Finding Fixed Points in the Epsilon expansion

Revisiting old results

Hugh Osborn based on work with Andy Stergiou Tom Steudtner Ian Jack

What is the space of CFTs in 3 and 4 and perhaps other dimensions?

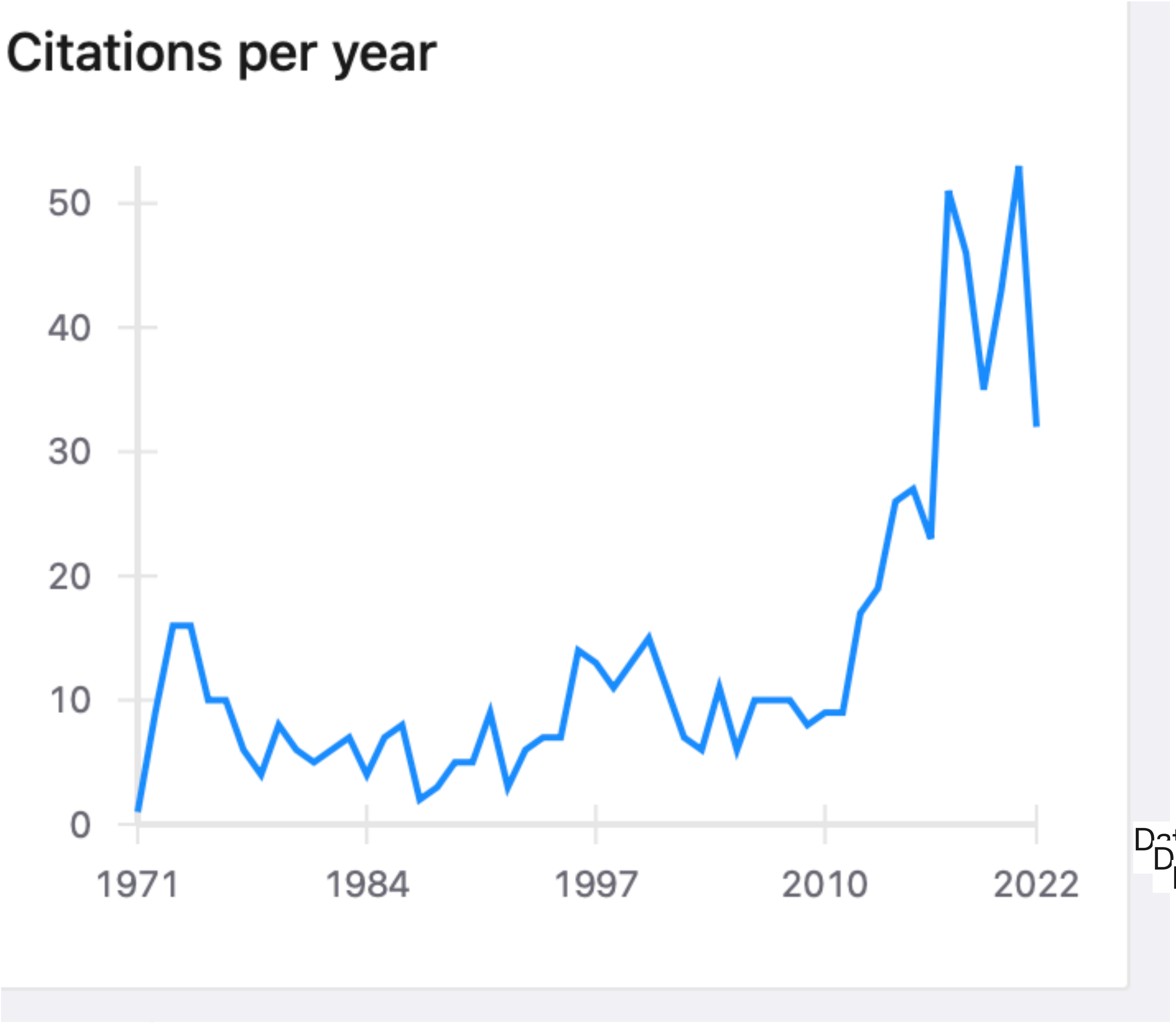
Very non trivial problem perhaps tractable with supersymmetry.

