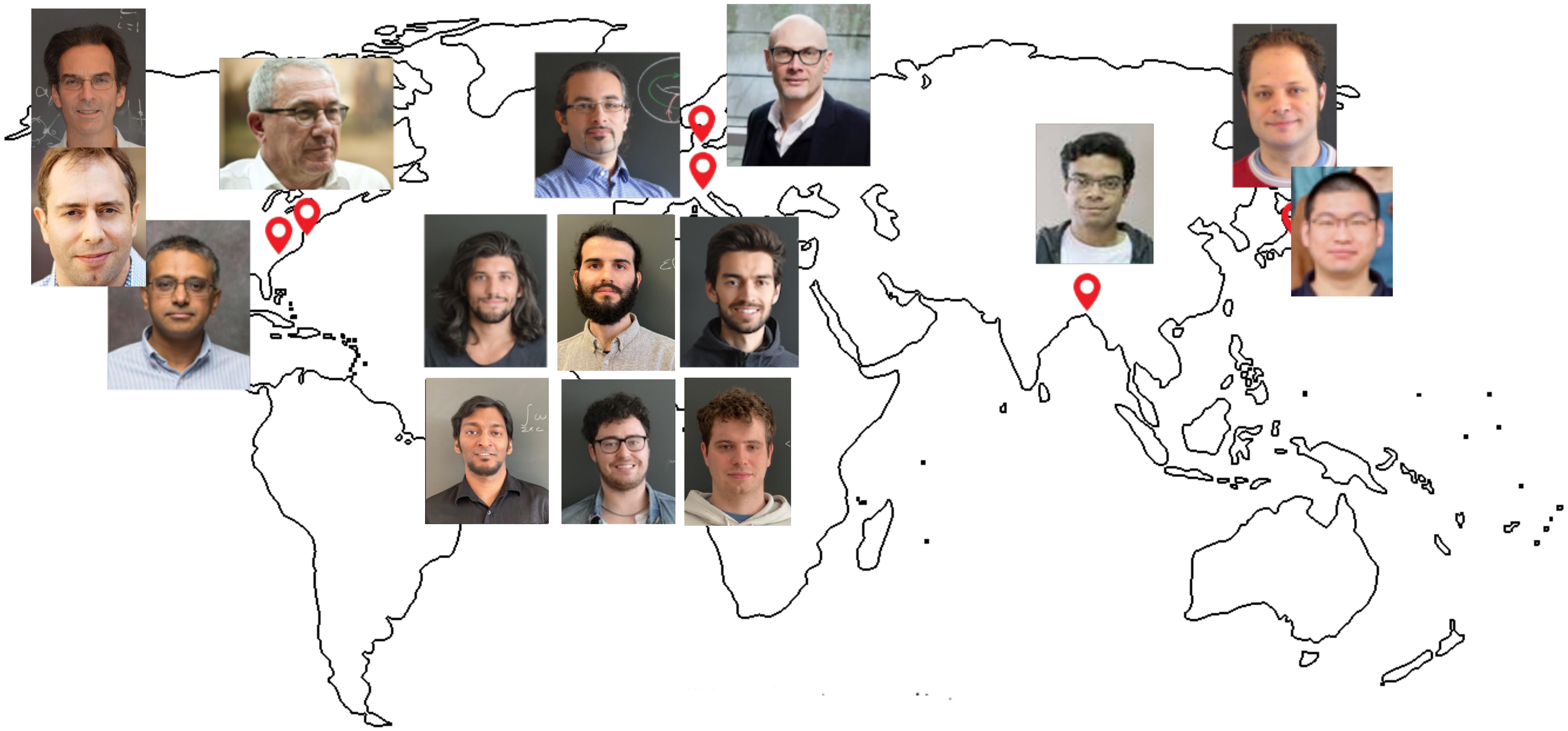
UNIVERSITÄT BERN
AEC
ALBERT EINSTEIN CENTER FOR FUNDAMENTAL PHYSICS

The Large Charge Expansion

Susanne Reffert
University of Bern

based on arXiv:1505.01537, 1610.04495, 1707.00710, 1809.06371, 1902.09542, 1905.00026, 1909.08642,1909.02571, 2008.03308, 2010.07942, 2102.12488, 2110.07617, 2110.07616, 2203.12624, 2211.15318, 2311.14793, 2403.18898 + work in progress

L. Alvarez-Gaume (SCGP), D. Banerjee (Kolkata), S. Beane (Washington), Sh. Chandrasekharan (Duke), N. Dondi (ICTP), S. Hellerman (IPMU), I. Kalogerakis (Bern), R. Moser (Bern), Daniil Krichevskiy (, O. Loukas, D. Orlando (INFN Torino), V. Pellizzani (Bern), F. Sannino (Odense/Napoli), T. Schmidt (Bern), Ian Swanson, M. Watanabe (Kyoto)



Introduction

Strongly coupled physics is notoriously difficult to access, especially analytically.

We do not have small parameters in which to do a perturbative expansion. Our most basic notions of field theory are of a perturbative nature.

Make use of symmetries, look at special limits/subsectors where things simplify.

Examples:

- large-N limit, ’t Hooft limit
- ε expansion
- supersymmetric sectors
- large spin
- integrability
- ….

Introduction

Study theories with a global symmetry group.
Hilbert space of the theory can be decomposed into sectors of fixed charge Q.
Study subsectors with large charge Q.

Best case scenario: Large charge Q becomes controlling parameter in a perturbative expansion!

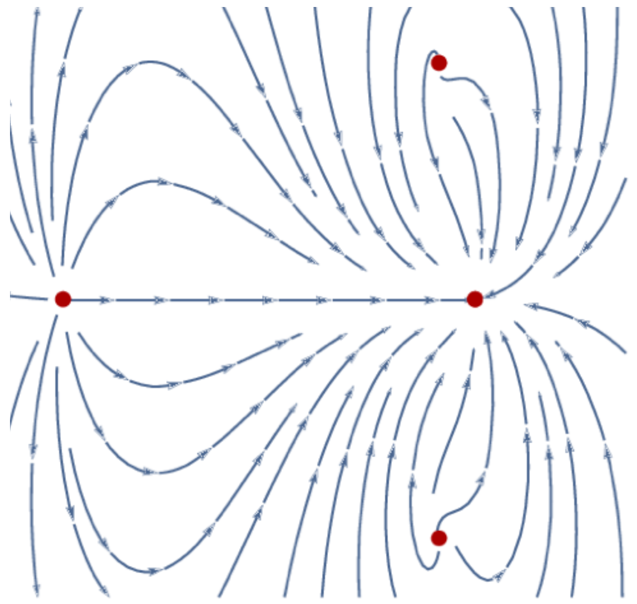
> Effective theory at large Q:
>
> vacuum + Goldstone + 1/Q-suppressed corrections

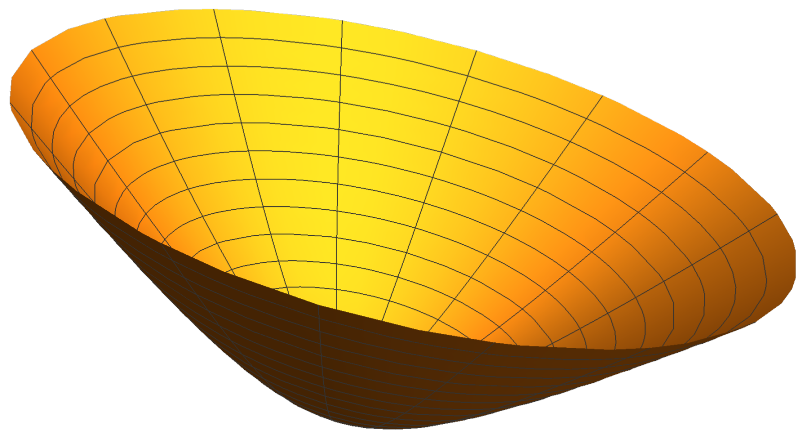
Working at large charge Q always leads to simplifications. For hard problems, large charge may however not be enough (combine with other limits, etc.)

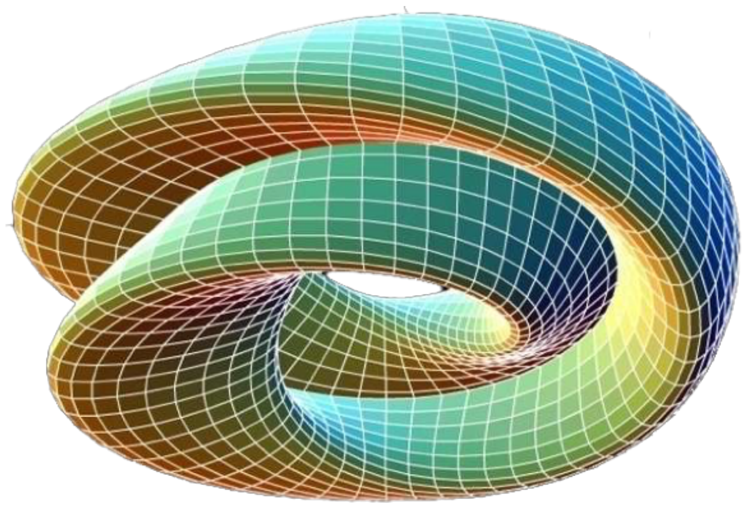
Introduction

Conformal field theories (CFTs) play an important role in theoretical physics:

- fixed points in RG flows
- critical phenomena
- quantum gravity (via AdS/CFT)
- string theory (WS theory)



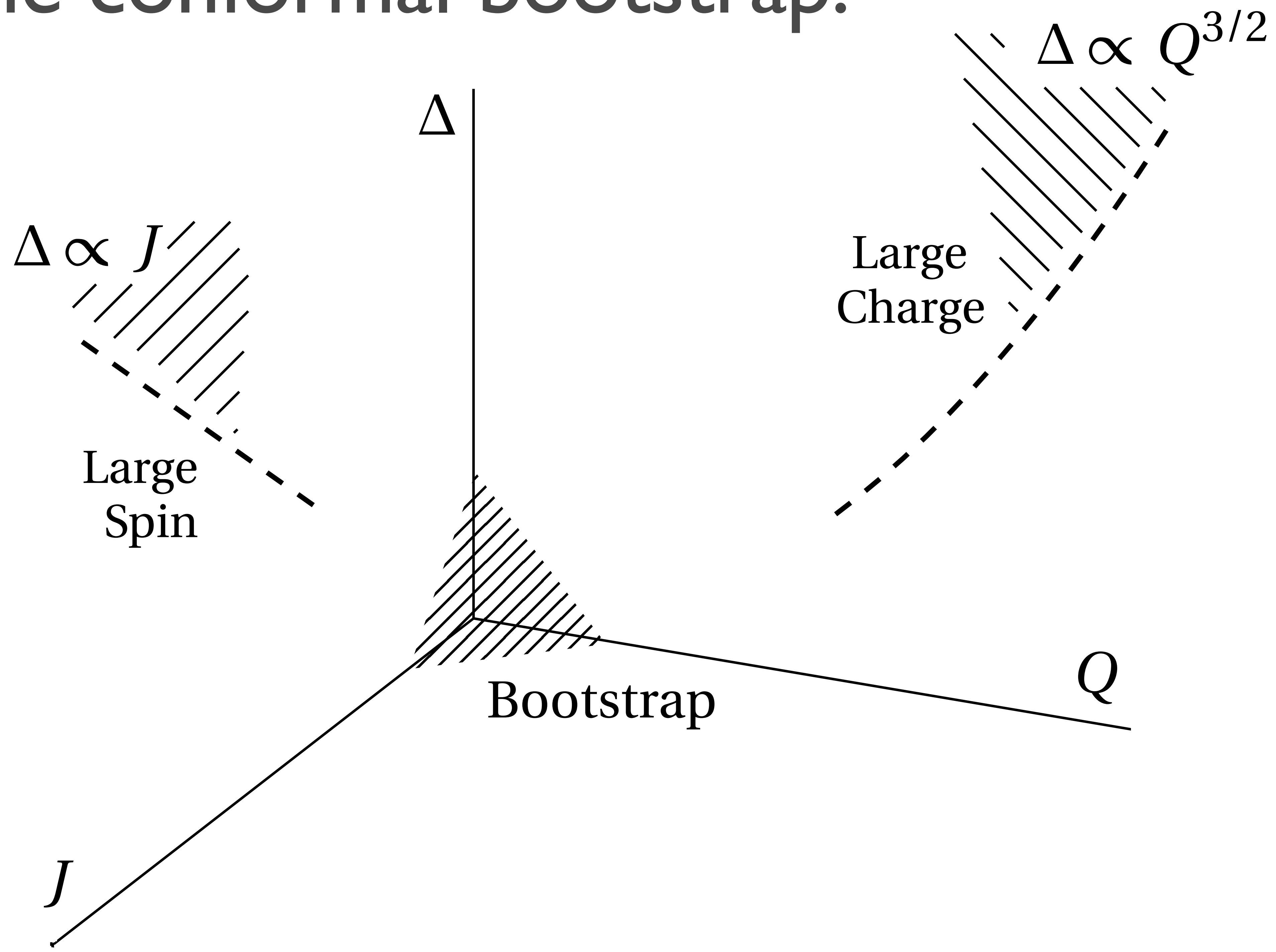




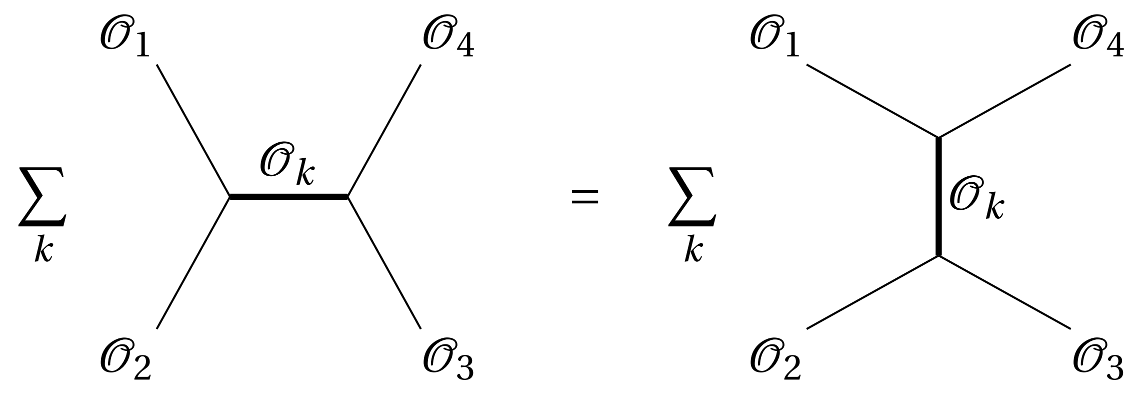
But: CFTs do not have any intrinsic scales, most have by naturalness couplings of O(1).
Possibilities: analytic (2d), conformal bootstrap (d>2), lattice calculations, non-perturbative methods…
Prime candidate for the large-charge approach.

Introduction

The large charge expansion is complementary to other CFT approaches like large spin and the conformal bootstrap:



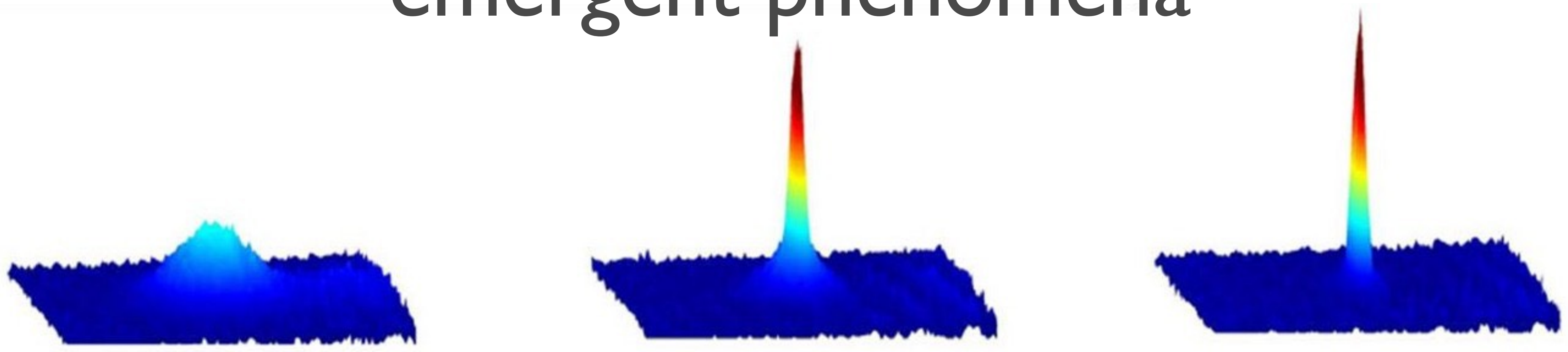
Booth bootstrap and large spin are based on crossing symmetry



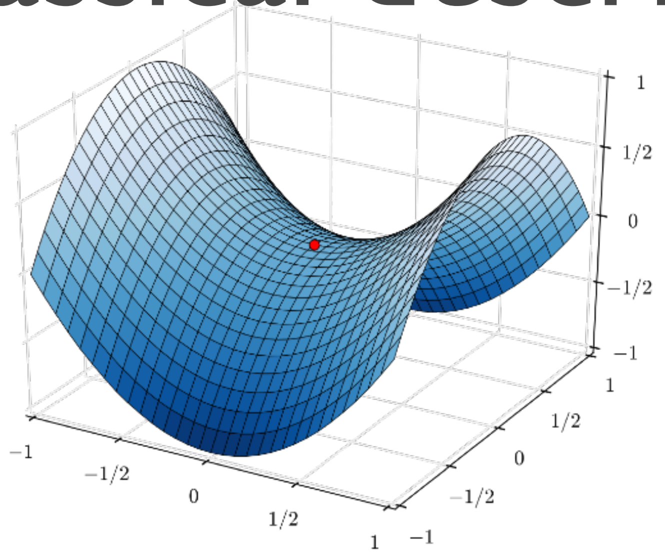
Introduction

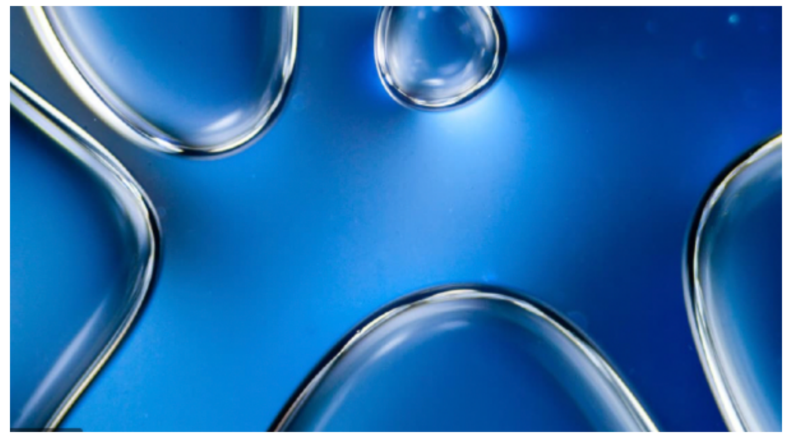
Consider systems with large quantum number
many degrees of freedom

emergent phenomena



semiclassical description





e.g. superfluid

works especially well for strongly coupled systems!

