Quasinormal modes of four-dimensional Schwarzschild (anti-)de Sitter black holes

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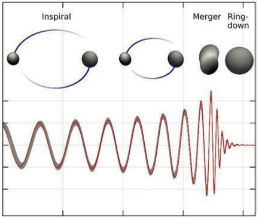
Introduction

- The recent experimental verification of gravitational waves renewed the interest in theoretical studies of General Relativity and black hole perturbation theory.
- In particular, we look for exact computational techniques to produce high-precision tests of General Relativity equations by computing analytical expressions for significant gravitational quantities.

The QNM frequencies

- The QNMs are quantized frequencies that can be seen as characteristic oscillations of black holes, and are responsible for the damped oscillations appearing, for example, in the ringdown phase of two colliding black holes.



- Mathematically, the QNMs arise in the analysis of a linear perturbation around fixed gravitational backgrounds. The perturbation usually obeys linear 2nd order differential equations with singularities, whose symmetry properties are dictated by the symmetries of the background. The quasinormal modes are obtained by imposing suitable boundary conditions to the perturbation fields.

Schwarzschild (anti-)de Sitter black holes

The relevant differential equation describing the conformally-coupled scalar perturbation around S(A)dS$_4$ black holes is described by a **Heun equation**

$$\left(\frac{d^2}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-t}\right)\frac{d}{dz} + \frac{\alpha\beta z - q}{z(z-1)(z-t)}\right)\psi(z) = 0, \tag{1}$$
$$\alpha + \beta + 1 = \gamma + \delta + \epsilon.$$

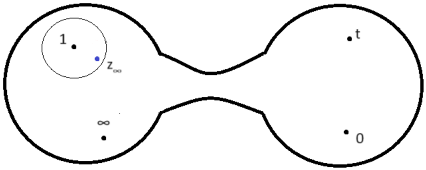
In the asymptotically de Sitter case, taking into account the boundary conditions, the connection coefficient between $\psi_{\rm in}^{\rm hor}$ and $\psi_{\rm in}^{\rm cosm.hor.}$ has to be set equal to zero:

$$\psi_{\rm in}^{\rm hor}(z) = \mathcal{M}_{\rm in,out}\psi_{\rm out}^{\rm cosm.hor.}(z) + \underbrace{\mathcal{M}_{\rm in,in}}_{=0}\psi_{\rm in}^{\rm cosm.hor.}(z). \tag{2}$$

Thanks to **Liouville CFT** (BPZ equation and crossing symmetry) and the **AGT correspondence**, we have concrete expressions for these connection coefficients.

Connection Problem for asymptotically AdS

In the asymptotically anti-de Sitter case, the second boundary condition is imposed at a regular point of the differential equation.



Therefore, the quantization condition involves not only the expressions of the connection coefficients but also the values of the local solutions (e.g. Heun functions) in the regular point:

$$\psi_{\text{in}}^{\text{hor}}(z) = \mathcal{M}_1\, \psi_1^{\text{sing.}}(z) + \mathcal{M}_2\, \psi_2^{\text{sing.}}(z)\Big|_{z=\text{AdS boundary}} = 0. \qquad (3)$$

The multi polylog method I

We divide the radial space in several regions, and expand in each region the differential equation and its wave solution in series in a small parameter α,

$$\psi(z) = f_0(z) + \sum_{K\geq 1} f_K(z)\alpha^K. \tag{4}$$

At each order in α, $\psi(z)$ is determined by a second-order equation

$$(f_K(z))'' + \varphi(z)\,(f_K(z))' + \nu(z) f_K(z) + \eta_K(z) = 0, \tag{5}$$

which we solve by using the method of variation of parameters.

The multi polylog method II

Let f_0, g_0 be the two solutions of the homogeneous part of (5). Then we write the generic solution to (5) as

$$f_K(z) = c_K g_0(z) - g_0(z) \int^z f_0(z') \frac{\eta_K(z')}{W_0(z')} \mathrm{d}z' + f_0(z) \int^z g_0(z') \frac{\eta_K(z')}{W_0(z')} \mathrm{d}z', \quad (6)$$

where W_0 is the Wronskian of the two leading order solutions

$$W_0(z) \equiv f_0(z)(g_0(z))' - (f_0(z))' g_0(z). \quad (7)$$

Imposing the boundary conditions and gluing together the local expansions in different regions, it is possible to fix the integration constants and obtain the analytic expansion of the QNMs.

