$$

2. Choose n_* to be a saddle point

$$f'(n_*) = 0 \tag{13}$$

3. Deform the contour $\mathcal{C}$ to run through the **dominant** saddle point

$$D(E) \approx \frac{e^{f(n_*)}}{\sqrt{2\pi f''(n_*)}} \tag{14}$$

4. Check that

$$f''(n_*) > 0 \tag{15}$$

Results

Example: 2 dimensions

$$S_n\Big|_{d=2} = E_0\,\frac{n+1}{n} \quad \Rightarrow \quad f(n) = n(E-E_0) + \frac{E_0}{n} \tag{16}$$

Dominant saddle point

$$n_* = \sqrt{\frac{E_0}{E-E_0}} \quad \Rightarrow \quad D(E) \approx \left[\frac{E_0}{(4\pi^2)E^3}\right]^{\frac{1}{4}} \exp\left(2\sqrt{E_0 E}\right) \tag{17}$$

Coincides with

- Cardy formula [Cardy, 1986]
- Leading-order expansion of the exact formula [Calabrese, Lefevre, 2008]

$$S(E) = \log D(E) \approx 2\sqrt{E_0 E} + \dots \tag{18}$$

High energies in general dimensions

Saddle point condition

$$\frac{2}{d-1}\left(\frac{\mathcal{E}(d)E}{2E_0}-1\right)=x_{n_*}^{d}-x_{n_*}^{d-2}\,,\qquad \mathcal{E}(d)\equiv 2-2\,\frac{d-1}{d-2}\left(\frac{d-2}{d}\right)^{\frac{d}{2}} \tag{19}$$

Work at high energies

$$\frac{E-E_0}{E_0}\gg 1 \tag{20}$$

Saddle point condition is solved at $n\to 0\,(x_n\gg 1)$ by

$$x_{n_*}^{(k)}\approx e^{2\pi i k/d}\left(\frac{2}{d-1}\frac{\mathcal{E}(d)E}{2E_0}\right)^{\frac{1}{d}} \tag{21}$$

Dominant saddle point is the real and positive solution $(k=0)$, gives

$$D(E)\approx\frac{1}{d\sqrt{\pi}}\left(\frac{\mathcal{E}(d)E^{d+1}}{(d-1)E_0}\right)^{-\frac{1}{2d}}\exp\left[2\left(\frac{E_0}{\mathcal{E}(d)}\right)^{\frac{1}{d}}\left(\frac{E}{d-1}\right)^{\frac{d-1}{d}}\right] \tag{22}$$

Generalized Cardy formula

$$S = \log D(E) \approx 2 \left(\frac{E_0}{\mathcal{E}(d)} \right)^{\frac{1}{d}} \left(\frac{E}{d-1} \right)^{\frac{d-1}{d}} + \dots \tag{23}$$

Consistency checks with CFT spectra:

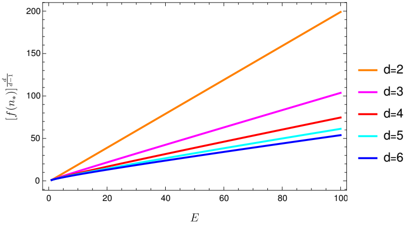
- Thermodynamic limit

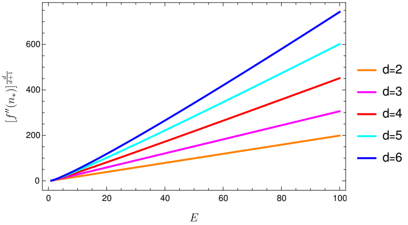
$$S \sim V T^{d-1} \,, \qquad E \sim V T^d \tag{24}$$

- Higher-dimensional Cardy formula for CFT spectra [Verlinde, 2000]
- Scaling obtained with modular forms [Shaghoulian, 2015]
- Scaling of CFTs obtained from thermal effective action [Benjamin, Lee, Ooguri, Simmons-Duffin, 2023]

Numerical analysis

$$D(E) \approx \frac{e^{f(n_*)}}{\sqrt{2\pi f''(n_*)}} \tag{25}$$





Asymptotic behaviour at high energies

$$f(n_*) \sim E^{\frac{d-1}{d}}, \qquad f''(n_*) \sim E^{\frac{d+1}{d}} \tag{26}$$

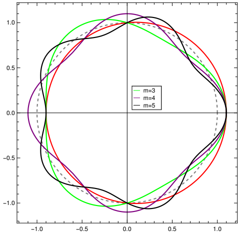
Extensions and conclusions

Shape deformations

[Bianchi, Chapman, Dong, Galante, Meineri, Myers, 2017] [SB, Bianchi, Chapman, Galante, 2022]

$$S(E) \approx 2 \left(\frac{E_0}{\mathcal{E}(d)} \right)^{\frac{1}{d}} \left(1 - \frac{\mathfrak{b}\mathcal{E}(d)}{2E_0} \frac{\pi^2 d C_T}{d+1} \right)^{\frac{1}{d}} \left(\frac{E-E_0}{d-1} \right)^{\frac{d-1}{d}} + \dots \quad (27)$$

$\mathfrak{b}$ parametrizes the deformation, C_T higher-dimensional central charge



Supersymmetric case

Supersymmetric Rényi entropies [Nishioka, Yaakov, 2013] [Nishioka, 2014]

$$S_n = \pi V_\Sigma \left(\frac{L}{\ell_{\rm P}}\right)^{d-1} \frac{n}{n-1} \left[1 + x_n(2 - x_n - 2x_n^{d-2})\right] \tag{28}$$

$$x_n = \frac{(d-2)n+1}{(d-1)n} \tag{29}$$

Microcanonical entropy:

$$S \propto \log D(E) \approx \begin{cases} 2\sqrt{E_0(E-E_0)} + \dots & \text{if } d=2 \\ \sqrt{\frac{12}{5}E_0(E-E_0)} + \dots & \text{if } d=3 \\ \left[\frac{2E_0}{\mathcal{E}_{\rm s}(d)}\left(\frac{E-E_0}{d-2}\right)^{d-2}\right]^{\frac{1}{d-1}} + \dots & \text{if } d>3 \end{cases} \tag{30}$$

Conclusions

- Systematic method to extract the density of states
- Generalization of the Cardy formula

$$S(E) \sim E^{\frac{d-1}{d}} \tag{31}$$

- Different scaling in the supersymmetric case

$$S(E) \sim \begin{cases} \sqrt{E} & \text{if } d = 2,3 \\ E^{\frac{d-2}{d-1}} & \text{otherwise} \end{cases} \tag{32}$$

Further developments:

- Entanglement spectra from symmetry-resolved Rényi entropies
- Contributions from boundary terms [Ohmori, Tachikawa, 2014] [Alba, Calabrese, Tonni, 2017]

Thank you!