Introduction

Is the microscopic theory accessible?

yes

- weakly coupled
- large-N limit
- ε expansion
- susy
- integrable

↓

- large Q + large N
- large Q + ε expansion
- large Q + susy
- large Q + integrability

↓

go beyond perturbation theory in 1/Q, calculate non-perturbative (exponential) corrections!

no

strongly coupled

↓

work @large Q

↓

large-Q EFT, expansion in 1/Q

Introduction

The seem to be 2 main categories of behavior for systems at large quantum number:

Superfluid

isolated vacuum

- Wilson-Fisher CFT
- NRCFT (unitary Fermi gas)
- N=2 SCFT in 3d
- asymptotically safe model in 4d
- NJL model

EFT of the moduli space

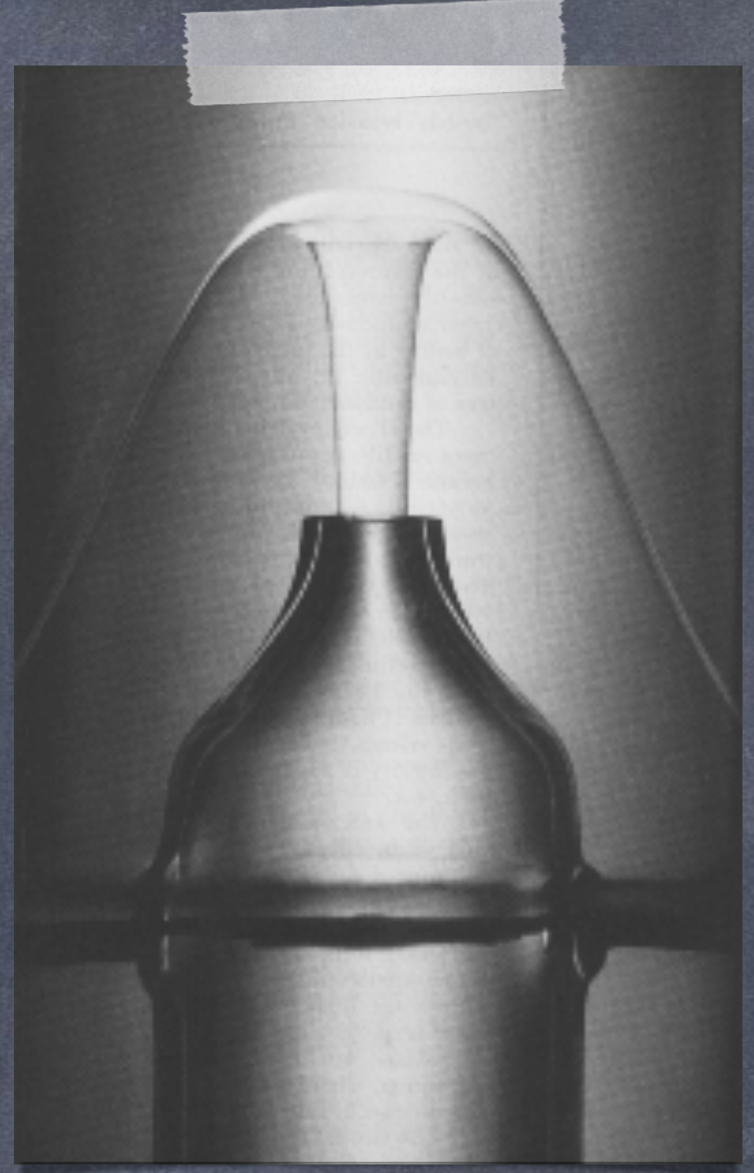
moduli space of vacua

- free boson
- N=2 theories in 4d

Introduction

To which models can we apply the large Q expansion?

- O(N) vector model in 3D
- NJL in 3D
- non-relativistic CFTs
- integrable models
- SCFTs
- …



The O(2) model

The O(2) model

Simplest example: O(2) model in (2+1) dimensions

$$\mathcal{L}_{\mathrm{UV}} = \partial_\mu \phi^* \, \partial^\mu \phi - g^2 (\phi^* \phi)^2$$

Flows to Wilson-Fisher fixed point in IR.

Assume that also the IR DOF are encoded by cplx scalar

$$\varphi_{\mathrm{IR}} = a \, e^{i\chi}$$ Global U(1) symmetry: $$\chi \to \chi + \mathrm{const.}$$

Look at scales: put system in box (2-sphere) of scale R

Second scale given by U(1) charge Q:

$$\rho^{1/2} \sim Q^{1/2}/R$$

Study the CFT at the fixed point in a sector with

$$\frac{1}{R} \ll \Lambda \ll \frac{Q^{1/2}}{R} \ll g^2$$

UV scale (g^2)

cut-off of effective theory (Λ)

The O(2) model

Fixing the charge breaks symmetries:

$$SO(3,2) \times O(2) \to SO(3) \times D \times O(2) \rightsquigarrow SO(3) \times D'$$

$D' = D - \mu O(2)$

Broken U(1) - superfluid!

Dynamics is described by a single Goldstone field χ:

$$\mathcal{L}_{LO} = k_{3/2}(\partial_\mu \chi \, \partial^\mu \chi)^{3/2}$$

← can get this purely by dimensional analysis

Lowest-energy solution: homogeneous ground state

$$\chi = \mu t,$$

← non-const. vev

Beyond LO: use dimensional analysis, parity and scale invariance to determine (tree-level) operators in effective action (Lorentz scalars of scaling dimension 3, including couplings to geometric invariants)

The O(2) model

Use ρ-scaling to determine which terms are not suppressed:

$$\partial\chi \sim \rho^{1/2}, \quad \partial\ldots\partial\chi \sim \rho^{-1/4}$$

Result for NLSM action in D=3:

LO Lagrangian — curvature coupling

$$\mathcal{L} = k_{3/2}(\partial_\mu\chi\partial^\mu\chi)^{3/2} + k_{1/2}R(\partial_\mu\chi\partial^\mu\chi)^{1/2} + \mathcal{O}(Q^{-1/2})$$

dimensionless parameters — suppressed by inverse powers of Q

Energy of classical ground state at fixed charge:

2 dimensionless parameters — cannot be calculated within EFT!

$$E_\Sigma(Q) = \frac{c_{3/2}}{\sqrt{V}}Q^{3/2} + \frac{c_{1/2}}{2}R\sqrt{V}Q^{1/2} + \mathcal{O}(Q^{-1/2})$$

dependence on manifold

The O(2) model

Expand action around GS to second order in fields: $\chi = \mu t + \hat{\chi}$

$$\mathcal{L} = k_{3/2}\mu^3 + k_{1/2}R\mu + (\partial_t\hat{\chi})^2 - \tfrac{1}{2}(\nabla_{S^2}\hat{\chi})^2 + \ldots$$

Compute zeros of inverse propagator for fluctuations and get dispersion relation:

$$\omega_{\vec{p}} = \frac{|\vec{p}|}{\sqrt{2}}$$ ← dictated by conf. invariance $1/\sqrt{d}$

⇒ χ is indeed a "conformal" Goldstone

Are also the quantum effects controlled?

Yes! All effects except Casimir energy of χ are suppressed (negative ρ-scaling).

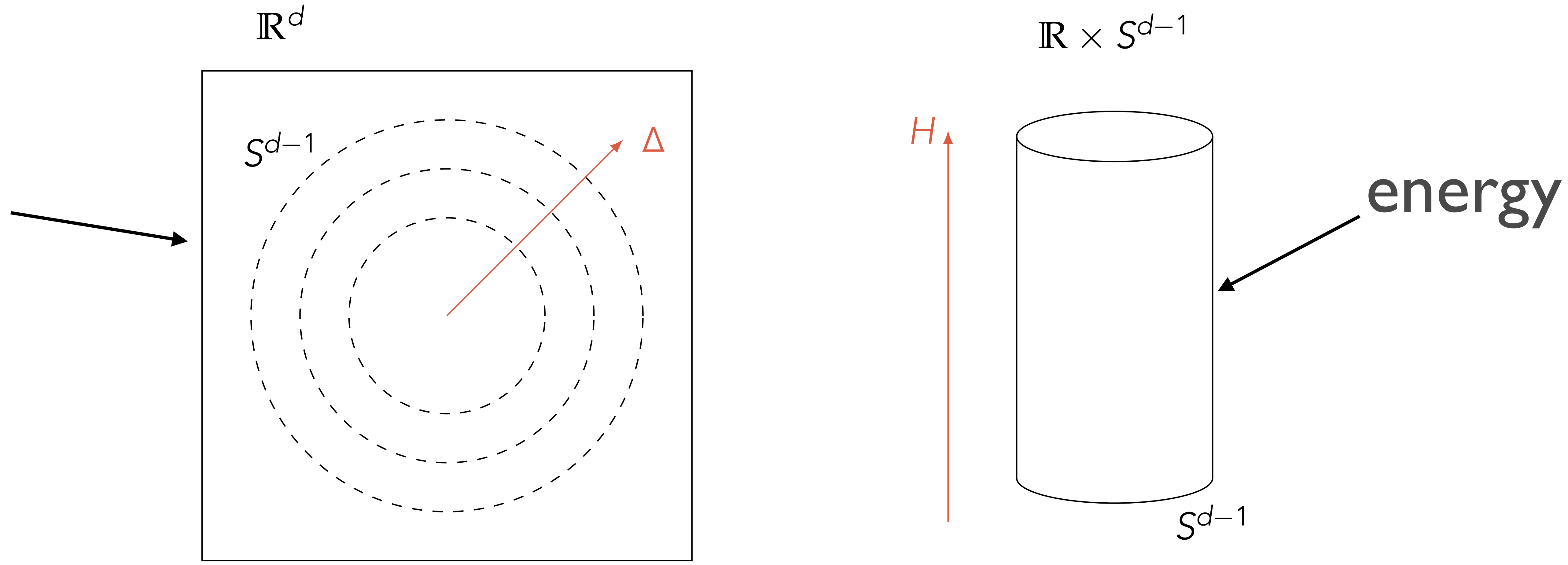
Effective theory at large Q:

vacuum + Goldstone + 1/Q-suppressed corrections

The O(2) model

We're ready to calculate observables:
CFT: conformal data (scaling dim. + 3pt coefficients)!

Use state-operator correspondence of CFT:



Scaling dimension of lowest operator of charge Q:

energy of class. ground state

$$D(Q) = R_0(E_0 + E_{Cas}) = c_{3/2}Q^{3/2} + c_{1/2}Q^{1/2} - 0.0937\cdots + \mathcal{O}(Q^{-1/2})$$

quantum correction from Casimir energy of Goldstone

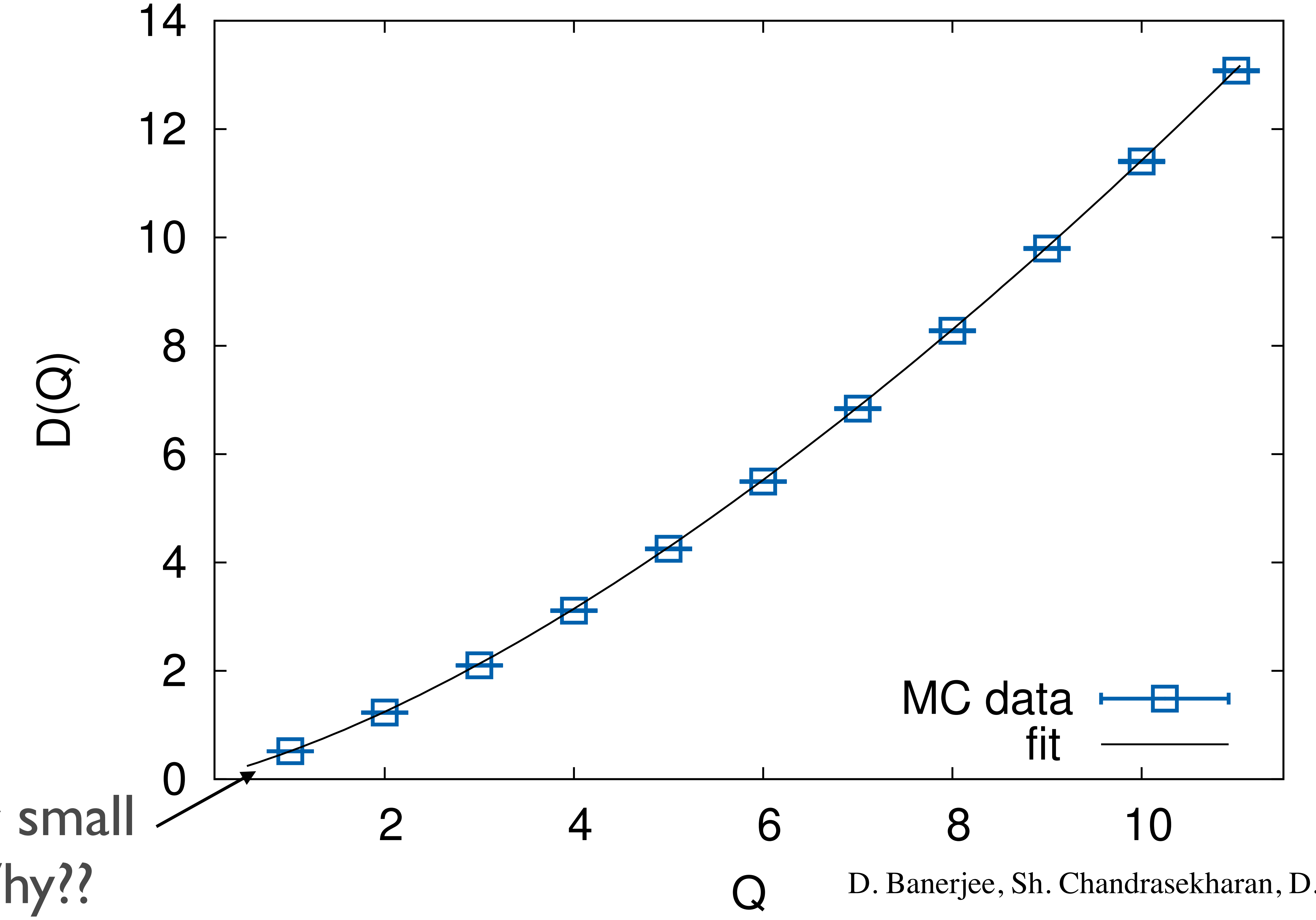
S. Hellerman, D. Orlando, S. R., M. Watanabe, arXiv:1505.01537 [hep-th]

The O(2) model

Testing our prediction:

$$D(Q) = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}(Q^{-1/2})$$

Independent calculation on the lattice:



works for small charge. Why??

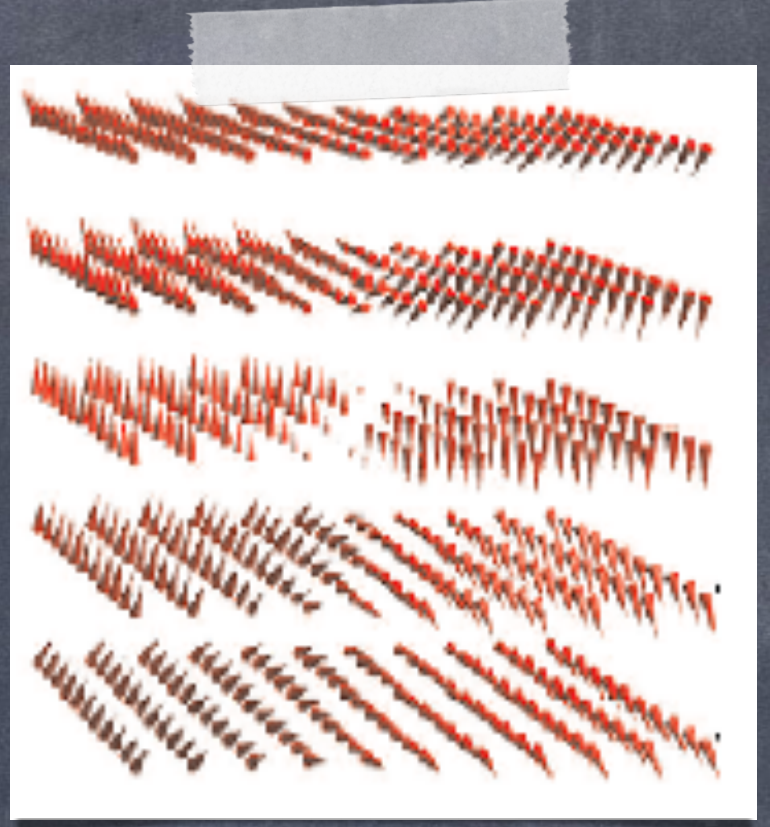
D. Banerjee, Sh. Chandrasekharan, D. Orlando [hep-th/1707.00711]

Excellent agreement!!

$c_{3/2} = 1.195(10)$

$c_{1/2} = 0.075(10)$

Large-charge expansion works extremely well for O(2).



