THE NUMERICAL BOOTSTRAP OF POINTS AND LINES

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To appear, with Bharathkumar Radhakrishnan

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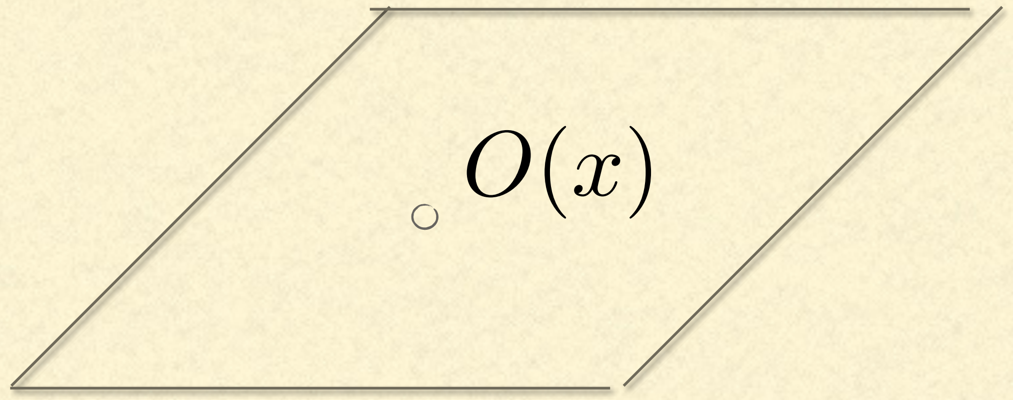
Overview

- Introduction: conformal field theories coupled to defects
- Conformal boundaries in two dimensions
- A positive semi-definite program
- A few plots
- Outlook

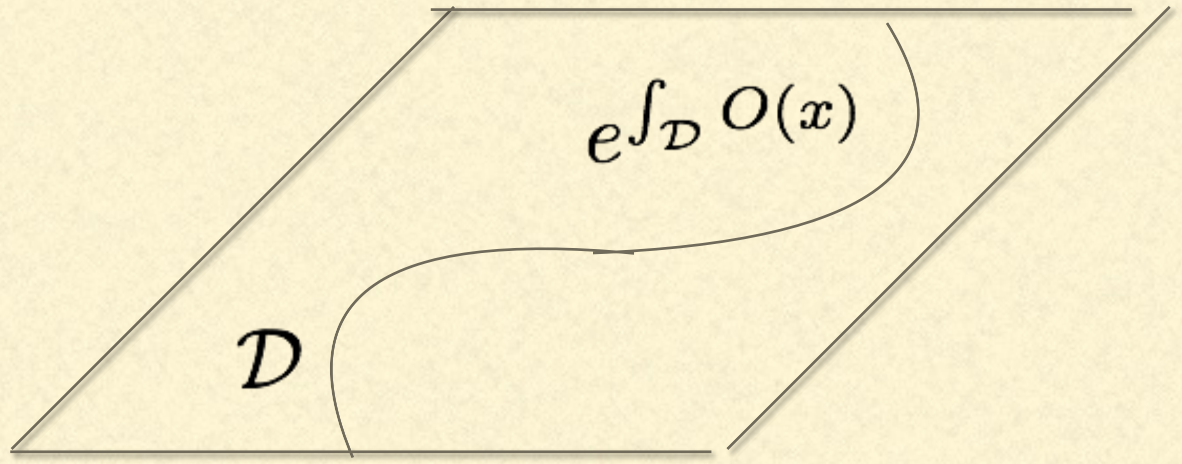
Introduction: conformal field theories coupled to defects

[Cardy, 1984]
[McAvity, Osborn, 1995]
…

Local probes in quantum field theory



Extended probes



Low energy: surfaces, interfaces, impurities in samples, vortices, quenches…

High energy: Wilson lines, monopoles, branes…

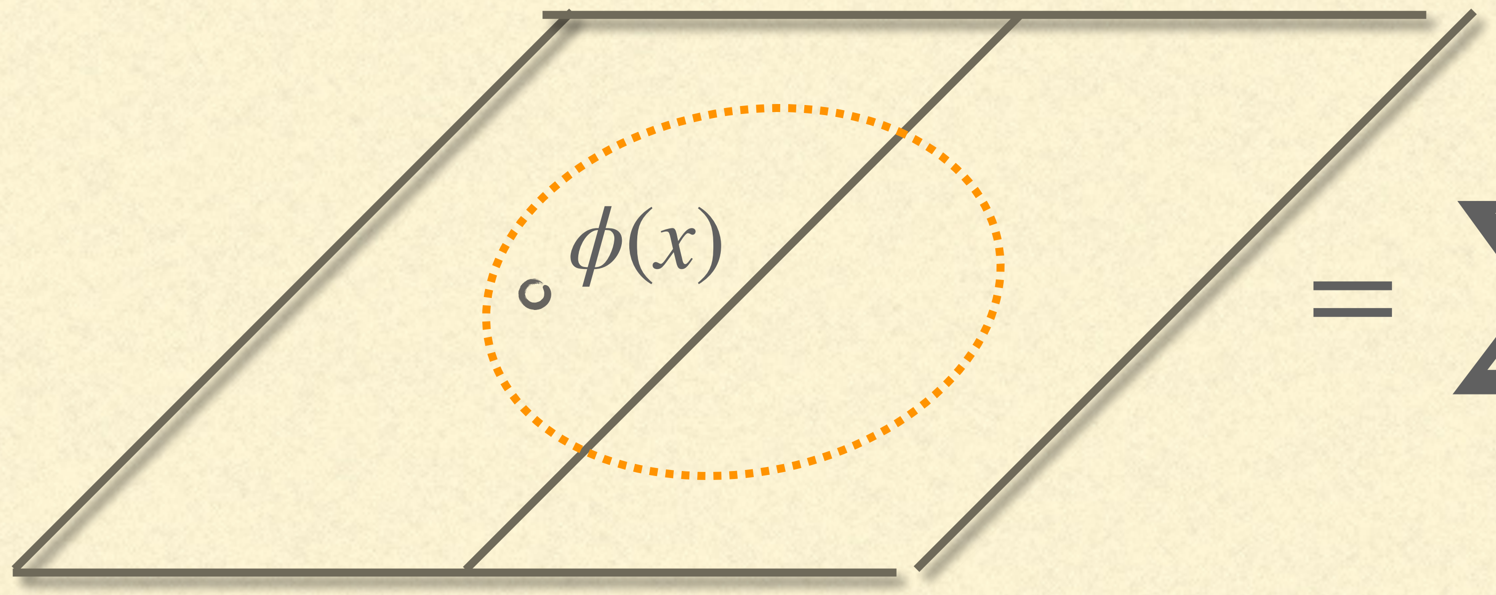
Common theme: defects arise when a heavy stable degree of freedom interacts with light excitations

Introduction: conformal field theories coupled to defects

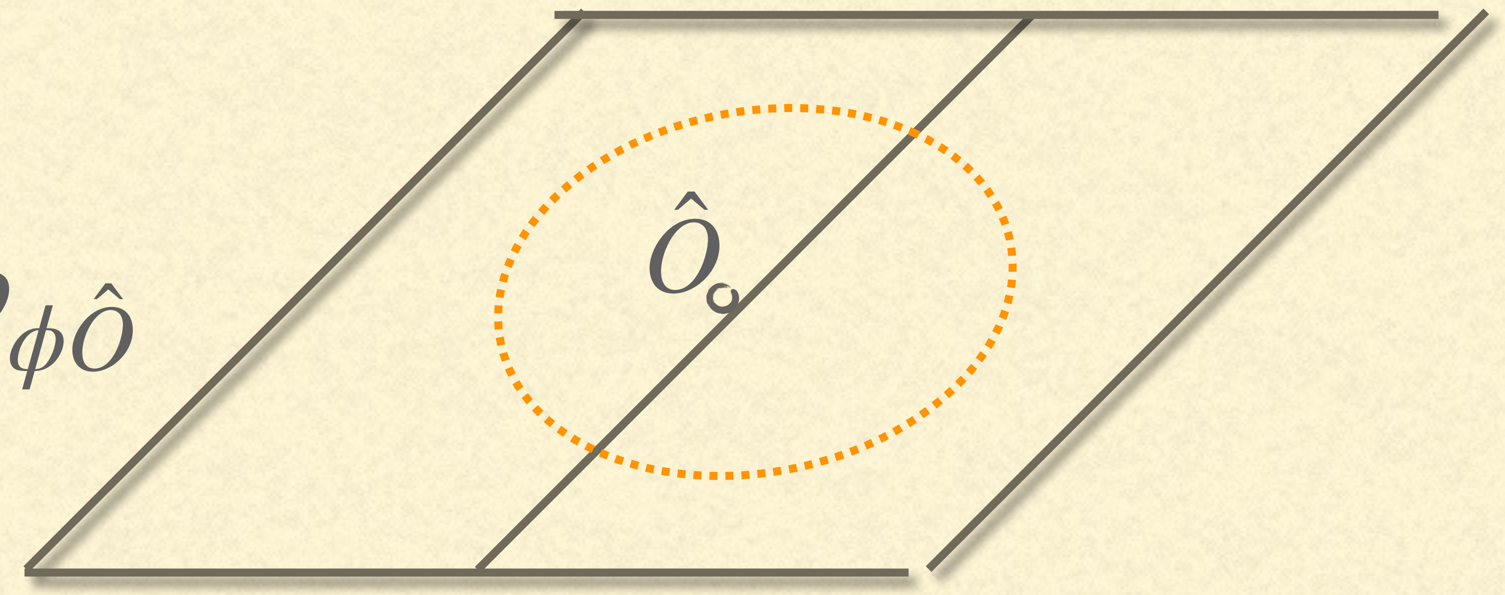
In a CFT, when the defect is conformal, the coupling to the bulk is captured by

the defect OPE: $\phi \sim \sum \hat{O}$

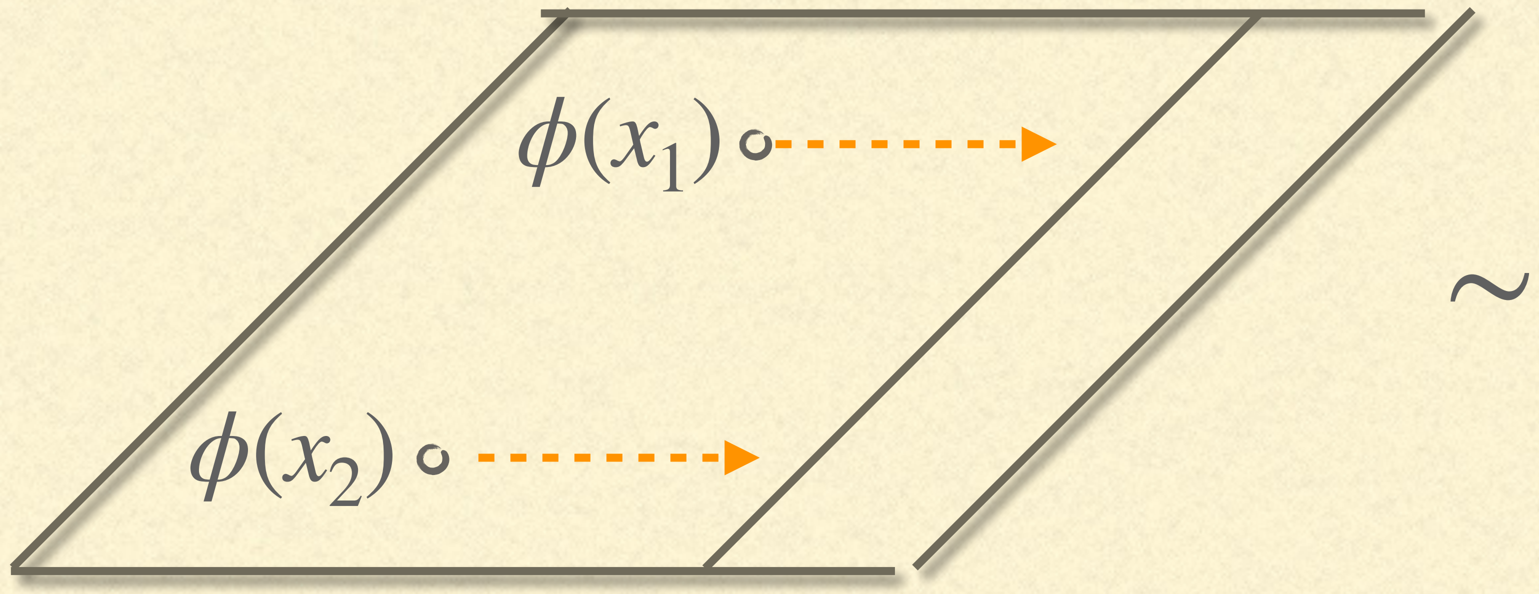
defect operators



$= \sum b_{\phi\hat{O}}$



Simplest correlator subject to crossing that probes $b_{\phi\hat{O}}$ is $\langle \phi\phi \rangle$



