# Wilson loop correlators at strong-coupling in $\mathcal{N} = 2$ quiver gauge theories

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SISSA, 22$^{nd}$ February 2024

This talk is mainly based on

A. Pini and P.V. "Wilson loop correlators at strong coupling in $\mathcal{N} = 2$ quiver gauge theories" arXiv: 2308.03848, JHEP 11 (2023) 003

# Motivation

- The analysis of the strong-coupling regime in an interacting gauge theory is a very difficult problem but, when there is a high amount of symmetry, remarkable progress can be made.
- In particular this happens for $\mathcal{N} = 4$ SYM theory, where many exact results have been found over the years, especially in the planar limit

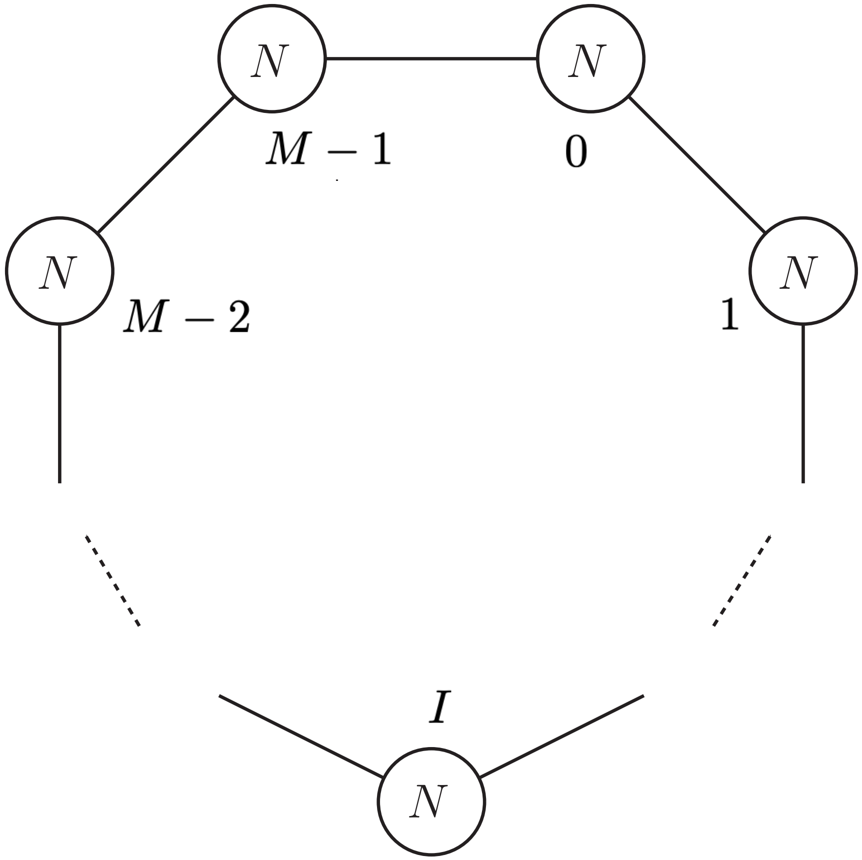
$$N \to \infty \quad \text{and} \quad \lambda \equiv g_{YM}^2 N \quad \text{fixed}$$

**Less is known about strong-coupling results in** 4d $\mathcal{N} = 2$

# $\mathcal{N} = 2$ quiver gauge theory

Gauge group: $\underbrace{SU(N) \times \cdots \times SU(N)}_{\text{M times}}$

- The links represent hypermultiplets in the bifundamental representation.
- It arises as a $\mathbb{Z}_M$ orbifold projection from $\mathcal{N} = 4$ SYM.
- This is a conformal gauge theory.
- We consider the symmetric configuration: $\lambda_I \equiv \lambda \; \forall \, I$.