Beyond O(2):
3d O(2N) vector model

Beyond O(2)

Where else can we apply the large-charge expansion?

Obvious generalization in 3d: O(2N) vector model
non-Abelian global symmetry group: new effects

Different symmetry breaking patterns possible, inhomogeneous ground states possible.

Homogeneous case: same form of ground state,

$$SO(3,2) \times O(2N) \to SO(3) \times D \times U(N) \to SO(3) \times D' \times U(N-1)$$

We expect dim[U(N)/U(N-1)] = 2N-1 Goldstone d.o.f.

On top of the conformal Goldstone of O(2), a new sector with N-1 non-relativistic type II Goldstones and N-1 massive modes with m=2μ appears.

The O(2N) vector model

Dispersion relation:

$$\omega = \frac{p^2}{2\mu} + \mathcal{O}\mu^{-3}$$

The non-relativistic Goldstones count double. Nielsen and Chadha; Murayama and Watanabe

Counting type I and type II modes, indeed,

$$1 + 2(N-1) = 2N - 1 = \dim(U(N)/U(N-1))$$

Non-relativistic Goldstones contribute to the conformal dimensions only at higher order.
The ground-state energy is again determined by a single relativistic Goldstone!

Same formula for scaling dimensions as for O(2):

N-dependent

universal for O(2N)

$$D(Q) = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}(Q^{-1/2})$$

verified at large N for CP(N-1) model de la Fuente

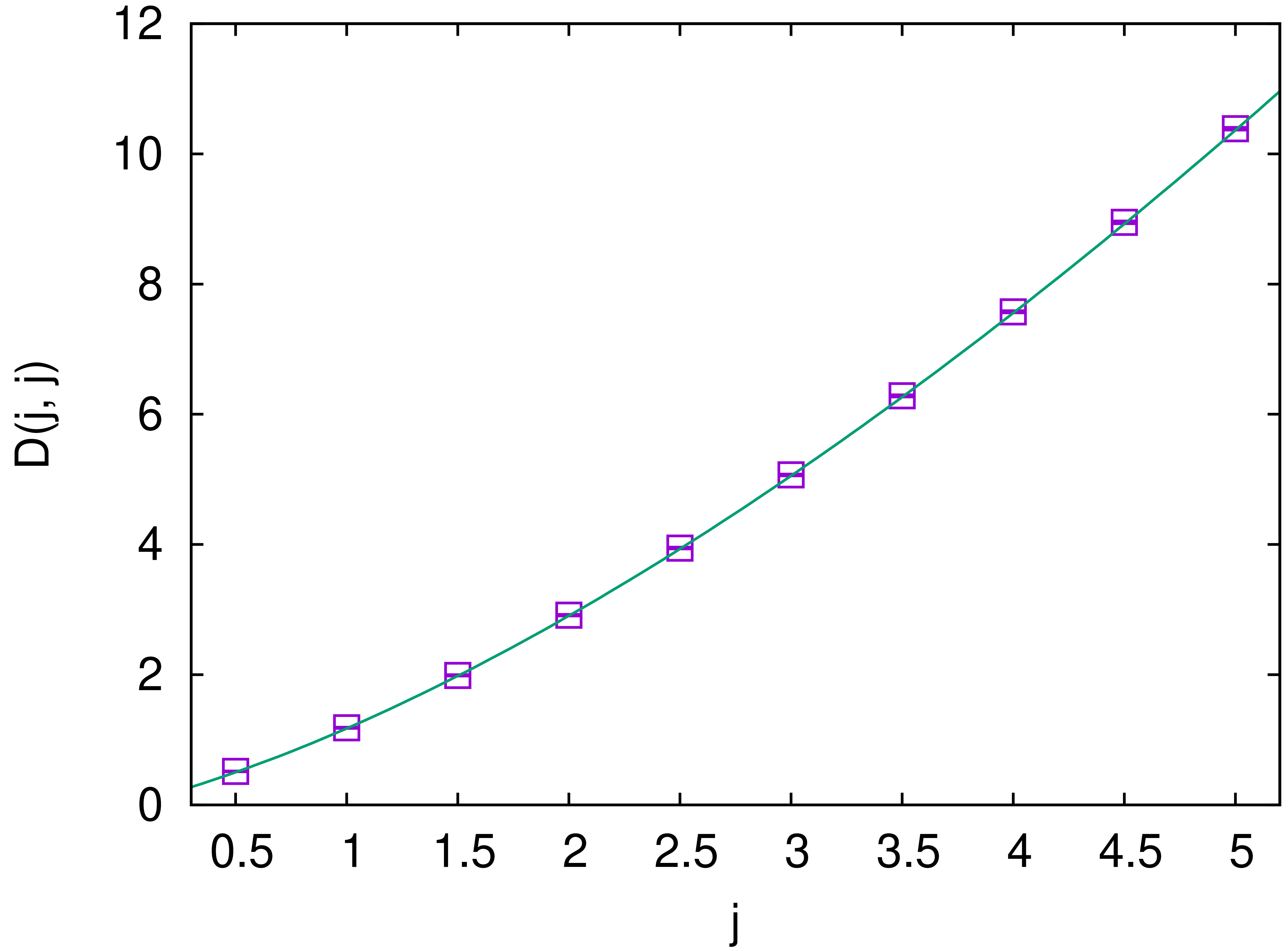
L. Alvarez-Gaume, O. Loukas, D. Orlando and S. R., arXiv:1610.04495 [hep-th]

The O(2N) vector model

Testing our prediction:

$$D(Q) = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}(Q^{-1/2})$$

Lattice data for O(4) model:



$$c_{3/2} = 1.068(4)$$
$$c_{1/2} = 0.083(3)$$

D. Banerjee, Sh. Chandrasekharan, D. Orlando, S.R. 1902.09542

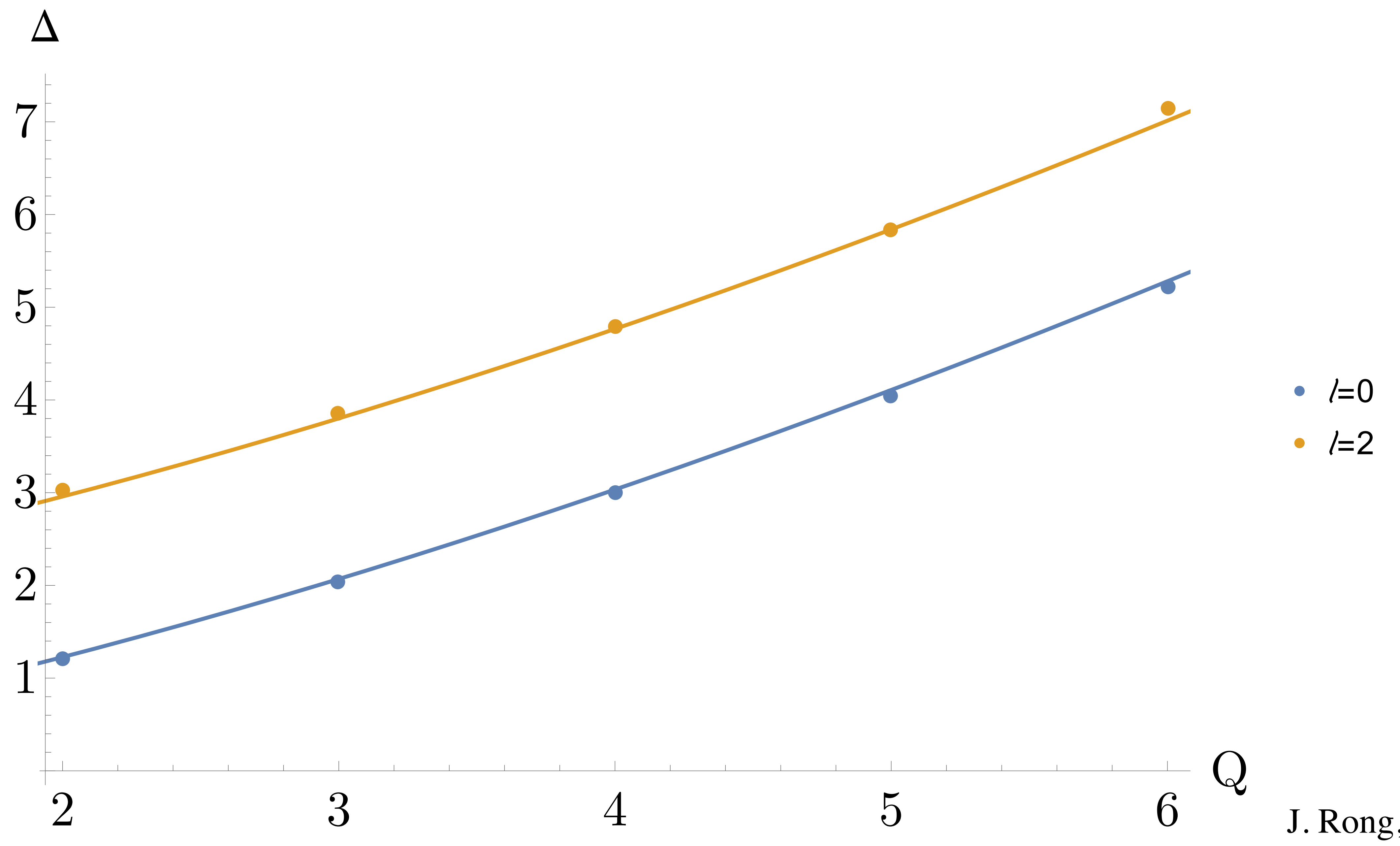
Again excellent agreement with large-Q prediction!

The O(2N) vector model

Testing our prediction:

$$D(Q) = \frac{c_{3/2}}{2\sqrt{\pi}} Q^{3/2} + 2\sqrt{\pi}\, c_{1/2} Q^{1/2} - 0.094 + \mathcal{O}(Q^{-1/2})$$

Numerical bootstrap data for O(3) model:



J. Rong, N. Su, 2311.00933

Again excellent agreement with large-Q prediction!

The large-N limit

Standard large-N methods, expand path integral at fixed charge around saddle point (no EFT!)

Extra control parameter at large N: can go further!

Start in the UV with

$$S[\phi_i] = \sum_{i=1}^{N} \int \mathrm{d}t\mathrm{d}\Sigma \left[g^{\mu\nu} (\partial_\mu^i \phi_i)^\dagger (\partial_\nu^i \phi_i) + r(\phi_i^\dagger \phi_i) + \frac{u}{2N} (\phi_i^\dagger \phi_i)^2 \right]$$

For r=R/8, this flows to the WF fixed pt in the IR, $u \to \infty$

Scaling dimension for Q/N>>1:

$$\frac{\Delta(Q)}{2N} = \frac{2}{3} \left(\frac{Q}{2N} \right)^{3/2} + \frac{1}{6} \left(\frac{Q}{2N} \right)^{1/2} - \frac{7}{720} \left(\frac{Q}{2N} \right)^{-1/2} - \frac{71}{181440} \left(\frac{Q}{2N} \right)^{-3/2} + \dots$$

L. Alvarez-Gaume, D. Orlando, S.R. 1909.02571

same Q-scaling as in EFT

Small charge limit, Q/N<<1:

engineering dimension of ϕ

$$\frac{\Delta(Q)}{Q} = \frac{1}{2} + \frac{4}{\pi^2} \frac{Q}{N} + \mathcal{O}\left(\frac{Q}{2N} \right)^2$$

In this limit, the operator of charge Q is φ^Q.

The large-N limit

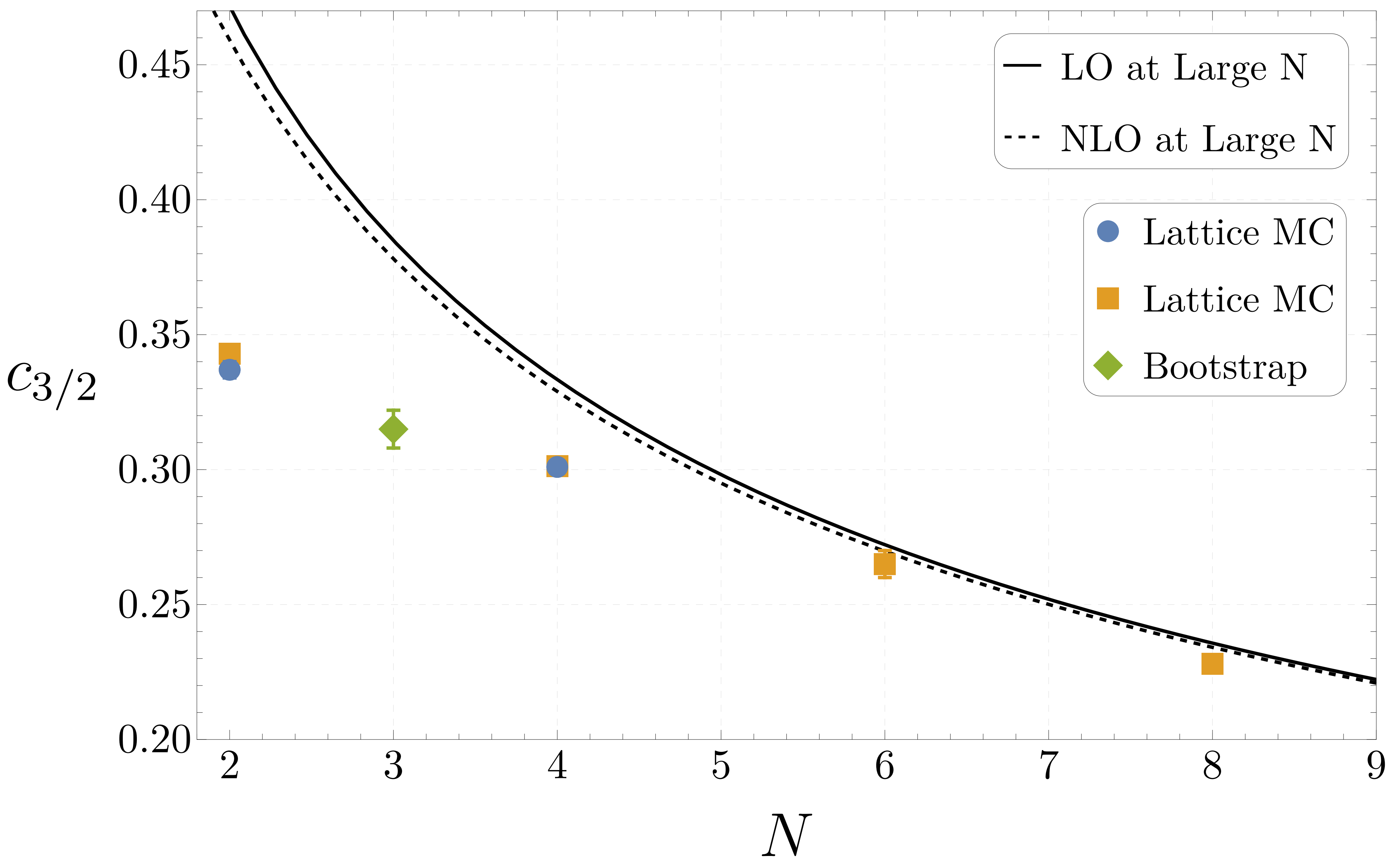
NLO in N: reproduce dispersion relations of Goldstones.

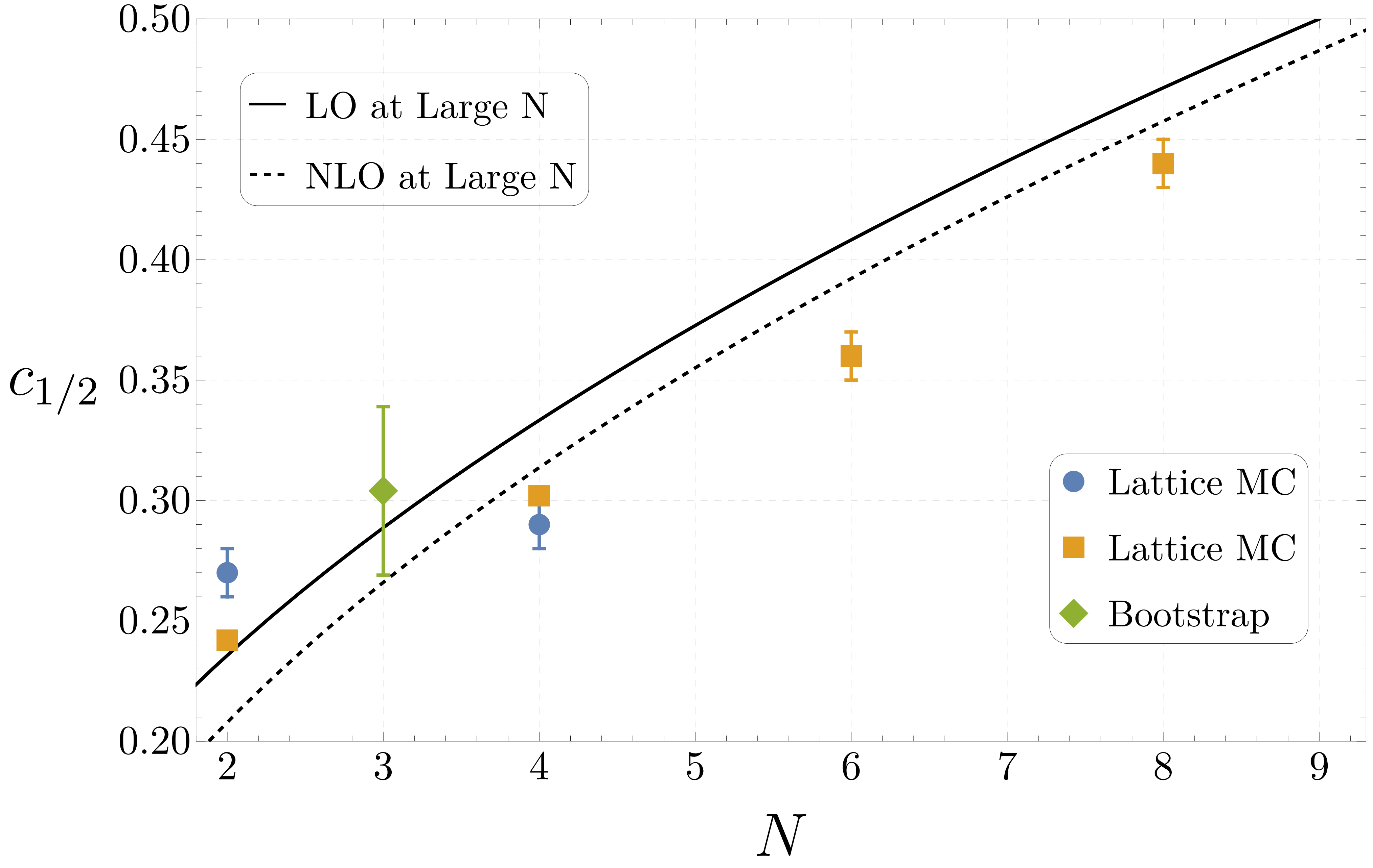
Find coefficients of the expansion (leading order in N):

$$c_{3/2} = \frac{1}{3}\sqrt{\frac{2}{N}} \qquad c_{1/2} = \frac{1}{3}\sqrt{\frac{N}{2}}$$

