Resurgence in CFT in 2D

Minimal models and beyond?

Tomás Reis
work with A. Bissi (ICTP, Uppsala, INFN), N. Dondi (ICTP, INFN), A. Piazza (SISSA, INFN), M. Serone (SISSA, INFN)

23rd September 2024,
TFI 2024: Theories of the Fundamental Interactions

SISSA & INFN, Trieste

SISSA

INFN
Istituto Nazionale di Fisica Nucleare
Laboratori Nazionali di Frascati

Introduction

Resurgence of resurgence

- Resurgence has been very successful in Quantum Mechanics and weak coupling expansions in Quantum Field Theory.
- A developing frontier for application of resurgence is CFTs. There have been developments in $\mathcal{N} = 4$ SYM and SCFTs [Dorigoni et al., Perlmutter et al.], 3d sigma models [Reffert et al.], and two dimensional CFTs.
- We want to focus on the last case, CFT$_2$ in the large central charge c expansion. We expand the direction of [Benjamin Collier Maloney Merulyia '23], which is related to the work of [Fiztpatrick Kaplan et al. '14 – '16]. Recently, a new paper of [Benjamin et al.] came out which is also related.

It was quantum gravity all along

- A big motivation is AdS/CFT. The large central charge expansion of the CFT is related to the weak coupling expansion (small G_N) in gravity. So by studying the more accessible CFT side we have a model for resurgence in the much harder graviton expansions in quantum gravity.
- Even though there are no actual gravitons in AdS_3, there is a small G_N perturbation theory from offshell virtual gravitons.
- Two dimensions has many powerful exact techniques. In particular, minimal models provide very simple cases where we can do the analysis thoroughly. The history of resurgence suggests that we should start from the simpler solvable models.

Resurgence in an instant

In Borel space nobody can hear you diverge

Many if not most series in QFT are asymptotic, i.e. divergent (Dyson 1953). Typically they are of the form:

$$F_N(g) = \sum_{k=1}^{N} a_k g^k, \qquad a_k \sim A^{-k} k! \quad k \gg 1. \tag{2.1}$$

In Borel space nobody can hear you diverge

Many if not most series in QFT are asymptotic, i.e. divergent (Dyson 1953). Typically they are of the form:

$$F_N(g) = \sum_{k=1}^{N} a_k g^k, \qquad a_k \sim A^{-k} k! \quad k \gg 1. \tag{2.1}$$

We can try **Borel (re)summation** (1899). The Borel transform of a series is given by

$$\varphi(g) \approx \sum_{k\geq 0} c_k g^k \to \widehat{\varphi}(\zeta) = \sum_{k\geq 0} \frac{c_k}{k!}\zeta^k \tag{2.2}$$

In Borel space nobody can hear you diverge

Many if not most series in QFT are asymptotic, i.e. divergent (Dyson 1953). Typically they are of the form:

$$F_N(g) = \sum_{k=1}^{N} a_k g^k, \qquad a_k \sim A^{-k} k! \quad k \gg 1. \tag{2.1}$$

We can try **Borel (re)summation** (1899). The Borel transform of a series is given by

$$\varphi(g) \approx \sum_{k \geq 0} c_k g^k \to \widehat{\varphi}(\zeta) = \sum_{k \geq 0} \frac{c_k}{k!} \zeta^k \tag{2.2}$$

If φ is *Borel summable*, we recover a well defined function $\varphi(z)$ from the Borel sum

$$s(\varphi)(g) = \int_0^\infty \mathrm{e}^{-\zeta} \widehat{\varphi}(g\zeta)\, d\zeta. \tag{2.3}$$

But in most physical theories this is not enough.

Or how I learned to stop worrying and love divergent series

If we Borel transform the example from before with $A > 0$

$$\sum_{k\geq 0}^{\infty}(A^{-k}k!)g^k \Rightarrow \widehat{F}(\zeta) = \frac{1}{1-\zeta/A}$$
$$\Rightarrow s(F)(\mathrm{e}^{+i\epsilon}g) - s(F)(\mathrm{e}^{-i\epsilon}g) = 2\pi\mathrm{i}Ag^{-1}\mathrm{e}^{-A/g} \quad (2.4)$$

There's an ambiguous residue!