$\sim \sum_{\hat{O}} b^2_{\phi\hat{O}}$

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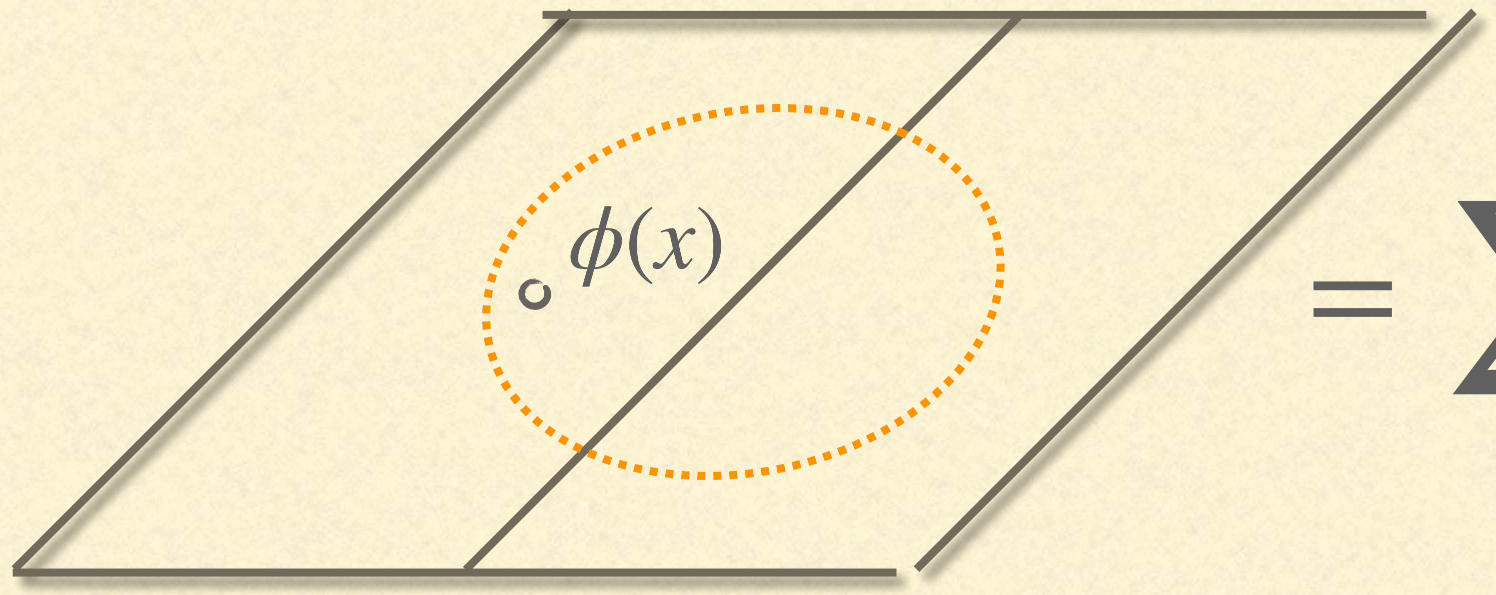

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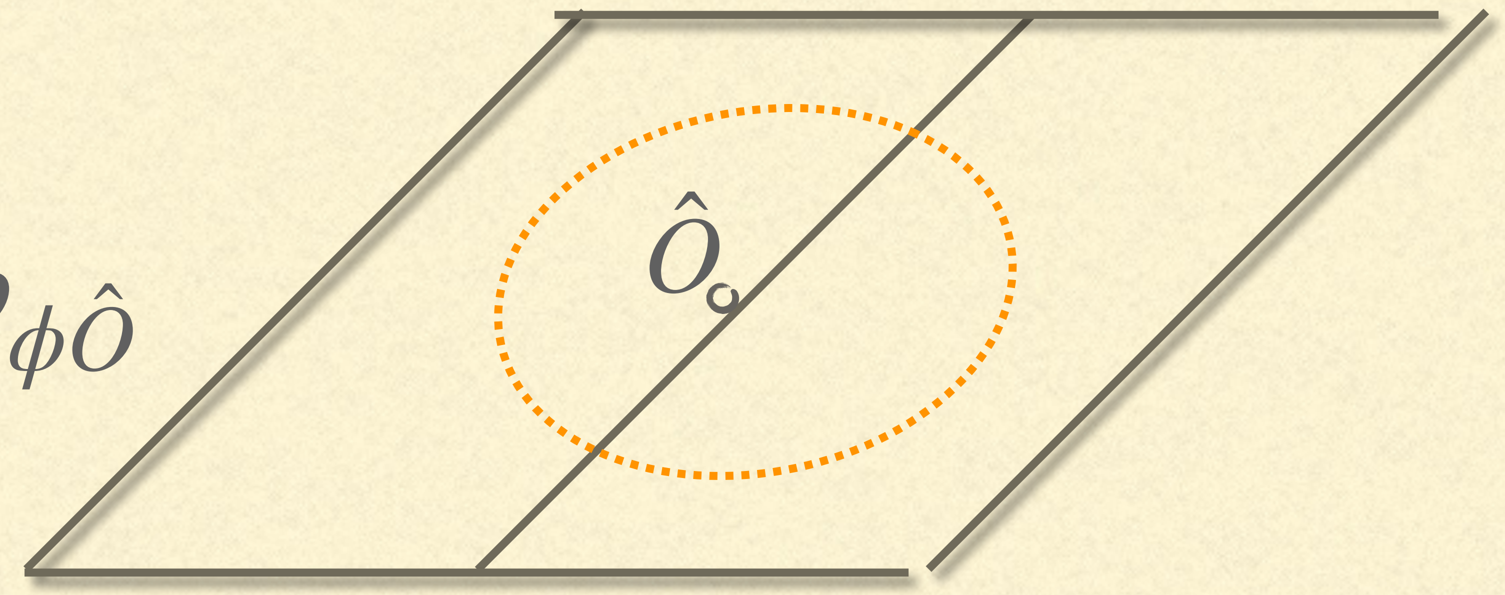
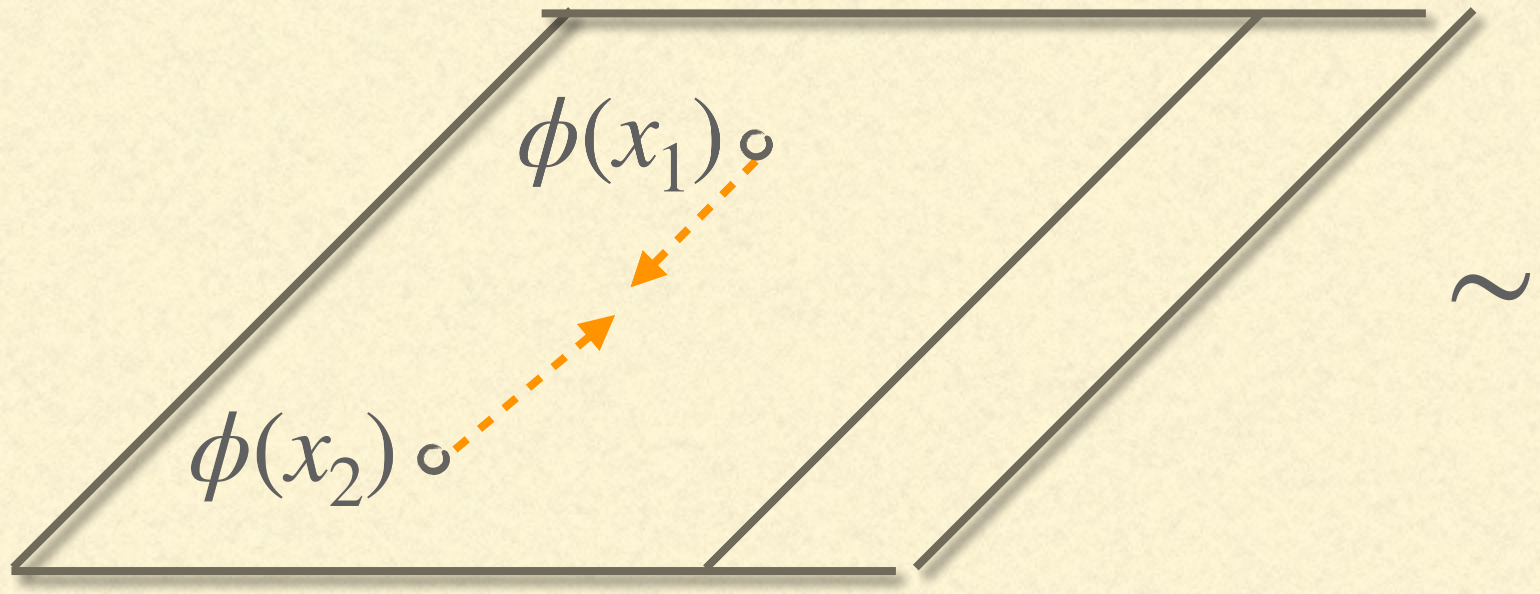


$b_{O1} \equiv a_O$

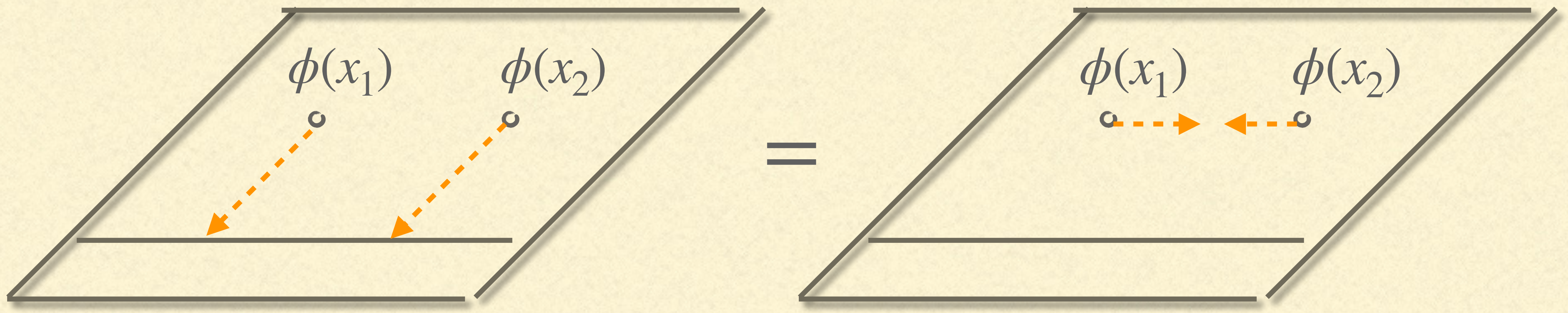
One-point function

Simplest correlator subject to crossing that probes $b_{\phi\hat{O}}$ is $\langle \phi\phi \rangle$



$\sim \sum_O c_{\phi\phi O} b_{O1}$

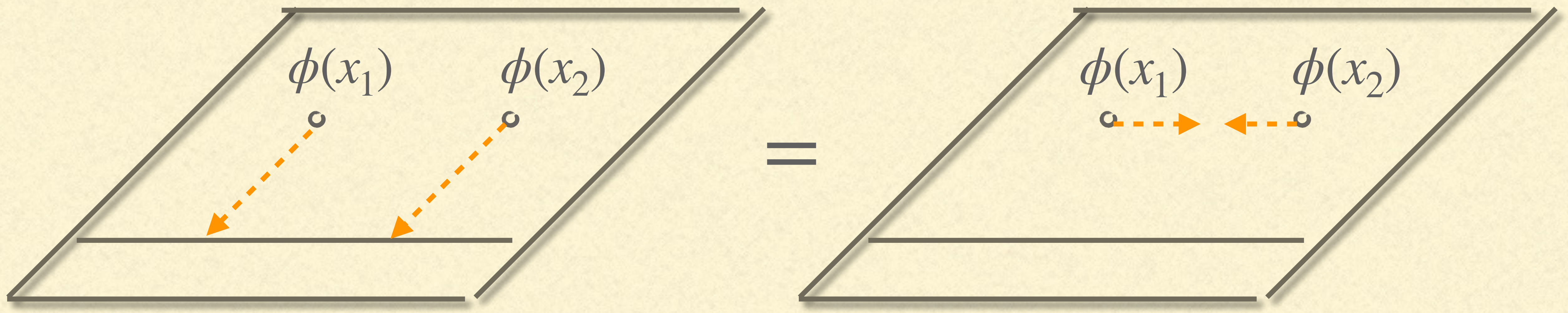
Introduction: conformal field theories coupled to defects



$$\sum_{\hat{O}} b^2_{\phi\hat{O}}\, \hat{g}_{\hat{O}}(x_i) = \sum_{O} c_{\phi\phi O} a_O\, g_O(x_i)$$

- Lack of positivity → Mixed correlators!
  - $c^2_{\phi\phi O}$ from the four-point function