Comparison of results:

L. Alvarez-Gaume, D. Orlando, S.R. 1909.02571
N. Dondi, G. Sberveglieri 2409.06781





D. Banerjee, Sh. Chandrasekharan, D. Orlando, S.R. 1902.09542 Singh, arXiv:2203.00059 [hep-lat] J. Rong, N. Su, 2311.00933

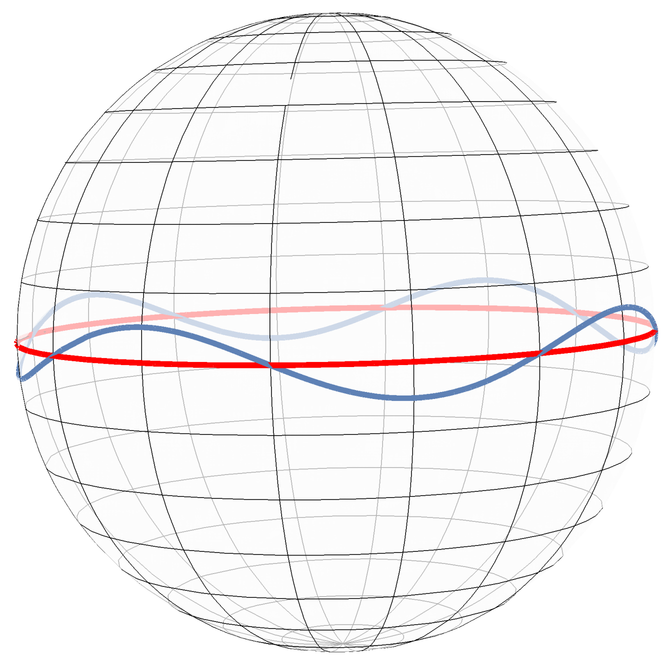
Resurgence analysis

Since we can compute all the coefficients of the large-Q expansion, we can do a resurgence analysis to relate the large and small-charge regimes. Asymptotic series which diverges as (2L)!

We can write the transseries. Find non-perturbative corrections:

$$e^{-2\pi k\sqrt{Q/(2N)}}$$

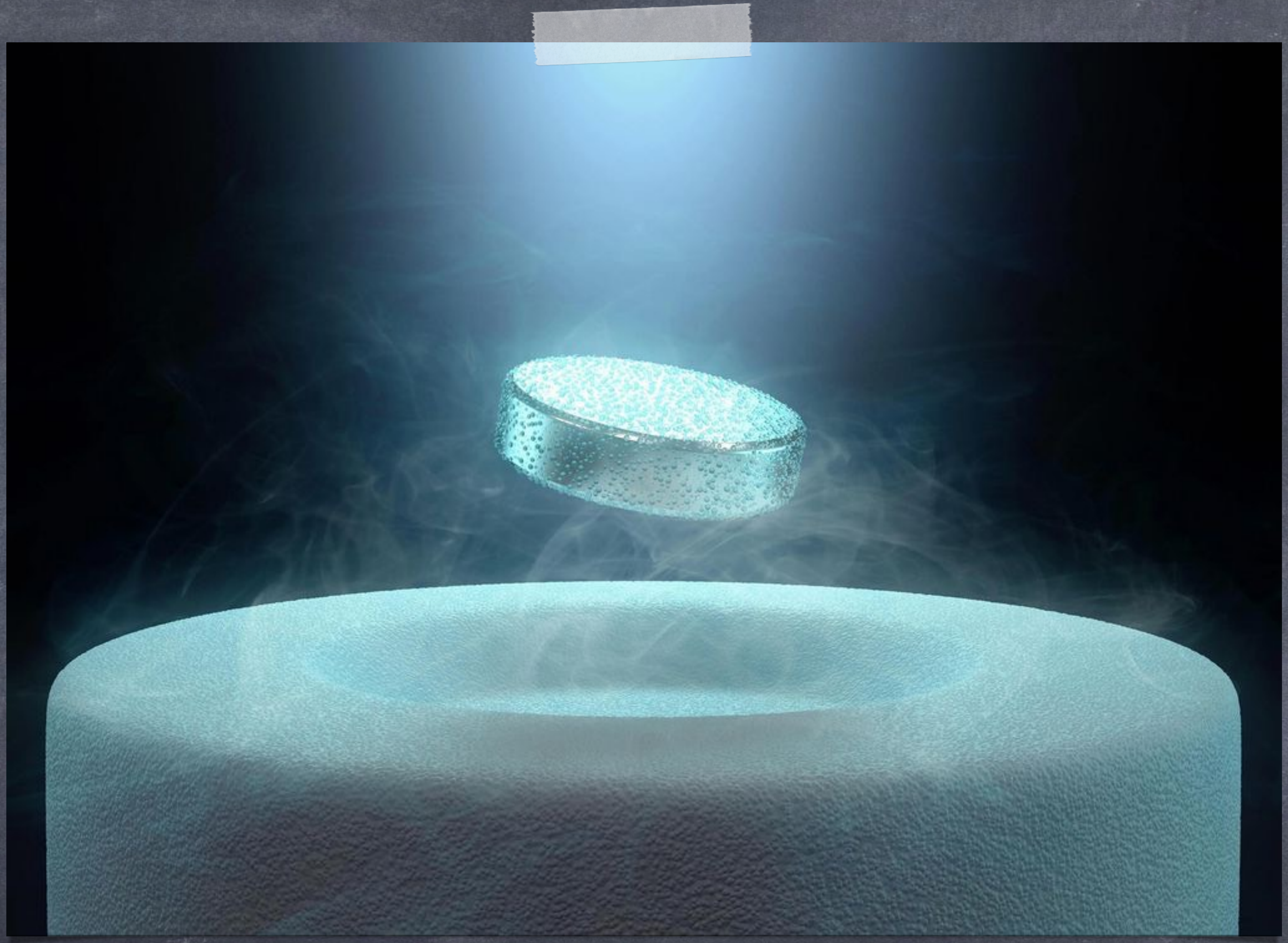
Geometric interpretation: particles of mass μ propagating on the equator of the 2-sphere.



CFT + resurgence: This picture must work for any N!

The optimal truncation is $\mathcal{O}(\sqrt{Q})$ terms. This explains why the comparison to the lattice calculation works so well.

A. Dondi, I. Kalogerakis, D.Orlando, S.R, arXiv: 2102.12488 [hep-th]



Fermions@large Q

Fermions@large Q

Will large Q work for fermionic models?

Antipin, Bersini, Panopoulos;

Let's start with the multicomponent Nambu-Jona-Lasinio (NJL) model, also known as the chiral Gross-Neveu (GN) model in 3D:

$$S_{\text{cGN}} = -\int \mathrm{d}^3x \left[\bar{\psi}_a i\partial\!\!\!/ \psi_a + \frac{g}{2N}\left((\bar{\psi}_a\psi_a)^2 + (\bar{\psi}_a i\gamma_5\psi_a)^2\right)\right]$$

There are two conserved currents:

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \qquad j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$$

We can study this model at large N with standard methods.

We find that only the axial charge gives rise to a condensate at criticality.

Scaling dimension:

large Q/N

$$\frac{\Delta}{N} = \frac{\sqrt{2}}{3}\left(\frac{Q}{\kappa N}\right)^{3/2} + \frac{1}{3\sqrt{2}}\left(\frac{Q}{\kappa N}\right)^{1/2} + \dots$$

small Q/N

$$= \frac{1}{2}\frac{Q}{N} + \frac{1}{\pi^2}\left(\frac{Q}{N}\right)^2 + \dots$$

Dondi, Hellerman, Kalogerakis, Moser, Orlando, S.R., 2211.15318

Fermions@large Q

Like for the scalar case, we get a condensate at fixed charge, but not WF universality class.

Can go to a different frame using the Pauli-Gürsey transformation:

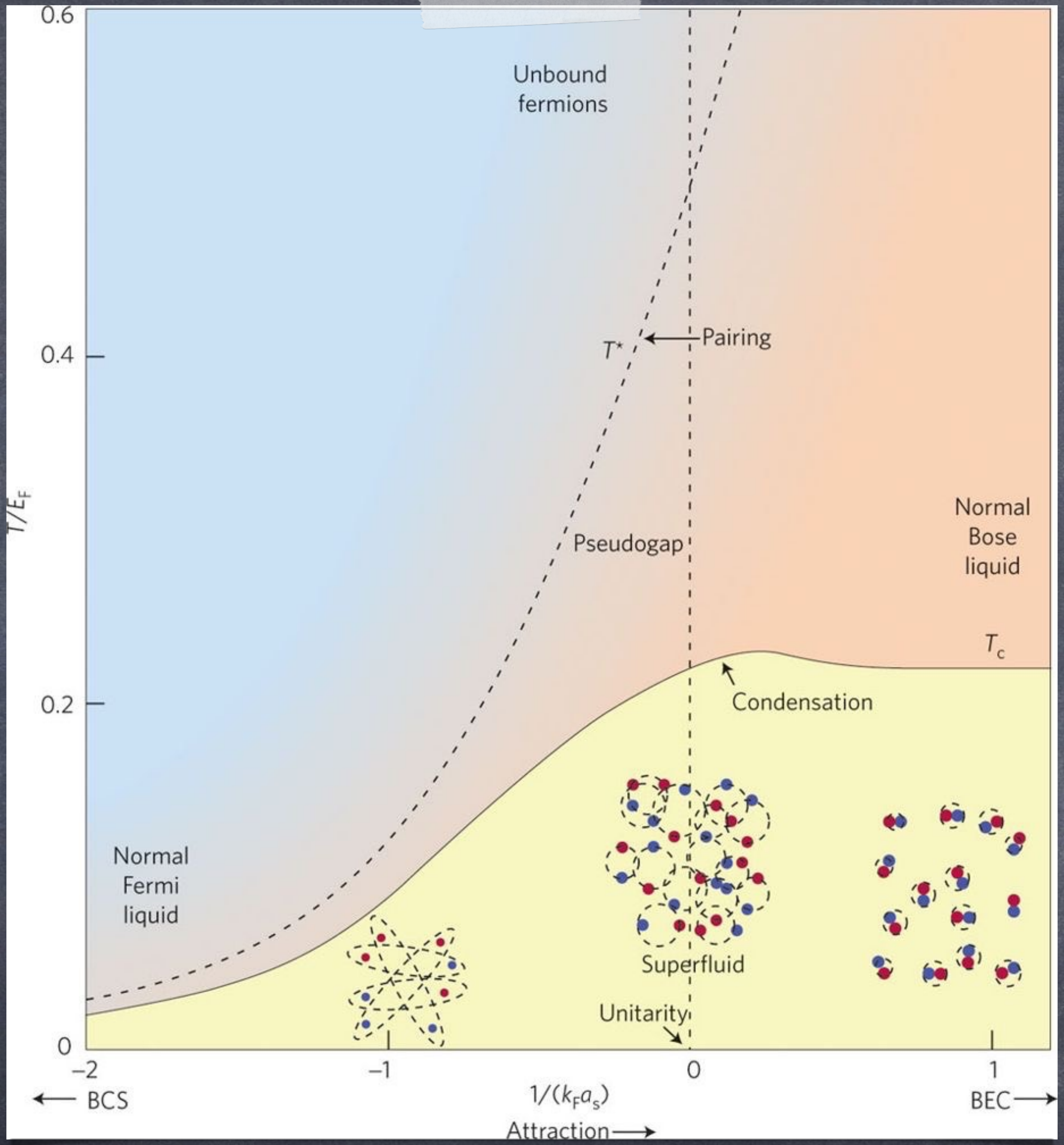
$$\psi_a \mapsto \frac{1}{2}(1-\gamma^5)\psi_a + \frac{1}{2}(1+\gamma^5)C\bar{\psi}_a^T$$

$$S_{\text{BCS}} = -\int \mathrm{d}^3x \left[\bar{\psi}_a i\partial\!\!\!/\psi_a + \frac{g}{2N}\left(\bar{\psi}_a C\bar{\psi}_a^T\right)\left(\psi_b^T C\psi_b\right)\right]$$

This model gives rise to superconductivity from Cooper pair formation!

The condensate consists of Cooper pairs - superconductor!

The end result is similar to the scalar case in the sense that we have an EFT in terms of Goldstones fluctuating around a condensate.

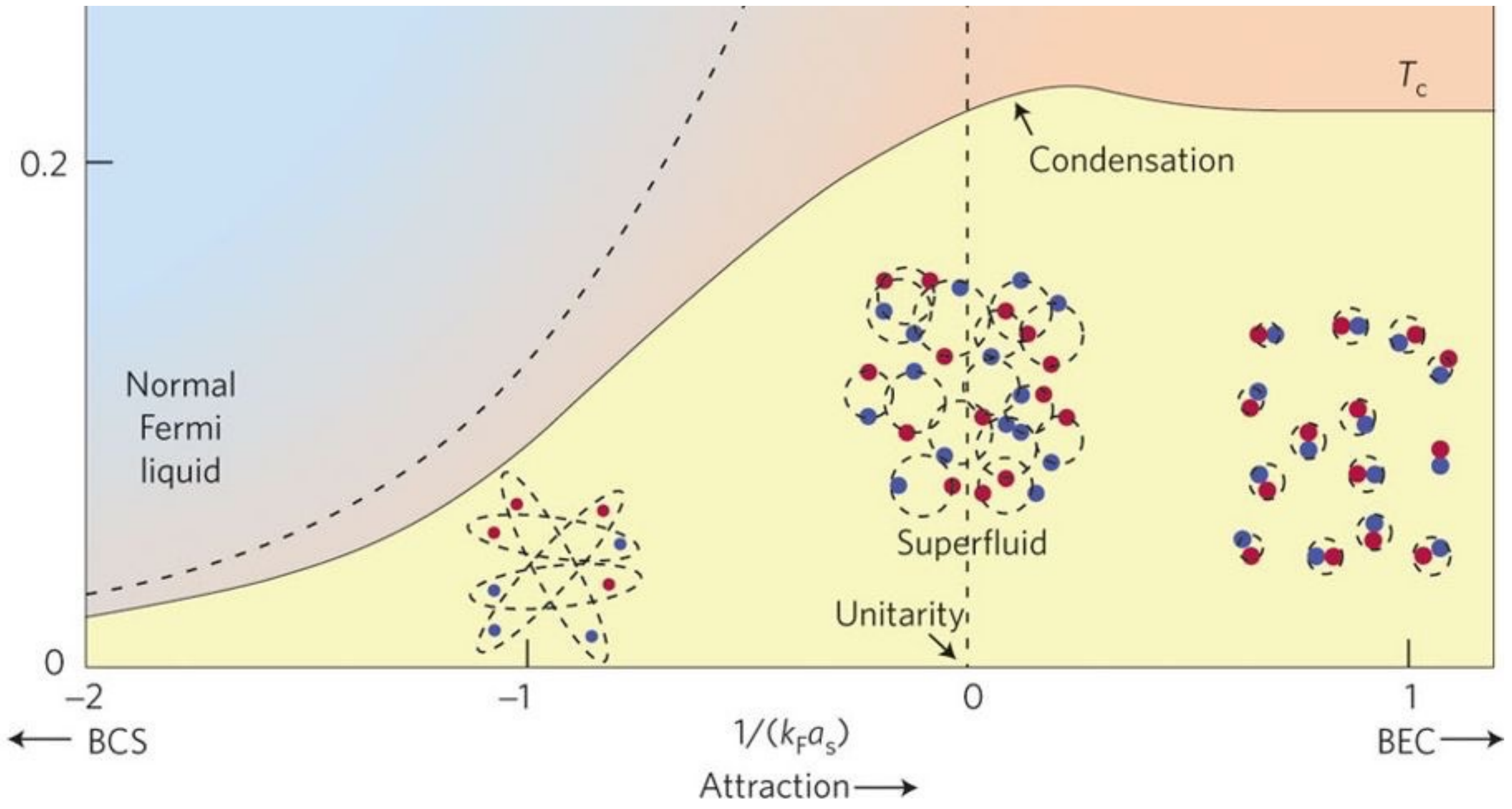


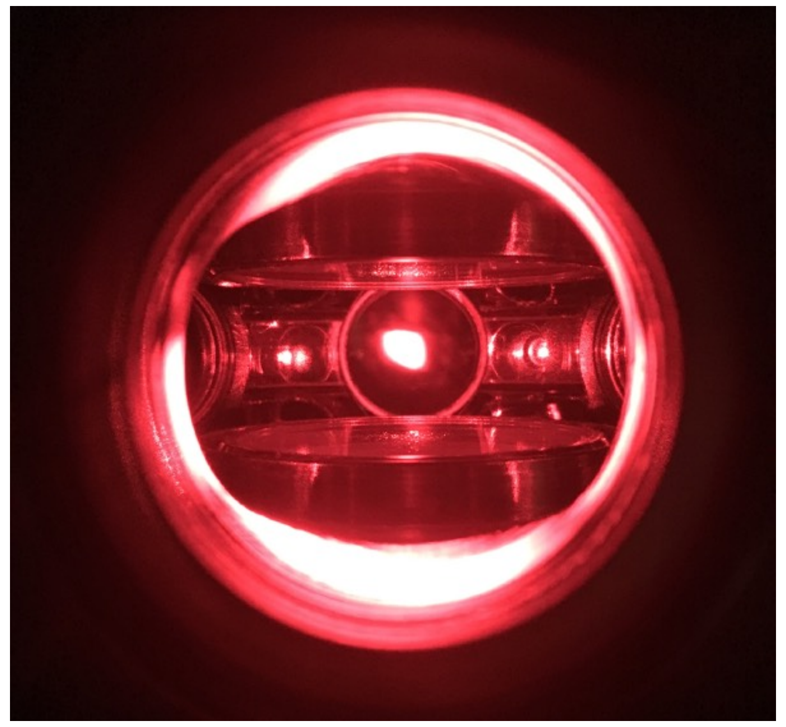
Non-relativistic CFTs

Nonrelativistic CFTs

Motivation: unitary Fermi gas (3+1)D

Can be realized in the lab via cold atoms in a trap. Tuning via Feshbach resonances: unitary point, correlation length = ∞, interaction length = 0





At unitary point: described by a non-relativistic superfluid.