# $\mathcal{N} = 2$ quiver gauge theory

Half-BPS circular Wilson loop

$$W^{(I)} \equiv \frac{1}{N} \operatorname{tr} \mathcal{P} \exp\left\{ g \oint_C d\tau \left[ iA_\mu^{(I)}(x)\dot{x}^\mu(\tau) + \frac{1}{\sqrt{2}}\left(\phi^{(I)}(x) + \overline{\phi}^{(I)}(x)\right)\right]\right\}$$

It is convenient to introduce the following change of basis for the operators

$$W_{\hat{\alpha}} = \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} \rho^{-\hat{\alpha} I}\, W^{(I)}\,, \qquad \hat{\alpha} = 0, \ldots, M-1$$

**Untwisted**

$$W_0 = \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} W^{(I)}$$

**Twisted**

$$W_\alpha = \frac{1}{\sqrt{M}} \sum_{I=0}^{M-1} \rho^{-\alpha I}\, W^{(I)}$$

$$\alpha = 1, \ldots, M-1$$

# $\mathcal{N} = 2$ quiver gauge theory

Untwisted Wilson loop correlators $\Rightarrow$ **planar equivalent** to $\mathcal{N} = 4$

$$\langle \underbrace{W_0 \, W_0 \; \cdots \; W_0}_{n} \rangle \simeq \left(\sqrt{M}\,\langle W \rangle_0\right)^n = \frac{2^n\, M^{n/2}}{\lambda^{n/2}} \left(I_1(\sqrt{\lambda})\right)^n .$$

[Rey, Suyama, 2011]
[Galvagno, Preti, 2021]

**n-point correlators of coincident Wilson loops**

$$\langle W_{\hat{\alpha}_1} \, W_{\hat{\alpha}_2} \ldots W_{\hat{\alpha}_n} \rangle \qquad \sum_{i=1}^{n} \hat{\alpha}_i = 0 \bmod M$$

⬇

**Supersymmetric localization** maps the computation of these correlators in the gauge theory to a **matrix model** on $\mathbb{S}^4$. [Pestun, 2007]

Matrix model representation for the 1/2-BPS Wilson loop [Pestun, 2007]

$$W_{\hat{\alpha}} = \frac{1}{N}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\lambda}{2N}\right)^{\frac{k}{2}} A_{\hat{\alpha},k}\,, \qquad A_{\hat{\alpha},k} = \frac{1}{\sqrt{M}}\sum_{I=0}^{M-1}\rho^{-\hat{\alpha}I}\,\mathrm{tr}\,a_I^k$$

$$A^{\dagger}_{\hat{\alpha},k} = A_{M-\hat{\alpha},k}$$

$$\langle W_{\hat{\alpha}_1}\dots W_{\hat{\alpha}_n}\rangle \quad \Rightarrow \quad \langle A_{\hat{\alpha}_1,k_1}\dots A_{\hat{\alpha}_n,k_n}\rangle$$

**Change the operator basis** $A_{\hat{\alpha},k} \longrightarrow P_{\hat{\alpha},k}$

$$A_{\alpha,k} = \left(\frac{N}{2}\right)^{\frac{k}{2}}\sum_{\ell=0}^{\lfloor\frac{k-1}{2}\rfloor}\sqrt{k-2\ell}\binom{k}{\ell}\mathcal{P}_{\alpha,k-2\ell}$$

# Interacting matrix model

- $$\left\langle P_{\hat\alpha,n} P^\dagger_{\hat\beta,m} \right\rangle = \delta_{\hat\alpha,\hat\beta} \mathsf{D}^{(\hat\alpha)}_{n,m}$$ $$\mathsf{D}^{(\hat\alpha)}_{n,m} \equiv \left( \frac{1}{1 - s_{\hat\alpha} \mathsf{X}} \right)_{n,m} \qquad s_{\hat\alpha} = \sin^2\left(\frac{\pi\,\hat\alpha}{M}\right)$$

[Billò, Frau, Galvagno, Lerda, Pini, 2021]

- $$\left\langle P_{\hat\alpha,n} P_{\hat\beta,m} P^\dagger_{\hat\gamma,p} \right\rangle = \frac{\delta_{\hat\alpha+\hat\beta,\hat\gamma}}{\sqrt{MN}} \mathsf{d}^{(\hat\alpha)}_k \mathsf{d}^{(\hat\beta)}_m \mathsf{d}^{(\hat\gamma)}_p$$ $$\mathsf{d}^{(\hat\alpha)}_k = \sum_{k'} \sqrt{k'} \mathsf{D}^{(\hat\alpha)}_{k,k'}$$

