Thermal one-point blocks

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Motivation

Study Conformal Field Theories

- Four-point functions
- Multipoints
- Defects
- Thermal CFTs:

  Universal properties (e.g. at high Δ)

  CFTs on other manifolds

Outline

- Casimir for spinning representations on $S^1 \times S^{d-1}$

- Solve it with expansion in T in $d = 3 \rightarrow$ one-point thermal blocks

- Computations of OPE coefficients (holography)

Different notion of temperature T

Classical statistical theory

- Relevant coupling in effective action
- Description of second-order phase transitions at critical temperature T_c
- Changing T means moving away from the CFT

QFT at finite temperature

- Compactified dimension $T \in S^1$
- Non-zero temperature physics of quantum critical points
- The only scale is T (or chemical potentials)

Intro to thermal CFTs

- Since conformal invariance is broken,

$$\langle\phi\rangle_\beta \neq 0 \qquad \text{for } \Delta \neq 0\,.$$

- For general thermal QFTs,

$$\langle\phi\rangle_\beta = \frac{1}{\mathcal{Z}}\text{tr}_{\mathcal{H}}\big(\phi\, e^{-\beta\mathcal{H}}\big) = \frac{1}{\mathcal{Z}}\sum_n \langle n|\phi\, e^{-\beta\mathcal{H}}|n\rangle\,.$$

	$S^1 \times \mathbb{R}^{d-1}$	$S^1 \times S^{d-1}$
$\langle\phi\rangle$	Fixed	$\sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}} \times$ Blocks
$\langle\phi\phi\rangle$	$\sum_{\mathcal{O}} \lambda_{\phi\phi\mathcal{O}} \times$ Blocks	...

High-temperature limit

CFT correlators on $S^1 \times S^{d-1} \xrightarrow{T\to\infty}$ correlators on $S^1 \times \mathbb{R}^{d-1}$

Invariance under $SO(d-1)$ implies: [Iliesiu, Kologlu, Mahajan, Perlmutter, Simmons-Duffin '18]

- Only STT representations

- By symmetry and dimensional analysis

$$\langle \phi^{\mu_1\dots\mu_l}(x)\rangle_\beta \xrightarrow{T\to\infty} \frac{b_\phi}{\beta^\Delta}(e^{\mu_1}\dots e^{\mu_l} - \text{traces})\,.$$

Definition of correlator

- Consider CFT on cylinder $\mathbb{R} \times S^{d-1}$
- One-point function at finite T (on $S^1 \times S^{d-1}$) as

$$G_1(x, \beta) = \langle \phi_{\text{cyl}}(x) \rangle_\beta = \text{tr}_{\mathcal{H}}\left(\phi_{\text{cyl}}(x) e^{-\beta D}\right)$$

- We want non-zero chemical potentials,

$$G_1(x, \beta, \mu_i) = \langle \phi_{\text{cyl}}(x) \rangle_{\beta, \mu_i} = \text{tr}_{\mathcal{H}}\left(\phi_{\text{cyl}}(x) e^{-\beta D} e^{-\mu_1 H_1} \ldots e^{-\mu_{d-2} H_{d-2}}\right),$$

with $H_1, \ldots, H_{d-2}$ Cartan generators of rotations.
We will denote $q = e^{-\beta}$ and $y_i = e^{-\mu_i}$.

Thermal blocks

Definition of blocks with projectors,

$$\lambda_{\mathcal{O}\mathcal{O}\phi} G_{\mathcal{O}}(x) = \mathrm{tr}\big(P_{\mathcal{O}} \phi(x) q^D y_1^{H_1} \dots y_{d-2}^{H_{d-2}}\big) \, .$$

Relation with OPE coefficients with shadow formalism,

$$\lambda_{\mathcal{O}\mathcal{O}\phi} G_{\mathcal{O}}(x) = q^{\Delta_\phi} y_1^{J_1} \dots y_{d-2}^{J_{d-2}} \int d^d x' \, \langle \tilde{\mathcal{O}}(x') \phi(x) \mathcal{O}(x'_{q,y}) \rangle \, .$$

Without chemical potentials μ_i:

- Lost most of information about OPE coefficients,

- No clear way to obtain Casimir.

Casimir with internal and external scalars

Fix $d = 3 \to$ only STT operators and one chemical potential.
[Gobeil, Maloney, Ng, Wu '18]

$$C_2 = q^2\partial_q^2 + y^2\partial_y^2 + q\partial_q + y\partial_y + \frac{(1+y)y\partial_y}{y-1} + \frac{(1+q)q\partial_q}{q-1} + \frac{(q+y)(q\partial_q - y\partial_y)}{q-y} + \frac{(1+qy)(q\partial_q + y\partial_y)}{qy-1} + \mathcal{F}_1(q,y,p)\mathcal{F}_2(\partial_p^2, \partial_p)\,,$$

with $\mathcal{F}_1$ rational function and $\mathcal{F}_2$ linear.