Effective action (small momentum expansion)

Son & Wingate

Nonrelativistic CFTs

What is a nonrelativstic CFT?

Non-relativistic systems are not invariant under the full conformal group.

Schrödinger algebra: contains the Galilean algebra with central extension (particle number) plus

scale transformation: $(t, x_i) \to (t', x_i') = (e^{2\tau} t, e^{\tau} x_i)$

special conf. transf: $(t, x_i) \to (t', x_i') = \left(\frac{t}{1+\lambda t}, \frac{x_i}{1+\lambda t} \right)$

real parameters

The Schrödinger Lagrangian (in d space-dim) is invariant under Schrödinger symmetry:

$$\mathcal{L}(\psi) = \frac{i}{2}(\psi^* \partial_t \psi - \psi \partial_t \psi^*) - \frac{\hbar}{2m} \partial_i \psi^* \partial_i \psi - \frac{k}{m} \hbar^{\frac{d-2}{d}} (\psi^* \psi)^{\frac{d+2}{d}}$$

scale

most general potential compatible with symmetry

Nonrelativistic CFTs

Let's build an EFT at large Q!

System has an inbuilt a global U(1) symmetry (charge=particle number).

Follow the same recipe as for O(2): $\psi = a\, e^{i\theta}$

The leading piece of the effective action for θ can be found by dimensional analysis:

$$\mathcal{L}^{(0)} = c_0\ \hbar^{(2-d)/2} m^{d/2} U^{(d+2)/2}$$

$$U = \partial_t\theta - \frac{\hbar}{2m}\,\partial_i\theta\,\partial_i\theta$$

Homogeneous ground state:

$$\theta = \mu t + \chi \qquad\qquad \mu = k\frac{d+2}{d}\frac{\hbar}{m}\rho^{2/d}$$

The first quantum correction to this (semi-classical) result is the Casimir energy, it goes as $Q^{1/d}$

S. Favrod, D.Orlando, S.R.1809.06371

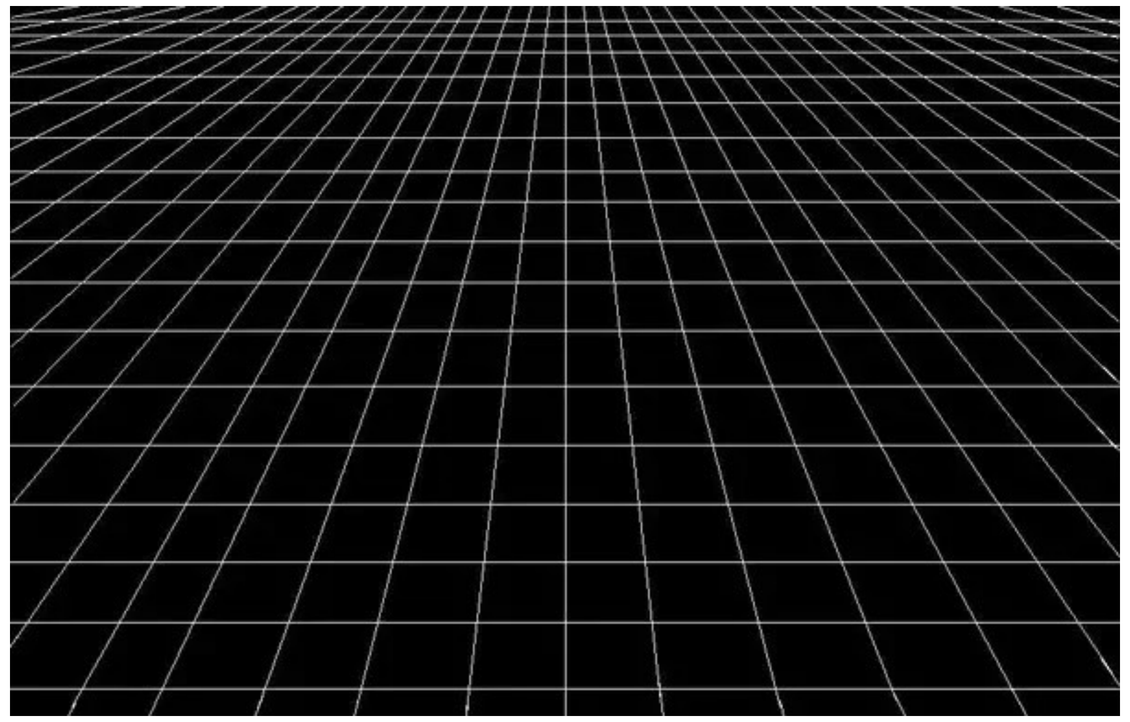
Nonrelativistic CFTs

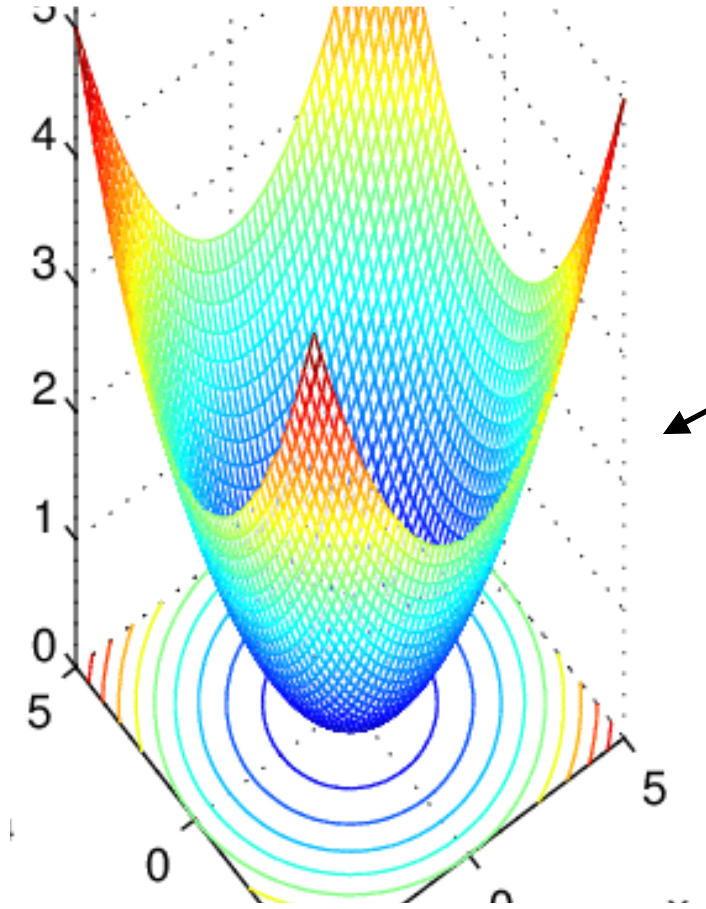
Also for NRCFTs, the form of the two-point function is fixed:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = c\,\delta_{\Delta_1,\Delta_2}\delta_{Q_1,-Q_2}\frac{\exp\left[\mathbf{i}Q_2\frac{|\vec{x}|^2}{2t}\right]}{(t_1-t_2)^{\Delta_1}}$$

There is also a state-operator correspondence for NRCFTs:

conformal dimension



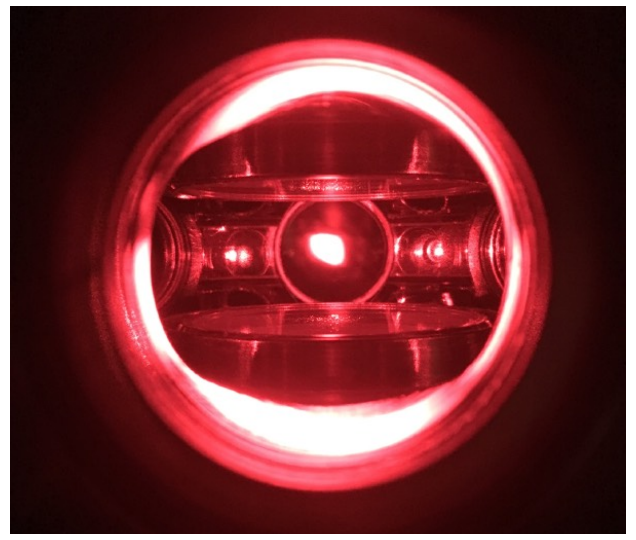


energy of system in harmonic trap

$A_0(\vec{x}) = \frac{m\omega^2}{2\hbar}|\vec{x}|^2$

Son and Nishida 0706.3746

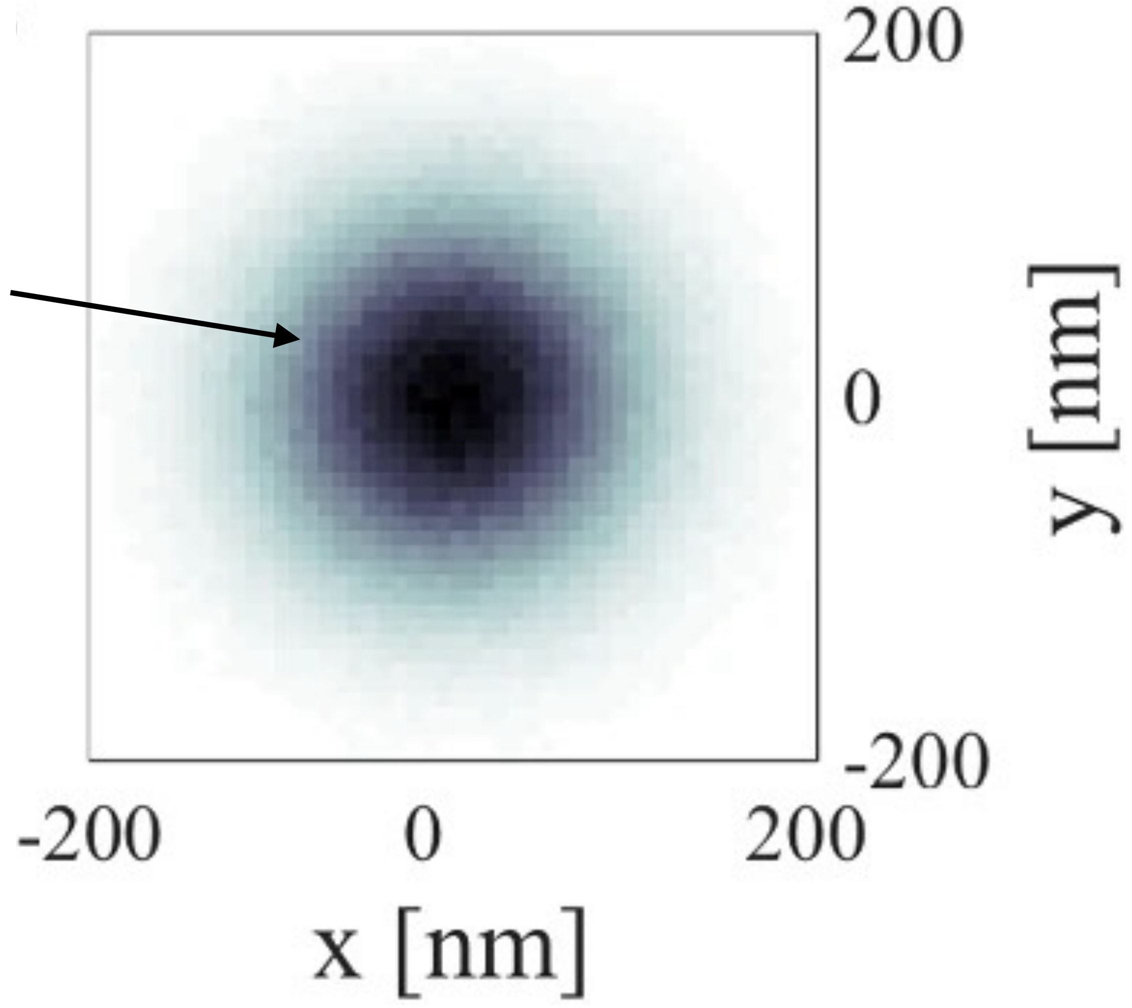
This interestingly corresponds to the situation in the lab!

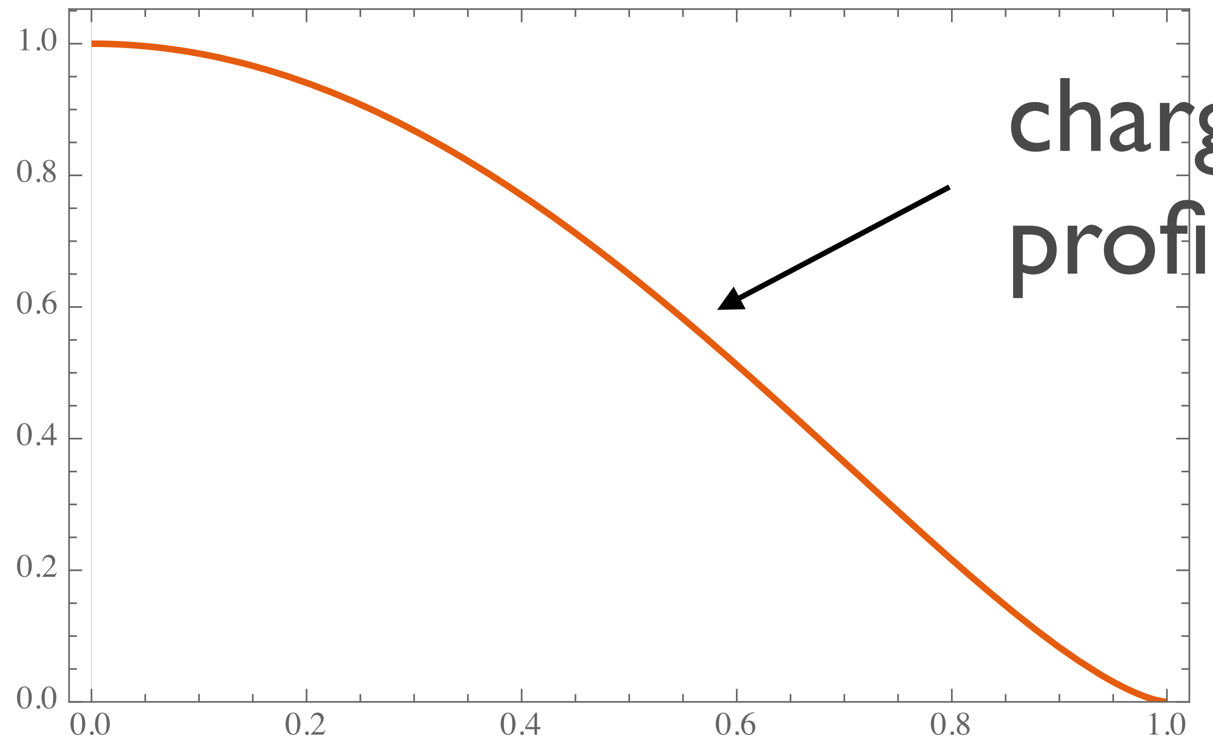


Nonrelativistic CFTs

Disadvantage: charge distribution is inhomogeneous (but spherically symmetric).

particle cloud





charge-density profile

Bulk EFT breaks down near the edge of the particle cloud.

Need to include terms describing the physics at the cloud edge.

dim. analysis

Bulk EFT:

$$\mathcal{L}_{LO} = c_0 U^{d/2+1}$$

$$U = \dot{\chi} - \frac{1}{2}r^2 - \frac{1}{2}(\partial_i \chi)^2$$

harmonic potential

Nonrelativistic CFTs

vev of U on ground state: $$\langle U \rangle = \mu - \frac{1}{2} r^2$$

Vanishes at the cloud edge, $$R_{\text{cl}} \equiv \sqrt{2\mu}.$$

LO scaling dimension:

$$\Delta(Q) = \frac{d}{d+1} \zeta Q^{(d+1)/d}$$

$$\zeta = \frac{1}{\sqrt{2\pi}} \left[\frac{\Gamma(d+1)}{\Gamma\left(\frac{d}{2}+2\right) c_0} \right]^{1/d}$$

Include higher-order terms in the EFT: only operator allowed besides U and its derivatives is

$$Z = \nabla^2 A_0 - \frac{1}{d} (\nabla^2 \chi)^2$$

All non-trivial composite operators that can appear have the form

$$\mathcal{O}^{(m,n)}_{\text{bulk}} \equiv c_{m,n} \cdot (\partial_i U)^{2m} Z^n U^{d/2+1-(3m+2n)}$$

Wilsonian coefficients ($c_{m,n}$); integers (m, n)

Kravec and Pal, 1809.08188

Nonrelativistic CFTs

Must also consider terms located at the cloud edge!
Most general form:

$$\mathcal{Z}^{(p)}_{\text{edge}} \equiv \kappa_p Z^p \delta(U) (\partial_i U)^{(d+4(1-p))/3}$$

integer

operator-valued delta-function

Wilsonian coefficient

Hellerman and Swanson, 2010.07967

The contributions of the bulk operators to Δ can have edge divergences if

- d is even
- the operator has positive Q-scaling

We can always regulate these divergences with an edge counter term of the same μ-scaling.

