Quantum groups as global symmetries

Bernardo Zan
University of Cambridge

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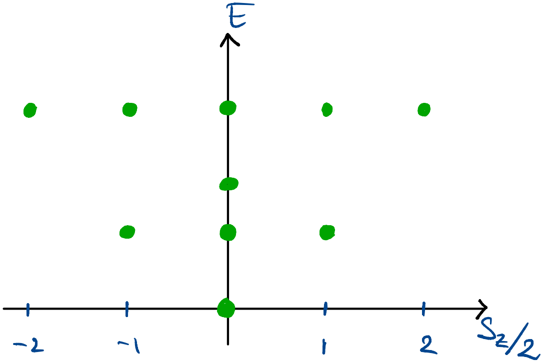
UNIVERSITY OF CAMBRIDGE

A $U_q(sl_2)$ symmetric spin chain

Early example of quantum group symmetry on the lattice [Pasquier, Saleur '90]

$$\mathcal{H}_{\mathrm{PS}} = \sum_{i=1}^{N-1} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \frac{q+q^{-1}}{2} \sigma_i^z \sigma_{i+1}^z \right) + \frac{q-q^{-1}}{2} \left(\sigma_1^z - \sigma_N^z \right)$$



$q = e^{i\theta}$, $\theta \in \mathbb{R}$.

Observe **degeneracies** in the spectrum, explained by

- $q = 1$: $SU(2)$ symmetry
- $q \neq 1$: $U_q(sl_2)$ symmetry

Quantum Groups in physics

Quantum groups, e.g. $U_q(sl_2)$, algebras that appear in several systems

- $1+1$ D spin chains, 2D statistical mechanical models (e.g. loop models) as **global symmetries**
- Integrability: Yang-Baxter equation
- In QFT, they appear more **indirectly**:
  - Crossing kernel of Virasoro blocks in minimal models $\to$ $6j$ symbols of $U_q(sl_2)$
  - Fusion rule for $SU(2)_k$ WZW models $\to$ fusion rules of $U_q(sl_2)$

Global Symmetries in the Continuum?

In 2d CFTs $U_q(sl_2)$ appears in a subtle way in theories with no $U_q(sl_2)$ global symmetry (e.g. Ising model).

$\mathcal{H}_{\text{PS}}$ is **critical**: described by CFT with $U_q(sl_2)$ global symmetry!

Questions:

- CFTs with $U_q(sl_2)$ symmetry?
- Why does this have to do with non-$U_q(sl_2)$ symmetric theories?
- Related somehow to non-invertible symmetries? In some loop models, they seem to explain the same phenomena [Read, Saleur '07] [Gorbenko, BZ '20] [Jacobsen, Saleur '23]

Quantum Group

A quantum group is not a group! Deformation of sl_2 algebra

$\underline{sl_2}$	$\underline{U_q(sl_2)}$
Generators: E , F , H (Raising → E, Lowering → F)	
• $[H, E] = 2E$	• $[H, E] = 2E$
• $[H, F] = -2F$	• $[H, F] = -2F$
• $[E, F] = H$	• $[E, F] = \frac{q^H - q^{-H}}{q - q^{-1}}$

Coproduct

Action of E, F, H on one spin/operator, how do they act on many? **Coproduct Δ**

sl_2 : $\Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X$, with $X = E, F, H$

Coproduct Δ needs to be **compatible with the commutation relations**, e.g. $\Delta([E, F]) = [\Delta(E), \Delta(F)]$.

$U_q(sl_2)$: deformed commutation relations $\rightarrow$ deformed coproduct. Many chocies

$$\Delta(E) = E \otimes \mathbb{1} + q^{-H} \otimes E$$
$$\Delta(F) = F \otimes q^{H} + \mathbb{1} \otimes F$$
$$\Delta(H) = H \otimes \mathbb{1} + \mathbb{1} \otimes H$$

Why not a group?

For sl_2, get group element by

$$g = e^{i\alpha X}$$

given $\Delta(X) = X \otimes \mathbb{1} + \mathbb{1} \otimes X$, coproduct acts as

$$\Delta(g) = g \otimes g$$

For $U_q(sl_2)$, non-trivial coproduct prevents us from building group-like element with these ingredients.

(non-invertible symmetries are still group-like, $\Delta(g) = g \otimes g$)

Representations

If q **not a root of unity**, $q^p \neq 1 \; \forall p \in \mathbb{Z}$, representations are the same as $su(2)$

$$|\ell, m\rangle\,, \qquad 2\ell \in \mathbb{Z}_{\geq 0}, \quad m = -\ell, \ldots, \ell$$

Generators

$$\begin{aligned} H\,|\ell, m\rangle &= 2m\,|\ell, m\rangle \\ E\,|\ell, m\rangle &\sim |\ell, m+1\rangle \\ F\,|\ell, m\rangle &\sim |\ell, m-1\rangle \\ E\,|\ell, \ell\rangle &= F\,|\ell, -\ell\rangle = 0 \end{aligned}$$

Clebsch-Gordan coefficients, $6j$-symbols, ...: now depend on q but work in the usual way.