Introduction: conformal field theories coupled to defects



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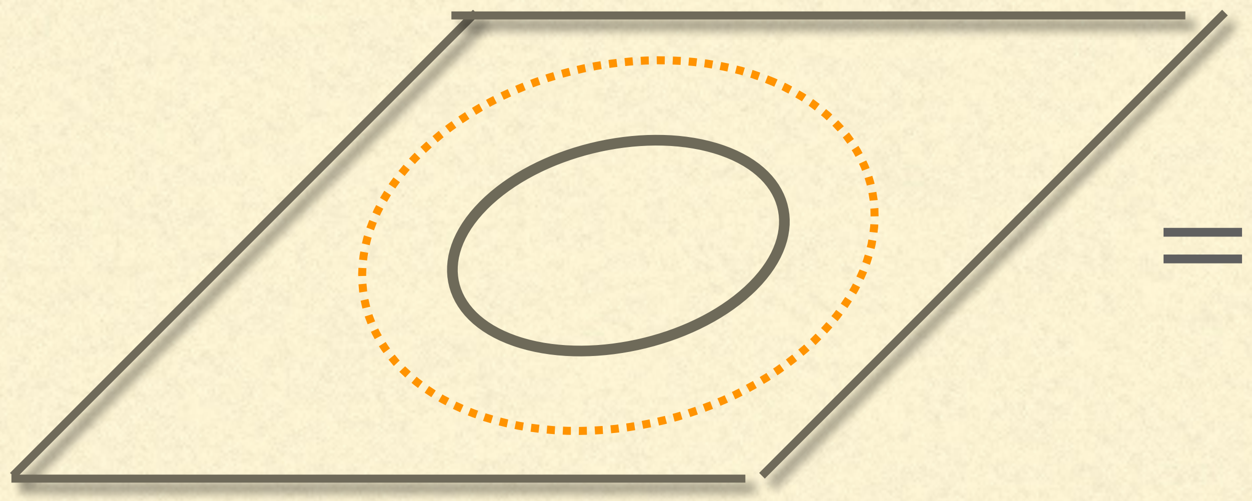
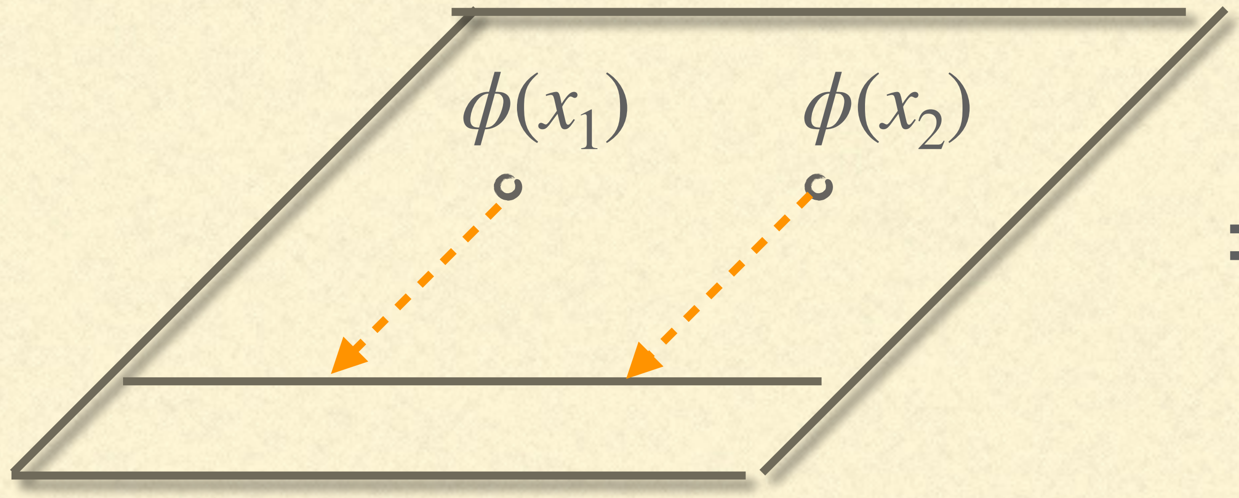
  - a^2_O from a correlator of two defects



$$= \sum_{O} a_O\, O$$

Introduction: conformal field theories coupled to defects



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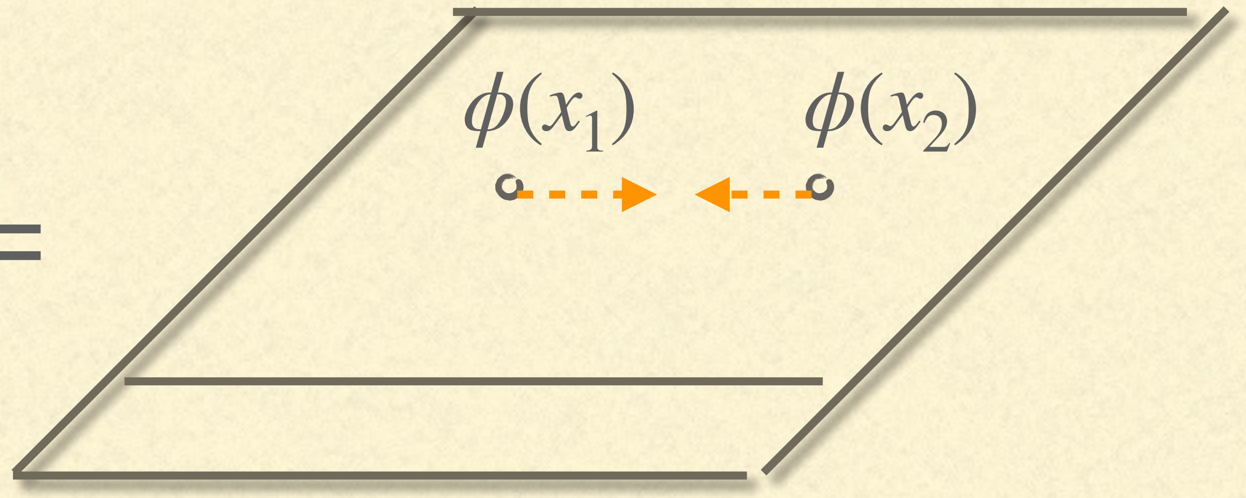

$$\sum_{\hat{O}} b^2_{\phi\hat{O}}\, \hat{g}_{\hat{O}}(x_i) = \sum_{O} c_{\phi\phi O} a_O\, g_O(x_i)$$

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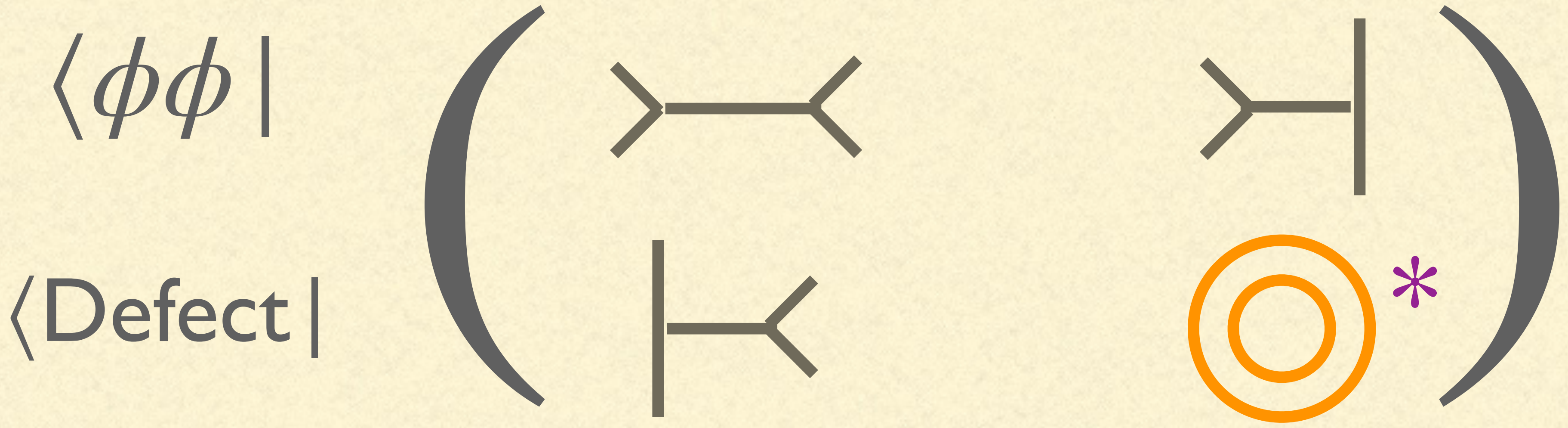
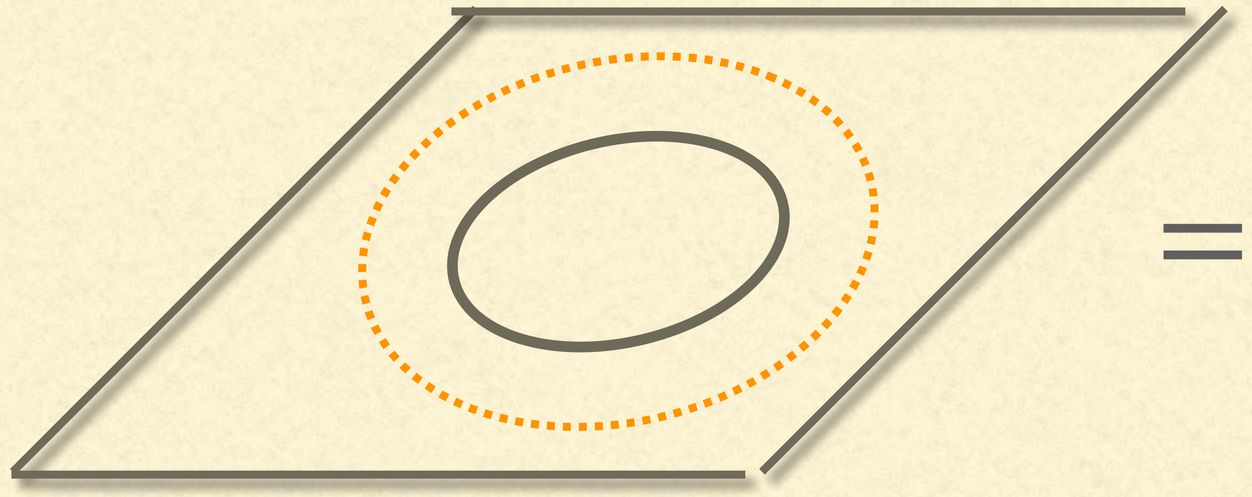




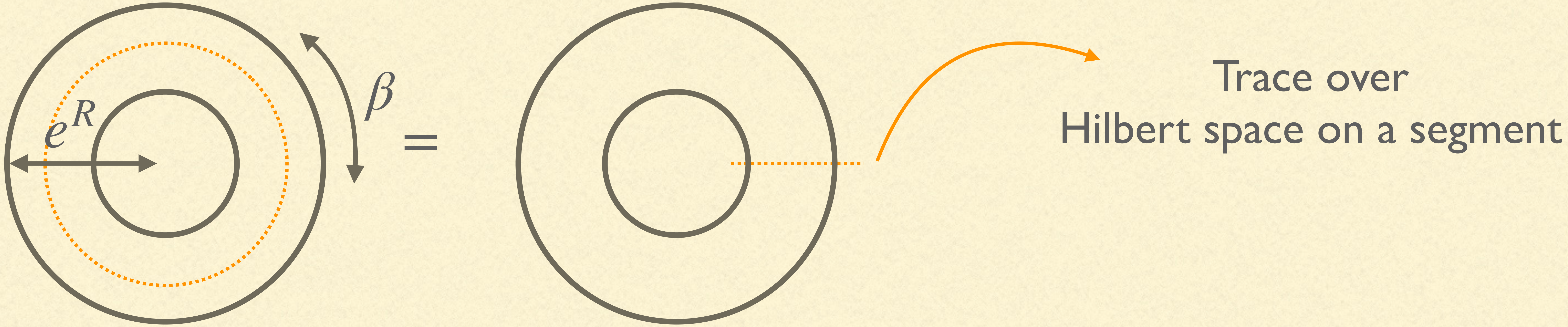
$$= \sum_O a_O\, O$$

- We need an equation for this correlator

* We cannot just put zero here

Conformal boundaries in two-dimensions

- In two dimensions, the annulus partition function provides us with an equation:



$$\langle \mathrm{B} | e^{-RH} | \mathrm{B} \rangle = \mathrm{Tr} e^{-\beta \hat{H}}$$

[Cardy, 1989]

[Friedan, Konechny, Schmidt-Colinet, 2012] [Collier, Mazac, Wang, 2021]

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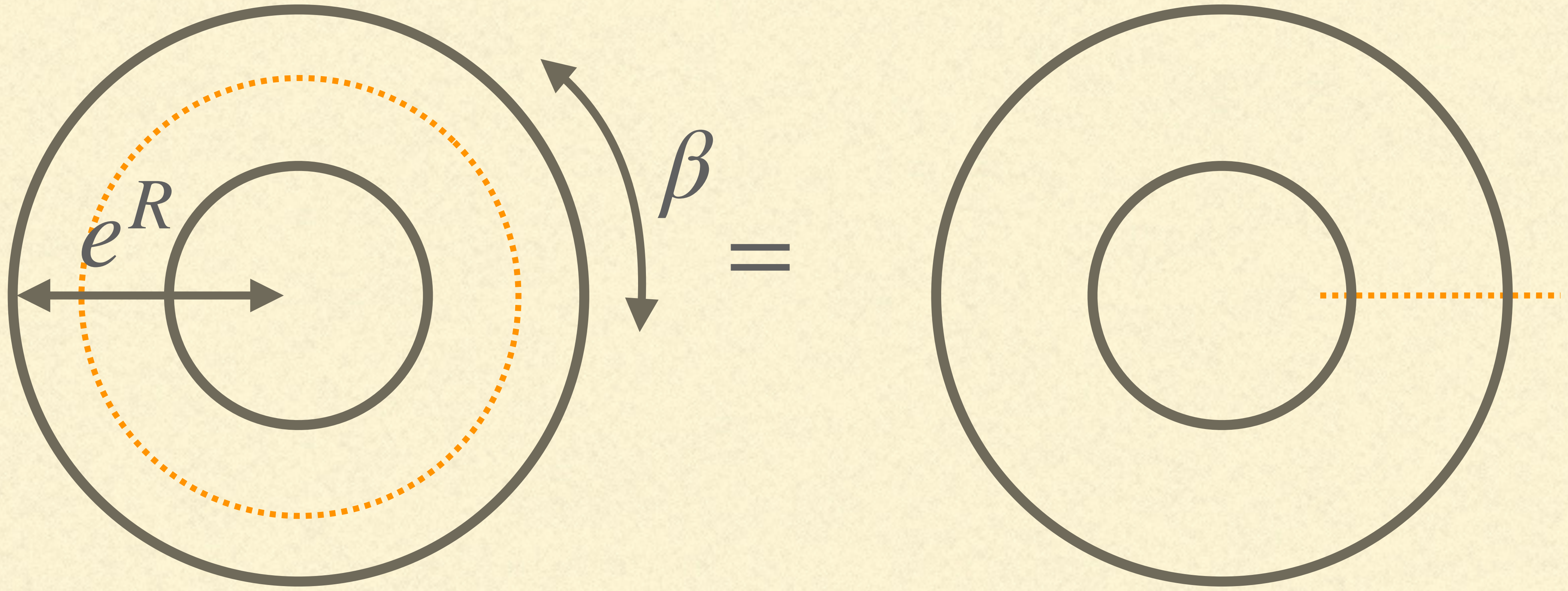