Or how I learned to stop worrying and love divergent series

If we Borel transform the example from before with $A > 0$

$$\sum_{k\geq 0}^{\infty}(A^{-k}k!)g^k \Rightarrow \widehat{F}(\zeta) = \frac{1}{1-\zeta/A}$$

$$\Rightarrow s(F)(\mathrm{e}^{+i\epsilon}g) - s(F)(\mathrm{e}^{-i\epsilon}g) = 2\pi\mathrm{i}Ag^{-1}\mathrm{e}^{-A/g} \quad (2.4)$$

There's an ambiguous residue! But the asymptotic series is not complete. The **trans-series** includes non-perturbative effects,

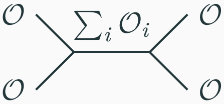
$$\Phi(g) = \sum_{k\geq 0} c_k g^k + \sum_i \mathcal{C}_i^{\pm}\mathrm{e}^{-A_i/g}g^{b_i}\sum_{k\geq 0} c_k^{(i)} g^k + \cdots \quad (2.5)$$

Crucially, the $\mathcal{C}_i^{\pm}$ are piece-wise constant in the complex g plane, compensating the Borel discontinuity.

Resurgence helps both **make sense** of what we know and **explore** what we don't know.

2d CFT at large central charge

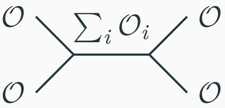
Fantastic Four point functions



The four-point function in a CFT can be expanded as an OPE,

$$\langle \mathcal{O}(0)\mathcal{O}(z)\mathcal{O}(1)\mathcal{O}(\infty)\rangle = |\mathcal{F}_{\mathbb{I}}(c,z)|^2 + \sum_i C_{\mathcal{O}\mathcal{O}i}\,|\mathcal{F}_i(c,z)|^2 \quad (3.6)$$

Fantastic Four point functions



The four-point function in a CFT can be expanded as an OPE,

$$\langle \mathcal{O}(0)\mathcal{O}(z)\mathcal{O}(1)\mathcal{O}(\infty)\rangle = |\mathcal{F}_{\mathbb{I}}(c,z)|^2 + \sum_i C_{\mathcal{O}\mathcal{O}i}\, |\mathcal{F}_i(c,z)|^2 \quad (3.6)$$

And it is also known that (in certain setups), the blocks $\mathcal{F}_i$ admit an expansion as an asymptotic series in $1/c$

$$\mathcal{F}_i(c,z) \sim \mathrm{e}^{cS_i(z)} \sum_{n\geq 0} \frac{f_n(z)}{c^n} \quad (3.7)$$

So the 4pt function has a trans-series structure. Can blocks “discover each other” through asymptotic behaviour?

Minimal models are constructed with a finite number of (degenerate) operators constrained by Virasoro symmetry. For the simplest case we take the operator $\phi_{2,1}$, whose OPE is constrained to be $\phi_{2,1} \times \phi_{2,1} = \phi_{1,1} + \phi_{3,1}$ (where $\phi_{1,1} = \mathbb{I}$)

$$\langle \phi_{2,1}\phi_{2,1}\phi_{2,1}\phi_{2,1} \rangle = |\mathcal{F}_{1,1}(c,z)|^2 + g(c)\,|\mathcal{F}_{3,1}(c,z)|^2 \tag{3.8}$$

We will take a correlation function of Virasoro primaries and analytically continue in c, thus the example is **non-unitary** (can we come back and connect to unitary minimal models? Yes*.).

The identity block is known exactly

$$\mathcal{F}_{1,1} = z^{1+\frac{3b^2}{2}}(1-z)^{-\frac{b^2}{2}}\,{}_2F_1\left(-b^2, 1+b^2, 2+2b^2; z\right), \tag{3.9}$$

where $c = 13 + 6(b^2 + b^{-2})$. From now on, we change $b^2 = 1/\epsilon - 3/2$, keep in mind $\epsilon \sim \mathcal{O}(1/c)$.