Multiple Polylogarithms in a single variable

The integrals in (6) are described in terms of multiple polylogarithms

$$\mathrm{Li}_{s_1,\ldots,s_k}(z) = \sum_{n_1>n_2>\cdots>n_k\geq 1}^{\infty} \frac{z^{n_1}}{n_1^{s_1}\ldots n_k^{s_k}}. \tag{8}$$

The latter satisfies the following relation for $s_1 \geq 2$:

$$z\,\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{Li}_{s_1,\ldots,s_k}(z) = \mathrm{Li}_{s_1-1,\ldots,s_k}(z) \tag{9}$$

and the following relation for $s_1 = 1$, $k \geq 2$:

$$(1-z)\frac{\mathrm{d}}{\mathrm{d}\,z}\mathrm{Li}_{1,s_2,\ldots,s_k}(z) = \mathrm{Li}_{s_2,\ldots,s_k}(z). \tag{10}$$

Summary of results I

- In the SdS case, the quantization condition depends only on the connection coefficient, and we found a branch of purely imaginary modes, providing analytical confirmation of the results obtained through previous numerical studies.

$$\omega^{(n,\ell,s)} = \sum_{k=0}^{\infty} \omega_k^{(n,\ell,s)} R_h^k, \quad \omega_k^{(n,\ell,s)} \in \mathrm{i}\,\mathbb{R}_{<0}. \tag{11}$$

- Because of the different boundary condition, the method is less effective in the anti-de Sitter case, therefore we switched to the *multi polylog method*.

$$\omega^{(n,\ell,s)} = \sum_{k=0}^{\infty} \omega_k^{(n,\ell,s)} R_h^k, \quad \mathrm{Im}\left(\omega^{(n,\ell,s)}\right) \sim R_h^{2\ell+2}. \tag{12}$$

Summary of results II

In the SAdS_4 case, we also considered the scalar sector of gravitational perturbation, in the large horizon regime. We computed the so-called low-lying frequencies, which, after taking the double scaling limit

$$R_h, \ell \to \infty, \quad \mathfrak{q} = \frac{2\,\ell}{3\,R_h} \text{ and } \mathfrak{w} = \frac{2\,\omega}{3\,R_h} \text{ finite,} \tag{13}$$

are related to the hydrodynamic sound mode on the thermal 3-dimensional CFT living on the boundary

$$\mathfrak{w} = \sum_{k\geq 1} \mathfrak{w}_k\, \mathfrak{q}^k. \tag{14}$$

We could compute more analytic corrections in the expansion of the frequency w.r.t. the results presently available in the literature, obtaining finite spin predictions for the dual 3d CFT.

Further Directions

- Both methods can be in principle applied also in different BH geometries (including charged and/or rotating BHs).
  We are applying the first method to SAdS_7, where the differential equation has five regular singularities, and the associated gauge theory is a $SU(2) \times SU(2)$ quiver theory.
- We are extending the multi-polylog method to problems involving the presence of irregular singularities.
  For example, in the study of quasinormal modes of Schwarzschild and Kerr black holes, the relevant differential equation is a **confluent Heun equation**. Around an irregular singularity, the expansion of the wave solution involves **multiple polyexponential** functions.

THANK YOU FOR YOUR ATTENTION!

Multiple Polyexponential functions

Let $s_1, \ldots, s_n \in \mathbb{Z}_{\geq 1}$. We define

$$eL_{s_1,\ldots,s_n}(z) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{k_1-1} \cdots \sum_{k_n=1}^{k_{n-1}-1} \frac{1}{k_1^{s_1} \ldots k_n^{s_n}} \frac{z^{k_1}}{k_1!}. \tag{15}$$

If $s_1 > 1$, the following relation holds:

$$z \frac{\mathrm{d}}{\mathrm{d}z} eL_{s_1,\ldots,s_n}(z) = eL_{s_1-1,\ldots,s_n}(z). \tag{16}$$

If $s_1 = 1$, the relation is more involved:

$$z \frac{\mathrm{d}}{\mathrm{d}z} eL_{1,s_2,\ldots,s_n}(z) = -eL_{s_2,\ldots,s_n}(z) - (-1)^n e^z \sum_{j_2=1}^{2^{s_2-1}} \cdots \sum_{j_n=1}^{2^{s_n-1}} eL_{\mathrm{op}(s_2)_{j_2},\ldots,\mathrm{op}(s_n)_{j_n}}(-z), \tag{17}$$

where, given $s \in \mathbb{N}$, we denoted with $\mathrm{op}(s)$ the set of ordered partitions of s. For example,

$$\mathrm{op}(1) = \{1\}, \quad \mathrm{op}(2) = \{2, (1,1)\}, \quad \mathrm{op}(3) = \{3, (2,1), (1,2), (1,1,1)\}.$$