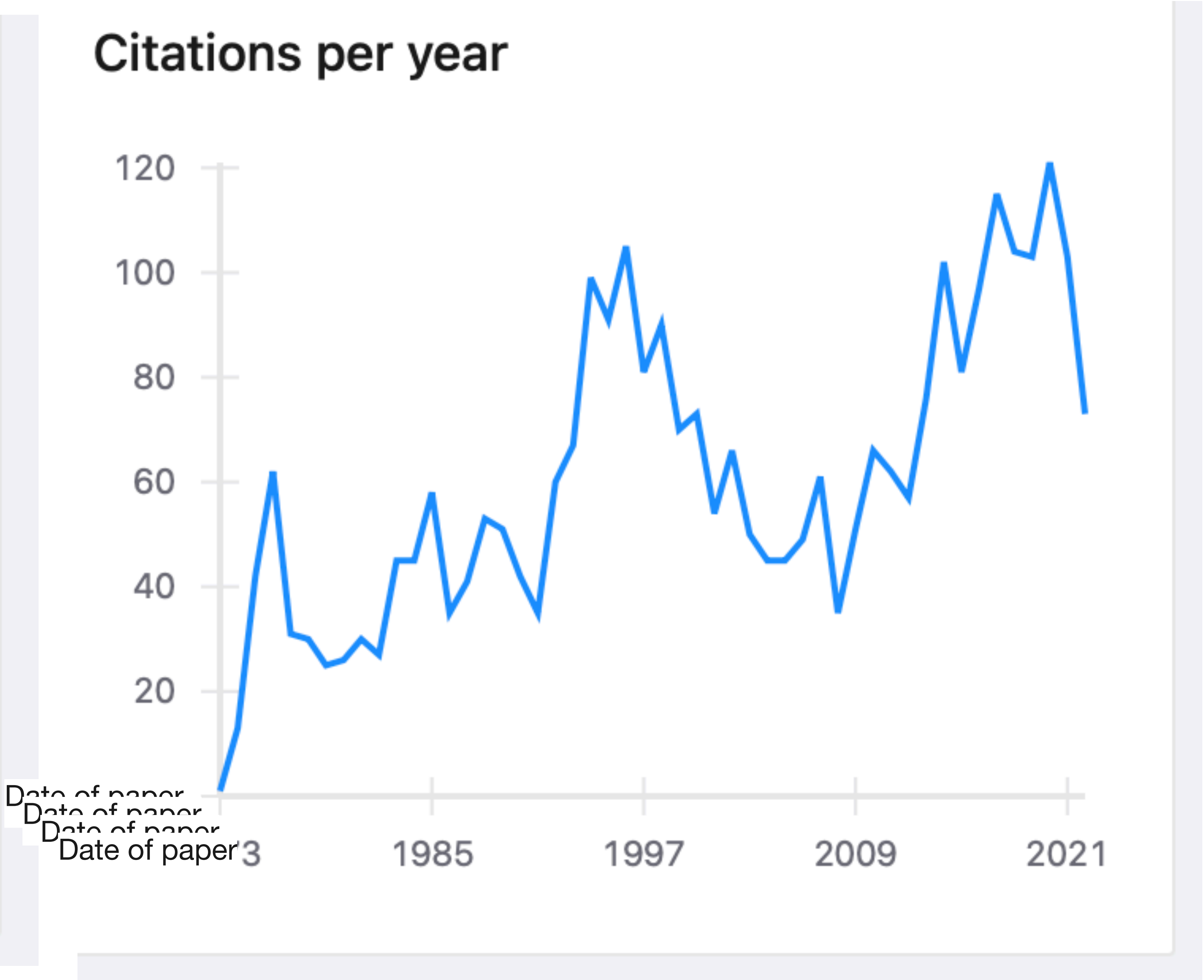
In general for any CFT with a relevant operator then perturbation in this operator lead to RG equations whose solutions may flow in the IR to new fixed points.
The flow may lead to decoupled theories, free theories or there may be no non trivial limit.

The epsilon expansion is a historically important procedure for finding fixed points.
Start from a free theory in $4-\epsilon$ dimensions. $\phi^4,\ \bar{\psi}\psi\phi$ are relevant.

Since the starting theory is free the expansion in ϵ can be carried out to pretty high order and sophisticated resummation techniques used,

Can use epsilon expansion to calculate critical exponents in specific theories.
Historically to 2,3 loops, nowadays to 6,7 loops in scalar theories.
Can also search for critical points more generally. Mostly need only lowest order contributions in the epsilon expansion.