While saddle point techniques are available, they are cumbersome. Instead, by using the BPZ differential equation

$$\left[\frac{\partial^2}{\partial z_1^2} + b^2 \sum_{i=2}^{4} \left(\frac{1}{z_1 - z_i} \frac{\partial}{\partial z_i} + \frac{h_{2,1}}{(z_1 - z_i)^2} \right) \right] \langle \phi_{2,1}\phi_{2,1}\phi_{2,1}\phi_{2,1} \rangle = 0, \tag{3.10}$$

one can find explicitly the asymptotic series at finite z,

$$\mathcal{F}_{1,1} \sim z^{-\frac{5}{4}+\frac{3}{2\epsilon}} (1-z)^{\frac{3}{4}-\frac{1}{2\epsilon}} \mathrm{e}^{\left(1-\frac{1}{\epsilon}\right) S_0(z)} A_0(z) \sum_{n\geq 0} f_n(r(z)) \epsilon^n \tag{3.11}$$

where

$$S_0(z) = \frac{1}{2} \log \left(\frac{27}{16} \frac{1-r(z)}{r(z)} \frac{z^2}{(z-1)^2} \right), \quad A_0(z) = \frac{1}{(1-(1-z)z)^{\frac{1}{4}}},$$

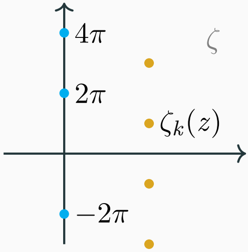
$$r(z) = \frac{(z-2)(z+1)(1-2z)}{4((z-1)z+1)^{3/2}} + \frac{1}{2}.$$

The Two Towers of Borel singularities

The Borel transform of the f_n series can be written analytically

$$\widehat{\varphi}(r(z),\zeta) = \frac{5r\zeta}{36}\,{}_2F_1\left(\frac{7}{6},\frac{11}{6};2;r(z)(1-e^{-\zeta})\right). \tag{3.12}$$

At a generic value of z, in the Borel plane dual to ϵ there are two families of singularities,



$$\zeta_k^i = (2k)\pi\mathrm{i}, \quad k \in \mathbb{Z}\backslash\{0\}$$

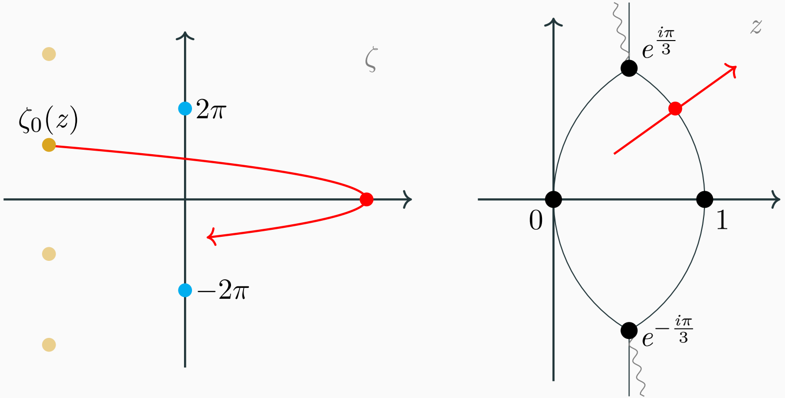
$$\zeta_k = \log\left(\frac{r(z)}{1-r(z)}\right) + (2k-1)\pi\mathrm{i},$$

$$k \in \mathbb{Z}$$

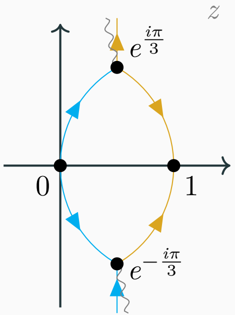
The Stokes jump at ζ_k give the same series with $z \to 1-z$.

From $\mathcal{B}$ to z

The map $r(z)$ leads to non-trivial lines in the z-plane. These lines happen when the a singularity crosses the positive real line. Similar to WKB (see Aoki et al.).