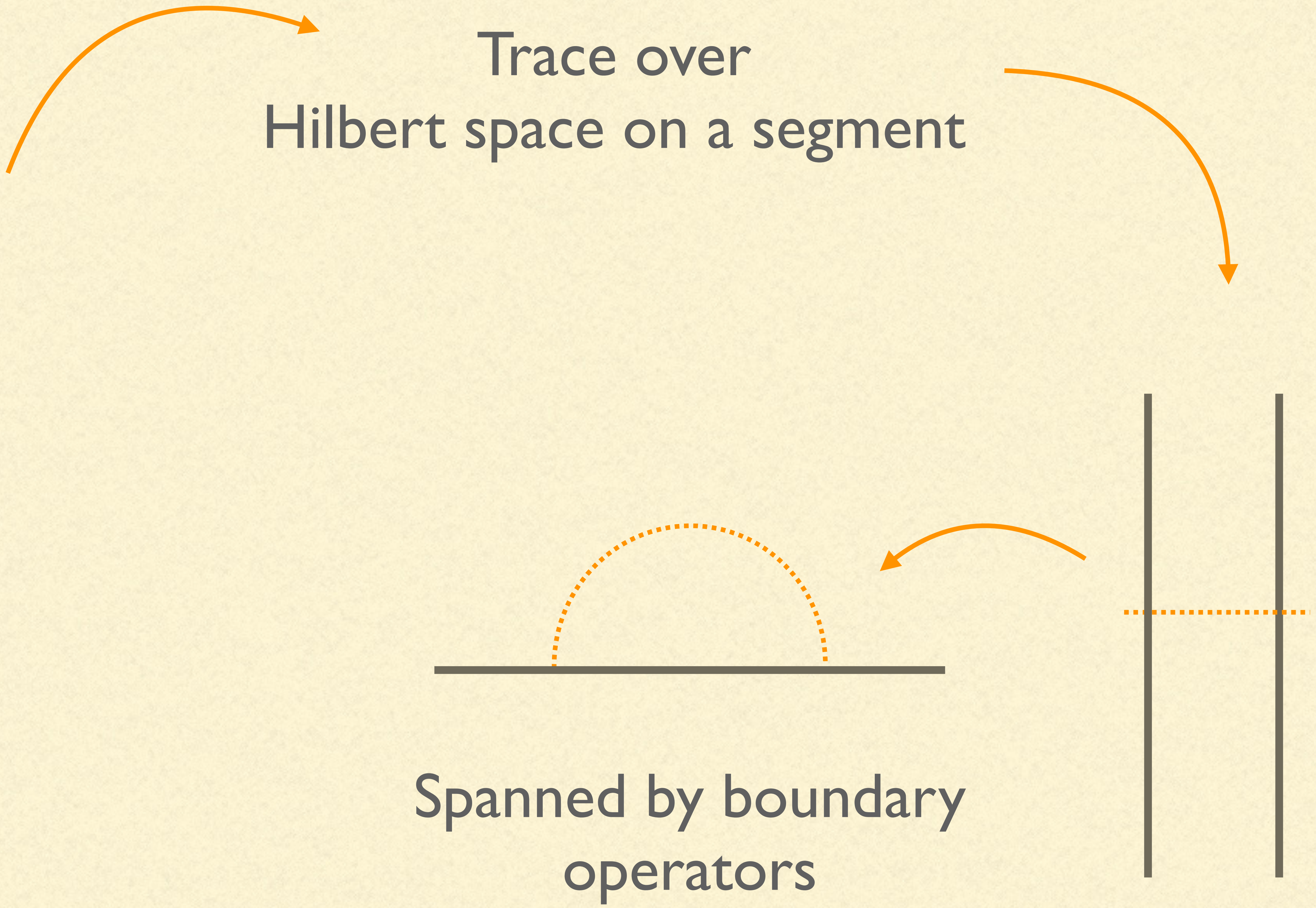

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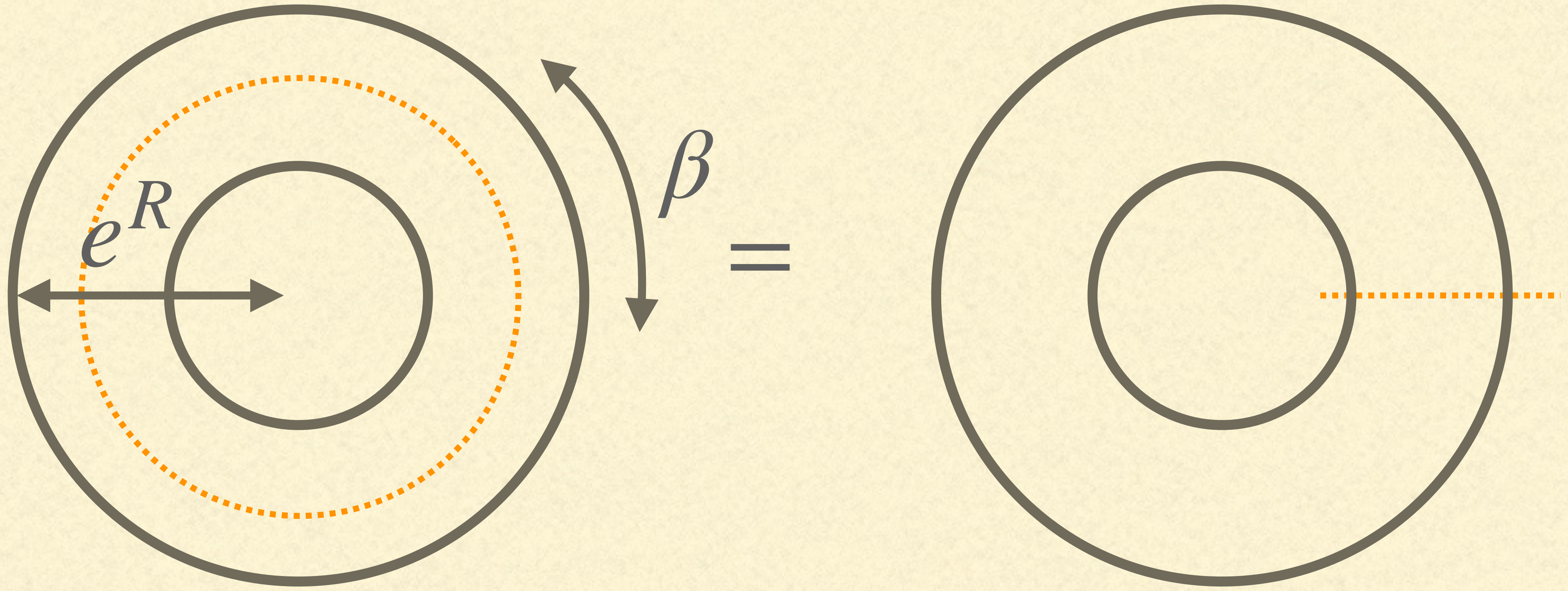
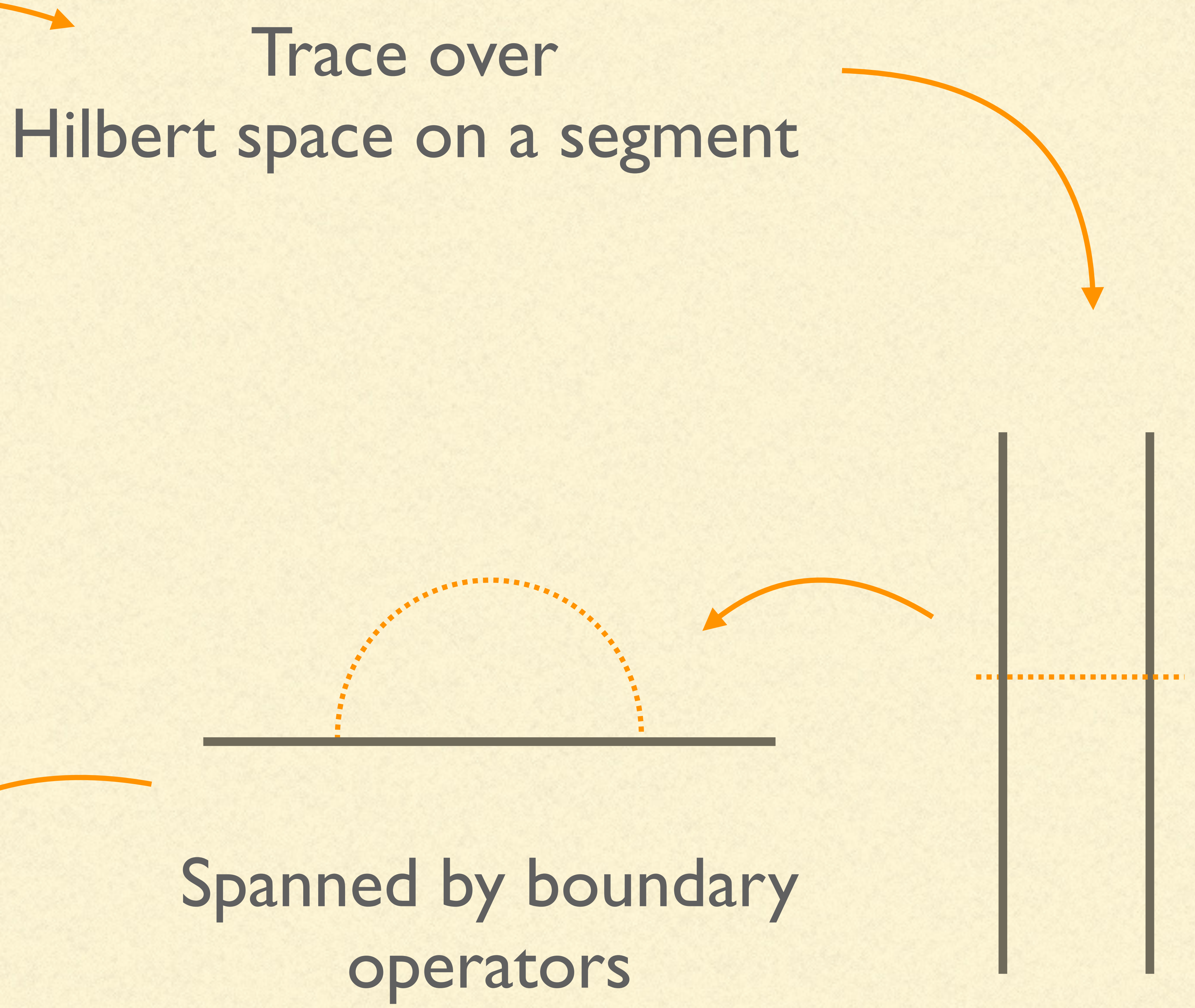




$$\langle \mathrm{B} | e^{-RH} | \mathrm{B} \rangle = \mathrm{Tr} e^{-\beta \hat{H}} \qquad \tau = \frac{2R}{\beta}$$

[Cardy, 1989]

$$\sum_{\Delta} a_{\Delta}^2 e^{-\pi\tau\left(\Delta - \frac{c}{12}\right)} = \sum_{h} n_h e^{-\frac{2\pi}{\tau}\left(h - \frac{c}{24}\right)}$$

[Friedan, Konechny, Schmidt-Colinet, 2012] [Collier, Mazac, Wang, 2021]

Conformal boundaries in two-dimensions

- The trace normalizes the Cardy state $|\mathrm{B}\rangle$

$$\langle \mathrm{B}| e^{-RH} |\mathrm{B}\rangle = \mathrm{Tr} e^{-\beta \hat{H}}$$

- Therefore the disk partition function is physical:

$$a_0 = \langle 0|\mathrm{B}\rangle = g$$

[Affleck, Ludwig 1991]

[Friedan, Konechny 2004]

  - monotonic under boundary RG flows
  - gives a boundary contribution to thermal and entanglement entropy:

$$S = \frac{c}{3} RT + \log g \qquad S_{EE} = \frac{c}{6} \log \frac{2L}{\epsilon} + \log g$$

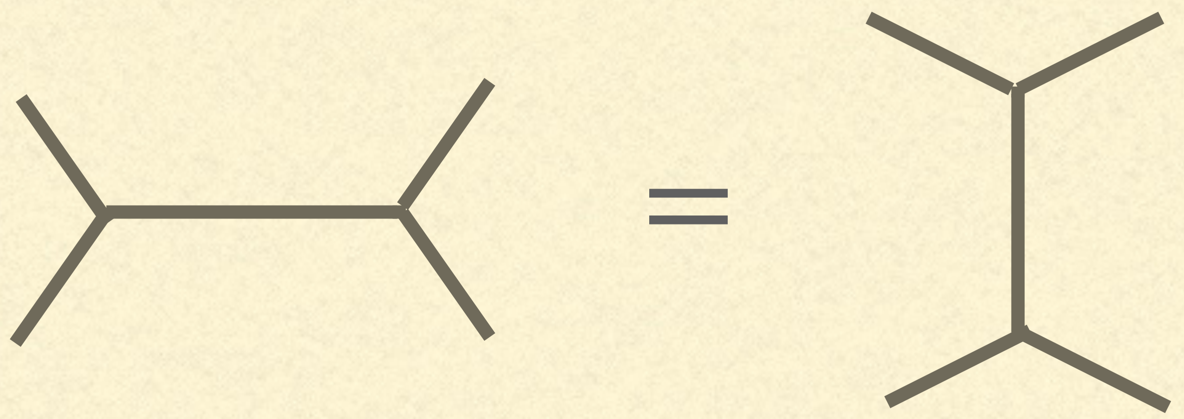
Conformal boundaries in two-dimensions

[Cardy, Lewellen 1991]

[Lewellen 1992]

- The other sewing relations:

