Rotating Metrics from Scattering Amplitudes in Arbitrary Dimensions

Meeting PRIN "String Theory as a bridge between Gauge Theories and Quantum Gravity"

Claudio Gambino, University of Rome "La Sapienza"

Based on [CG, P. Pani, F. Riccioni, 2403.XXXXX]

February 23th, 2024

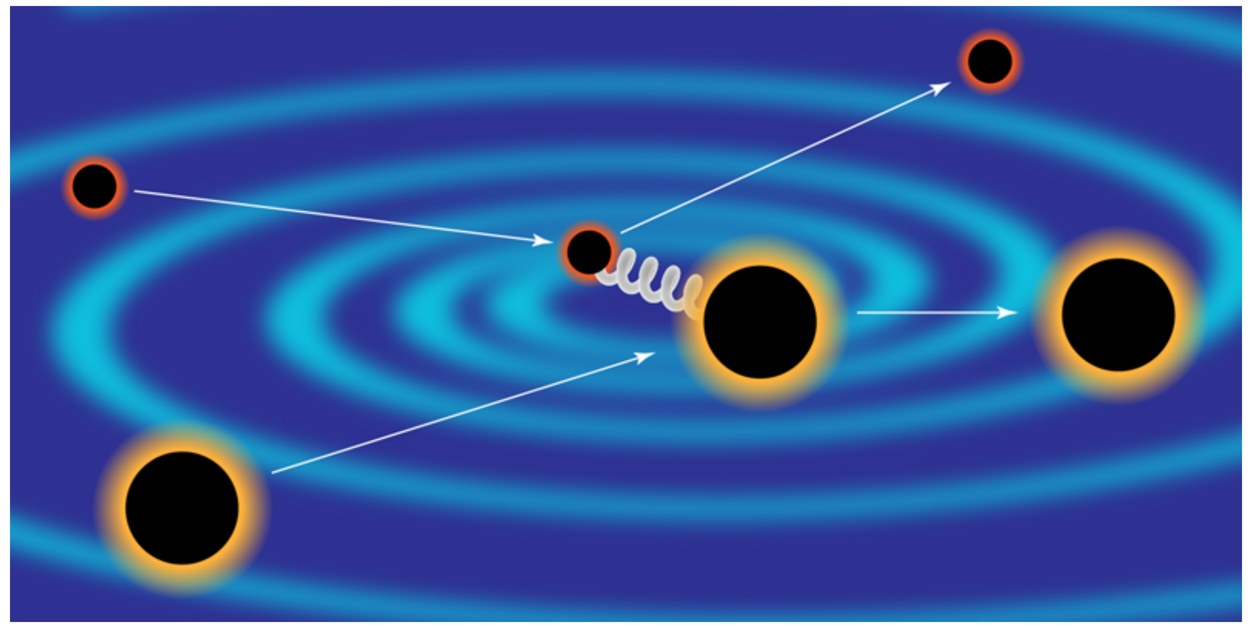
SAPIENZA
UNIVERSITÀ DI ROMA

INFN
Istituto Nazionale di Fisica Nucleare

Introduction

[Donoghue, 2211.09902]
[Bjerrum-Bohr, Planté, Vanhove, 2212.08957]

Nowadays we accept the idea that **GR is the classical limit of an effective quantum theory of gravity**, in which the Einstein-Hilbert action is the leading order in a higher-derivative expansion.

$\lim_{\hbar \to 0}$  $= \textit{Gravity}$

It has been proven that a static metric in arbitrary dimensions is recovered from 3-point amplitudes of massive scalars emitting gravitons.