Critical Exponents in 3.99 Dimensions*

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THE RENORMALIZATION GROUP AND THE ϵ EXPANSION*

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The start of the epsilon expansion

50 + 3 year of epsilon expansion

Kompaniets and Panzer 2017

Table XI. Estimates for critical exponents in $D = 3$ dimensions of the $O(n)$ vector model. Results from the conformal bootstrap and Monte Carlo techniques are listed first (we tried to collect the most accurate predictions in each case). Our estimates from the 5- and 6-loop ε-expansions are shown next. For comparison of the resummation methods, we display the 5-loop results (from ε-expansion without $D = 2$ boundary conditions) according to [76].

		$n=0$	$n=1$	$n=2$	$n=3$	$n=4$
η		0.031043(3)[a]	0.036298(2)[101]	0.0381(2)[38]	0.0378(3)[79]	0.0360(3)[b]
	ε^6	0.0310(7)	0.0362(6)	0.0380(6)	0.0378(5)	0.0366(4)
	ε^5	0.0314(11)	0.0366(11)	0.0384(10)	0.0382(10)	0.0370(9)
	[76]	0.0300(50)	0.0360(50)	0.0380(50)	0.0375(45)	0.036(4)
ν		0.5875970(4)[54]	0.629971(4)[101]	0.6717(1)[38]	0.7112(5)[37]	0.7477(8)[c]
	ε^6	0.5874(3)	0.6292(5)	0.6690(10)	0.7059(20)	0.7397(35)
	ε^5	0.5873(13)	0.6290(20)	0.6687(13)	0.7056(16)	0.7389(24)
	[76]	0.5875(25)	0.6290(25)	0.6680(35)	0.7045(55)	0.737(8)
ω		0.904(5)[d]	0.830(2)[65]	0.811(10)[e]	0.791(22)[e]	0.817(30)[e]
	ε^6	0.841(13)	0.820(7)	0.804(3)	0.795(7)	0.794(9)
	ε^5	0.835(11)	0.818(8)	0.803(6)	0.797(7)	0.795(6)
	[76]	0.828(23)	0.814(18)	0.802(18)	0.794(18)	0.795(30)

[a] From $\gamma = 1.156953(1)$ [53] and $\nu = 0.5875970(4)$ [54] via $\gamma = \nu(2-\eta)$ in (10).
[b] Given in [79] and compatible with 0.0365(10) [77] and $y_h = (5-\eta)/2 = 2.4820(2)$ in [60].
[c] From $y_t = 1/\nu = 1.3375(15)$ in [60], compatible with $\nu = 0.749(2)$ [77] and 0.750(2) [79].
[d] Computed from $\omega\nu = \Delta = 0.531(3)$ according to [13] and $\nu = 0.5875970(4)$ in [54].
[e] These are the results given as $\Delta_{S'} = 3 + \omega$ in [40, Table 2].

$n = 0$ (self-avoiding walks): polymers [58],

$n = 1$ (Ising universality class): liquid-vapour transitions, uniaxial magnets,

$n = 2$ (XY universality class): superfluid λ-transition of helium [107],

$n = 3$ (Heisenberg universality class): isotropic ferromagnets,

$n = 4$: finite temperature QCD with two light flavours [133].

