Framed DDF Operators and Minimally Off-shell Solutions of Virasoro Constraints in Bosonic String Theory

A Short Presentation

Dripto Biswas
(In collaboration with Igor Pesando)

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University of Turin
Department of Physics

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Outline

Introduction

What are DDF operators/states?

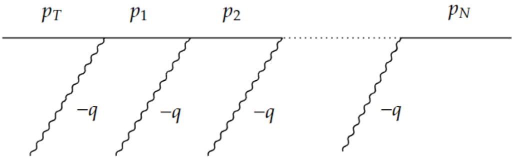

Figure: Pictorial representation of the standard DDF formulation.

Mathematical definition

- Starting from a tachyon vertex operator,

$$V_T(x;p_T) := c(x) : e^{2ip_T\cdot L(z)} : \tag{1}$$

- **Choose** a null vector q_μ such that, $2\alpha' p_T \cdot q = 1$. For simplification later on, one usually also chooses,

$$p_T^+ = p_T^- = \frac{1}{\alpha'};\ p_T^i = 0;\ q^+ = q^i = 0;\ q^- \neq 0. \tag{2}$$

- Successively scatter photons of momentum $-q_\mu$, polarisation $\epsilon_\mu^{(i)}(q,p_T)$ and pick out the on-shell part (pole) of the resulting state (i.e. take the OPE!). Summarising, we get

$$\boxed{A_n^i(q,p_T) = i\sqrt{\frac{2}{\alpha'}}\oint_{z=0}\frac{dz}{2\pi i}\,\epsilon^{(i)}(q,p_T)\cdot\partial_z L\, e^{2in\,q\cdot L(z)}} \tag{3}$$

DDF v.s *Framed* DDF operators (FDDF)

Definition of FDDF operators

- Introduce a vielbein, $E^{\underline{\mu}}_{\mu}$ and its inverse $E^{\mu}_{\underline{\mu}}$, such that

$$E^{\underline{\mu}}_{\mu} E^{\underline{\nu}}_{\nu} \eta_{\underline{\mu}\underline{\nu}} = g_{\mu\nu}, \tag{4}$$

where, $g_{\mu\nu}$ is the metric in the Action.

- Framed DDF operators in the ghost number sector 0:

$$\boxed{\underline{A}^{i}_{n}(E) = i\sqrt{\frac{2}{\alpha'}} \oint_{z=0} \frac{dz}{2\pi i} : \partial_z \underline{L}^{i}(z) e^{in\frac{\underline{L}^{+}(z)}{\alpha' \underline{p}^{+}_{0}}} :}. \tag{5}$$

Brower operators:

$$\underline{A}^{-}_{n}(E) = i\sqrt{\frac{2}{\alpha'}} \oint_{z=0} \frac{dz}{2\pi i} : \left[\partial_z \underline{L}^{-}(z) - i\frac{n}{4\underline{p}^{+}_{0}} \frac{\partial^2_z \underline{L}^{+}}{\partial_z \underline{L}^{+}} \right] e^{in\frac{\underline{L}^{+}(z)}{\alpha' \underline{p}^{+}_{0}}} : , \tag{6}$$

$$\tilde{\underline{A}}^{-}_{n}(E) = \underline{A}^{-}_{n}(E) - \frac{1}{\underline{\alpha}^{+}_{0}} \mathcal{L}_n(E) + \frac{D-2}{24} \frac{1}{\underline{\alpha}^{+}_{0}} \delta_{n,0}, \tag{7}$$

$$\mathcal{L}_m(E) = \frac{1}{2} \sum_{j=2}^{D-1} \sum_{l\in\mathbb{Z}} : \underline{A}^{j}_{l}(E)\, \underline{A}^{j}_{m-l}(E) : \tag{8}$$

Properties

DDF

- **Algebra:** $[A^i_m(q, p_T),\, A^j_n(q, p_T)] = m\,\delta^{ij}\,\delta_{m+n,0}$. Derivation uses relations between q, p_T, ϵ explictly.
- **Conformal properties:** $[L_n, A^i_m(q, p_T)] \propto \sin(2\pi m\alpha' p_0 \cdot q)$.
- **Mass-shell:** $A^i_{-n}\,|k_T\rangle$ has momentum $p_n = k_T - nq \implies -\alpha' p_n^2 = n - 1$

FDDF

- **Algebra:** $[\underline{A}^i_m(E),\, \underline{A}^j_n(E)] = m\,\delta_{m+n,0}\delta^{ij}$.
- **Conformal properties:** $[L_n, \underline{A}^i_m] = 0$. $[L_n, \underline{\tilde{A}}^-_m] = 0$.
- **Mass-shell:** Need not satisfy! Only $\underline{p}^+_0 \neq 0$.