[Billò, Frau, Lerda, Pini, PV, 2022]

- Higher-pt functions factorized à la Wick in terms of 2- and 3-pt.

$$\mathsf{X}_{n,m} = -8(-1)^{\frac{n+m+2nm}{2}} \sqrt{n\,m} \int_0^\infty \frac{dt}{t} \frac{e^t}{(e^t-1)^2} J_n\left(\frac{t\sqrt{\lambda}}{2\pi}\right) J_m\left(\frac{t\sqrt{\lambda}}{2\pi}\right)$$

Exact dependence on $\lambda$ through the X-matrix

# 2-point functions of coincident twisted Wilson loops

Exact expression in the 't Hooft limit

$$\langle W_\alpha W_\alpha^\dagger \rangle \simeq \frac{1}{N^2} \sum_{k=2}^{\infty} \sum_{\ell=2}^{\infty} I_k(\sqrt{\lambda}) I_\ell(\sqrt{\lambda}) \sqrt{k\,\ell}\; \mathsf{D}_{k,\ell}^{(\alpha)}$$

in $\mathcal{N} = 4$ SYM$(S_{int} \to 0)$ this quantity becomes

$$W_{conn}^{(2)}(\lambda) \equiv \langle W\, W \rangle_0 - \langle W \rangle_0^2 \simeq \frac{\sqrt{\lambda}}{2N^2} I_1(\sqrt{\lambda}) I_2(\sqrt{\lambda})$$

Ratio: $$\frac{\langle W_\alpha W_\alpha^\dagger \rangle}{W_{conn}^{(2)}(\lambda)} \equiv 1 + \Delta w^{(\alpha)}(M, \lambda)$$

$$\frac{I_k(\sqrt{\lambda})}{I_1(\sqrt{\lambda})} \underset{\lambda\to\infty}{\sim} \sum_{s=0}^{\infty} \frac{Q_{2s}(k)}{\lambda^{s/2}} \quad \Rightarrow \quad \Delta w^{(\alpha)}(M,\lambda) \underset{\lambda\to\infty}{\sim} \frac{2}{\sqrt{\lambda}} \sum_{P=0}^{\infty} \frac{\mathcal{S}^{(P)}(s_\alpha)}{\lambda^{P/2}}$$

$\mathcal{S}^{(P)}(s_\alpha) \quad \Rightarrow \quad$ Bessel operator

$\Downarrow$

[Beccaria, Korchemsky, Tseytlin, 2023]