We extend this program in the case of **spinning geometries at quadrupole order**:

- **No Birkhoff theorem for stationary objects.**
- **No black-hole uniqueness in $D > 4$.**

[Donoghue, gr-qc/9310024]
[Mougiakakos, Vanhove, 2010.08882]

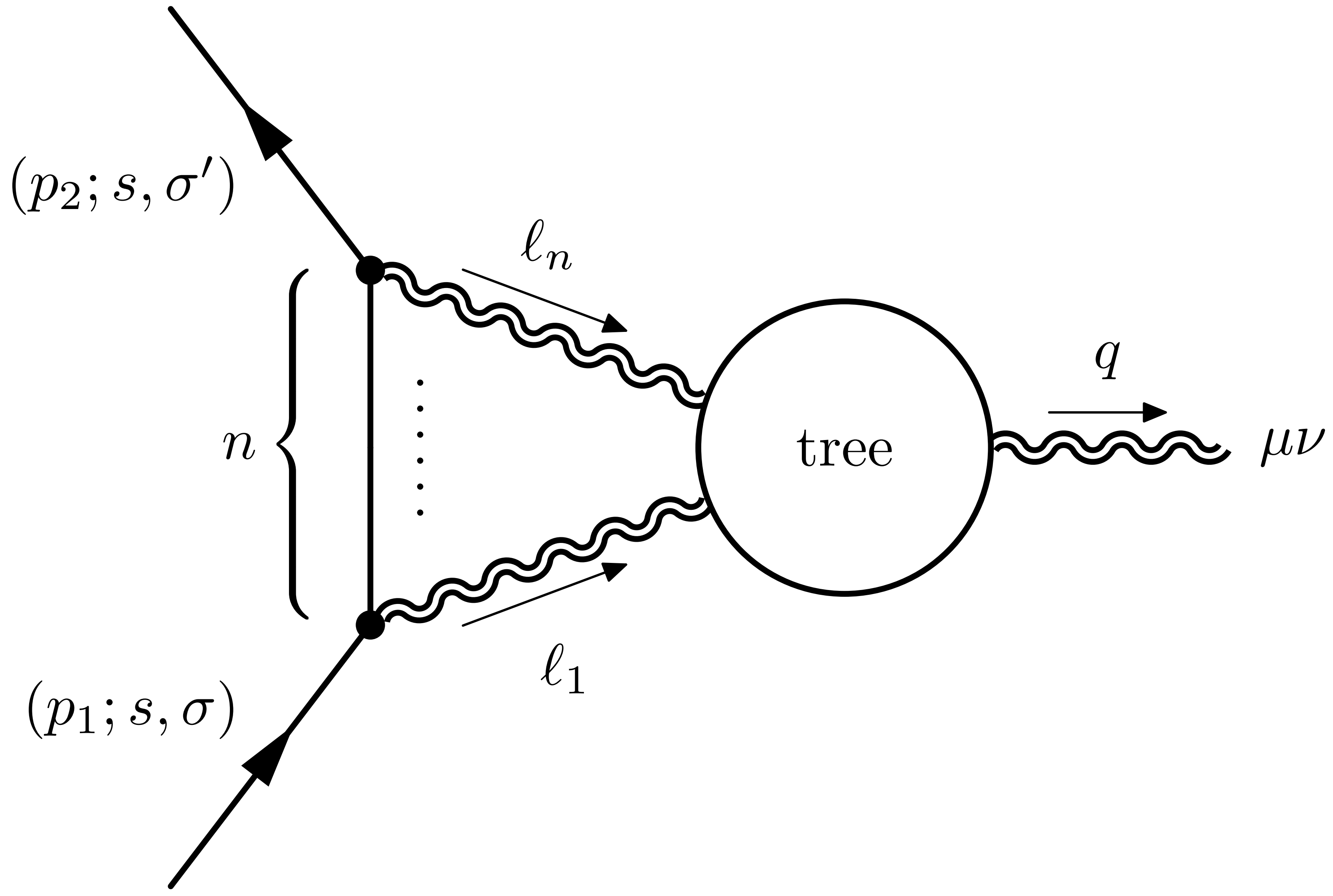
Metrics from scattering amplitudes

Consider a spin-s field coupled to gravity $$S = \int d^{d+1}x \left(-\frac{2}{\kappa^2}\sqrt{-g}R + \mathscr{L}_m(\Phi_s, g_{\mu\nu}) \right)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} = \eta_{\mu\nu} + \kappa \sum_{n=1}^{+\infty} h_{\mu\nu}^{(n)}$$