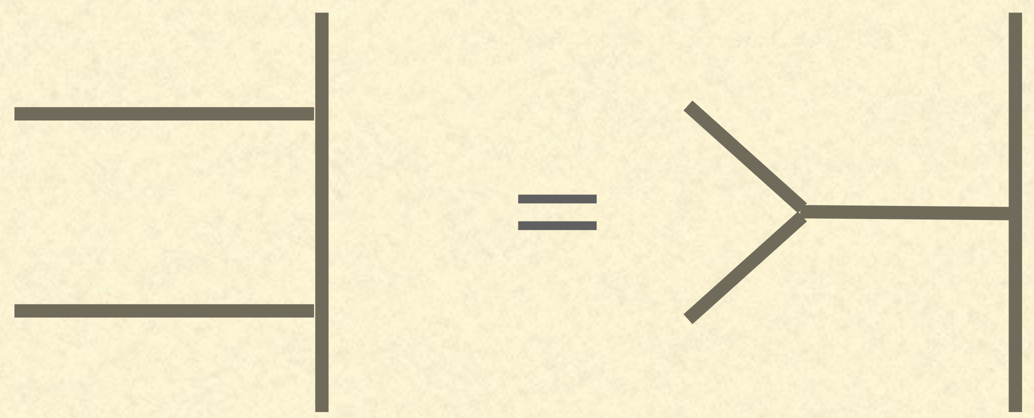
[Recknagel, Schomerus, BCFT and worldsheet approach to D-branes, 2013]



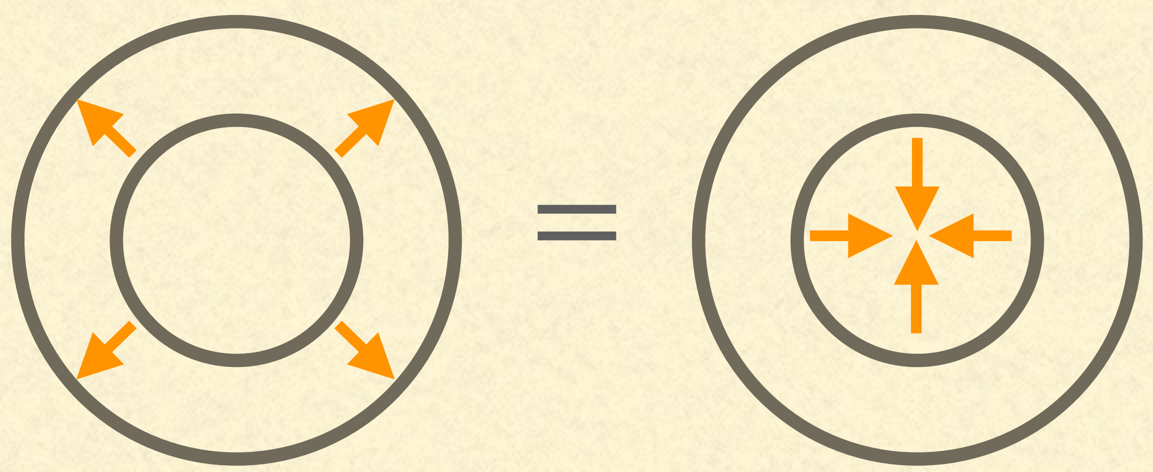
$$\sum_{\Delta\ell} c^2_{\Delta\ell} F_{\Delta\ell}(z,\bar{z}) = 0$$

$$F_{\Delta\ell}(z,\bar{z}) = v^{\Delta_\phi} g_{\Delta\ell}(z,\bar{z}) - u^{\Delta_\phi} g_{\Delta\ell}(1-z,1-\bar{z})$$

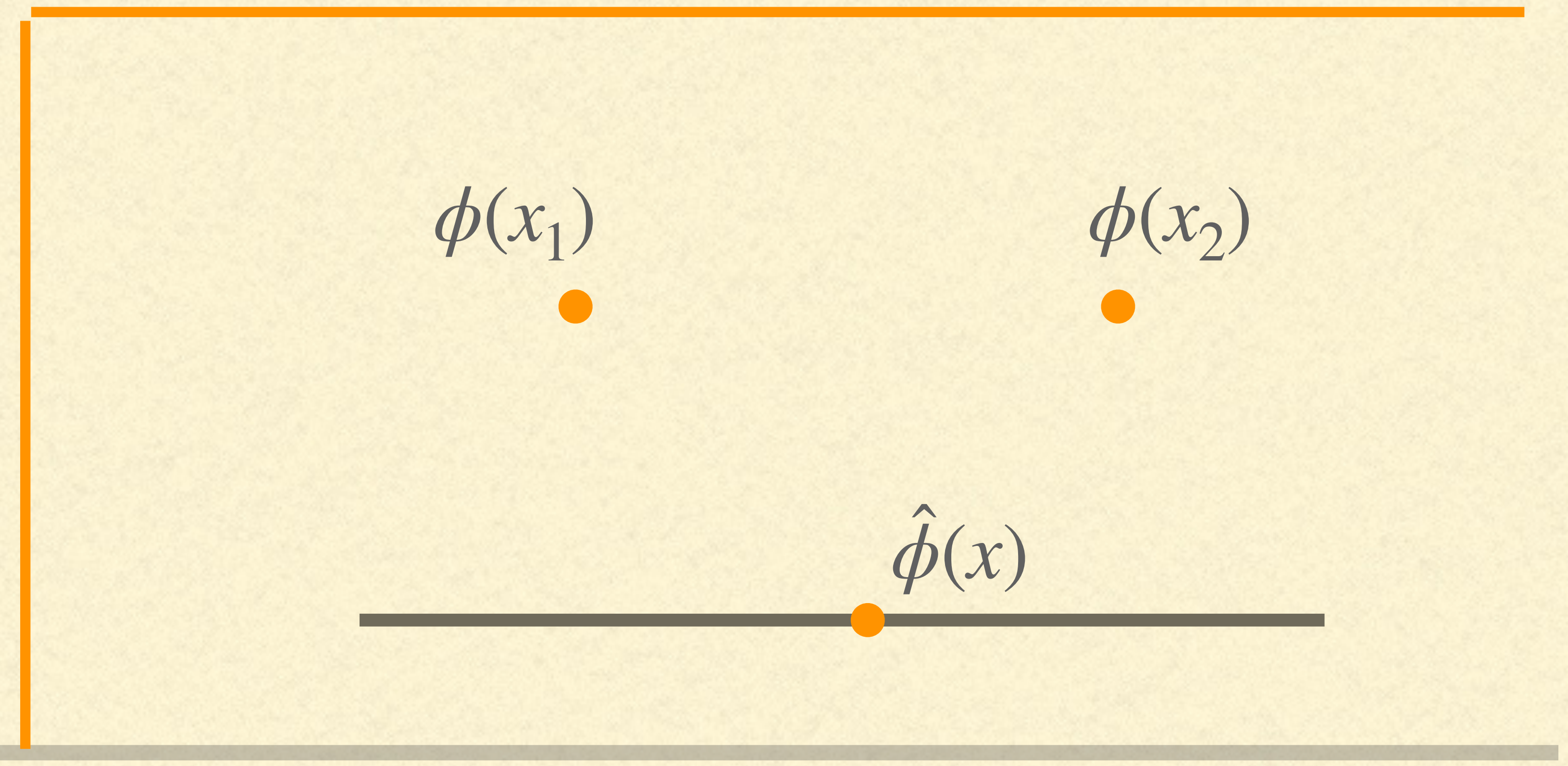
$$g_{\Delta,\ell}(z,\bar{z}) = z^{\frac{\Delta-\ell}{2}}\,\bar{z}^{\frac{\Delta+\ell}{2}}\,\kappa_{\frac{\Delta-\ell}{2}}(z)\,\kappa_{\frac{\Delta+\ell}{2}}(\bar{z})$$



$$\sum_h b_h^2 \xi^{-h} \kappa_h(-1/\xi) = \sum_\Delta c_{\Delta 0} a_\Delta \xi^{-\Delta_\phi + \frac{\Delta}{2}} \kappa_{\frac{\Delta}{2}}(-\xi)$$



$$\sum_h n_h \chi_h(1/\tau) = \sum_\Delta a_\Delta^2 \chi_{\frac{\Delta}{2}}(\tau)$$



A positive semi-definite program

- Putting all together:

$$
\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} F_{00}(z,\bar{z}) & \frac{1}{2} \\ \frac{1}{2} & \chi_0(\tau) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \sum_{\Delta \neq 0} \begin{pmatrix} c_{\Delta\ell} & a_\Delta \end{pmatrix} \begin{pmatrix} F_{\Delta\ell}(z,\bar{z}) & \frac{1}{2}\delta_{\ell,0}\,\xi^{\frac{\Delta}{2}}\,\kappa_{\frac{\Delta}{2}}(-\xi) \\ \frac{1}{2}\delta_{\ell,0}\,\xi^{\frac{\Delta}{2}}\,\kappa_{\frac{\Delta}{2}}(-\xi) & \delta_{\ell,0}\,\chi_{\frac{\Delta}{2}}(\tau) \end{pmatrix} \begin{pmatrix} c_{\Delta\ell} \\ a_\Delta \end{pmatrix}
$$

$$
-\sum_h \left[\frac{n_h}{g^2} \chi_h(1/\tau) + b_h^2\, \xi^{\Delta_\phi - h}\, \kappa_h(-1/\xi) \right] = 0\ .
$$

- We use sl(2) blocks when bootstrapping mixed correlators
- Bootstrap strategy: given a trial spectrum, find a linear combination of derivatives evaluated at a point, which makes all matrices positive semi-definite (and one positive definite):

$$
1 + \sum_{\Delta > \Delta_{gap}} \begin{pmatrix} c_{\Delta\ell} & a_\Delta \end{pmatrix} M(\Delta) \begin{pmatrix} c_{\Delta\ell} \\ a_\Delta \end{pmatrix} - \sum_{h > h_{gap}} \left[\frac{n_h}{g^2} m_a(h) + b_h^2\, m_{2pt}(h) \right] > 0\ ,
$$

$$
M(\Delta) \succeq 0\ ,\ m_a(h) \leq 0\ ,\ m_{2pt}(h) \leq 0
$$

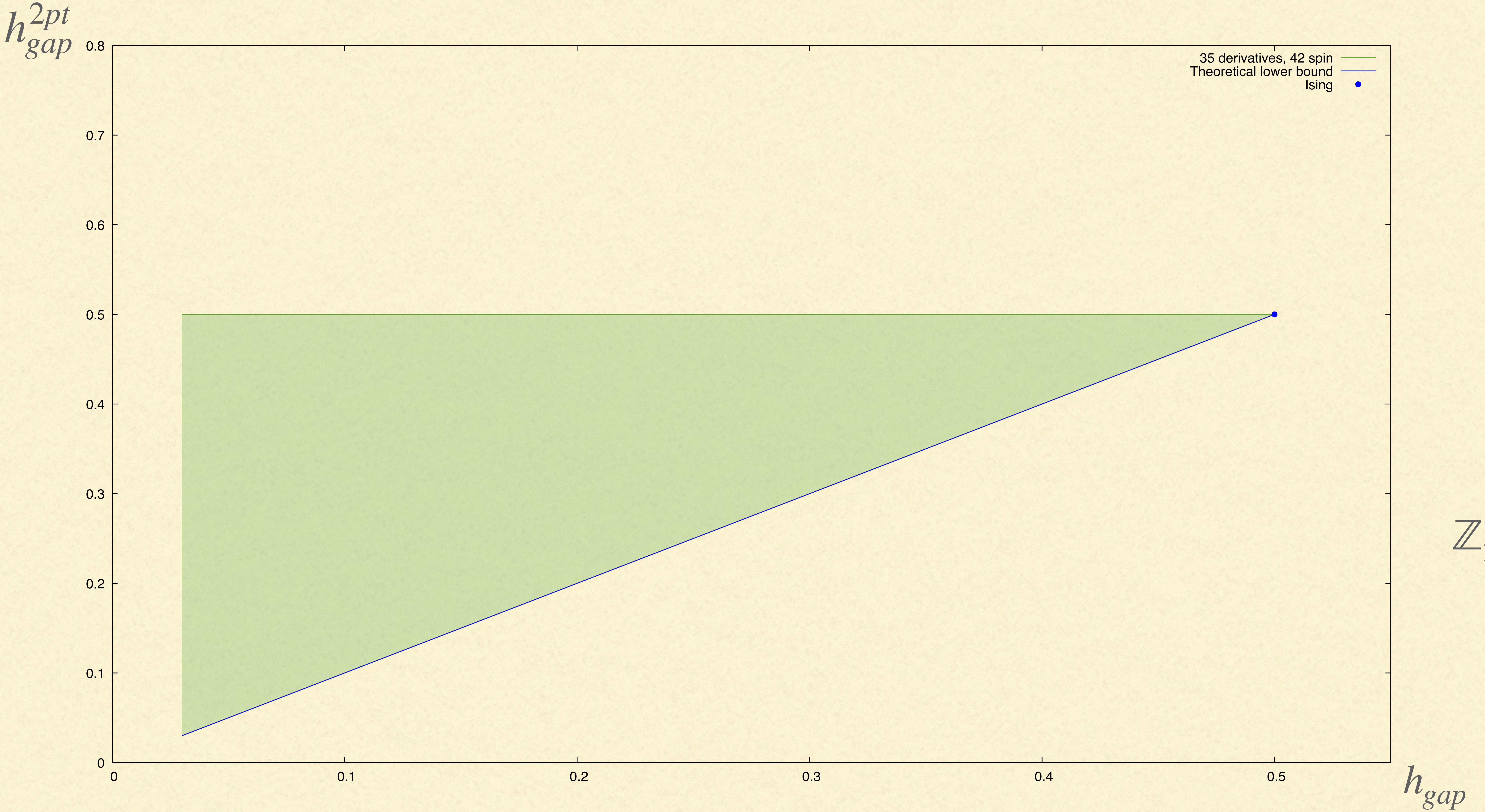
The parameter space and some technicalities

Δ_ϕ	Dimension of the external operator
Δ_{gap}	Dimension of the first unknown bulk operator
h_{gap}	Dimension of the first unknown operator on the boundary
h_{gap}^{2pt}	Dimension of the first unknown boundary operator in ϕ boundary OPE
c	Central charge