$$\Delta_\phi = \tfrac{1}{2}(d - 2 + \eta)$$

$$\nu = \frac{1}{d - \Delta_{\phi^2}}$$

$$\omega = \Delta_{\phi^4} - d$$

Free fields

$$\Delta_\phi = \tfrac{1}{2}(d-2)$$

$$\Delta_{\phi^2} = d - 2$$

$$\Delta_{\phi^4} = 2(d-2)$$

$$\eta = 0, \ \nu = \tfrac{1}{2}, \ \omega = -\epsilon$$

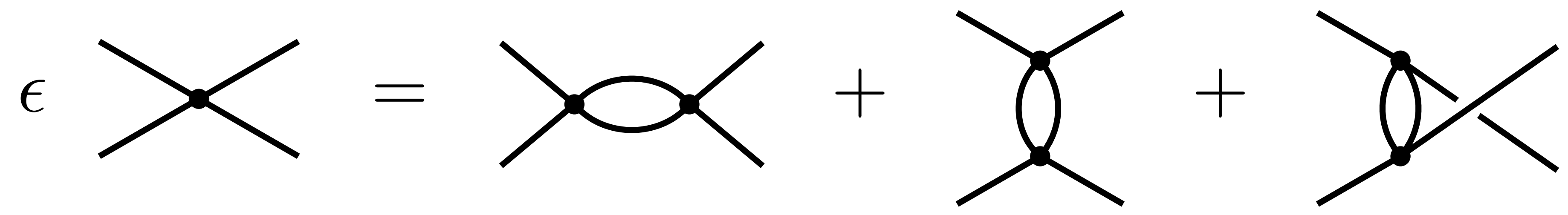
General scalar theories n_s component real scalars ϕ_i

$V(\phi) = \frac{1}{24}\lambda_{ijkl}\,\phi_i\phi_j\phi_k\phi_l$

λ_{ijkl} symmetric tensor $\mathcal{N} = \frac{1}{24}n_s(n_s+1)(n_s+2)(n_s+3)$ components

Lowest order RG equations for a fixed point

$\epsilon\,\lambda_{ijkl} = \lambda_{ijmn}\lambda_{klmn} + \lambda_{ikmn}\lambda_{jlmn} + \lambda_{ilmn}\lambda_{jkmn}$



$\mathcal{N}$ coupled quadratic equations, up to $2^{\mathcal{N}}$ solutions, want real solutions. Many solutions involve decoupled theories

Solutions covariant under $O(n_s)$ strongest case $O(n_s)$ symmetric theory

$\mathcal{N} - \frac{1}{2}n_s(n_s-1) = 1,\ 4,\ 12,\ 29, \qquad n_s = 1,\ 2,\ 3,\ 4$

At a non trivial fixed point $\epsilon\,V(\phi) = \frac{1}{8}\sum_{ij}\big(\lambda_{ijkl}\phi_k\phi_l\big)^2 > 0$ If $\epsilon > 0$

$V(\phi) = \frac{1}{8}\lambda(\phi_i\phi_i)^2 \quad \lambda_* = \dfrac{\epsilon}{n_s+8}$ Heisenberg fixed point, $n_s = 1$ Ising, $n_s = 2$ XY model

Many solutions for fixed points with different symmetry groups $H \subset O(n_s)$ are known

In applications H is chosen depending on the particular problem, then $V(\phi)$ is a sum of of quartic monomials in ϕ invariant under H. Generally the condition that there is a single quadratic invariant $\phi_i\phi_i$ is imposed and ϕ_i forms an irreducible representation of H. Potentially there can be many couplings.

All possible subgroups with a single quadratic invariant were an analysed for $n_s = 4$ by Brezin et al (1985) and for $n_s = 6$ by Hatch et al (1986), pretty non trivial.

Landau criterion, only first order transitions if there is a cubic invariant?

General analysis, decompose λ_{ijkl} into spin 0, spin 2 $d_{2,ij}$ and spin 4 tensors $d_{4,ijkl}$

$$a_0 = \lambda_{iijj}, \qquad a_2 = ||d_2||^2, \qquad a_4 = ||d_4||^2, \qquad S_{n_s} = ||\lambda||^2$$

$d_{4,ijkl}$ defines a relevant operator for $n_s > 4$.

At any fixed points there are various bounds

$$a_0 \leq \frac{n_s(n_s+2)}{n_s+8}\epsilon,$$

$$a_2 \geq 0 \quad \Rightarrow \quad S_{n_s} = ||\lambda||^2 \leq \frac{n_s}{8}\,\epsilon^2 - \frac{1}{2n_s}\big(a_0 - \tfrac{1}{2}n_s\,\epsilon\big)^2 \leq \frac{n_s}{8}\,\epsilon^2$$

Rychkov & Stergiou, Hogervorst & Toldo

$a_2 > 0$ implies > 1 quadratic invariants, S_{n_s} bound restricts numbers of possible solutions.

Different fixed points are characterised by stability matrix eigenvalues $\kappa\,\epsilon$, $-1 \leq \kappa \leq 1$.