graviton — nPM order

$$\rightarrow \quad h_{\mu\nu}^{(n)}(x) = -\frac{\kappa}{2}\int \frac{d^d\vec{q}}{(2\pi)^d}\frac{e^{i\vec{q}\cdot\vec{x}}}{\vec{q}^{\,2}}\left(T_{\mu\nu}^{(n-1)}(q) - \frac{1}{d-1}\eta_{\mu\nu}T^{(n-1)}(q)\right)$$



$$= -\frac{i\kappa}{2}T_{\mu\nu}^{(n-1)}(q)\delta_{\sigma\sigma'}$$

We want to compute $T_{\mu\nu}^{(n)}(q)$ through scattering amplitudes

[Bern, Luna, Roiban, Shen, Zeng, 2005.03071]

Dressed vertex

In the **stationary** and **classical limit** it is verified that $|p_1\rangle = |p_2\rangle + O(\hbar)$

Leading to the definition of the spin tensor $S^{\mu\nu}$ as the classical limit of the Lorentz generators $M^{\mu\nu}$

normalization coefficient

$$\langle p_2; s, \sigma' | M^{\mu\nu} | p_1; s, \sigma\rangle = S^{\mu\nu}\langle p_1; s, \sigma' | p_1; s, \sigma\rangle + O(\hbar) = S^{\mu\nu}C(s)\delta_{\sigma\sigma'} + O(\hbar^0)$$

$$\langle p_2; s, \sigma' | (\tau_{\Phi^2 h})^{\mu\nu} | p_1; s, \sigma\rangle = \langle p_1; s, \sigma' | (\tau_{\Phi^2 h})^{\mu\nu} | p_1; s, \sigma\rangle + O(\hbar) = \hat{\tau}^{\mu\nu}_{\Phi^2 h}(q, S)\delta_{\sigma\sigma'} + O(\hbar)$$

In $D > 4$ the angular momentum is an anti-symmetric rank-2 tensor

Spin-1

$$\mathbb{S}^{\mu\nu}_{a,b} = S^{\mu\nu}\delta_{ab} + O(\kappa)$$

$$S = \int d^Dx\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 V_\mu V^\mu + K_1\, R\, V^\alpha \left(\mathbb{S}^{\mu\nu}\mathbb{S}_{\mu\nu}\right)_{\alpha\beta} V^\beta + K_2\, R_{\mu\nu}\, V^\alpha \left(\mathbb{S}^{\mu\lambda}\mathbb{S}_\lambda{}^\nu\right)_{\alpha\beta} V^\beta + \cdots \right)$$

$$\hat{\tau}^{\mu\nu}_{V^2h}(q,S) = -\frac{i\kappa}{2}\Big(2\, m^2\, \delta^\mu_0\delta^\nu_0 - i\, m\, q_\lambda\left(S^{\mu\lambda}\delta^\nu_0 + S^{\nu\lambda}\delta^\mu_0\right) - H_1\, q_\rho q_\sigma S^{\mu\rho}S^{\nu\sigma} + H_2\, \delta^\mu_0\delta^\nu_0 q_\rho q_\sigma S^{\rho\lambda}S_\lambda{}^\sigma$$

$$+ C_1\, S^{\rho\sigma}S_{\rho\sigma}q^\mu q^\nu + C_2\left(\eta^{\mu\nu}q_\rho q_\sigma S^{\rho\lambda}S^\sigma{}_\lambda - q^\lambda\left(q^\mu S_{\lambda\sigma}S^{\nu\sigma} + q^\nu S_{\lambda\sigma}S^{\mu\sigma}\right)\right)\Big)$$

$K_i = 0$ → **Minimal Limit** → ***The "simplest" metric***

Minimal Limit

$H_1 = 1 \quad H_2 = 0$

$C_1 = 0 \quad C_2 = 0$

The "simplest" metric

(metric associated to a minimally coupled field)

[Bjerrum-Bohr, Donoghue, Holstein, hep-th/0211071]