Quantization condition for SdS_4 QNMs

The quantization condition in the SdS_4 case is given by

$$\sum_{\sigma=\pm} \frac{\Gamma(1+2a_t)\Gamma(-2a_1)\Gamma(-2\sigma v)\Gamma(1-2\sigma v)}{\prod_{\pm}\Gamma(1/2-\sigma v+a_t\pm a_0)\prod_{\pm}\Gamma(1/2-\sigma v-a_1\pm a_\infty)} t^{\sigma v} e^{-\frac{\sigma}{2}\partial_v F(t)} = 0, \tag{18}$$

where

$$\begin{aligned} a_0 &= \frac{1-\gamma}{2}, \qquad & a_1 &= \frac{1-\delta}{2}, \\ a_t &= \frac{1-\epsilon}{2}, \qquad & a_\infty &= \frac{\alpha-\beta}{2}, \end{aligned} \tag{19}$$

$$F(t) = \frac{\left(4v^2-4a_0^2+4a_t^2-1\right)\left(4v^2+4a_1^2-4a_\infty^2-1\right)}{8-32v^2} t + \mathcal{O}\left(t^2\right),$$

$$v = \pm\left\{\sqrt{-\frac{1}{4}-u+a_t^2+a_0^2} + \frac{\left(\frac{1}{2}+u-a_t^2-a_0^2-a_1^2+a_\infty^2\right)\left(\frac{1}{2}+u-2a_t^2\right)}{2(1+2u-2a_t^2-2a_0^2)\sqrt{-\frac{1}{4}-u+a_t^2+a_0^2}} t + \mathcal{O}(t^2)\right\},$$

with $u = \dfrac{-2q+2t\alpha\beta+\gamma\epsilon-t(\gamma+\delta)\epsilon}{2(t-1)}$.

QNMs results for Schwarzschild de Sitter

The results for the imaginary part of the quasinormal mode frequencies $\omega_{n,\ell,s}$ in the SdS_4 case, for $n = 0$, are

$$\begin{aligned}
\mathrm{Im}\left(\omega_{0,0,0}\right) = & -1 - \frac{5}{8}R_h^2 - 3R_h^3 - \left[\frac{1287}{128} + 2\log\left(2R_h\right)\right] R_h^4 + \mathcal{O}\left(R_h^5\right), \\
\mathrm{Im}\left(\omega_{0,1,1}\right) = & -2 - \frac{7}{12}\,R_h^2 + \frac{7123}{1728}\,R_h^4 + 8\,R_h^5 + \left[\frac{2\,757\,809}{124\,416} + \frac{32}{3}\log\left(2R_h\right)\right] R_h^6 + \mathcal{O}\left(R_h^7\right), \\
\mathrm{Im}\left(\omega_{0,2,2}\right) = & -3 - \frac{27}{40}R_h^2 + \frac{51\,423}{16\,000}R_h^4 - \frac{72\,333\,747}{3\,200\,000}R_h^6 - \frac{72}{5}R_h^7 + \\
& + \left[\frac{60\,278\,884\,503}{512\,000\,000} - \frac{144}{5}\log\left(2R_h\right)\right] R_h^8 + \mathcal{O}\left(R_h^9\right).
\end{aligned} \tag{20}$$

Example of expansion of wave solution in Schwarzschild anti-de Sitter

For $\ell = s = 1$ and $n = 0$,

$$f_0^L(z) = \frac{(z-2)z^2}{2(z-1)^3},$$

$$f_1^L(z) = \frac{3(z-2)z^2\log(z)}{4\left(\sqrt{z}-1\right)^3\left(\sqrt{z}+1\right)^3} - \frac{\left(3\pi z^4 - 6\pi z^3 - 8iz + 4i\right)\log(z-1)}{4\pi\left(\sqrt{z}-1\right)^3\left(\sqrt{z}+1\right)^3 z} - \frac{4iz^3 + 5\pi z^3 - 12iz^2 + 8iz - 16\pi z + 8\pi}{8\left[\pi\left(\sqrt{z}-1\right)^3\left(\sqrt{z}+1\right)^3 z\right]},$$

$$\begin{aligned} f_2^L(z) = & -\frac{3i(z-2)z^2\mathrm{Li}_2(1-z)}{2\left(\pi\left(\sqrt{z}-1\right)^3\left(\sqrt{z}+1\right)^3\right)} + \frac{9(z-2)z^2\log^2(z)}{16\left(\sqrt{z}-1\right)^3\left(\sqrt{z}+1\right)^3} + \\ & + \frac{(3\pi+4i)\left(3\pi z^4 - 6\pi z^3 - 8iz + 4i\right)\log^2(z-1)}{16\pi^2\left(\sqrt{z}-1\right)^3\left(\sqrt{z}+1\right)^3 z} - \frac{3\left(4iz^4 + 3\pi z^4 - 8iz^3 - 6\pi z^3 - 8iz + 4i\right)\log(z-1)\log(z)}{8\pi\left(\sqrt{z}-1\right)^3\left(\sqrt{z}+1\right)^3 z} + \dots \end{aligned}$$