For non trivial fixed points there is always one $\kappa = 1$, if more than one there are decoupled theories, $\lambda_{1,ijkl}, \lambda_{2,ijkl}, \lambda_1 \cdot \lambda_2 = 0$, if $\kappa = -1$ there is a decoupled free field. Generically for H a continuous Lie group of dimension d_H there will be $d_{O(n_s)} - d_H$ zero κ.

There may be κ which area only zero at lowest order in ϵ.

At a bifurcation where new fixed points emerge or annihilate there are additional zero κ.

Of course degeneracies of eigenvalues κ correspond to representations of H .

If $a_2 = 0,\ S_{n_s} = \frac{1}{8}n_s(1-\kappa^2),\ a_0 = \frac{1}{2}n_s(1+\kappa)$ for some κ .

When the S_{n_s} bound is attained there is a bifurcation.

Known fixed points

$a_2 = 0$

1 $\mathrm{O}(n_s)$

2 Cubic symmetry C_{n_s} $V_{C_{n_s}}(\phi) = \frac{1}{8}\lambda\,(\phi_i\phi_i)^2 + \frac{1}{24}g\sum_i \phi_i^4$

3 Tetrahedral symmetry $T_{n_s\pm}$ $\alpha = 1, \ldots, n_s + 1$ $\varphi_\alpha,\ \sum_\alpha \varphi_\alpha = 0$ are the vertices of a n_s dimensional hypertetrahedron. This theory has cubic invariants.

4 $MN_{r,s}$ $n_s = rs,\ \vec{\varphi}_a, r\ s$ dimensional vectors $V_{MN}(\varphi) = \frac{1}{8}\,\lambda\,(\vec{\varphi}^{\,2})^2 + \frac{1}{24}\,g\sum_a (\vec{\varphi}_a^{\,2})^2$

5 Bifundamental $n_s = rs,\ O(r)\times O(s)$ symmetry $R_{rs} = r^2 + s^2 - 10rs - 4(r+s) + 52 \geq 0$

6 Bifundamental complex $U(r)\times U(s)$ symmetry $S_{rs} = r^2 + s^2 - 10rs + 24 \geq 0$

New CFT fixed points with $a_2 = 0$ can be obtained by taking p copies of the same CFT, with fields $\varphi_a,\ a = 1, \ldots, p$ imposing $\mathcal{S}_p$ symmetry and perturbing with $\sum_{a\neq b} {\varphi_a}^2 {\varphi_b}^2$

The symmetry group is the wreath product $H \wr \mathcal{S}_p = H^p \rtimes \mathcal{S}_p$

Conjecture: for n_s prime and $a_2 = 0$ there are only $\mathrm{O}(n_s)$, cubic and tetrahedral fixed points.

These include all fixed points found by looking for subgroups of $O(4), O(6)$

For $a_2 > 0$ theories can be constructed in terms of perturbations of combinations of theories with $a_2 = 0$.

The simplest examples are biconical theories with $CFT_1 + CFT_2$ perturbed by $\phi_1^2 \phi_2^2$

This example gives two quadratic forms, but there may be many

Open problem.[4] Construct a fully interacting $N \geqslant 2$ scalar one-loop fixed point in $4 - \varepsilon$ dimensions with real couplings and just $\mathbb{Z}_2$ symmetry, or prove that all such fixed points have strictly larger symmetry.

[4] A bottle of Dom Pérignon champagne will be awarded for a solution of this problem. for collecting the prize.

Rychkov and Stergiou

When can the Rychkov Stergiou bound be saturated ?

$$||\lambda||^2 = \frac{1}{8} n_s \, \epsilon^2 \,, \quad a_0 = |\lambda| = \tfrac{1}{2} n_s \, \epsilon \,, \quad a_2 = 0 \,, \quad a_4 = ||d_4||^2 = \frac{n_s(n_s - 4)}{8(n_s + 2)} \, \epsilon^2$$

This might be a solvable problem

Known examples

$$n_s = 4 \quad O(4)$$

$$n_s = 5 \quad \mathcal{S}_6 \times \mathbb{Z}_2$$

Bifundamental theories, solve a diophantine equation

$$n_s = mn \quad O(m) \times O(n)/\mathbb{Z}_2 \,, \quad (m,n) = (73,7), \ldots$$

$$n_s = 2mn \quad U(m) \times U(n)/U(1) \,, \quad (m,n) = (49,5), \ldots$$

$$n_s = 4mn \quad Sp(m) \times Sp(n)/\mathbb{Z}_2 \,, \quad (m,n) = (37,4), \ldots$$