Gauge equivalence of different frames: An example

On-shell gauge equivalence ($N = 1$)

- The same (physical) $N = 1$ state with momentum k_μ and polarization ϵ_μ are related in two different frames $E, \hat{E}$ as,

$$\underline{\epsilon}_{(\hat{E})i}\underline{A}^i_{-1}(\hat{E})|\underline{k}_{T(\hat{E})}\rangle = \underline{\epsilon}_{(E)i}\underline{A}^i_{-1}(E)|\underline{k}_{T(E)}\rangle + L_{-1}\,\xi|k\rangle, \tag{9}$$

$$\implies \underline{\epsilon}_{(\hat{E})i}\hat{E}^i_\mu = \underline{\epsilon}_{(E)i}E^i_\mu + \xi k_\mu. \tag{10}$$

- This is a set of D equations in $D-1$ unknowns, i.e. $\underline{\epsilon}_{(E)i}$ and ξ. The system is soluble on-shell where $k^2 = 0$ and $k \cdot \epsilon = 0$. In fact contracting with k and (an auxiliary null vector) $\bar{k}$ ($k \cdot \bar{k} = -1$) we get

$$\underline{\epsilon}_{(\hat{E})i}\underline{k}^i_{(\hat{E})} = \underline{\epsilon}_{(E)i}\underline{k}^i_{(E)} + \xi k^2 \quad \Rightarrow \quad 0 = \xi k^2,$$

$$\underline{\epsilon}_{(\hat{E})i}\underline{\bar{k}}^i_{(\hat{E})} = \underline{\epsilon}_{(E)i}\underline{\bar{k}}^i_{(E)} - \xi. \tag{11}$$

? What happens when we go off-shell and $k^2 \neq 0$ anymore?

Off-shell gauge equivalence ($N = 1$)

Problem: The system (10) is not solvable off-shell!

Solution: Use the (improved) Brower states generated by $\underline{\tilde{A}}^-$.

- The same (off-shell) $N = 1$ state with momentum k_μ and polarization ϵ_μ are related in two different frames $E, \hat{E}$ as,

$$|k, \epsilon_{(\hat{E})}\rangle = |k, \epsilon_{(E)}\rangle + \zeta\, \underline{\tilde{A}}^-_{-1}(E)|\underline{k}_T\rangle. \tag{12}$$

- Then we get,

$$\underline{\epsilon}_{(\hat{E})i}\, \Pi^{\underline{i}}_{\mu}(\hat{E}) = \underline{\epsilon}_{(E)i}\, \Pi^{\underline{i}}_{\mu}(E) + \zeta \left(E^{\underline{-}}_{\mu} + \frac{-2\underline{k}^-_{T+1}\underline{k}^+ + 2\underline{\vec{k}}^2}{2(\underline{k}^+)^2} E^{\underline{+}}_{\mu} - \frac{\underline{k}^j}{\underline{k}^+} E^{\underline{j}}_{\mu} \right), \tag{13}$$

where, $\Pi^{\underline{i}}_{\mu} = E^{\underline{i}}_{\mu} - \frac{\underline{p}_0^i}{\underline{p}_0^+} E^{\underline{+}}_{\mu}$; $\underline{k}^-_{T+N} = \underline{k}^-_T + \frac{N}{2\alpha' \underline{k}^+}$.