QNMs results for Schwarzschild anti-de Sitter

In all the computed cases, the imaginary part is delayed with respect to the real one, and it does not appear before order $2\ell + 2$ in R_h:

$$\mathrm{Im}\left(\omega_{n,\ell,s}\right) \sim R_h^{2\ell+2}. \tag{21}$$

For example, for $\ell = s = 1$ and $n = 0$,

$$\mathrm{Re}\left(\omega_{0,1,1}\right) = 3 - \frac{4}{\pi} R_h + \left(\frac{27}{8} - \frac{140}{3\,\pi^2}\right) R_h^2 - \left(3\,\pi - \frac{601}{12\,\pi} - \frac{18}{\pi}\log(2) + \frac{2020}{3\,\pi^3} - \frac{168}{\pi^3}\zeta(3)\right) R_h^3 + \mathcal{O}\left(R_h^4\right),$$

$$\mathrm{Im}\left(\omega_{0,1,1}\right) = -\frac{16}{\pi} R_h^4 - \left(24 + \frac{96}{\pi^2}\right) R_h^5 - \left(60\,\pi + \frac{579}{\pi} - \frac{264}{\pi}\log(2\,R_h) + \frac{11\,536}{9\,\pi^3} - \frac{1344}{\pi^3}\zeta(3)\right) R_h^6 + \mathcal{O}\left(R_h^7\right).$$

The irrational numbers entering these QNM frequencies are $\log(2)$, π, and Euler sums.

Results for QNMs Scalar sector of gravitational perturbations

We obtained the frequency corrections $\omega_0, \ldots, \omega_6$ in the expansion $\omega = \sum_{K \geq 0} \omega_K / R_h^K$:

$$
\begin{aligned}
\omega_0 =& \sqrt{\frac{m+2}{2}}, \quad \omega_1 = -\frac{im}{6}, \\
\omega_2 =& \frac{\sqrt{2}\, m}{36\sqrt{m+2}} + \frac{m\sqrt{m+2}}{108\sqrt{2}} \left[15 + \sqrt{3}\,\pi - 9\, \log(3) \right], \\
\omega_3 =& -\frac{m(m+2)}{18\sqrt{3}} \left[\mathrm{Li}_{1,1}(u_1, u_1) + u_1 \mathrm{Li}_{1,1}(u_2, u_1) - u_2 \mathrm{Li}_{1,1}(u_1, u_2) \right] + \\
& + \frac{m(m+2)}{1296\sqrt{3}} \left[\pi^2 - 6i\,\pi \log(3) + 9(u_2 - 3\,u_1) \log(3)^2 \right] + \frac{im(m+3)}{162} \left[9 + \sqrt{3}\,\pi - 9 \log(3) \right].
\end{aligned}
\tag{22}
$$

Results for QNMs II

Upon taking the scaling limit

$$R_h \to \infty, \quad \ell \to \infty, \quad \frac{2\,\ell}{3\,R_h} \to \mathfrak{q}, \tag{23}$$

where $\mathfrak{q}$ stays constant, and rescaling the frequency as

$$\mathfrak{w} = \frac{2\,\omega}{3\,R_h}, \tag{24}$$

we obtain an expansion of $\mathfrak{w}$ in $\mathfrak{q}$

$$\mathfrak{w} = \sum_{k\geq 1} \mathfrak{w}_k\, \mathfrak{q}^k, \tag{25}$$

(up to $\mathfrak{w}_7$) reproducing the results for the QNM frequencies of the M2-brane in the AdS_4 background which are directly linked to hydrodynamics.

Results for QNMs III

The numerical values of these coefficients are

$$\begin{aligned}
\mathfrak{w}_1 &= \frac{1}{\sqrt{2}}, \\
\mathfrak{w}_2 &= -\frac{i}{4}\ , \\
\mathfrak{w}_3 &= 0.155473446153645..., \\
\mathfrak{w}_4 &= 0.0676903888847266...\ \cdot i, \\
\mathfrak{w}_5 &= -0.0107334169576 92..., \\
\mathfrak{w}_6 &= 0.0139595436599 02...\ \cdot i, \\
\mathfrak{w}_7 &= -0.0166158146267 11...\ .
\end{aligned} \tag{26}$$

These alternate between real and imaginary parts, precisely as predicted by previous works.

[S. Grozdanov, P. K. Kovtun, A. O. Starinets and P. Tadic, JHEP 11 (2019) 097, [1904.12862]]

[S. Grozdanov, P. K. Kovtun, A. O. Starinets and P. Tadic, Phys. Rev. Lett. 122 (2019) 251601, [1904.01018]]