**Analytic computation at strong coupling**

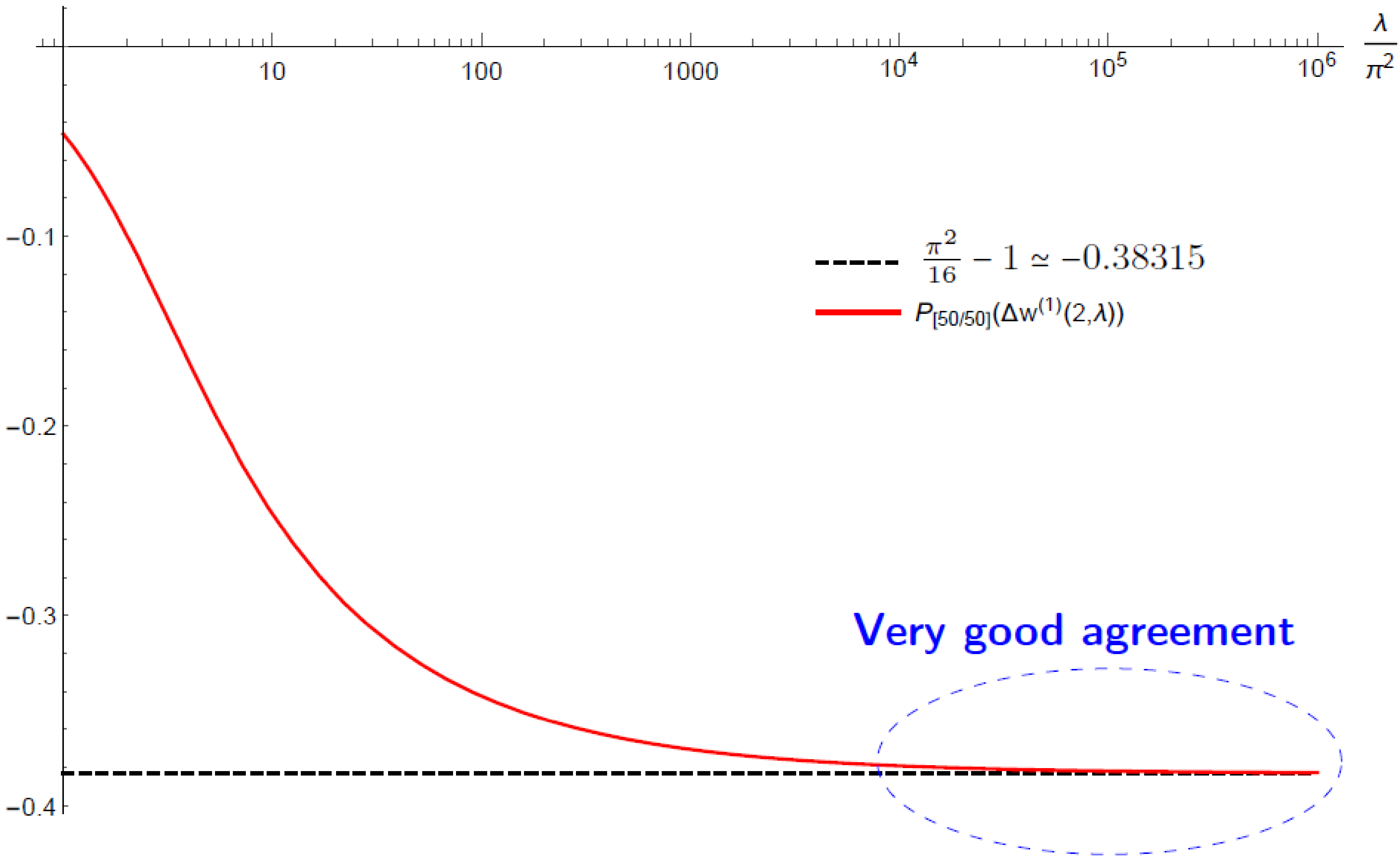
$$1 + \Delta w^{(\alpha)}(M,\lambda) \underset{\lambda\to\infty}{\sim} \left(\frac{\mathcal{I}_0(s_\alpha)}{\sqrt{s_\alpha}\,2}\right)^2 + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

[Pini, PV, 2023]

$$\mathcal{I}_0(s_\alpha) = \int_0^\infty \frac{dz}{\pi} z\partial_z \log\left(1 - s_\alpha \sinh\left(\frac{z}{2}\right)^{-2}\right)$$

E.g. $\mathbb{Z}_2$ quiver: $\mathcal{I}_0(1) = \frac{\pi}{2}$

# Numerical check for $\mathbb{Z}_2$ quiver



# 3-point functions

Ratio: $$\frac{\langle W_{\hat{\alpha}_1}\, W_{\hat{\alpha}_2}\, W^{\dagger}_{\hat{\alpha}_1+\hat{\alpha}_2}\rangle}{\sqrt{M}\, W^{(3)}_{conn}(\lambda)} \equiv 1 + \Delta w^{(\hat{\alpha}_1,\hat{\alpha}_2)}(M,\lambda)$$

$$W^{(3)}_{conn} \equiv \langle\, W\; W\; W\,\rangle_0 - 3\,\langle\, W\rangle_0 \langle W\; W\rangle_0 + 2\,\langle W\rangle_0^3$$

Studying this ratio in terms of the Bessel operator

$$1 + \Delta w^{(\alpha_1,\alpha_2)}(M,\lambda) \underset{\lambda\to\infty}{\sim} -\frac{\delta_{\alpha_1+\alpha_2,\alpha_3}}{8} \prod_{p=1}^{3} \frac{\mathcal{I}_0(s_{\alpha_p})}{\sqrt{s_{\alpha_p}}} + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

E.g. $\mathbb{Z}_3$ quiver: $1 + \Delta w^{(1,1)}(3,\lambda) \underset{\lambda\to\infty}{\sim} \left(\frac{4\pi}{9\sqrt{3}}\right)^3$

[Pini, PV, 2023]

# Conclusions and outlook

- We derived an expression valid for all values of $\lambda$ in the 't Hooft limit and found analytically

Simple rule at strong coupling $\quad W_{\alpha_i} \to -\frac{\mathcal{I}_0(s_{\alpha_i})}{2\sqrt{s_{\alpha_i}}}$

- It would be interesting to study **non-planar** corrections to these observables and also **subleading corrections in the strong-coupling expansion**.