Expanding the metric in a multipole series, we get

$$h_{\mu\nu}^{(n)} = \sum_{j=0}^{2s} h_{\mu\nu}^{(n,j)} = h_{\mu\nu}^{(n,\text{monopole})} + h_{\mu\nu}^{(n,\text{dipole})} + h_{\mu\nu}^{(n,\text{quadrupole})} + \cdots$$

Spin-1  Quadrupole

The monopole and the dipole of any metric in asymptotically cartesian coordinates are unique!

$$h_{00}^{(1,0)}(r) = -\frac{4(d-2)}{d-1} Gm\,\rho(r)$$

$$h_{0i}^{(1,0)}(r) = 0$$

$$h_{ij}^{(1,0)}(r) = -\frac{4\delta_{ij}}{d-1} Gm\,\rho(r)$$

&

$$h_{00}^{(1,1)}(r) = 0$$

$$h_{0i}^{(1,1)}(r) = -\frac{2(d-2)x^k S_k^i}{r^2} G\,\rho(r)$$

$$h_{ij}^{(1,1)}(r) = 0$$

Where $\rho(r) = \dfrac{\Gamma(\frac{d}{2}-1)\pi^{1-d/2}}{r^{d-2}}$

The first non-trivial multipole order to look for to resolve the structure of different matter configurations is the *quadrupole*

$$h^{(1,2)}_{00}(r) = \frac{2(d-2)\Big(H_2(d-2)+H_1\Big)}{d-1}\,\frac{r^2 S_{k_1k_2}S^{k_1k_2} - d\,x^{k_1}x^{k_2}S_{k_1}{}^{k_3}S_{k_2k_3}}{mr^4}G\rho(r)$$

$$h^{(1,2)}_{0i}(r) = 0$$

$$h^{(1,2)}_{ij}(r) = -\frac{2(d-2)}{(d-1)mr^4}\Bigg(-C_1(d-1)d\,x_ix_jS_{k_1k_2}S^{k_1k_2} - r^2(d-1)\Big(2C_2+H_1\Big)S_{ik}S_j^{\;k}$$
$$+r^2\Big(C_1(d-1)+H_1-H_2\Big)S_{k_1k_2}S^{k_1k_2}\delta_{ij} + d\,C_2(d-1)x^{k_1}S_{k_1k_2}\Big(x_jS_i^{\;k_2}+x_iS_j^{\;k_2}\Big)$$
$$+d\,x^{k_1}x^{k_2}\Big((d-1)H_1S_{ik_1}S_{jk_2}+(H_2-H_1)S_{k_1}{}^{k_3}S_{k_2k_3}\delta_{ij}\Big)\Bigg)G\rho(r)$$

This metric depends on four different arbitrary parameters. Are they all physical?

Spoiler, no!

Eliminating gauge parameters

Consider an infinitesimal coordinate transformation as $x' = x + \xi(x)$ such that the metric transforms like $h'_{\mu\nu} = h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi \mu)$.

By definition in the harmonic gauge $\Box x^\mu = 0$, we can define a coordinate transformation inside the gauge if

$$\Box x'^\mu = 0 \quad \rightarrow \quad \Box \xi^\mu = 0$$

harmonic function in arbitrary dimensions

$$\Box \rho(r) = 0 \longrightarrow \xi^i = \frac{G}{m}\left(2C_2\, S^{ik} S_k{}^j + C_1\, S^{lm} S_{lm} \delta^{ij}\right)\partial_j \rho(r) \quad \& \quad \xi^0 = 0$$

With this coordinate transformation we can cancel the C_i's from the metric $\longrightarrow$ **H_1 and H_2 are physical parameters**