This gives rise to log(Q) terms in Δ.

Nonrelativistic CFTs

Additionally, there is a universal log(Q)-term from the Casimir energy in odd d.

$$E^{d=2}_{\text{Casimir}} = -0.294159\ldots$$
$$E^{3+2\epsilon}_{\text{Casimir}} = -\frac{1}{2\sqrt{3}\epsilon} + \text{regular}$$
$$\Delta(Q)\Big|_{Q^0} = \frac{1}{3\sqrt{3}}\log(Q) + \text{const.}$$

D. Orlando, V. Pellizzani, S. R., 2010.07942

protected by scale invariance

Scaling dimension in d=3:

bulk contributions

edge contributions

bulk+edge

Casimir energy

$$\Delta(Q) = Q^{12/9}\left[a_1 + \frac{a_2}{Q^{6/9}} + \frac{a_3}{Q^{12/9}} + \ldots\right] + Q^{5/9}\left[b_1 + \frac{b_2}{Q^{2/9}} + \frac{b_3}{Q^{4/9}} + \ldots\right] + Q^{-2/9}\left[d_1 + \frac{d_2}{Q^{2/9}} + \frac{d_3}{Q^{4/9}} + \ldots\right] + \frac{1}{3\sqrt{3}}\log Q + \text{const.}$$

S. Hellerman, D. Orlando, V. Pellizzani, S. R., I. Swanson, 2111.12094

Nonrelativistic CFTs

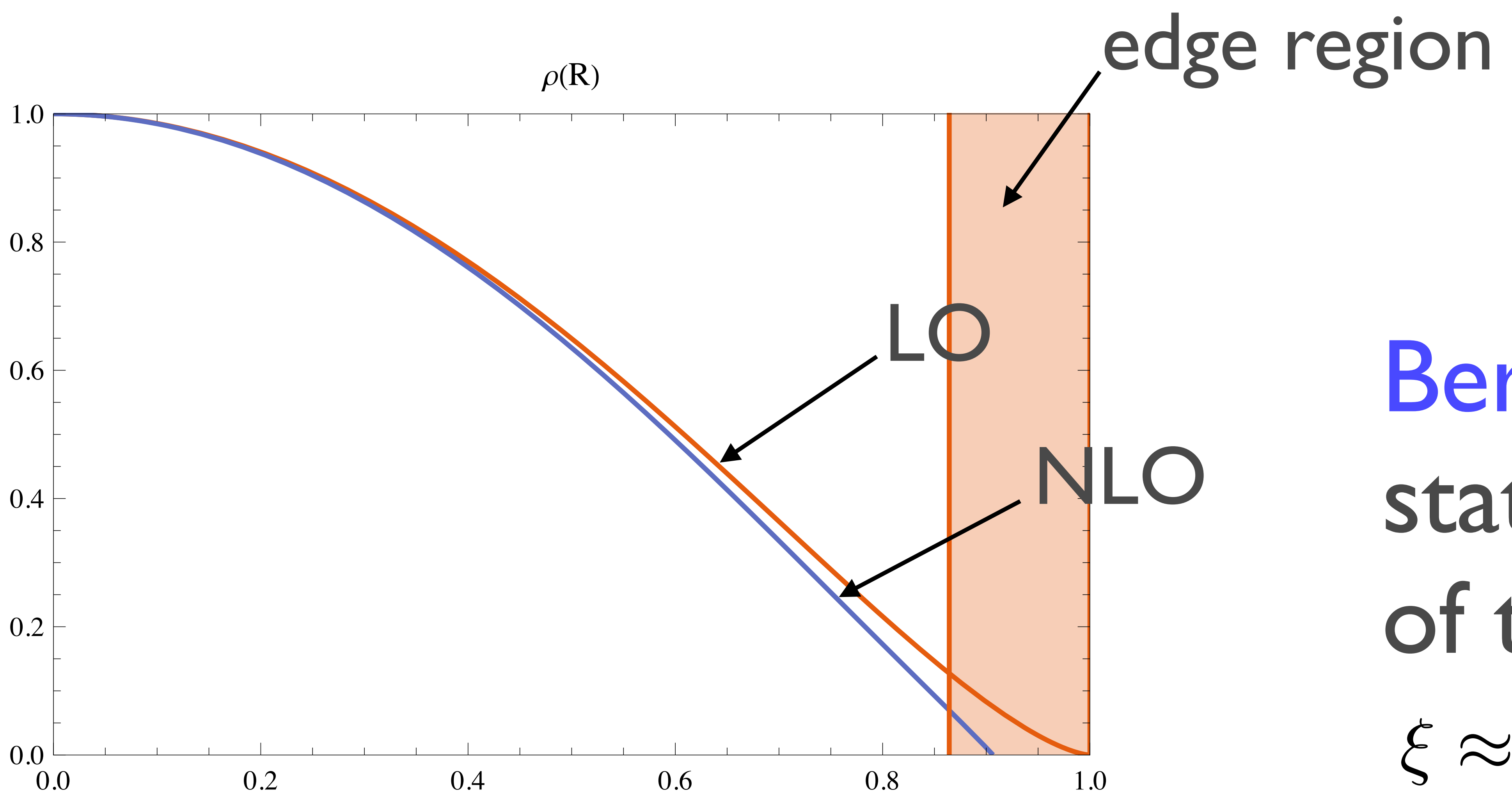
Large-N treatment:

Analogous to relativistic case (Stratonovich transform, integrating out fermions, evaluate functional determinant)

Much harder - problem is not homogeneous - vev of collective field non-constant!

Perform gradient expansion

- reproduce the terms in the EFT (both bulk and boundary)
- can compute the Wilsonian coefficients in the bulk



$$\frac{\Delta}{N} = 0.8313 \left(\frac{Q}{N}\right)^{4/3} + 0.26315 \left(\frac{Q}{N}\right)^{2/3} + \dots$$

S. Hellerman, D. Orlando, V. Pellizzani, S. R., I. Swanson, 2311.14793

Bertsch parameter: ratio between the ground-state energy of the Fermi gas at unitarity and that of the noninteracting Fermi gas: $c_0 = \frac{2^{5/2}}{15\pi^2 \xi^{3/2}}$

$\xi \approx 0.5906\dots$ reproduces mean-field value

$\xi_{\text{exp}} \approx 0.37\dots$

Nonrelativistic CFTs

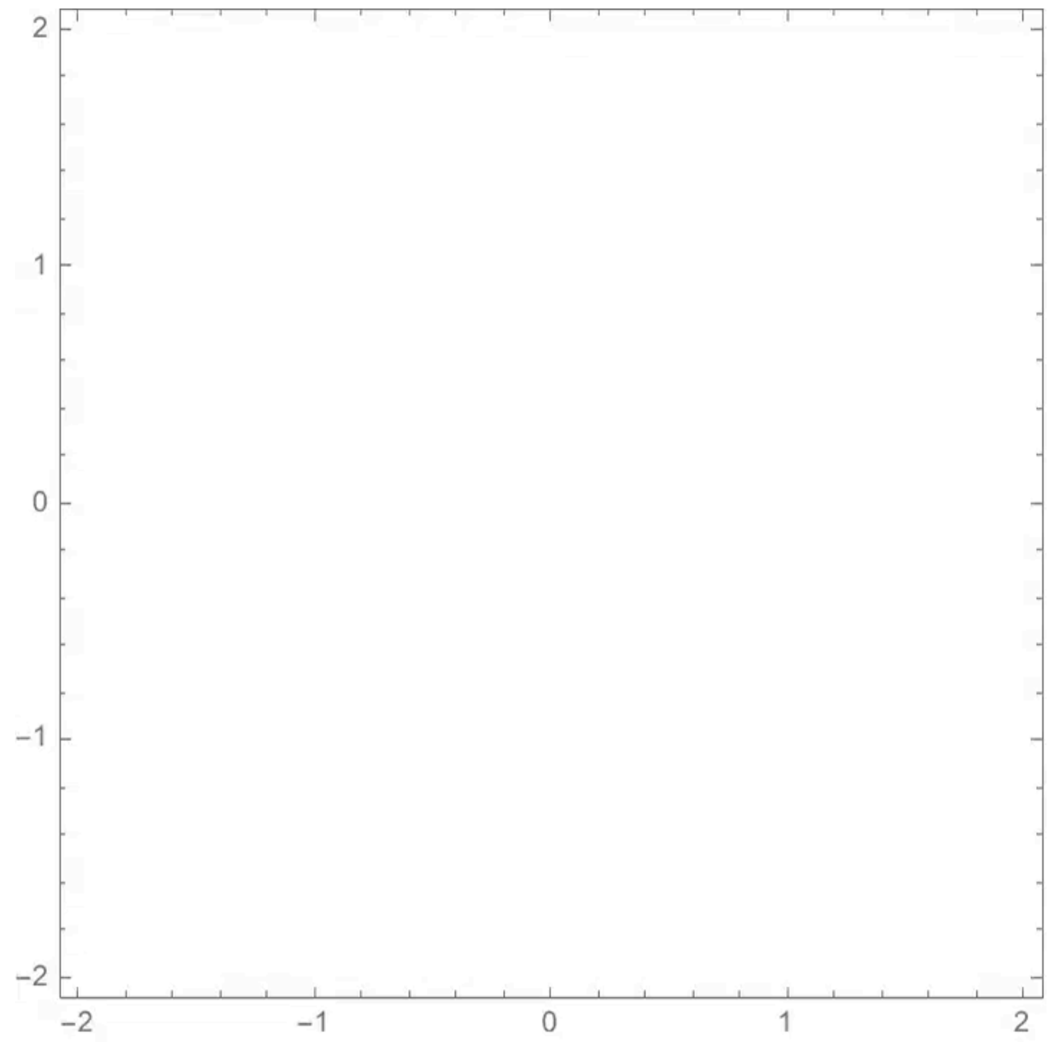
Nuclear physics:

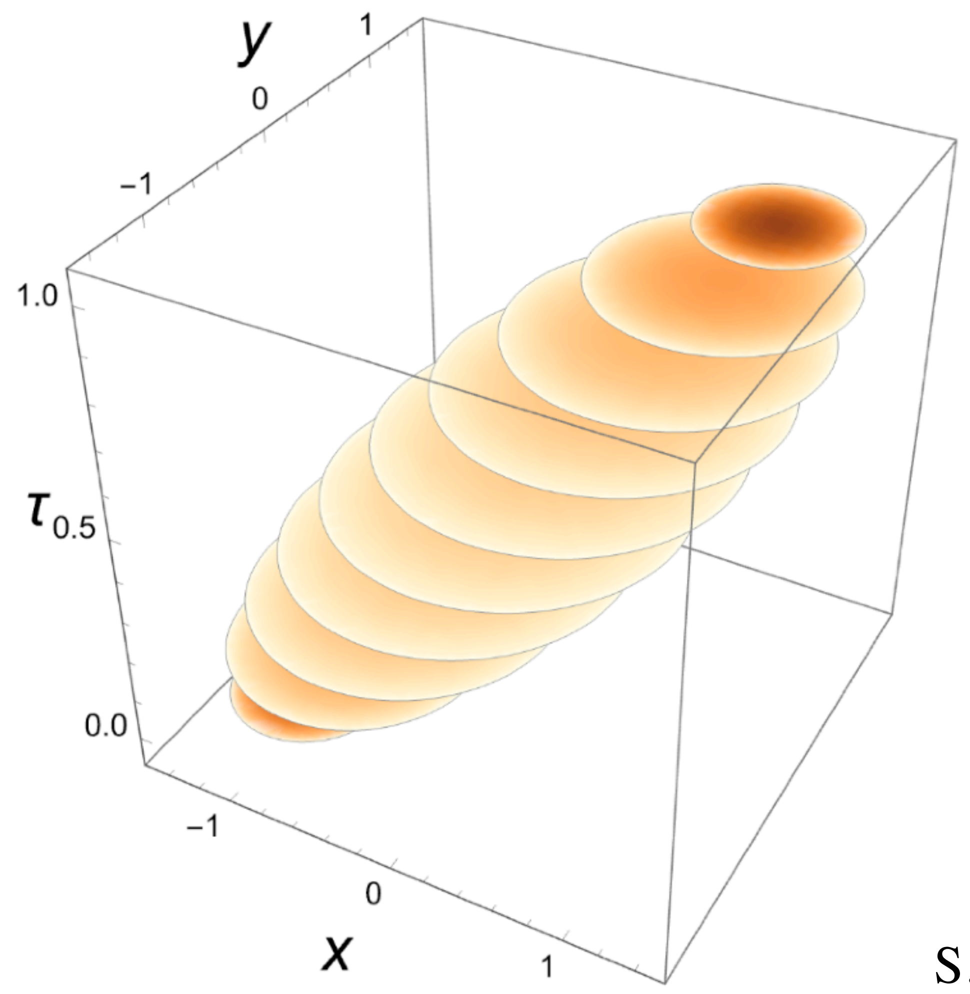
Consider system with only neutrons: neutron-neutron scattering length very large, system is near unitarity: described by NRCFT (same EFT as unitary Fermi gas - non-relativistic superfluid) with small range and scattering length corrections. "un-unclear physics" - nuclear physics w/o nucleons

Hammer and Son; Dutta Chowdhuri, Mishra, Son

Calculate n-pt correlation functions at large Q directly from insertions in the path integral

2-pt fn: Droplet of superfluid evolving between insertion points.





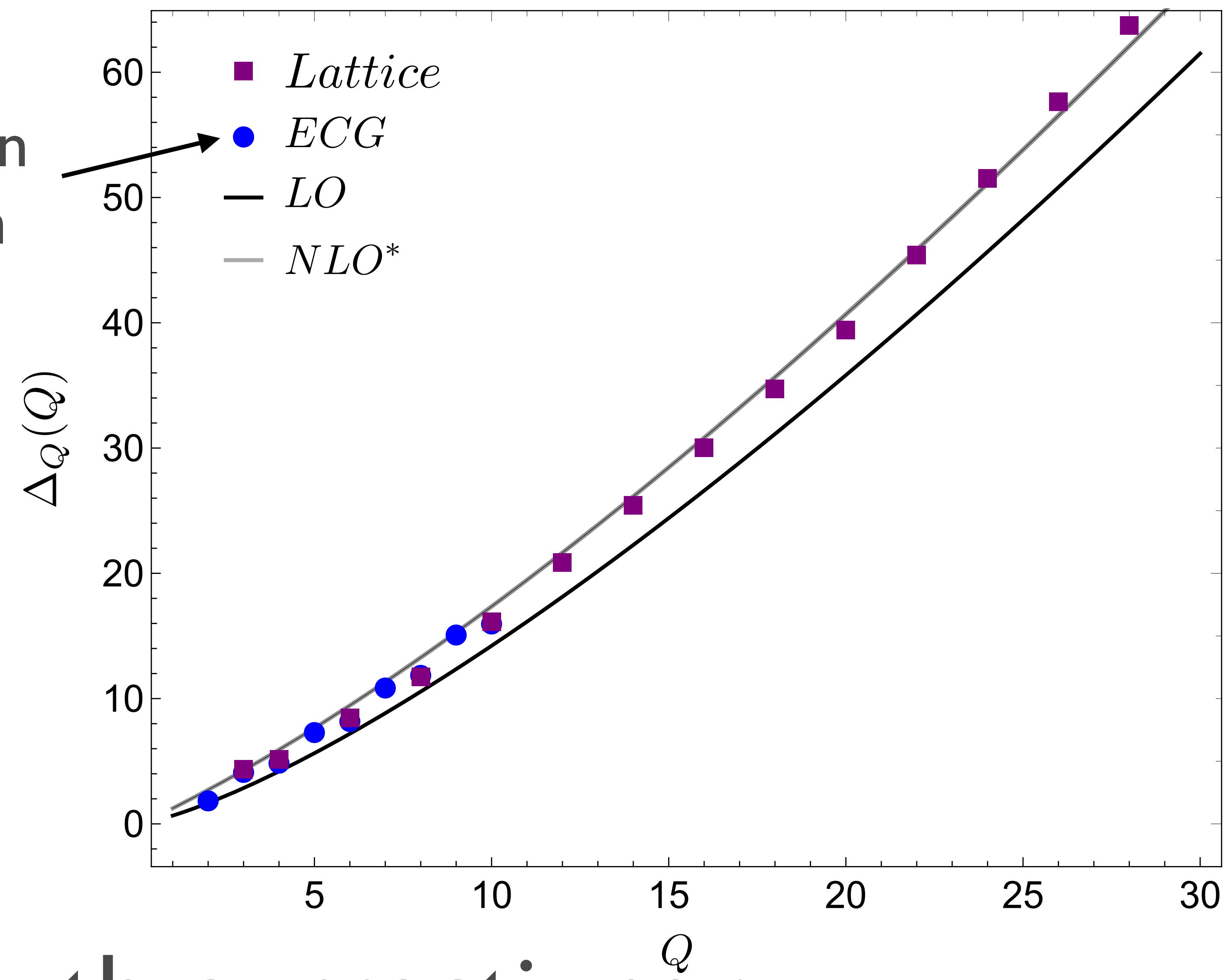
S. Beane, D. Orlando, S. R., 2403.18898

Nonrelativistic CFTs

Conformal dimension at unitarity:

$$\Delta_Q(Q) = \frac{3^{4/3}}{4}\xi^{1/2} Q^{4/3} - 3^{2/3}\sqrt{2}\pi^2 \xi\, c_{\rm NLO}\, Q^{2/3} + \mathcal{O}(Q^{5/9}) + \ldots + \frac{1}{3\sqrt{3}} \log Q$$

correlated Gaussian basis set expansion



$$\xi = 0.372(5)$$

$$c_{\rm NLO} = -0.0537(2)$$

Endres, Kaplan, Lee, Nicholson; Yin, Blume

Range and scattering length corrections :