These jumps are locally similar to the Airy function, when appropriately normalized.

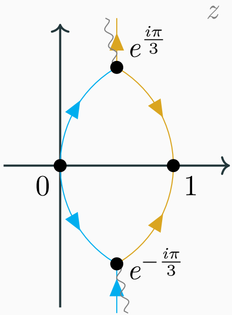


$$\begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_+ = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_-$$

$$\begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_+ = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_-$$

Love is in the Airy

These jumps are locally similar to the Airy function, when appropriately normalized.



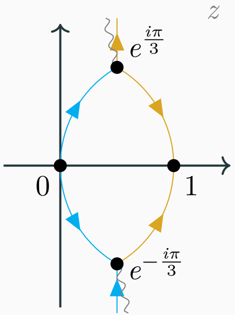
$$\begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_+ = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_-$$

$$\begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_+ = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_-$$

What is the physics of $\varphi(1-z)$?

Love is in the Airy

These jumps are locally similar to the Airy function, when appropriately normalized.



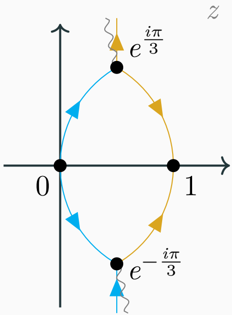
$$\begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_+ = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_-$$

$$\begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_+ = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_-$$

What is the physics of $\varphi(1-z)$? If we take $z \to 0$, we have $\varphi(1-z) \sim z^{-\frac{1}{\epsilon}+\frac{3}{4}}$ which predicts the conformal weight of the $(3,1)$ block!

Love is in the Airy

These jumps are locally similar to the Airy function, when appropriately normalized.



$$\begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_+ = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_-$$

$$\begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_+ = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} \varphi(z) \\ \varphi(1-z) \end{pmatrix}_-$$

What is the physics of $\varphi(1-z)$? If we take $z \to 0$, we have $\varphi(1-z) \sim z^{-\frac{1}{\epsilon}+\frac{3}{4}}$ which predicts the conformal weight of the $(3,1)$ block! With the exact answer, we check that it is the perturbative series of the trans-series for $\mathcal{F}_{3,1}$, when z is small.

Resurgence bootstrap?

If we write the four point function in terms of the asymptotic series

$$\langle \phi_{2,1}\phi_{2,1}\phi_{2,1}\phi_{2,1}\rangle = \\ f(\epsilon)\begin{pmatrix}\varphi(z)\\ \varphi(1-z)\end{pmatrix}^{\dagger}\cdot\begin{pmatrix}1 & \csc\left(\frac{\pi}{\epsilon}\right)\\ \csc\left(\frac{\pi}{\epsilon}\right) & 1\end{pmatrix}\cdot\begin{pmatrix}\varphi(z)\\ \varphi(1-z)\end{pmatrix}. \quad (3.13)$$

As is well known, the 4pt function in this case can be fixed by demanding single valuedness. This can be extracted directly from the resurgence language in a compact basis.

How much can we constraint observables in more general cases using resurgence?

Conclusion

Summary

- Resurgence can unlock new insights in CFTs and help understand asymptotic series in $1/c$.
- In simple examples, we can see that different conformal blocks have information about each other which is visible in the resurgent properties of their large c expansion.
- You can read important constraints on physical observables from resurgence.
- We expect this to be related to resurgent properties of asymptotic series of AdS gravity.

Future directions

- We still have more to do! Other types of correlaiton functions (heavy-heavy-light-light for example), more complicated operators (e.g. $\phi_{3,1}$) where not everything is analytically available, unitary non-minimal models (numerically through Zamolodchikov recursion relations, an analysis already initiated in Benjamin et al. for $z \to 0$).
- A more specific holographic interpretation of these relations could give insights into resurgence in quantum gravity. In Benjamin et al., they identify a Borel singularity which they associate to unphysical excess angle geometry. Could there be more? Can black holes be seen?
- Are some of these insights valuable for higher dimensions?

Fin

Grazie!