Rotating metric at quadrupole order

$$h_{00}^{(1,2)} = \frac{2(d-2)\Big(H_2(d-2)+H_1\Big)}{d-1}\,\frac{r^2 S_{k_1k_2}S^{k_1k_2} - d\,x^{k_1}x^{k_2}S_{k_1}{}^{k_3}S_{k_2k_3}}{mr^4}G\rho(r)$$

$$h_{ij}^{(1,2)} = -\frac{2(d-2)}{(d-1)mr^4}\Bigg(-r^2(d-1)H_1S_{ik}S_j^{\,k} + r^2\Big(H_1-H_2\Big)S_{k_1k_2}S^{k_1k_2}\delta_{ij}$$

$$+d\,x^{k_1}x^{k_2}\Big((d-1)H_1S_{ik_1}S_{jk_2} + (H_2-H_1)S_{k_1}{}^{k_3}S_{k_2k_3}\delta_{ij}\Big)\Bigg)G\rho(r)$$

We are observing for the first time that in GR there exist two independent quadrupole moments! But we can say more…

Multipole expansion in arbitrary dimensions

We conjecture the existence of a new "gauge invariant" multipole tensor never observed before

$$g_{00} = 1 - \frac{4(d-2)}{d-1} Gm\,\rho(r) + \frac{2(d-2)}{d-1} \sum_{\ell=2}^{+\infty} \frac{Gm\,\rho(r)}{r^{\ell}} \left(\mathbb{M}^{(\ell)}_{A_\ell} N_{A_\ell} + \cdots \right)$$

$$g_{0i} = -2(d-2) \frac{Gm\,\rho(r)}{r} \left(\frac{1}{m} S^{ik} \frac{x_k}{r} \right) - 2(d-2) \sum_{\ell=3}^{+\infty} \frac{Gm\,\rho(r)}{r^{\ell}} \left(\mathbb{J}^{(\ell)}_{i,A_\ell} N_{A_\ell} + \cdots \right)$$

$$g_{ij} = -\delta_{ij} - \frac{4}{d-1} \delta_{ij} Gm\,\rho(r) + \frac{2(d-2)}{d-1} \sum_{\ell=2}^{+\infty} \frac{Gm\,\rho(r)}{r^{\ell}} \left(\mathbb{G}^{(\ell)}_{ij,A_\ell} N_{A_\ell} + \cdots \right)$$

Anything which is gauge dependent

$$N_{A_\ell} = \frac{x_{a_1} \cdots x_{a_\ell}}{r^{\ell}}$$

$$\mathbb{M}^{(2)}_{a_1 a_2} = -\frac{d}{m^2} \Big(H_2(d-2) + H_1 \Big) S_{a_1}{}^{k} S_{a_2 k}$$

$$\mathbb{G}^{(2)}_{ij,a_1 a_2} = -\frac{d}{m^2} \Big((d-1) H_1 S_{i a_1} S_{j a_2} + (H_2 - H_1) S_{a_1}{}^{k} S_{a_2 k} \delta_{ij} \Big)$$

$D = 4$ is a special case since we can rewrite $S^{ij} = \epsilon^{ijk} S_k$

$$\mathbb{M}^{(2)}_{a_1 a_2} N_{a_1 a_2} \Big|_{d=3} = 3(H_1 + H_2)(S \cdot x)^2 + \cdots \qquad \mathbb{G}^{(2)}_{ij, a_1 a_2} N_{a_1 a_2} \Big|_{d=3} = \delta_{ij} \, \mathbb{M}^{(2)}_{a_1 a_2} N_{a_1 a_2} \Big|_{d=3} + \cdots$$

In $D > 4$ there are two independent quadrupole moments

but

In $D = 4$ there is only one quadrupole moment

We can extend this result at every multipole order, and saying that ***in GR there are three different kinds of multipoles:***

Mass multipoles, Current multipoles… and new ones, Space multipoles(?)