- $D - 1$ equations in $D - 1$ unknowns! Therefore,

$$\zeta = \underline{\epsilon}_{(\hat{E})i}\, \Pi^{\underline{i}}_{\mu}(\hat{E})\, E^{\mu}_{\underline{-}}, \quad \underline{\epsilon}_{(E)i} = \underline{\epsilon}_{(\hat{E})j}\, \Pi^{\underline{j}}_{\mu}(\hat{E})\, E^{\mu}_{\underline{i}} - \frac{\underline{k}_i}{\underline{k}^+}\, \underline{\epsilon}_{(\hat{E})j}\, \Pi^{\underline{j}}_{\mu}(\hat{E})\, E^{\mu}_{\underline{-}}. \tag{14}$$

Amplitudes using *(F)DDF*

Chaotic DDF scattering amplitudes

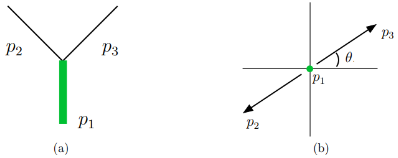

Figure: Fig. 7 of [1]: (a) 3-pt scattering process with 1 HES. (b) Kinematics in the scattering plane.

- It was found in [1] and subsequently studied in [2, 3, 4, 5], that the 3-point scattering amplitude involving a Highly Excited String (HES) DDF state exhibits 'chaos'.
-

$$\mathcal{A} \propto \prod_m [\sin(\pi m p_3 \cdot q)]^{n_m}; \quad \sum_m n_m = N \tag{15}$$

Plots of *chaotic* 3-point amplitudes

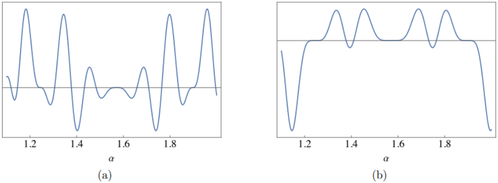

Figure: Fig. 9 of [1]. (a) $N = 50$ state with $n_{11} = n_7 = n_5 = 1, n_4 = 4, n_3 = 3, n_1 = 2$. (b) Slightly different $N = 50$ state with $n_{12} = n_6 = 1$ and $n_{11} = n_7 = 0$, and other occupation numbers unaffected.

- A detector moving in the θ-direction will detect erratic peaks and zeros of scattered tachyons (in this case).
- The authors in [4, 5] proposed new measures for the 'chaotic' behavior by studying the spacing of the amplitude peaks.

? What could be the possible reasons for this behavior?

3-point scattering involving massive $N = 6$ scalar

(In collaboration with Raffaele Marotta)

- We suspect that the 'chaos' is due to the 'spin content' of the HES involved in the scattering.
- To see this in an example, we consider the $N = 6$ transverse scalar formed using FDDF operators as,

$$|S_6\rangle = \Big[c_1(5,1) + c_2(4,2) + c_3(3,3) + c_4(1,1)(3,1) + c_5(1,1)(2,2) + c_6(2,1)^2 + c_7(1,1)^3\Big]\,|k_T\rangle\,, \tag{16}$$

  where, $(m,n) = \underline{A}^j_{\ -m} \cdot \underline{A}^j_{\ -n}$.
- For generic c_i and spacetime dim. D, $|S_6\rangle$ transforms as a tensor under $SO(D-1)$. Only in $D = 26$ and for the particular choice (upto rescaling) $(c_1, c_2, c_3, c_4, c_5, c_6, c_7) = (-2/7, 4, -10/3, 1, -9/14, -2/7, -5/84)$, $|S_6\rangle$ is a scalar!

Amplitude computation

- We define the coherent open-string state as

$$|\{\lambda_n^i\}, \underline{k}_T\rangle \equiv e^{\sum_{i=2}^{D-2}\sum_{n=1}^{\infty}\lambda_n^i \underline{A}_{-n}^i}|\underline{k}_T\rangle. \tag{17}$$

- We use the Sciuto-Della Selva-Saito (SDS) to get the corresponding vertex operator and insert it in an M-point correlator to get,

$$\langle S_1..S_M\rangle = \exp\left[\sum_{t=1}^{M}\sum_{i}\sum_{n\in\mathbb{N}}\lambda_{[t]n}^i I_{t,n} + \frac{1}{2}\sum_{n,m\in\mathbb{N}}\sum_{i}\sum_{r,t}\lambda_{[r]n}^i\lambda_{[t]m}^i J_{r,n;t,m}\right], \tag{18}$$

where, $I_{t,n}$ and $J_{r,n;t,m}$ are complex integrals that have been computed explicitly (at least for $M=3$).

- For $M=3$ on-shell amplitudes, there is no integral over the vertex operator insertions x_i. We just sum over the other Chan-Paton ordering to get the amplitude.