- It would be nice to check our localization results from the gravity dual through AdS/CFT correspondence.

*Thanks for your attention!*

# Matrix model

In the quiver gauge theory we have in the large-N limit

$$\mathcal{Z} = \int \prod_{I=0}^{M-1} da_I \ \mathrm{e}^{-\mathrm{tr}\, a_I^2 - S_{\rm int}} \cancel{|Z_{inst}|^2}$$

where $a_I \equiv a_I^b T_b$ are $N \times N$ traceless Hermitean matrices and

$$S_{int} = \sum_{\hat\alpha=0}^{M-1} \left[ s_{\hat\alpha} \sum_{m=2}^{\infty} \sum_{k=2}^{2m} \left(\frac{\lambda}{8\pi^2 N}\right)^m \mathcal{F}_{m,k} (\mathrm{tr}\, a_I^{2m-k} - \mathrm{tr}\, a_{I+1}^{2m-k})(\mathrm{tr}\, a_I^k - \mathrm{tr}\, a_{I+1}^k) \right]$$

$$\mathcal{F}_{m,k} = 2(-1)^{k+m} \binom{2m}{k} \frac{\zeta_{2m-1}}{m} \quad , \quad s_{\hat\alpha} = \sin^2\left(\frac{\pi\,\hat\alpha}{M}\right)$$

Hence for a generic function $f$

$$\langle f \rangle = \frac{\int \prod_{I=0}^{M-1} da_I \ \mathrm{e}^{-\mathrm{tr}\, a_I^2 - S_{int}}\, f}{\int \prod_{I=0}^{M-1} da_I \ \mathrm{e}^{-\mathrm{tr}\, a_I^2 - S_{int}}} = \frac{\langle e^{-S_{int}}\, f \rangle_0}{\langle e^{-S_{int}} \rangle_0} \ ,$$

where $\langle\ \rangle_0$ stands for the expectation value in the free matrix model.

# $\mathcal{P}_{\alpha,k}$ operators

$$S_{int} = -\frac{1}{2}\sum_{\hat{\alpha}=0}^{M-1}\sum_{n,m} s_{\hat{\alpha}}\, P^{\dagger}_{\hat{\alpha},n}\, \mathsf{X}_{n,m}\, P_{\hat{\alpha},m}$$

[Beccaria, Billò, Galvagno, Hasan, Lerda, 2020]

$\mathcal{P}_{\alpha,k}$ operators $\longrightarrow$ Normal-ordered operators in the free matrix model

$\uparrow$

**Gram-Schmidt orthogonalization procedure**

The change of basis between the $A_{\alpha,k}$ and the $\mathcal{P}_{\alpha,k}$ operators reads

$$A_{\alpha,k} = \left(\frac{N}{2}\right)^{\frac{k}{2}} \sum_{\ell=0}^{\lfloor\frac{k-1}{2}\rfloor} \sqrt{k-2\ell}\begin{pmatrix} k \\ \ell \end{pmatrix} \mathcal{P}_{\alpha,k-2\ell}$$