$U_q(sl_2)$ symmetric CFTs

Require $U_q(sl_2)$ to be an internal symmetry

- **commutes** with spacetime symmetries, $[U_q(sl_2), \text{Virasoro}] = 0$
- operators transform under $U_q(sl_2)$: $\mathcal{O}_\ell^m(x)$ is in representation $|\ell, m\rangle$.

Correlation functions obey **Ward identities**!

$$\langle X \cdot (\mathcal{O}_1 \ldots \mathcal{O}_n) \rangle = 0 \qquad X = E, F, H$$

Consequence: $\mathcal{O}_i$ **cannot be mutually local!**

QFT in Euclidean space: for mutually local operators

$$\langle \ldots \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\rangle = \langle \ldots \mathcal{O}_2(x_2)\mathcal{O}_1(x_1)\ldots\rangle$$

Example: $\ell = \frac{1}{2}$ representation of $U_q(sl_2)$, $\mathcal{O}_\pm \equiv \mathcal{O}^{\pm\frac{1}{2}}_{\frac{1}{2}}$. Ward identity

$$\langle F \cdot (\mathcal{O}_+(x)\mathcal{O}_+(y))\rangle = 0$$

using $\Delta(F) = F \otimes q^H + \mathbb{1} \otimes F$ get

$$q \underbrace{\langle \mathcal{O}_-(x)\mathcal{O}_+(y)\rangle}_{(-1)^{2s}\langle \mathcal{O}_-(y)\mathcal{O}_+(x)\rangle} + \langle \mathcal{O}_+(x)\mathcal{O}_-(y)\rangle = 0$$

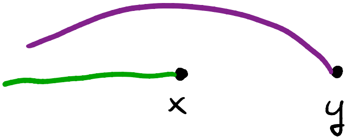
with $s = h - \bar{h}$ the spacetime spin.

$$\langle \mathcal{O}_+(x)\mathcal{O}_-(y)\rangle = -q(-1)^{2s}\langle \mathcal{O}_-(y)\mathcal{O}_+(x)\rangle$$

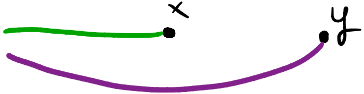
For $q \neq \pm 1$, either $s \neq \mathbb{Z}/2$ or operators do not commute. In any case lose mutual locality

Topological lines

Easiest way to lose locality: operators are **endpoints** of lines. Lines are **topological**



$$\langle \mathcal{O}_1(x)\, \mathcal{O}_2(y) \dots \rangle$$



$$\langle \mathcal{O}_2(y)\, \mathcal{O}_1(x) \dots \rangle$$

(An object exists to swap operators, the $\mathcal{R}$-matrix)

An example

XXZ_q: deformation of XXZ with non-local interactions [Grosse, Pallua, Prester, Raschhofer '94]. Central charge and spectrum are known

$$q = e^{i\pi\frac{\mu}{\mu+1}} \qquad c = 1 - \frac{6}{\mu(\mu+1)} \qquad \text{with } \mu \in \mathbb{R}_+$$

operators are in the Kac table (easier to study).

Non-trivial check I: spacetime and $U_q(sl_2)$ spin satisfy constraints given by existence of topological lines + $\mathcal{R}$-matrix.

Non-trivial check II: the theory is crossing symmetric. Found all OPE coefficients by two different method

- Bootstrap approach à la BPZ
- Coulomb gas approach

Well defined $U_q(sl_2)$ symmetric CFT!

Connection with unitary theories

Some explanation of the appearence of $U_q(sl_2)$ in minimal models

- For integer μ (c as unitary minimal models), the theory has a **closed subsector**. E.g. $c = 1/2$, $\mathrm{XXZ}_q \supset$ fermionic formulation of the Ising model $(\mathbb{1}, \psi, \bar{\psi}, \varepsilon)$.

- Explains the **appearence of** $6j$**-symbol** in crossing kernel. E.g. operators with weights $(h_{1,s}, h_{1,1} = 0)$ in XXZ_q: crossing symmetry of four point function

$$\mathcal{F}^{(t)}_{1,s'}(z) = \sum_s \underbrace{\frac{C_{(1,s_1),(1,s_2),(1,s_j)}\,C_{(1,s_3),(1,s_4),(1,s_j)}}{C_{(1,s_2),(1,s_3),(1,s_k)}\,C_{(1,s_4),(1,s_1),(1,s_k)}}}_{\text{OPE coefficients}} \underbrace{\begin{Bmatrix} \frac{s_1-1}{2} & \frac{s_2-1}{2} & \frac{s-1}{2} \\ \frac{s_3-1}{2} & \frac{s_4-1}{2} & \frac{s'-1}{2} \end{Bmatrix}_q}_{6j-\text{symbol}} \mathcal{F}^{(s)}_{1,s}(z)$$

t-channel Virasoro block $h = h_{1,s'}$

s-channel Virasoro block

Future directions

Lesson learned: QFTs can have quantum group as a global symmetry!

Open questions:

- Understanding relation to topological defect lines e.g. in minimal models
- Focused on CFT, but can deform XXZ_q by relevant perturbation preserving $U_q(sl_2)$. Integrable
- Generalization to $U_q(sl_{N\geq 3})$
- Codimension 2 operators in higher dimensions