Plot of the amplitude

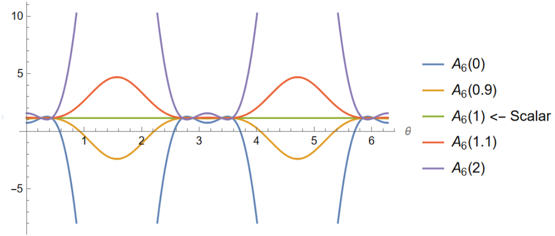

Figure: Plot of the amplitude with c_4 taking 5 values. The sensitivity of the amplitude zeros and peaks to the c_i is high.

- On-shell, the scattering amplitude of 3 scalars, $|S_6\rangle$ and two tachyons is a number (no angular dependence) as shown by the green line in the figure.
- Shifting a single parameter by a small amount turns $|S_6\rangle$ into a large 'mix of spins'.
- The amplitude develops peaks as the 'spin content' changes! For large N, $|S_N\rangle$ has very complicated numerical coefficients involved.

An example: two scalars at $N = 8, 10$

$$
\begin{aligned}
|S_8\rangle =& 36828\ (1,1)^2\ (2,2) + 41888\ (1,1)\ (2,1)^2 + 2745\ (1,1)^4 \\
&+19472\ (3,1)^2 + 66000\ (2,2)\ (3,1) - 70608\ (1,1)^2\ (3,1) \\
&-187264\ (2,1)\ (4,1) + 440192\ (1,1)\ (3,3) + 78848\ (2,1)\ (3,2) \\
&+111808\ (1,1)\ (5,1) + 1392160\ (4,4) - 564256\ (1,1)\ (4,2)
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
|S_{10}\rangle =& 593415\ (1,1)^3\ (2,2) + 868520\ (1,1)^2\ (2,1)^2 + 33291\ (1,1)^5 \\
&- 1157970\ (1,1)^3\ (3,1) + 883080\ (1,1)\ (2,2)^2 + 2622680\ (2,1)^2\ (2,2) \\
&+1174080\ (1,1)\ (2,1)\ (3,2) + 3107520\ (1,1)\ (3,1)^2 - 1771280\ (2,1)^2\ (3,1) \\
&+4587240\ (2,2)\ (3,3) + 9687360\ (1,1)^2\ (3,3) + 809280\ (3,2)^2 \\
&+10403520\ (3,2)\ (4,1) - 5470400\ (1,1)\ (2,1)\ (4,1) - 10663920\ (3,1)\ (3,3) \\
&- 3264520\ (2,2)\ (4,2) - 12196000\ (1,1)^2\ (4,2) - 5282480\ (4,1)^2 \\
&+59537420\ (1,1)\ (4,4) + 21296960\ (2,1)\ (4,3) + 11254000\ (3,1)\ (4,2) \\
&- 451840\ (3,1)\ (5,1) - 7603520\ (2,2)\ (5,1) + 2070100\ (1,1)^2\ (5,1) \\
&+158253184\ (5,5) - 74087040\ (1,1)\ (5,3) - 30050720\ (2,1)\ (5,2) \\
&- 176529120\ (6,4) + 13053600\ (1,1)\ (6,2) + 7112640\ (2,1)\ (6,1) \\
&+3818560\ (8,2) + 9447520\ (7,3) - 211840\ (1,1)\ (7,1)
\end{aligned} \tag{20}
$$

An example: $|S_{10}\rangle$ coefficients contain large primes!