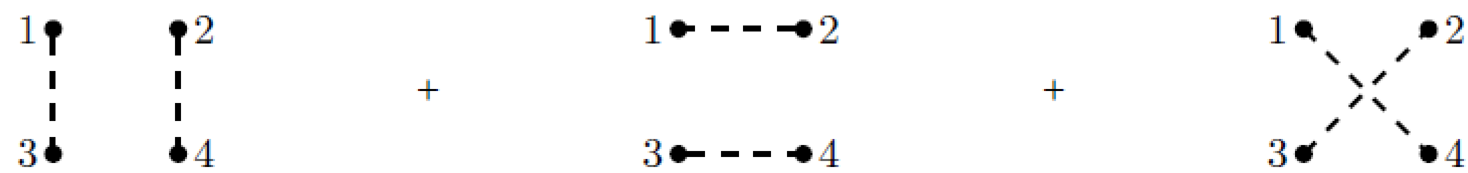
# n-point functions

**2-point** $\langle A_{\hat{\alpha}_1,k} A^{\dagger}_{\hat{\alpha}_2,\ell} \rangle_0 \simeq N^{\frac{k+\ell}{2}} \delta_{\hat{\alpha}_2,\hat{\alpha}_2}$

**3-point** $\langle A_{\hat{\alpha}_1,k} A^{\dagger}_{\hat{\alpha}_2,\ell} A_{\hat{\alpha}_3,p} \rangle_0 \simeq N^{\frac{k+\ell+p}{2}-1} \delta_{\hat{\alpha}_2+\hat{\alpha}_2,\hat{\alpha}_3}$

**Higher-point** $\longrightarrow$ Factorization à la Wick

Example $\langle \hat{A}_{\alpha_1,k_1} \hat{A}_{\alpha_2,k_2} \hat{A}_{\alpha_3,k_3} \hat{A}_{\alpha_4,k_4} \rangle_0 \simeq$




$\langle \hat{A}_{\alpha_1,k_1} \hat{A}_{\alpha_3,k_3} \rangle_0 \langle \hat{A}_{\alpha_2,k_2} \hat{A}_{\alpha_4,k_4} \rangle_0$ $\quad \langle \hat{A}_{\alpha_1,k_1} \hat{A}_{\alpha_2,k_2} \rangle_0 \langle \hat{A}_{\alpha_3,k_3} \hat{A}_{\alpha_4,k_4} \rangle_0$ $\quad \langle \hat{A}_{\alpha_1,k_1} \hat{A}_{\alpha_4,k_4} \rangle_0 \langle \hat{A}_{\alpha_2,k_2} \hat{A}_{\alpha_3,k_3} \rangle_0$

[Billò, Frau, Lerda, Pini, PV, 2022]

$n \geq 4$-**point functions** of $P_{\hat{\alpha},k}$ inherit properties of $A_{\hat{\alpha},k} \Rightarrow$ Wick's contractions

# Bessel operator

The **truncated Bessel operator** is defined as

$$\mathsf{K}\, f(x) = \int_0^\infty dy\, \mathcal{K}(x,y)\, \chi\left(\frac{\sqrt{y}}{2g}\right) f(y)\,, \qquad \chi\left(\frac{\sqrt{y}}{2g}\right) = -\sinh\left(\frac{\sqrt{y}}{4g}\right)^{-2}$$

where the **kernel** is given by

$$\mathcal{K}(x,y) = \sum_{k\geq 1} \psi_k(x)\psi_k(y)$$

and the orthonormal basis is defined as

$$\psi_k(x) = (-1)^{\frac{k}{2}(k-1)}\sqrt{k}\,\frac{J_k(\sqrt{x})}{\sqrt{x}}$$

[Belitsky, Korchemsky, 2020]

# Bessel operator

The Bessel operator can be realized as a semi-infinite matrix on the space of functions spanned by $\psi_n(x)$ and its matrix elements

$$\mathsf{K}_{n,m} = \langle \psi_n | \mathsf{K} | \psi_m \rangle$$

can be written as

$$\mathsf{K}_{n,m} = (-1)^{\frac{n+m+2nm}{2}} \sqrt{n\, m} \int_0^\infty \frac{dx}{x} \chi\left(\frac{\sqrt{x}}{2g}\right) J_n(\sqrt{x})\, J_m(\sqrt{x})$$

[Beccaria, Korchemsky, Tseytlin, 2022]

which exactly correpond to the matrix elements of X after the change of variable $\sqrt{x} = 2gt$, where $g = \frac{\sqrt{\lambda}}{4\pi}$ is a rescaling of the 't Hooft coupling.