- We use sl(2) blocks when bootstrapping mixed correlators
- To avoid discretizing the spectrum, take derivative around $\rho(z_\star),\ \rho_{2pt}(\xi_\star),\ \tau_\star$:

$$\frac{\rho(z_\star)}{\rho_{2pt}(\xi_\star)} = e^{\pi\tau_\star} \qquad z_\star = \frac{1}{2} \qquad \tau_\star = 1 \qquad \longrightarrow \qquad \xi_\star = 0.03$$

A few plots $c = 1/2$



$$\Delta_\phi = \Delta_\sigma = 1/8$$

$$a_\sigma = 0$$

$$\Delta_{gap} = 1$$

$\mathbb{Z}_2$ symmetric boundary state is extremal

- Rigorous bounds from a two-point function with a boundary

The boundary conditions for a free boson

- The compact free boson

$$c = 1$$

$$\phi \sim \phi + 2\pi R$$

$$R > \sqrt{2}$$

$$\Delta - \ell = \left(\frac{m}{R} + w\frac{R}{2}\right)^2$$

$$\Delta + \ell = \left(\frac{m}{R} - w\frac{R}{2}\right)^2$$

$$m, w \in \mathbb{Z}$$

$$U(1)_m \times U(1)_w$$

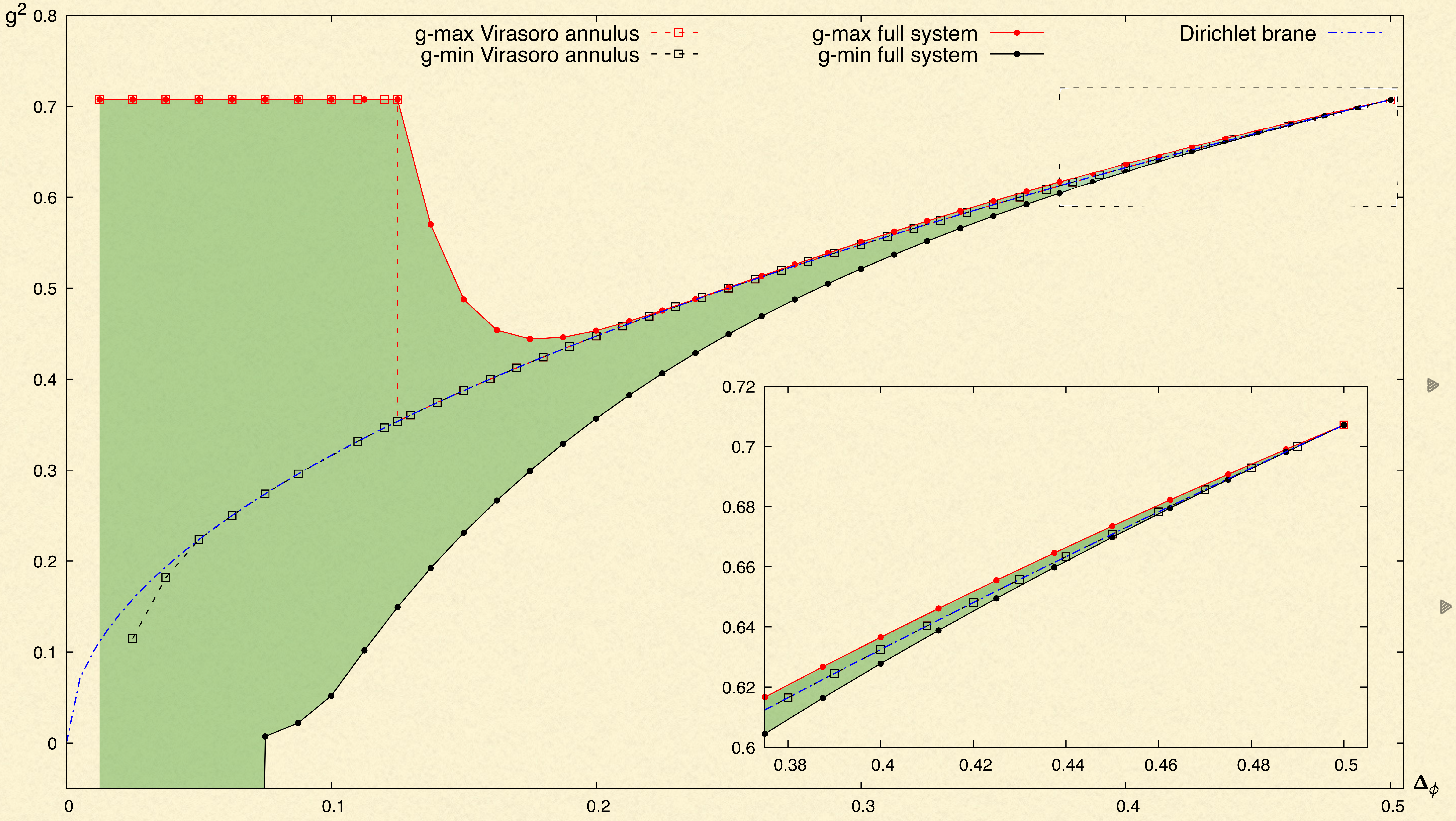
$(\partial\phi, \bar{\partial}\phi)$ → $U(1)_m$, $(\partial\phi, -\bar{\partial}\phi)$ → $U(1)_w$

- Boundary conditions at generic radius*

				$U(1)$ → $(\hat{m}, h_{gap})$		One-point functions
$\bar{\partial}\phi \leftrightarrow -\bar{\partial}\phi$	$\lvert D, \varphi\rangle$	$\varphi \sim \varphi + 2\pi R$	$g = \sqrt{\frac{1}{R}}$	$(0,1)$	$\left(1, \frac{R^2}{2}\right)$	$a_{m,0} \neq 0$
$R \leftrightarrow 2/R$						
$x_0 \leftrightarrow \tilde{x}_0$	$\lvert N, \tilde{\varphi}\rangle$	$\tilde{\varphi} \sim \tilde{\varphi} + 4\pi/R$	$g = \sqrt{\frac{R}{2}}$	$(0,1)$	$\left(1, \frac{2}{R^2}\right)$	$a_{0,w} \neq 0$

- Boundaries for the $\mathbb{Z}_2$ orbifold at generic radius $\phi \sim -\phi$
  - There is no $U(1)$ symmetry
  - Additional bulk scalar at $\Delta = \frac{1}{8}$
  - Additional (regular and fractional) branes
- Some more exceptional branes at special radii

A few plots $c = 1$



- Assumptions:

$$a_\phi \neq 0$$

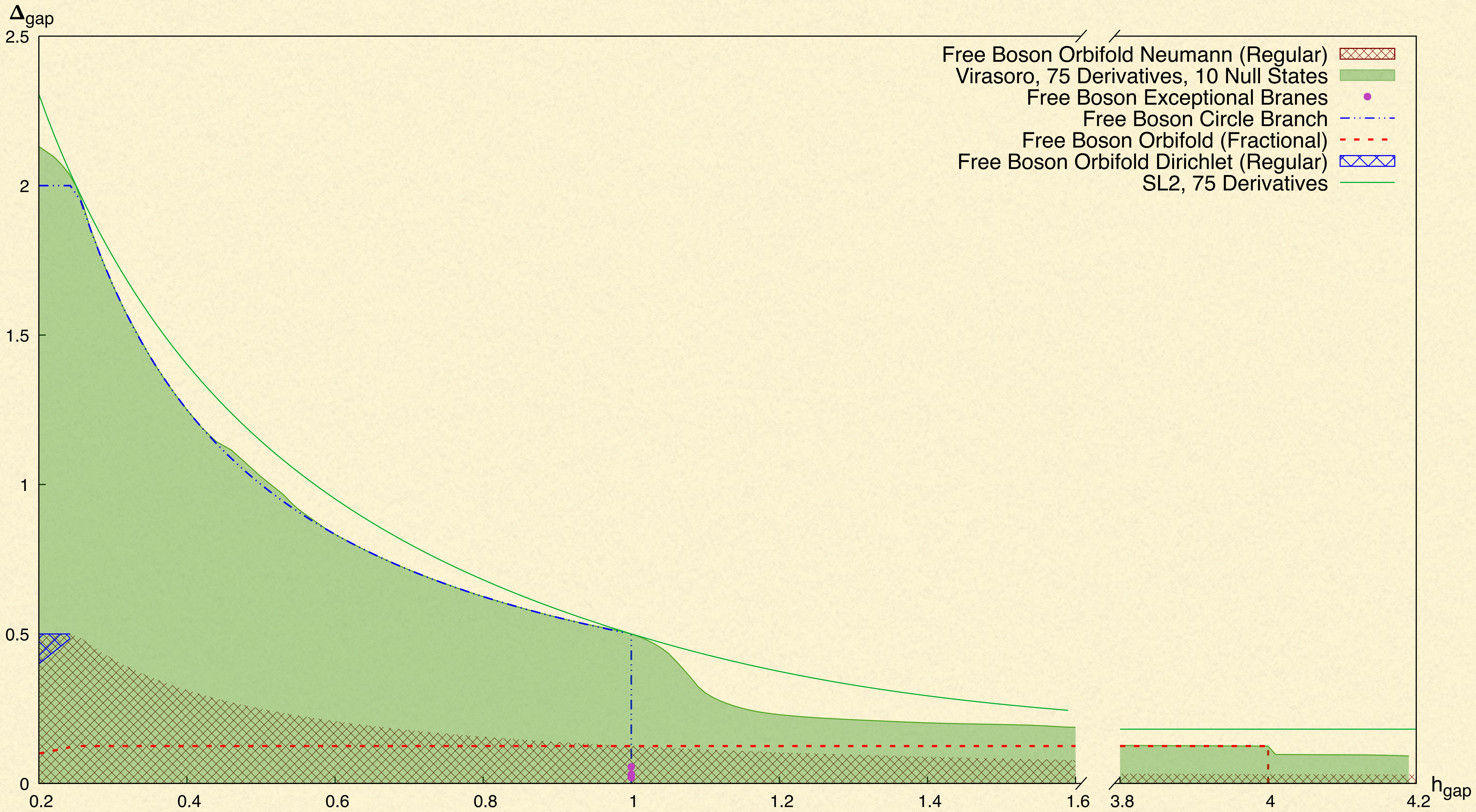
$$\Delta_{gap} = 4\Delta_\phi$$

- Sufficient to isolate the D-brane!

- The full system does not improve over the annulus

A few plots $c = 1$

- Annulus only



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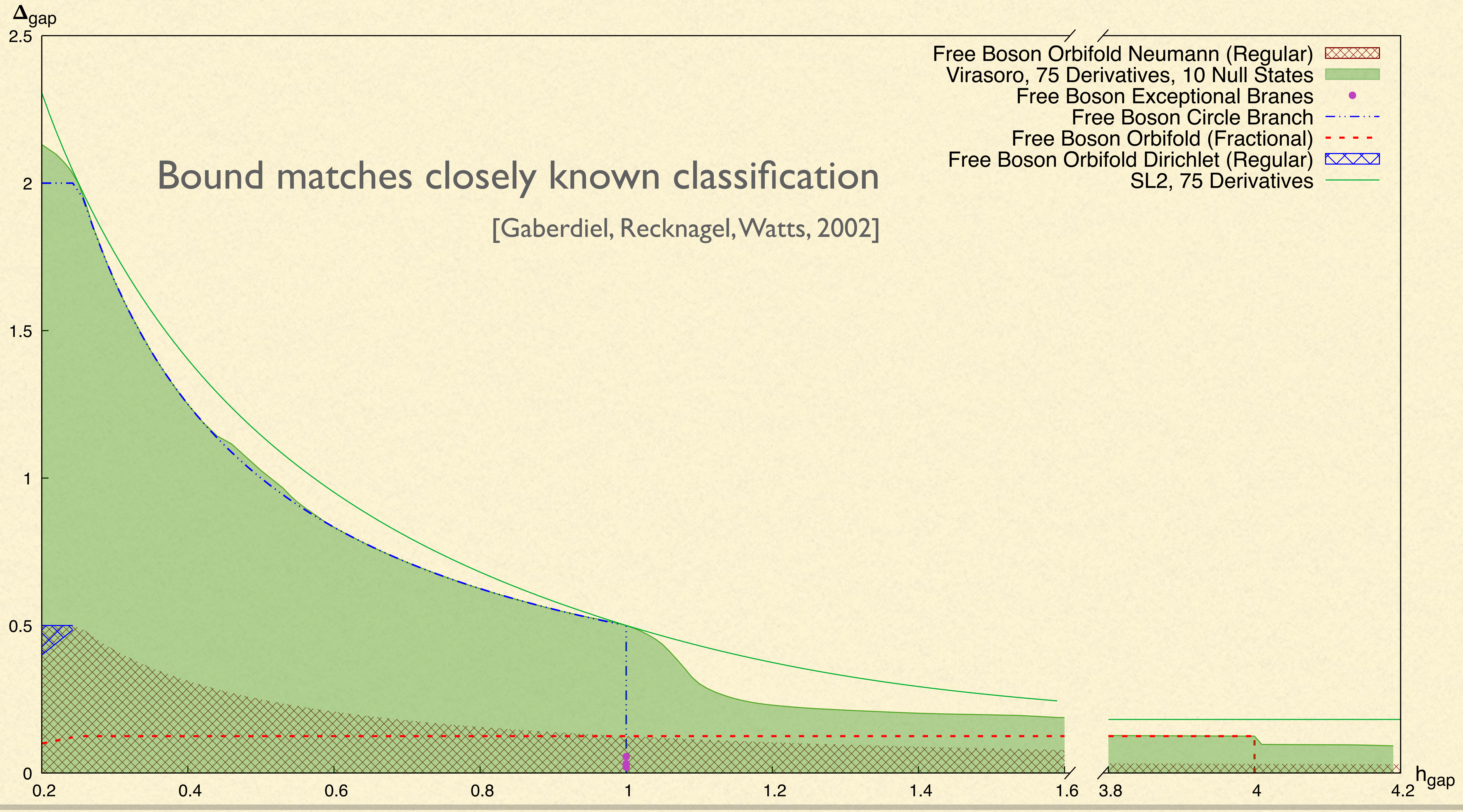