[Hartle, Thorne, Astrophys.J. 153 (1968) 807]

Matching the Hartle-Thorne metric in $D = 4$

To test our formalism we match the amplitude-based metric with the one associated with the most generic rotating solution at quadrupole order (HT) in *harmonic gauge*.

$$S_{ij} = \begin{pmatrix} 0 & J & 0 \\ -J & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad a = J/m$$

$$g_{tt}^{(HT)} = -1 + \frac{2GM}{r} - \frac{a^2GM\,k\,(3\cos(2\theta)+1)}{2r^3} + \cdots$$

$$g_{t\phi}^{(HT)} = -\frac{2aGM\sin^2(\theta)}{r} + \cdots$$

$$g_{rr}^{(HT)} = 1 + \frac{2GM}{r} - \frac{(a^2GM\,k\,(3\cos(2\theta)+1))}{2r^3} + \frac{4G^2M^2}{r^2} + \cdots$$

$$g_{\theta\theta}^{(HT)} = r^2 - \frac{a^2GM\,k\,(3\cos(2\theta)+1)}{2r} + \cdots$$

$$g_{\phi\phi}^{(HT)} = r^2\sin^2(\theta) - \frac{\left(a^2GM\,k\,(3\cos(2\theta)+1)\sin^2(\theta)\right)}{2r} + \cdots$$

This metric has only one quadrupole moment, parametrized by k, and we are able to reproduce it by fixing

$$H_1 + H_2 = k$$

We do not need to fix H_1 and H_2 independently!

Kerr limit of the Hartle-Thorne metric

For a specific value of the quadrupole we can recover the Kerr metric.

Kerr Limit

$k = 1$

corresponds to → $H_1 = 1 - H_2$

The Kerr metric, ***the only black hole solution in*** $D = 4$, is reproduced by an infinite number of non-minimal actions and by the **minimally coupled theory**.

Minimal limit → $(H_1 = 1) + (H_2 = 0) = 1$

"Simplest" metric in $D = 4$ = **Kerr black hole**

[Myers, Perry, Annals Phys. 172 (1986) 304]

Myers-Perry black holes in $D = 5$

Myers-Perry solutions are a class of black holes defined in arbitrary dimensions constructed in such a way that the limit $D = 4$ corresponds to Kerr.

$$ds^2 = -dt^2 + \frac{\mu}{\Sigma}\left(dt + a\,\sin^2\theta\,d\phi_1 + b\,\cos^2\theta\,d\phi_2\right)^2 + \frac{r^2\Sigma}{\Pi - \mu r^2}dr^2$$
$$+\Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta\,d\phi_1^2 + (r^2 + b^2)\cos^2\theta\,d\phi_2^2$$

$$\Sigma = r^2 + a^2\cos^2\theta + b^2\sin^2\theta$$
$$\Pi = (r^2 + a^2)(r^2 + b^2)$$

The solution now has two independent angular momenta since the group of the rotation $SO(4)$ has two Casimir

$$\frac{1}{m}S_{ij} = \frac{2}{3}\begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}$$

We need to fix H_1 and H_2 independently!

$$H_1 = \frac{3}{8} \quad H_2 = \frac{15}{16}$$

The Myers-Perry solution is not the "simplest" one

Simplest metric in higher dimension

If the simplest metric in $D = 4$ is the Kerr black hole, to what it does correspond in higher dimensions?

The simplest metric in arbitrary dimension

$$h_{00}^{(1,2)} = \frac{2(d-2)}{d-1}\,\frac{r^2 S_{k_1k_2}S^{k_1k_2} - d\,x^{k_1}x^{k_2}S_{k_1}{}^{k_3}S_{k_2k_3}}{mr^4}\,G\rho(r)$$

$$h_{ij}^{(1,2)} = -\frac{2(d-2)}{(d-1)mr^4}\Bigg(-r^2(d-1)S_{ik}S_j{}^{k} + r^2 S_{k_1k_2}S^{k_1k_2}\delta_{ij}$$

$$+d\,x^{k_1}x^{k_2}\Big((d-1)S_{ik_1}S_{jk_2} - S_{k_1}{}^{k_3}S_{k_2k_3}\delta_{ij}\Big)\Bigg)G\rho(r)$$

Take home message

In literature is known that in GR in $D = 4$ there are ***two*** class of multipoles. We proved that in $D > 4$ there are ***three*** of them.

A Kerr black hole is the simplest metric in $D = 4$. What is the simplest metric in $D > 4$? Is it a fundamental concept?

Thank you!