dimensionless Wilsonian parameters

$$\mathcal{L}_{\rm SB} = g_1\, a^{-1} m U^2 + g_2\, a^{-2} m^{1/2} U^{3/2} + h_1\, r m^2 U^3 + h_2\, r^2 m^{5/2} U^{7/2} + \ldots$$

scattering length

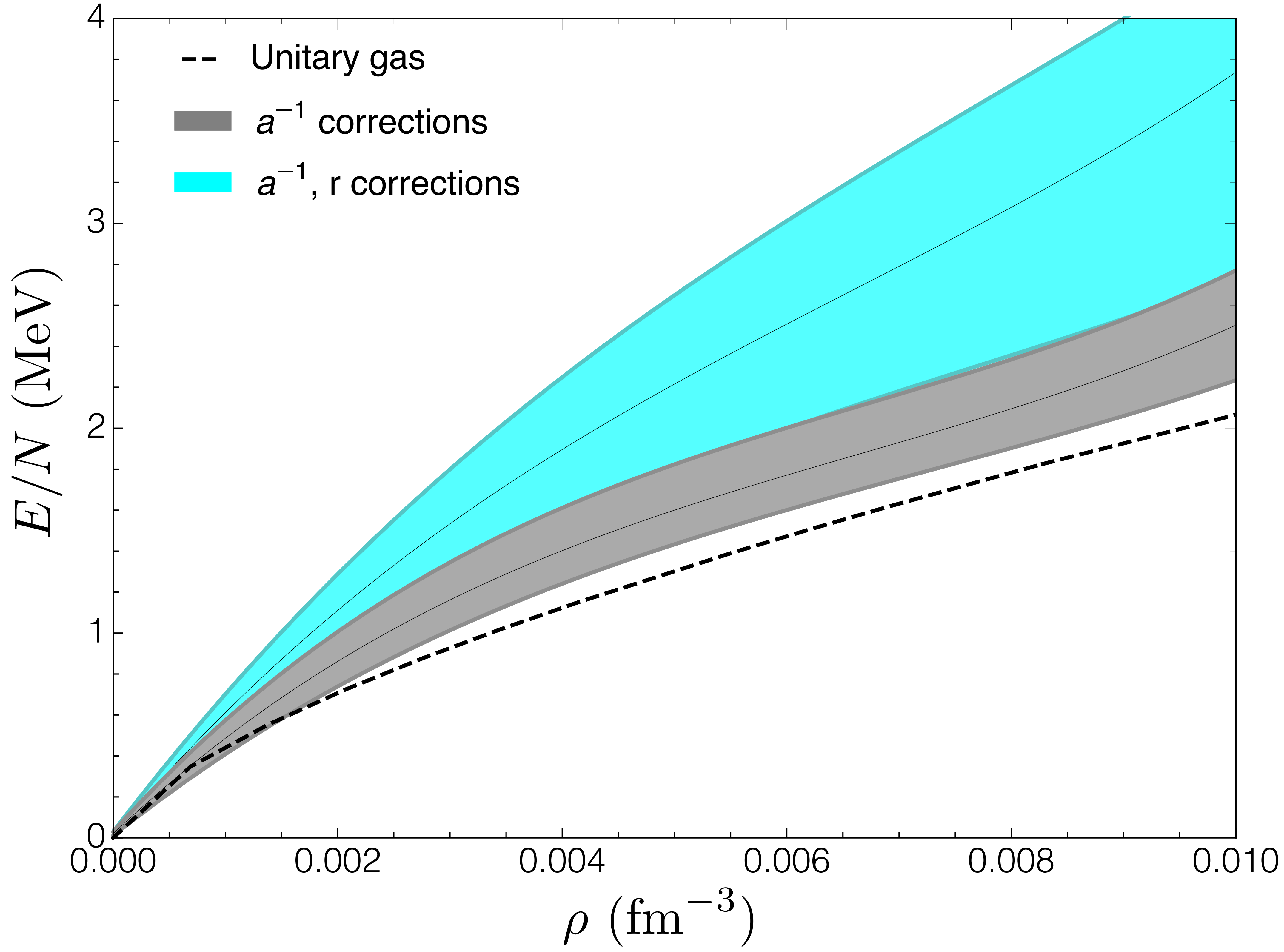
effective range

Nonrelativistic CFTs

Energy per particle :

scattering length — effective range

$$E/N = \frac{3}{5}\frac{k_F^2}{2M}\left(\xi - \frac{\zeta}{k_F a} - \frac{\zeta_2}{k_F^2 a^2} + \ldots + \eta\, k_F r + \eta_2\, k_F^2 r^2 + \ldots\right)$$



Values of coefficients extracted from numerical data in the literature

S. Beane, D. Orlando, S. R., to appear

Summary

Summary

Concrete examples where a strongly-coupled CFT simplifies significantly at large charge.

O(2N) model in 3d: in the limit of large U(1) charge Q, we computed the conformal dimensions in a controlled perturbative expansion:

- Excellent agreement with lattice results for O(2), O(4)
- large Q and large N: path integral at saddle pt., more control than in EFT, can calculate coefficients
- can follow the flow away from conformal point, find the full effective potential

NJL model: similar results, condensate due to Cooper pairs.

Summary

Many other interesting applications!

NRCFTs are also highly suited for the large-charge approach.

U(1) symmetry: particle number

Examples:

- unitary Fermi gas (4D)
- nuclear reactions involving neutrons in the end state
- anyons (3D)

State-operator correspondence involves harmonic potential.

Can compute 2- and 3-point functions in the limit of large Q

Further directions

- Further study of supersymmetric models at large R-charge (higher-dim. moduli spaces)

  Hellerman, Maeda, Orlando, Reffert, Watanabe; Argyres et al.

- Connection to holography (gravity duals)

  Loukas, Orlando, Reffert, Sarkar; De la Fuente, Zosso; Giombi, Komatsu, Offertaler; Perlmutter et al.

- Operators with spin; connection to large-spin results

  Cuomo, de la Fuente, Monin, Pirtskhalava, Rattazzi; Cuomo

- Use/check large-charge results in conformal bootstrap

  Jafferis and Zhiboedov; Rong and Su

- Further lattice simulations: inhomogeneous sector, general O(N)

  Chandrasekharan et al.; Singh

- CFTs in other dimensions (2, 5, 6)

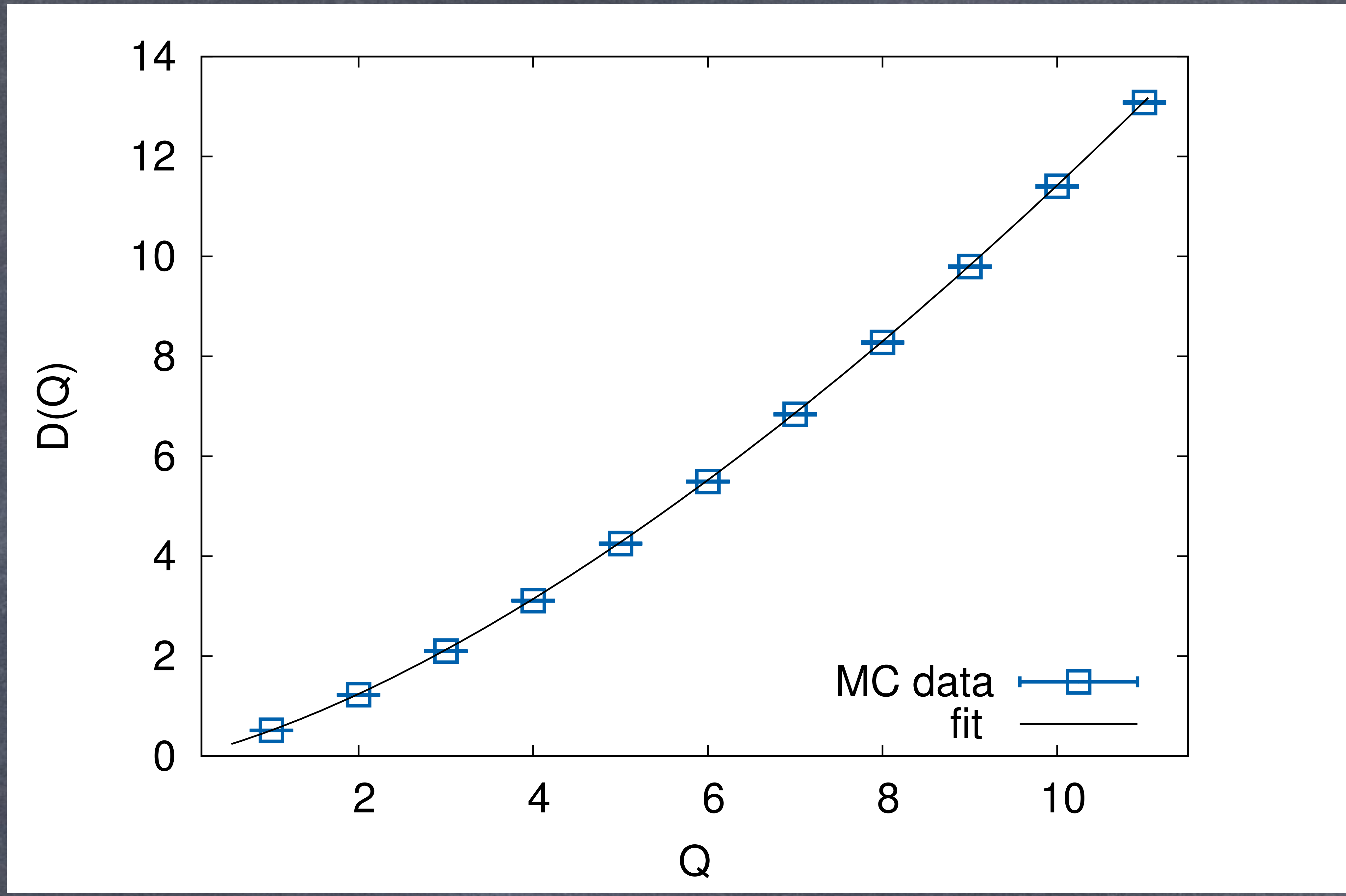
  Komargodski, Mezei, Pal, Raviv-Moshe; Araujo, Celikbas, Reffert, Orlando; Moser, Orlando, Reffert

- Integrability and large Q

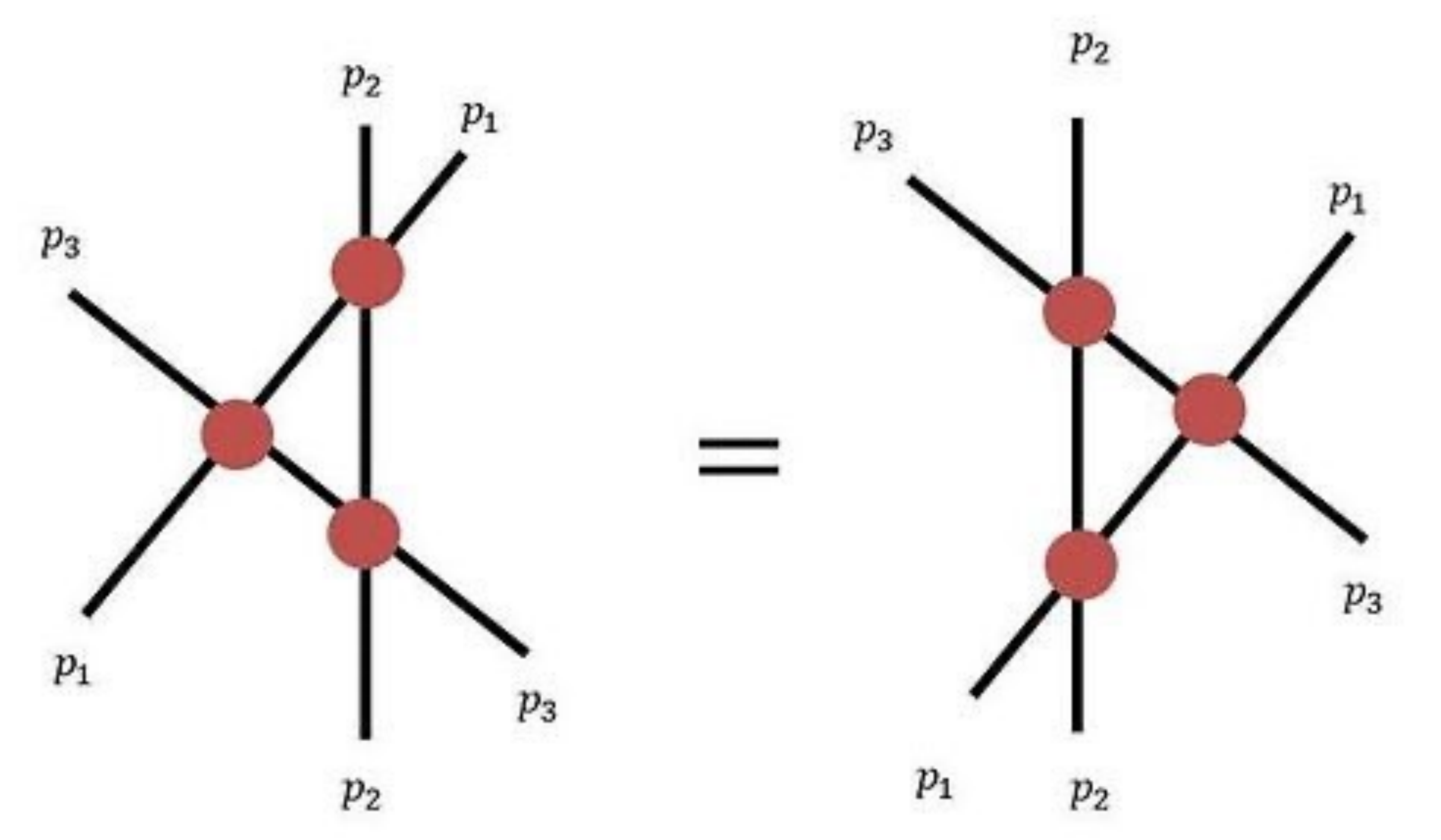
  Dodelson, Hellerman, Watanabe, Yamazaki

Further directions

- Chern-Simons matter theories @large charge
  Watanabe
- 4-ε expansion @large charge
  Arias-Tamargo, Rodriguez-Gomez, Russo; Badel, Cuomo, Monin, Rattazzi; Watanabe; Antipin et al.
- going away from the conformal point
  Orlando, Reffert, Sannino; Orlando, Pellizzani, Reffert
- non-relativistic CFTs
  Favrod, Orlando, Reffert; Kravec, Pal; Orlando, Pellizzani, Reffert; Hellerman, Swanson; Pellizzani
- Boundary CFTs at large Q
  Cuomo, Mezei, Raviv-Moshe
- Swampland, weak gravity conjecture
  Aharony, Palti; Antipin et al. Orlando, Palti
- Study fermionic theories. Can large-charge approach be used for QCD (e.g. large baryon number)?
  Komargodski, Mezei, Pal, Raviv-Moshe; Antipin, Bersini, Panopoulos; Dondi, Hellerman, Kalogerakis, Moser, Orlando, Reffert;
- Gauge theories @large charge, Standard Model
  Antipin, Bersini, Sannino et al.



Thank you for your attention!



Integrability and Large Charge

Integrability and Large charge

The large charge-expansion can be applied to integrable models to actually solve them.

Just like combining large Q with large N gave us more control, also combining large Q with integrability gives us extra control.

This has been done for several examples.

Sometimes, integrability emerges in the large-charge sector.

I'll briefly review 2d models:

- CFT: SU(2) WZW model and its marginal deformation
- massive integrable case: YB deformed SU(2) PCM

Integrable systems in 2d

In 2d CFTs, the U(1) sector decouples from the full dynamics at large charge.
It cannot be used to write an EFT as a large-charge expansion that controls the dynamics.

Komargodski, Mezei, Pal, Raviv-Moshe, 2112.12583

It is however possible to use the large-Q expansion to simplify models with a known NSLM description:
Work in a double-scaling limit (large Q and controlling scale), use e.g. WKB approximation to compute conformal dimensions.

We find that the scaling dimension of the lowest operator of charge Q takes the form of an expansion in 1/Q starting at $\mathcal{O}(Q^2)$.

Can verify these result in the case of solvable models.

Integrable systems in 2d

Example: SU(2) WZW model.

$$S = \frac{k}{16\pi}\int dz\, d\bar{z}\, \mathrm{Tr}\left[\partial^\mu g^{-1}\, \partial_\mu g\right] + k\Gamma,$$

$$\Gamma = -\frac{i}{24\pi}\int d^3y\, \epsilon_{\alpha\beta\gamma}\, \mathrm{Tr}\left[g^{-1}\, \partial^\alpha g g^{-1}\, \partial^\beta g g^{-1}\, \partial^\gamma g\right]$$

WZW models admit a geometrical description for $k \to \infty$

SU(2) WZW: NLSM on target space S^3

Global SU(2)xSU(2) symmetry, can fix 2 charges (left and right U(1))

In the limit $k \gg Q, \bar{Q} \gg 1$, we find using the WKB approximation

$$\Delta = \frac{(Q+\bar{Q})(Q+\bar{Q}+2)}{2k}$$

Araujo, Celikbas, Orlando, S.R., 2112.03286

Integrable systems in 2d

Continuous line of marginal deformations generated by

$$\int dz\, d\bar{z}\, J_0^3 \bar{J}_0^3$$

Breaks global symmetry down to U(1)xU(1)

Scaling dimension of lowest charged operator:

$$\Delta = \frac{(Q+\bar{Q})(Q+\bar{Q}+2)}{2(k+2)} + \frac{1-\lambda^2}{2k}\left(\frac{Q^2}{\lambda^2} - \bar{Q}^2\right)$$

Can verify by specializing to the fixed-charge sector in the exact partition function!

Interesting approach to study more general model without known exact solution.

Integrable systems in 2d

Integrability is an accidental property of generic 2d O(2)-symmetric asymptotically free theories when the charge density is much larger than the dynamical scale.

Exists infinite tower of higher spin conserved currents in the most generic EFT at large chemical potential.

Dodelson, Hellerman, Watanabe, Yamazaki, 2310.01823

Integrable systems in 2d

We have seen that for 2d CFTs, the U(1) sector does not control the dynamics. Let's instead study a massive case that is integrable.