Ansatz

$$\sum_{a,b,c} f_{a,b,c}\, q^a y^b p^c \quad \text{with } a \geq 0\,, b \in [-a, a]\,, c \in [0, 2a]\,.$$

Spherical functions

- How to deal with general spinning representations?

- Spherical functions over (G, K) with K subgroup of G

- Blocks $\longleftrightarrow$ spherical functions $\Longrightarrow$ tools from Harmonic analysis

	G	K
Conformal blocks [Schomerus, Sobko, Isachenkov '16] [Burić, Schomerus '22]	$SO(d+1,1)$	$SO(1,1) \times SO(d)$
Partial waves [Burić, FR, Vichi '23]	$SO(d-1)$	$SO(d-2)$
Thermal blocks	$SO(d+1,1) \times SO(d+1,1)$	$SO(d+1,1)$

Spinning Casimir

- Radial component map $\rightarrow$ reducing Casimir to four-variable (q, y, p, z) differential equation:

$$C_2 = q^2\partial_q^2 + y^2\partial_y^2 + q\partial_q + y\partial_y + \frac{(1+y)y\partial_y}{y-1} + \frac{(1+q)q\partial_q}{q-1} + \frac{(q+y)(q\partial_q - y\partial_y)}{q-y} + \frac{(1+qy)(q\partial_q + y\partial_y)}{qy-1} + \mathcal{F}_1(q, y, p, z, \sqrt{1+z^2}, \sqrt{1-p^2})\mathcal{F}_2(\partial_p^2, \partial_p, \partial_z^2, \partial_z, \partial_p\partial_z)\,,$$

with $\mathcal{F}_1$ rational function.

Weight-shifting operator

$$\hat{q}_{\Delta,l}C_2(\Delta, l) = C_2(\Delta - 1, l + 1)\hat{q}_{\Delta,l}$$

Spinning ansatz

$$\sum_{a,b,c} f_{a,b,c,d} q^a y^b p^c z^d (1+\sqrt{1-p^2})(1+\sqrt{1+z^2})\,, \quad \text{with } a \geq 0\,,$$
$$b \in [-a - l_{\mathcal{O}}, a + l_{\mathcal{O}}]\,, c \in [0, 2(a + l_\phi + l_{\mathcal{O}})]\,, d \in [0, l_\phi]\,.$$

Example

Order q for $l_\phi = l_{\mathcal{O}} = 1$ (and $\Delta_\phi = \Delta_{\mathcal{O}} = 3$),

$$\frac{1-y^2}{y} + \frac{q(1-y)}{16y^2}\left(9p^2(y-1)^2(y+1) + 2(1+y)\Big(5 + y(14+5y)\Big)\right.$$
$$\left. + 3p\sqrt{1-p^2}(y-1)\Big(2(y^2-1)z + (2+y)(1+2y)\sqrt{1+z^2}\Big)\right) + o(q)$$

Potential applications involving $\langle T \rangle_{\beta,\mu}$

General CFTs

From [Iliesiu, Kologlu, Mahajan, Perlmutter, Simmons-Duffin '18],

$$\langle T^{00} \rangle_\beta = \frac{1}{\text{vol}(S^{d-1})} \frac{\partial}{\partial \beta} \log \mathcal{Z} \,, \quad \mathcal{Z} = \sum_{\mathcal{O}} \chi_{\mathcal{O}}(e^{-\beta}) \,.$$

- $T \to \infty$
- Knowledge of spectrum $\to b_T$.

Blocks for $\langle T \rangle_{\beta,\mu}$ and spectrum $\to$ OPE coefficients.

Holography

- Consider a AdS Kerr black hole in the bulk

$$(m, J) \to (\Omega, T) \sim (\mu, T)$$

- Asymptotic expansion of the bulk metric [de Haro, Skenderis, Solodukhin '00]

- Compute $\langle T \rangle_{\beta,\mu}$ [Cardoso, Dias, Hartnett, Lehner, Santos '13]

- One-point block expansion for the stress tensor

- OPE coefficients for the dual CFT.

Conclusions

Results

- Casimir for general spinning representations

- Solving perturbatively in q

- Thermal blocks for one-point functions

Future plans

- Computation of OPE coefficients both in holographic setting or not

- Higher-points: Crossing equations? Positivity?

Thanks!