$$
\begin{aligned}
|S_{10}\rangle = {} & 3^2 * 5 * 13187\ (1,1)^3\ (2,2) + 2^3 * 5 * 21713\ (1,1)^2\ (2,1)^2 + 3^5 * 137\ (1,1)^5 \\
& -2 * 3 * 5 * 11^3 * 29\ (1,1)^3\ (3,1) + 2^3 * 3^2 * 5 * 11 * 223\ (1,1)\ (2,2)^2 + 2^3 * 5 * 173 * 379\ (2,1)^2\ (2,2) \\
& +2^6 * 3 * 5 * 1223\ (1,1)\ (2,1)\ (3,2) + 2^6 * 3^2 * 5 * 13 * 83\ (1,1)\ (3,1)^2 - 2^4 * 5 * 7 * 3163\ (2,1)^2\ (3,1 \\
& +2^3 * 3 * 5 * 7 * 43 * 127\ (2,2)\ (3,3) + 2^6 * 3 * 5 * 10091\ (1,1)^2\ (3,3) + 2^6 * 3^2 * 5 * 281\ (3,2)^2 \\
& +2^6 * 3 * 5 * 10837\ (3,2)\ (4,1) - 2^6 * 5^2 * 13 * 263\ (1,1)\ (2,1)\ (4,1) - 2^4 * 3^3 * 5 * 4937\ (3,1)\ (3,3) \\
& -2^3 * 5 * 7 * 89 * 131\ (2,2)\ (4,2) - 2^5 * 5^3 * 3049\ (1,1)^2\ (4,2) - 2^4 * 5 * 7 * 9433\ (4,1)^2 \\
& +2^2 * 5 * 149 * 19979\ (1,1)\ (4,4) + 2^6 * 5 * 66553\ (2,1)\ (4,3) + 2^4 * 5^3 * 17 * 331\ (3,1)\ (4,2) \\
& -2^8 * 5 * 353\ (3,1)\ (5,1) - 2^6 * 5 * 23761\ (2,2)\ (5,1) + 2^2 * 5^2 * 127 * 163\ (1,1)^2\ (5,1) \\
& +2^7 * 179 * 6907\ (5,5) - 2^7 * 3 * 5 * 47 * 821\ (1,1)\ (5,3) - 2^5 * 5 * 7^2 * 3833\ (2,1)\ (5,2) \\
& -2^5 * 3 * 5 * 61 * 6029\ (6,4) + 2^5 * 3^2 * 5^2 * 7^2 * 37\ (1,1)\ (6,2) + 2^6 * 3 * 5 * 31 * 239\ (2,1)\ (6,1) \\
& +2^6 * 5 * 11933\ (8,2) + 2^5 * 5 * 137 * 431\ (7,3) - 2^7 * 5 * 331\ (1,1)\ (7,1)
\end{aligned}
\tag{21}
$$

Ongoing and future work

- Given that FDDF are good conformal operators, one can explicitly compute minimally off-shell amplitudes (ongoing work).
- It would be interesting to study the on-shell 'chaotic' behaviour of the amplitudes for HES and its quantitative dependence on the 'spin content' of the DDF states involved (ongoing+future).
- A saddle point analysis of the correlator integrals would shed light on higher-order correlator structures as the explicit integrals are complicated (ongoing).
- We also hope to soon extend the FDDF formalism to include superstrings (future).
- Relation between massive string scattering amplitudes and those involving black holes using FDDF, possibly along the lines of Amati and Russo [6], and related works (future).

References

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[4] Massimo Bianchi, Maurizio Firrotta, Jacob Sonnenschein, and Dorin Weissman. Measure for Chaotic Scattering Amplitudes. *Phys. Rev. Lett.*, 129(26):261601, 2022.

[5] Massimo Bianchi, Maurizio Firrotta, Jacob Sonnenschein, and Dorin Weissman. Measuring chaos in string scattering processes. *Phys. Rev. D*, 108(6):066006, 2023.

[6] D. Amati and J.G. Russo. Fundamental strings as black bodies. *Physics Letters B*, 454(3):207–212, 1999.

Thank you!

More about FDDF amplitudes $(M = 3)$

Single integral:

$$I_{t,n} = (-1)^{n-1} \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (n\rho_{t+1\ t} + k) \times \left(\frac{x_{t;t+1}}{x_{t;t-1}}\right)^{-n\rho_{t+1\ t}} \left(\frac{x_{t;t+1}}{x_{t-1;t+1}}\right)^{-n}, \qquad (22)$$

Double Integral $(t = r)$:

$$J_{t,m;r,n} = \frac{mn}{m+n} \rho_{t+1\ t}(\rho_{t+1\ t} + 1) I_{t,m} I_{r,n}. \qquad (23)$$