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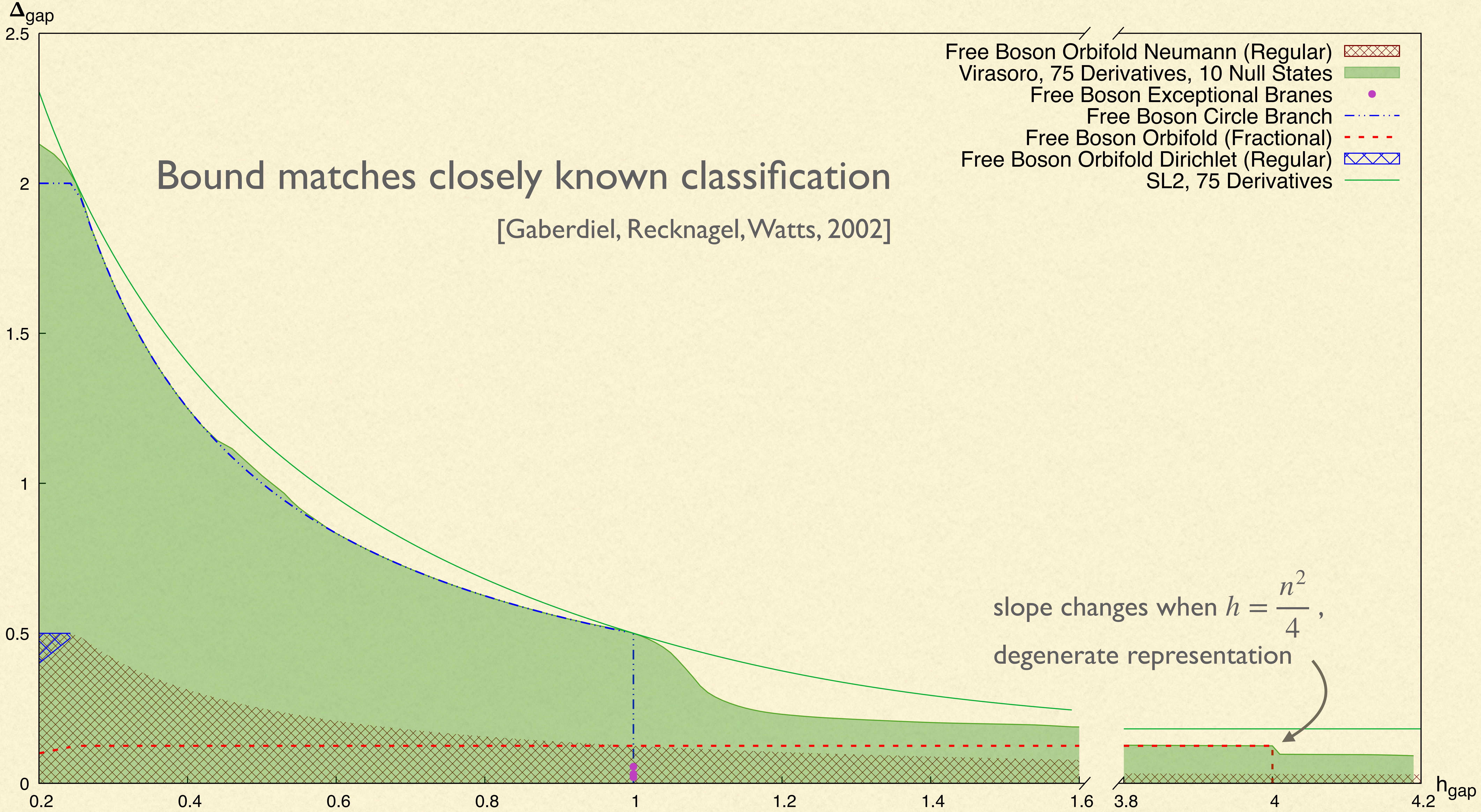

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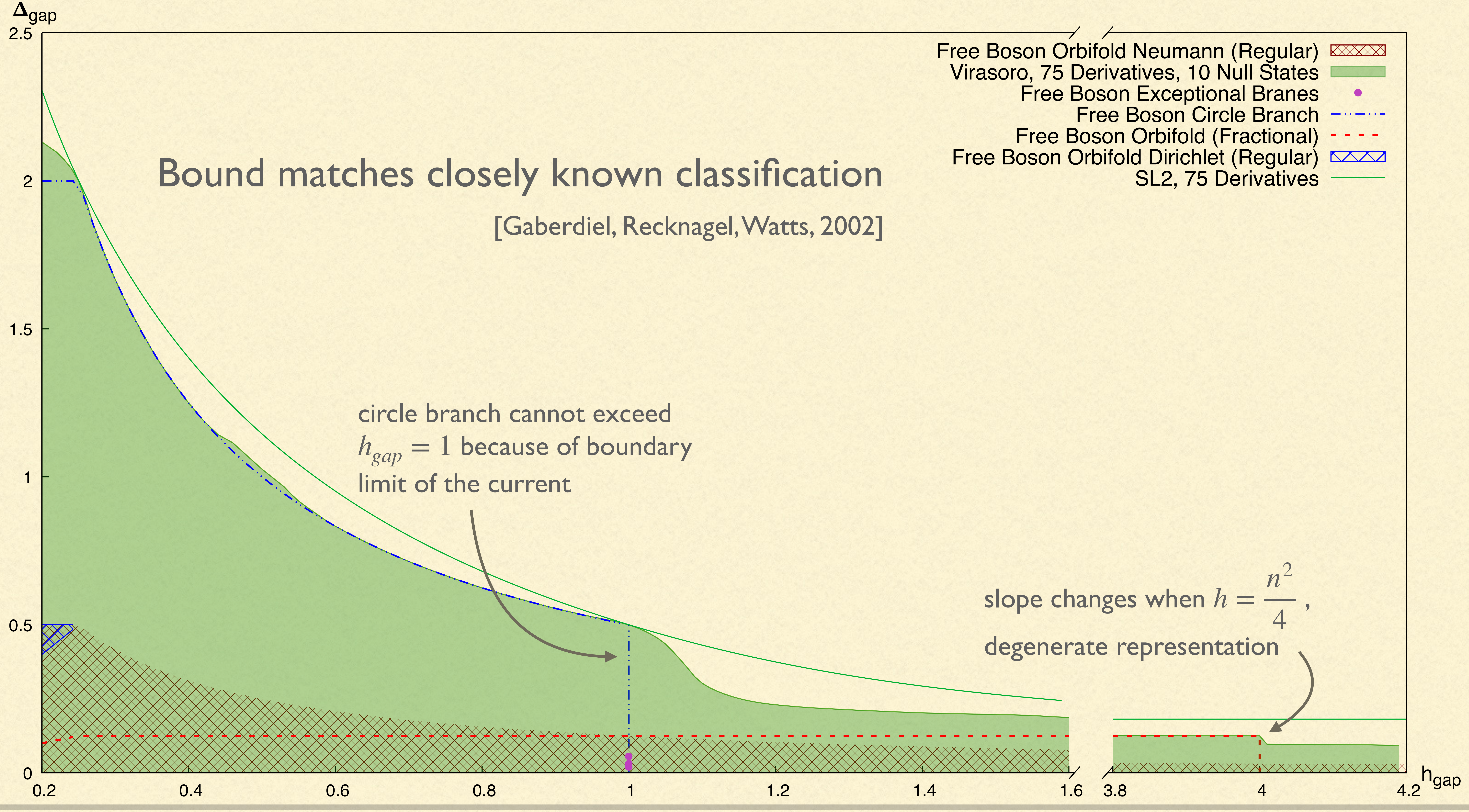

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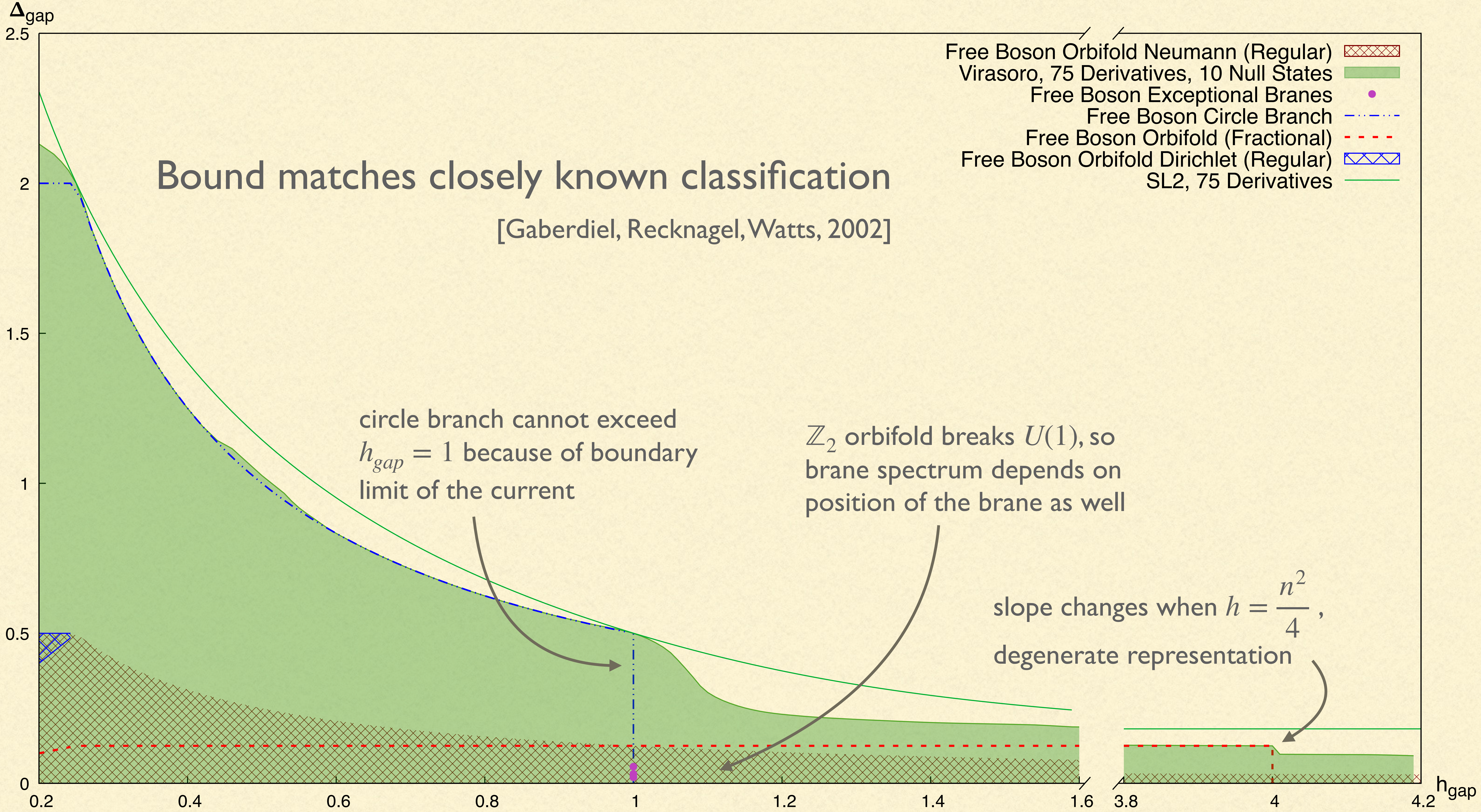

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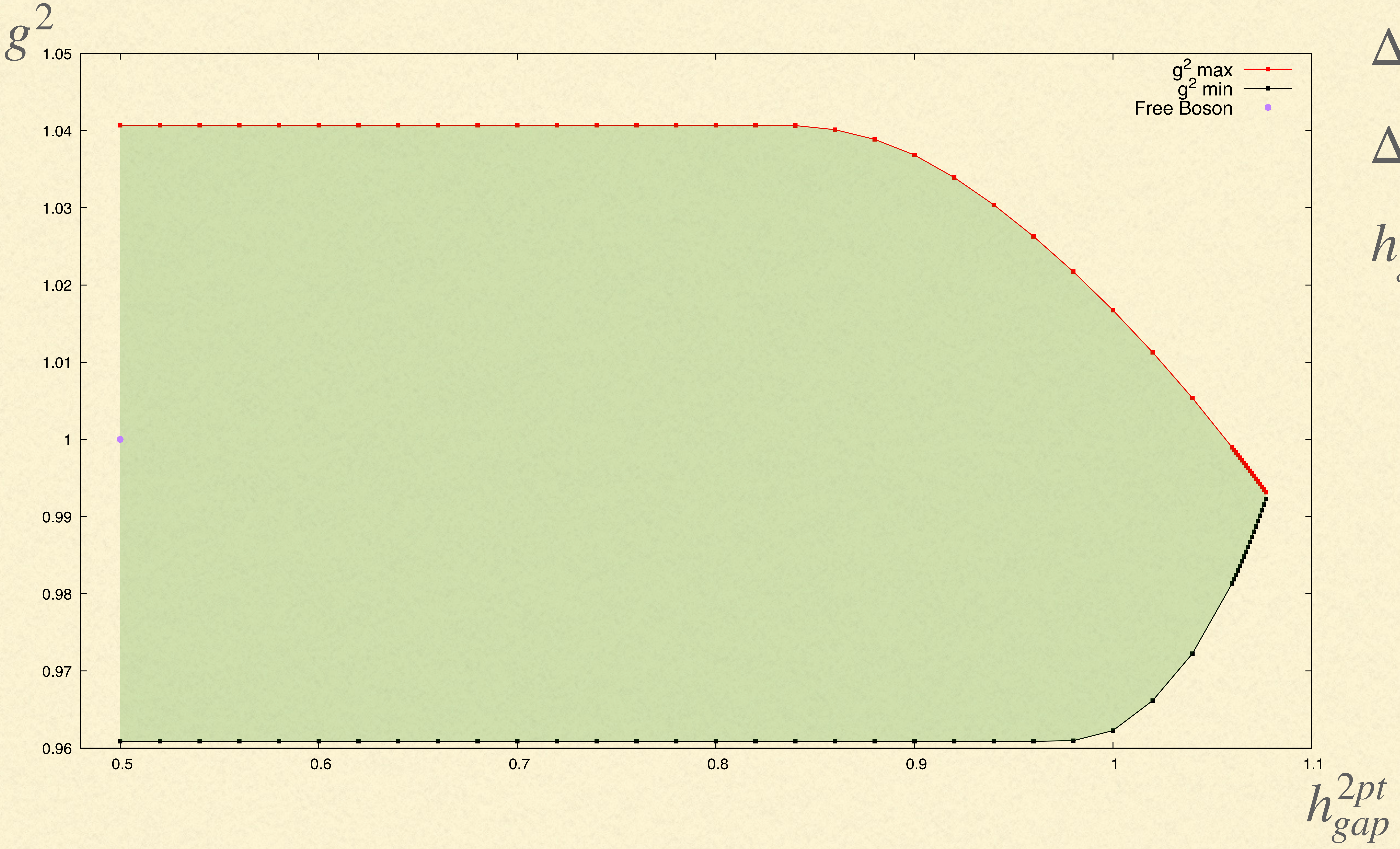