# Strong-coupling Wilson loop correlators

The Wilson loop correlators can be written in terms of the function

$$\mathcal{S}^{(P)}(s_\alpha) = \sum_{L+J=P} S^{(L,J)}(s_\alpha) = \sum_{L+J=P} \Big( S^{(L,J)}_{\text{odd}}(s_\alpha) + S^{(L,J)}_{\text{even}}(s_\alpha) \Big)$$

where

$$S^{(L,J)}_{\text{even}}(s_\alpha) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} \sqrt{2k}\, Q^{(1)\,\text{even}}_{2L}(k)\, \sqrt{2\ell}\, Q^{(2)\,\text{even}}_{2J}(\ell) \langle \psi_{2k} \Big| \frac{s_\alpha \mathsf{X}}{1 - s_\alpha \mathsf{X}} \Big| \psi_{2\ell} \rangle$$

and it turns out that

$$\sum_{n=1}^{\infty} \sqrt{2n}\, Q^{(1)\,\text{even}}_{2L}(n)\, \psi_{2n}(x) = \\ \left[(-2)^{L-1}(x\partial_x)^L + (-2)^{L-3}(2L - L^2)(x\partial_x)^{L-1} + \ldots\right] J_1(\sqrt{x})\,,$$

$$\sum_{n=1}^{\infty} \sqrt{2n}\, Q^{(2)\,\text{even}}_{2L}(n)\, \psi_{2n}(x) = \\ \left[(-2)^{L-1}(x\partial_x)^L + (-2)^{L-3}(-3 + 2L - L^2)(x\partial_x)^{L-1} + \ldots\right] J_1(\sqrt{x})$$

# Strong-coupling Wilson loop correlators

so that the *building blocks* of these correlators can be expressed in terms of the following matrix elements

$$w_{n,m}(s_\alpha) \equiv \langle (x\partial_x)^n J_1(\sqrt{x}) \Big| \frac{s_\alpha \mathsf{X}}{1 - s_\alpha \mathsf{X}} \Big| (x\partial_x)^m J_1(\sqrt{x}) \rangle$$

For instance

$$\mathcal{S}^{(0)}(s_\alpha) = S^{(0,0)}(s_\alpha) = \frac{1}{4}\left(w_{0,0}^{(1)}(s_\alpha) + w_{0,0}^{(2)}(s_\alpha)\right)$$

$$\mathcal{S}^{(1)}(s_\alpha) = S^{(1,0)}(s_\alpha) + S^{(0,1)}(s_\alpha) = \frac{1}{8}\, w_{0,0}^{(1)}(s_\alpha) - \frac{5}{8}\, w_{0,0}^{(2)}(s_\alpha) - \frac{1}{2}\left(w_{1,0}^{(1)}(s_\alpha) + w_{0,1}^{(1)}(s_\alpha) + w_{1,0}^{(2)}(s_\alpha) + w_{0,1}^{(2)}(s_\alpha)\right)$$

# Strong-coupling Wilson loop correlators

They satisfy two differential equations, i.e.

$$\partial_g\, w_{0,n}(s_\alpha) = -8g\int_0^\infty dz\, z^2 q_0\,(z,g,s_\alpha)\, q_n(z,g,s_\alpha)\,\partial_z(s_\alpha\chi(z))$$

$$q_{n+1}(z,g,s_\alpha) = -\frac{1}{4}q_0(z,g,s_\alpha)w_{0,n}(s_\alpha) + \frac{1}{2}g\partial_g q_n(z,g,s_\alpha)$$

[Beccaria, Korchemsky, Tseytlin, 2023]

where $q_0(z,g,s_\alpha)$ is a known solution of a different differential equation. As a final step one can prove that the WL correlators can be written in terms of the generating function of these coefficients $G(s_\alpha,x,y)$ evaluated in $x=y=-2\pi$

$$\sum_{P=0}^\infty \frac{\mathcal{S}^{(P)}(s_\alpha)}{\lambda^{P/2}} \sim \sum_{n=0}^\infty\sum_{m=0}^\infty \frac{\omega_{n,m}(s_\alpha)}{(-2\pi)^n(-2\pi)^m}$$

which takes the precise value

$$G(s_\alpha,-2\pi,-2\pi) = \frac{\pi}{s_\alpha}\mathcal{I}_0(s_\alpha)^2 - 4\pi$$