We can start from the thermodynamic Bethe ansatz equations - the thermodynamic limit is actually a large-charge limit!

$$\chi[\theta] - \int_{-B}^{B} K\left[\theta - \theta'\right] \chi\left[\theta'\right] d\theta' = m \cosh[\theta], \quad \theta^2 < B^2$$

From here, we can get the energy density which is in turn related to the free energy by a Legendre transform.

By studying the large ρ, or equivalently, large B asymptotics, Volin found an expansion of the energy density in terms of 1/B for the O(N) vector model - secretly a large-charge expansion.

Volin, 0904.2744

Integrable systems in 2d

Can apply Volin's method to other integrable systems, e.g. the Yang-Baxter deformed principal chiral model for SU(2):

$$\mathscr{L}_{\zeta,\eta}[g] = \mathrm{Tr}\left(g^{-1}\partial_+ g \frac{1}{1-\eta R - \zeta R^g} g^{-1}\partial_- g\right).$$

Work at η small, ζ=0 and B large → perturbative expansion in asymptotically free theory

$$k^2(e/\rho^2) = \alpha + \frac{\alpha^2}{2} + \alpha^3\left(\frac{1}{4} + \frac{1}{6\tilde{p}}\right) + \alpha^4\left(\frac{5}{16} - \frac{3\zeta(3)}{32} + \frac{\log 2}{6\tilde{p}}\right) \qquad \propto \frac{1}{\eta}$$
$$+ \alpha^5\left(\frac{53}{96} - \frac{9\zeta(3)}{64} + \frac{1 - \frac{4}{3}\log 2 + 2(\log 2)^2}{16\tilde{p}}\right) + \dots$$

Next step: find renormalon contributions to the free energy (poles or branch cuts in the Borel plane)

Ashwinkumar, Orlando, S.R., Sberveglieri, to appear

N=2 SCFT

Let's start with the SCFT case. Things are very different for SCFTs with a moduli space.

How can we write an EFT? Need extra ingredient.
Make use of SUSY properties.

Simplest case: systems with a 1-dim. moduli space on the Coulomb branch. The charge that is taken to be large is the R-charge and we want to write the EFT of the Coulomb branch.

Since we are in D=4, there is a Weyl anomaly, which must be reproduced in the EFT.

1D Coulomb branch: the EFT at large charge is encoded by single vector multiplet.

SCFTs at large R-charge

Coulomb branch is generated by $\mathcal{O}$.

2-pt functions are a solved problem for BPS operators, scaling dimension goes like Q

Compute 3-pt function on a conformally flat 4D space:

$$\langle \mathcal{O}^{n_1}(x_1)\mathcal{O}^{n_2}(x_1')\bar{\mathcal{O}}^{n_3}(x_2)\rangle = \frac{C^{n_1,n_2,\overline{n_1+n_2}}}{|x_1-x_2|^{2n_1 D_{\mathcal{O}}}|x_1'-x_2|^{2n_2 D_{\mathcal{O}}}}$$

Notice that the R-charge $Q_{\mathcal{O}} \propto D_{\mathcal{O}}$

OPE of chiral primaries is non-singular. Choose

$$x_1 = x_1' \qquad \mathcal{O}^{n_1}(x_1)\mathcal{O}^{n_2}(x_1) = \mathcal{O}^{n_1+n_2}(x_1)$$

3-pt function becomes a 2-pt function:

$$C^{n',n-n',\bar{n}} = |x_1-x_2|^{2nD_{\mathcal{O}}}\langle \mathcal{O}^n(x_1)\bar{\mathcal{O}}^n(x_2)\rangle = e^{q_n-q_0} \qquad e^{q_0} = Z$$

Write EFT controlled by n as $Q = nD_{\mathcal{O}}$

SCFTs at large R-charge

1D Coulomb branch: EFT encoded by single vector multiplet.

Assume for now that free theory for cplx scalar is dominating in the large-Q expansion:

CFT: no dependence on A

$$S = \int_{\mathbb{R}^4} \mathrm{d}^4 x \frac{\mathrm{Im}(\tau)}{4\pi} \partial_\mu A \, \partial^\mu \bar{A} + \dots$$

cplx scalar of vector multiplet

Introduce $\phi = \sqrt{\frac{\mathrm{Im}(\tau)}{4\pi}} A \qquad \mathcal{O} = N_{\mathcal{O}} \, \phi^{D_{\mathcal{O}}}$

Now we can write down our 2-pt function:

$$\langle \mathcal{O}^n(x_1) \bar{\mathcal{O}}^n(x_2) \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, \mathcal{O}^n(x_1) \mathcal{O}^n(x_2) e^{-S_{\text{free}}}$$

$$Z = \int \mathcal{D}\phi \, e^{-S_{\text{free}}}$$

SCFTs at large R-charge

Rewrite
$$\int \mathcal{D}\phi\, \mathcal{O}^n(x_1)\mathcal{O}^n(x_2)e^{-S_{\text{free}}} = \int \mathcal{D}\phi\, e^{-(S_{\text{free}}+S_{\text{sources}})}$$

$$Q = nD_{\mathcal{O}} \quad \mathcal{O} = N_{\mathcal{O}}\,\phi^{D_{\mathcal{O}}}$$

$$S_{\text{free+sources}} = -2Q\log N_{\mathcal{O}} + \int \mathrm{d}^4x\left[\partial_\mu\phi\,\partial^\mu\bar\phi - Q\log\phi\,\delta(x-x_1) - Q\log\bar\phi\,\delta(x-x_2)\right]$$

Minimize to find the fixed-charge ground state:

$$\phi(x) = \frac{e^{i\beta_0|x_1-x_2|}}{2\pi(x-x_2)^2}\sqrt{Q} \qquad \bar\phi(x) = \frac{e^{-i\beta_0|x_1-x_2|}}{2\pi(x-x_1)^2}\sqrt{Q}$$

Find value of the full action at the minimum:

$$\begin{aligned} S_{\text{full}} &= Q\left[-2\log N_{\mathcal{O}} + 1 + 2\log(2\pi)\right] - Q\log Q \\ &= k_1 Q - Q\log Q + 2Q\log|x_1-x_2| + \mathcal{O}(Q^0) \end{aligned}$$

→ directly gives leading term in Q-expansion

$$|x_1-x_2|^{2nD_{\mathcal{O}}}\langle\mathcal{O}^n(x_1)\bar{\mathcal{O}}^n(x_2)\rangle = e^{q_n-q_0} \qquad e^{q_0} = Z$$

$$q_n = Q\log Q + k_1 Q + \mathcal{O}(Q^0)$$

SCFTs at large R-charge

So far: used only free kinetic term.
In general, there will be higher-order corrections.

One can show that:

- all manifestly superconformal terms will give a contribution that is subleading in Q
- theories with a 1D Coulomb branch have no other F-terms

Only other possible term is the Wess-Zumino term in the bosonic action.

Necessary to compensate Weyl-anomaly mismatch between CFT and EFT:

Euler density

$$\mathcal{L}_{WZ} = -\tau\, 2\alpha\, E_4(g)$$

Calculate on S^4

$$\alpha = \frac{1}{2}(a_{CFT} - a_{EFT})$$

SCFTs at large R-charge

Contribution to action:

$\tau \sim -\log|\phi|$

$$S_{WZ}\Big|_{cl} = -(\alpha + \tfrac{1}{2})\log Q$$

quantum corrections in $S_{\text{free}} + S_{\text{WZ}}$

Full result:

$$q_n - q_0 = Q\log Q + k_1 Q + (\alpha + \tfrac{1}{2})\log Q + \sum_{m=1}^{\infty} \frac{k_m(\alpha)}{Q^m}$$

Can compute the $k_m(\alpha)$ perturbatively by expanding

$$\phi = \phi_{cl} + \phi_{fluc}$$

Just like in the O(2) model, 1/Q is the loop-counting parameter for the theory of ϕ_{fluc}

SCFTs at large R-charge

$$q_n - q_0 = \underbrace{Q \log Q + k_1 Q + (\alpha + \tfrac{1}{2}) \log Q}_{\text{class. ground state}} + \sum_{m=1}^{\infty} \frac{k_m(\alpha)}{Q^m}$$

quantum corr. in $S_{\text{free}} + S_{\text{WZ}}$ (for $k_m(\alpha)$)

Can in principle proceed order by order to compute quantum corrections.

Order 1/Q: $k_1(\alpha) = \frac{1}{2}(\alpha^2 + \alpha + \frac{1}{6})$

There's a better way!

Use recursion relation for theories with marginal coupling.

$$\partial\bar{\partial} q_n = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}$$

Baggio, Niarchos, Papadodimas;
Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski, Pufu;

Toda lattice equation.

Look for solution with EFT-inspired form

$$q_n = Q\, f(\tau, \bar{\tau}) + k_0(\tau, \bar{\tau}) + Q \log Q + (\alpha + \tfrac{1}{2}) \log Q + \sum_{m=1}^{\infty} \frac{k_m(\alpha)}{Q^m}$$

independent of τ (for $k_m(\alpha)$)

SCFTs at large R-charge

Solve recursion (using the result for $k_1(\alpha)$):

$$q_n = 2n\,A(\tau,\bar{\tau}) + B(\tau,\bar{\tau}) + \log\Gamma(2n+\alpha+1)$$

theory dependent ($A(\tau,\bar{\tau})$, $B(\tau,\bar{\tau})$)

universal, valid for any theory, depends only on α ($\log\Gamma(2n+\alpha+1)$)

Logic:

- EFT works for any theory (incl. non-Lagrangian)
- can solve it order by order via Feynman diagrams
- for Lagrangian theories, we can use the recursion relation
- result is valid for all theories, as it is independent of τ.

SCFTs at large R-charge

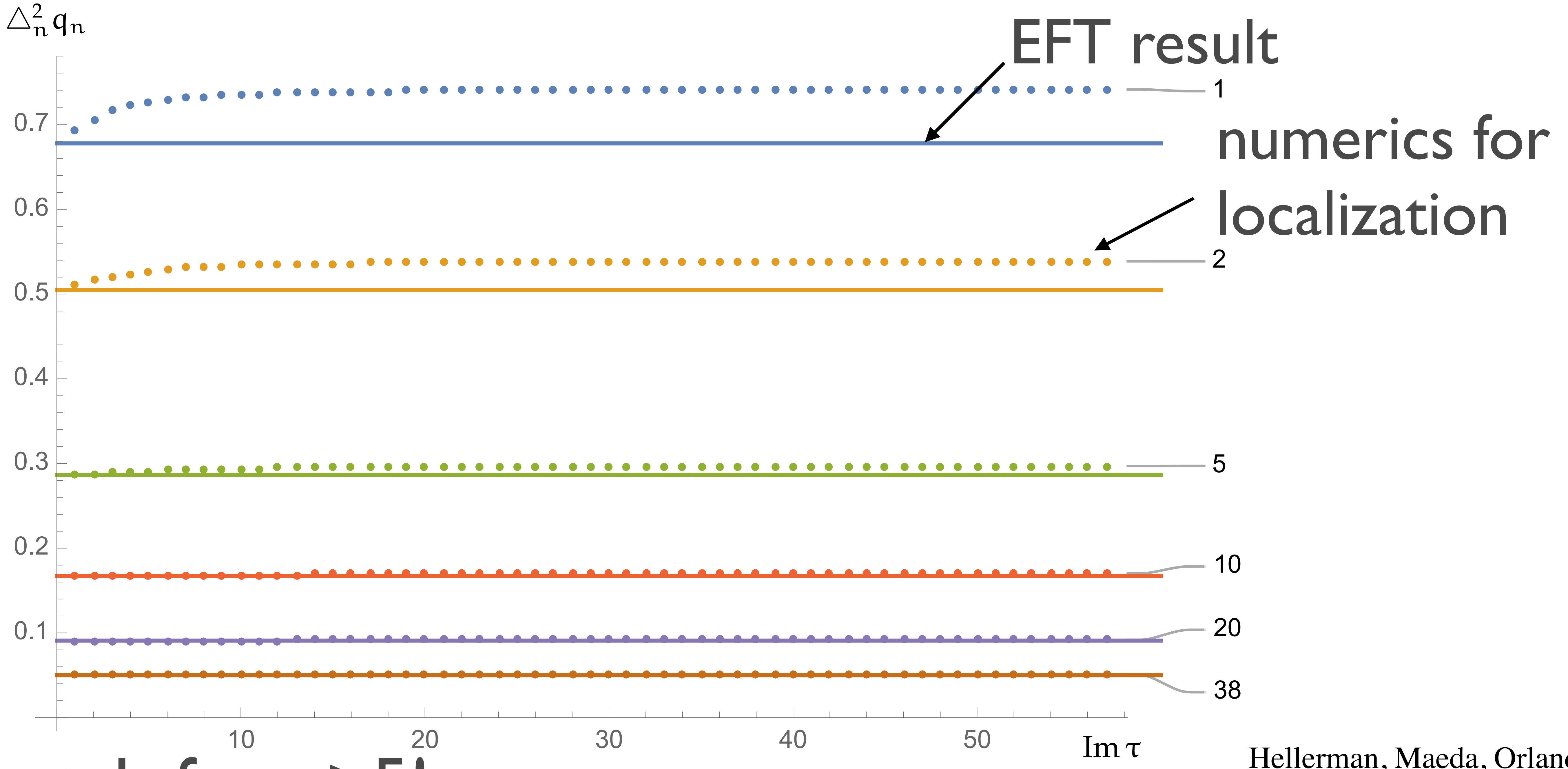
Solve known recursion:

$$q_n = 2n\, A(\tau, \bar{\tau}) + B(\tau, \bar{\tau}) + \log \Gamma(2n + \alpha + 1)$$

universal

For the case of SU(2) gauge theory with 4 flavors, we can compare our EFT results to localization results:

Gerchkovitz, Gomis, Ishtiaque, Karasik, Komargodski, Pufu;



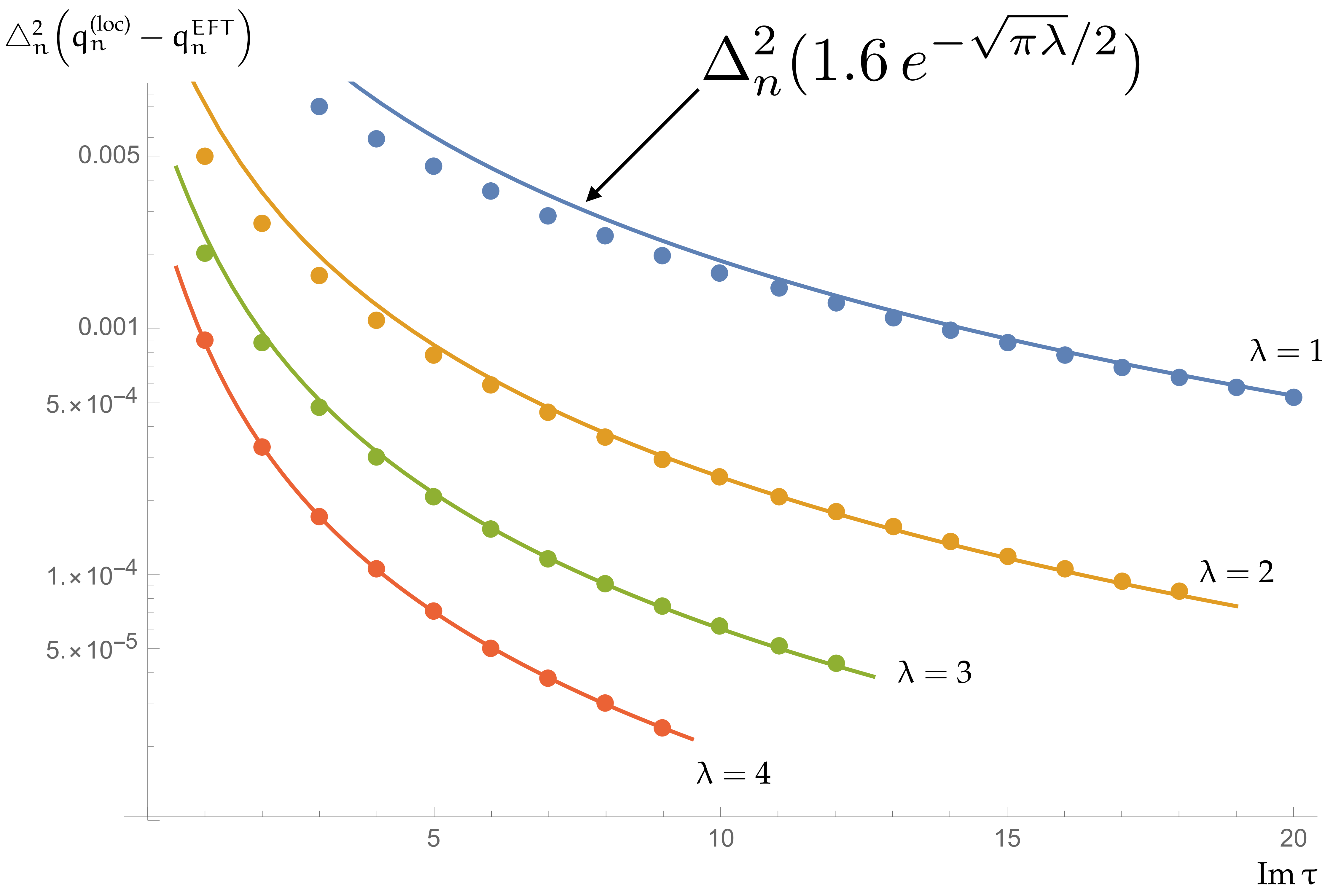
Hellerman, Maeda, Orlando, S.R., Watanabe, 1804.01535

Extremely good match for n>5!
EFT predictions have been verified.

Grassi, Komargodski, Tizzano;

SCFTs at large R-charge

We can even estimate the exponential corrections due to the propagation of massive BPS particles:



Can be computed explicitly!

Hellerman, Maeda, Orlando, S.R., Watanabe, 2005.03021
Hellerman,Orlando, 2103.05642
Hellerman, 2103.09312