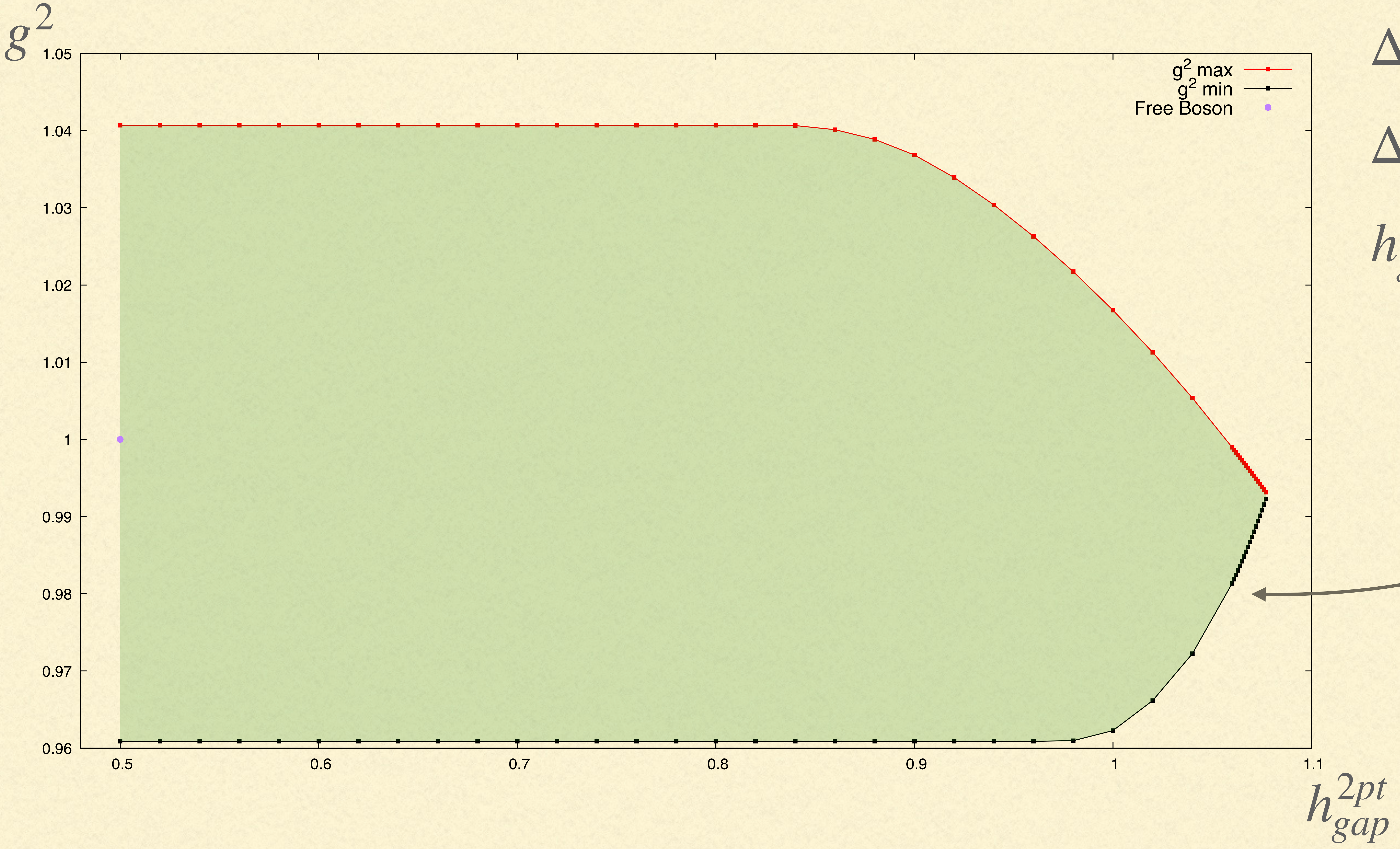
D-brane at $R = 1$

$\Delta_\phi = 1/4$ first winding mode

$\Delta_{gap} = 1$ first momentum mode

$h_{gap} = 1/2$

A few plots $c = 1$



D-brane at $R = 1$

$\Delta_\phi = 1/4$ first winding mode

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Genuine bounds from the two-point function

A few plots $c = 1$

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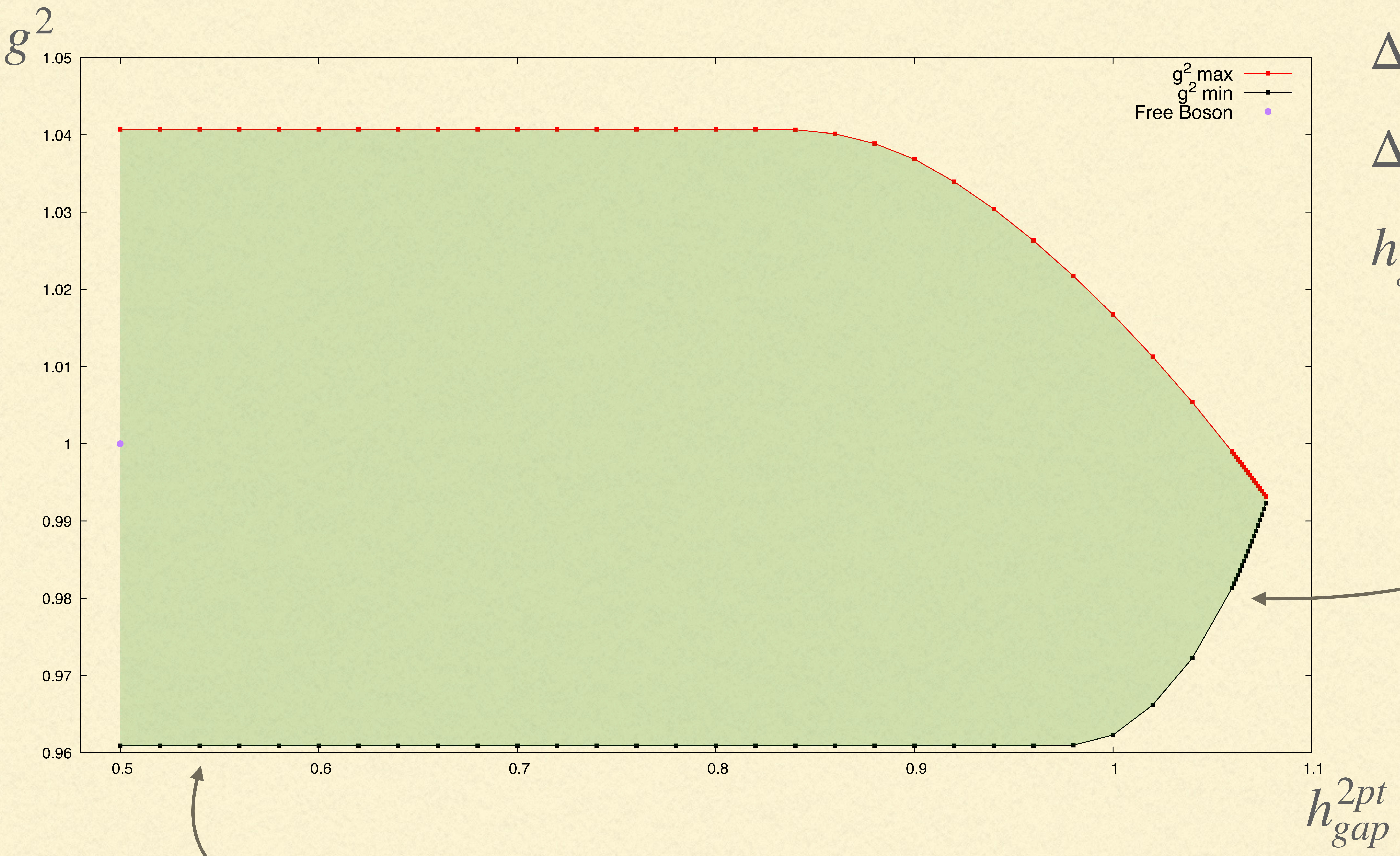
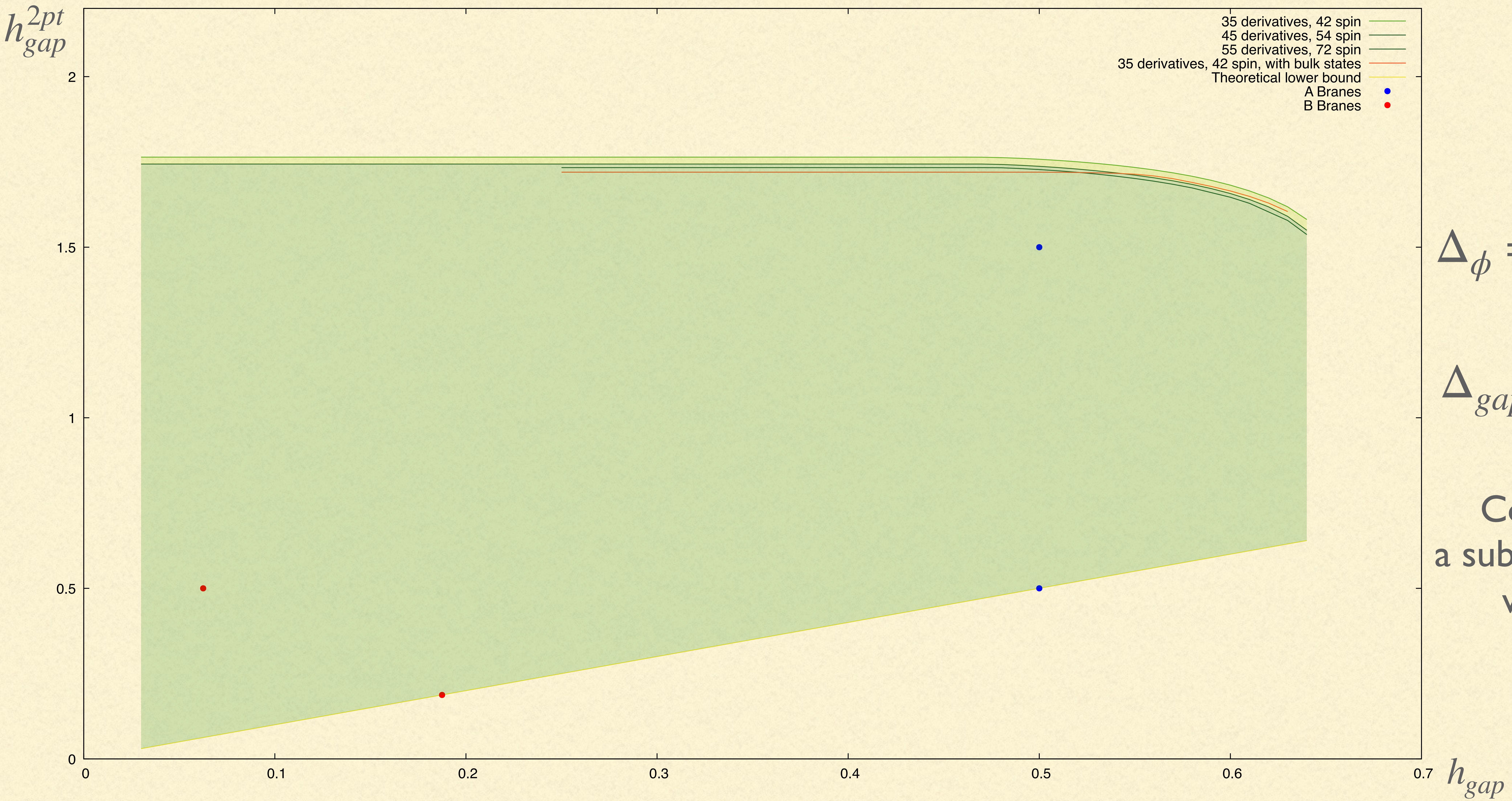


Genuine bounds from the two-point function

Fake, functional-dependent solution, with no bulk operators in $\phi \times \phi$ beyond the identity

A few plots $c = 3/2$



$$\Delta_\phi = \Delta_{j=1/2} = \frac{3}{8}$$

$$\Delta_{gap} = \Delta_{j=1} = 1$$

Compatible with a subset of boundaries where $a_\phi = 0$

Outlook

- Include more low lying states in the bulk
- Explore parameter space
- Virasoro
- Include more external operators
- Include all the sewing relations: nice challenge for the multipoint bootstrap
- Interfaces
- Higher dimensions


Thank you!