Double Integral $(t \neq r)$:

$$J_{1\ n;2\ m} = (-1)^{n+1}\, x_{13}^{-n\rho_{31}}\, x_{23}^{-m\rho_{32}}\, x_{12}^{-m(1+\rho_{21})-m(1+\rho_{12})} \left(\frac{x_{13}}{x_{23}}\right)^{-n+m} \sum_{k=1}^{T} k \binom{-n\rho_{31}}{n-k} \binom{-m\rho_{32}}{m-k},$$

$$J_{1\ n;3\ m} = (-1)^{m+1}\, x_{12}^{-n\rho_{21}}\, x_{23}^{-m\rho_{23}}\, x_{13}^{-m(1+\rho_{31})-m(1+\rho_{13})} \left(\frac{x_{12}}{x_{23}}\right)^{-n+m} \sum_{k=1}^{T} k \binom{-n\rho_{21}}{n-k} \binom{-m\rho_{23}}{m-k},$$

$$J_{2\ n;3\ m} = (-1)^{n+1}\, x_{12}^{-n\rho_{12}}\, x_{13}^{-m\rho_{13}}\, x_{23}^{-m(1+\rho_{32})-m(1+\rho_{23})} \left(\frac{x_{12}}{x_{13}}\right)^{-n+m} \sum_{k=1}^{T} k \binom{-n\rho_{12}}{n-k} \binom{-m\rho_{13}}{m-k},$$

where, $\rho_{t;r} = \underline{k}^{+}_{[t]} / \underline{k}^{+}_{[r]}$ and $T = \min(m, n)$.

$(N = 2)$ solution of Virasoro Constraints

At $N = 2$ level, there are two DDF states: $\underline{A}^i_{-2}\,|\underline{k}_T\rangle$ and $\underline{A}^i_{-1}\underline{A}^j_{-1}\,|\underline{k}_T\rangle$.

1st case:

$$\underline{A}^i_{-2}|\underline{k}_T\rangle = \left(T^{(i)}_\mu\,\alpha^\mu_{-2} + S^{(i)}_{\mu\nu}\,\alpha^\mu_{-1}\alpha^\nu_{-1}\right)|k_\nu\rangle, \tag{24}$$

$$\Longrightarrow \boxed{T^{(i)}_\mu = \Pi^i_\mu(E);\;\; S^{(i)}_{\mu\nu} = \Pi^i_{(\mu}\,E^+_{\nu)}\frac{2}{\sqrt{2\alpha'}\underline{k}^+_T}}, \tag{25}$$

$T^{(i)}_\mu$ is transverse, but $S^{(i)}_{\mu\nu}$ is not!

2nd case:

$$\underline{A}^i_{-1}\underline{A}^j_{-1}\,|k_T\rangle = \left(T^{(ij)}_\mu\,\alpha^\mu_{-2} + S^{(ij)}_{\mu\nu}\,\alpha^\mu_{-1}\alpha^\nu_{-1}\right)|k_\nu\rangle, \tag{26}$$

$$\Longrightarrow \boxed{T^{(ij)}_\mu = \frac{\delta^{ij}}{2\sqrt{2\alpha'}\underline{k}_{T-}}E^+_\mu;\quad S^{(ij)}_{\mu\nu} = \Pi^i_{(\mu}\,\Pi^j_{\nu)} + \frac{\delta^{ij}}{2(\sqrt{2\alpha'}\underline{k}^+_T)^2}E^+_\mu\,E^+_\nu.} \tag{27}$$

For $i \neq j$ they are TT. But for $i = j$, not TT!

An algebra homomorphism between lightcone and covariant string operators

- For the operator algebras $X = \{\alpha^i_{n(lc)}\}$ and $Y = \{\underline{A}^i_n(E), \underline{\tilde{A}}^-_n(E)\}$ there are infinitely many *injective homomorphisms* $i(E) : X \to Y$ such that,

$$\alpha^i_{n(lc)} \overset{i(E)}{\to} \underline{A}^i_n(E) \tag{28}$$

- To be comprehensive, $i(E)$ is inversely related to an *isomorphism* $f(E) : Y/ker(f) \to X$ such that,

$$\begin{aligned} [\underline{A}^i_n(E)] &\overset{f(E)}{\to} \alpha^i_{n(lc)} \\ [\underline{\tilde{A}}^-_n(E)] &\overset{f(E)}{\to} 0 \end{aligned} \tag{29}$$

- This can be extended to include the zero modes - but they depend on the states of the Fock Space. Hence we have a vector space homomorphism.