There are an infinite sequence of solutions in each case

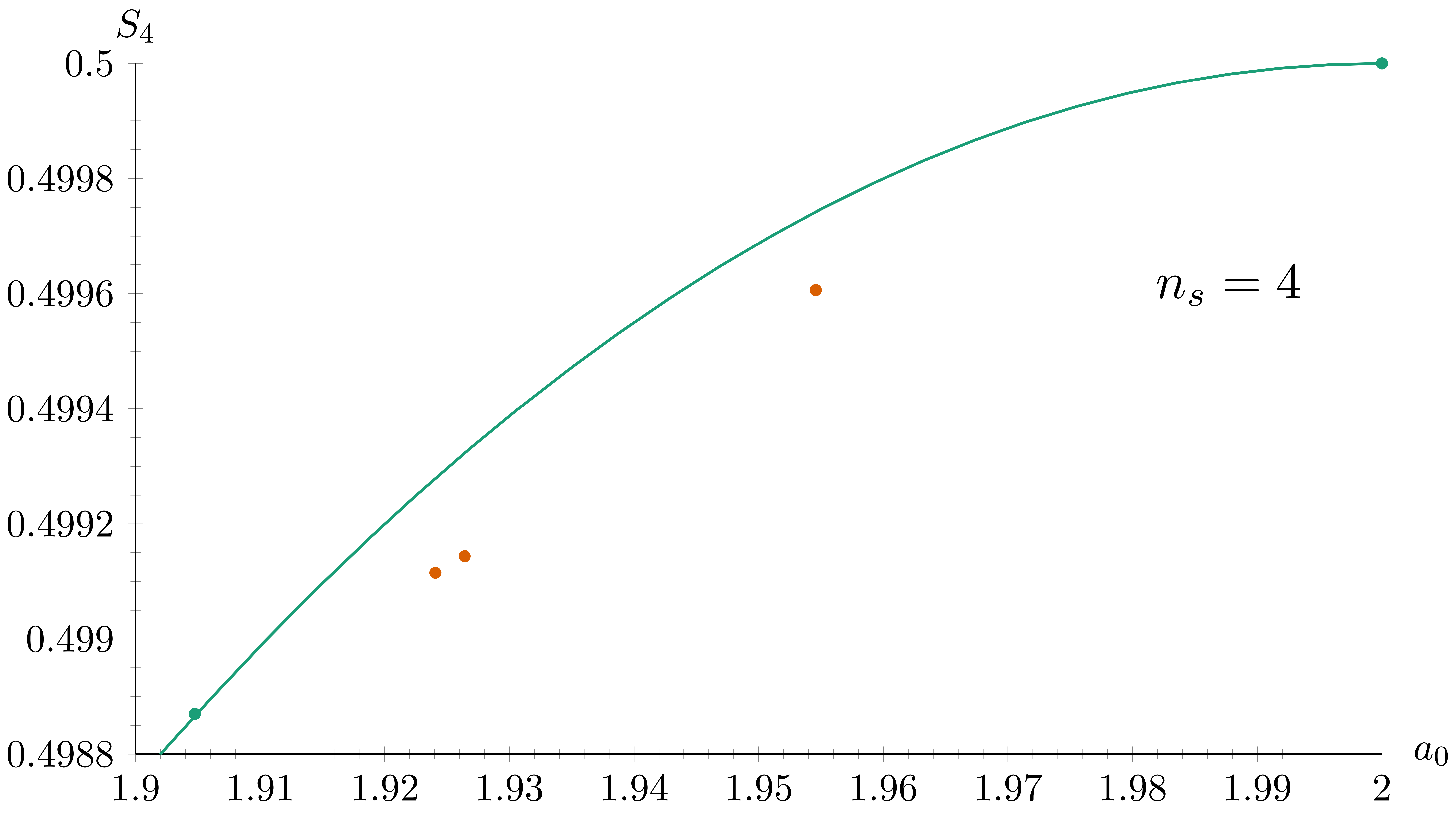
The fields are real, complex, quaternionic

Another example was found by Christian Jepsen by looking at trifundamental theories with $O(n_1) \times O(n_2) \times O(n_3)$ symmetry.

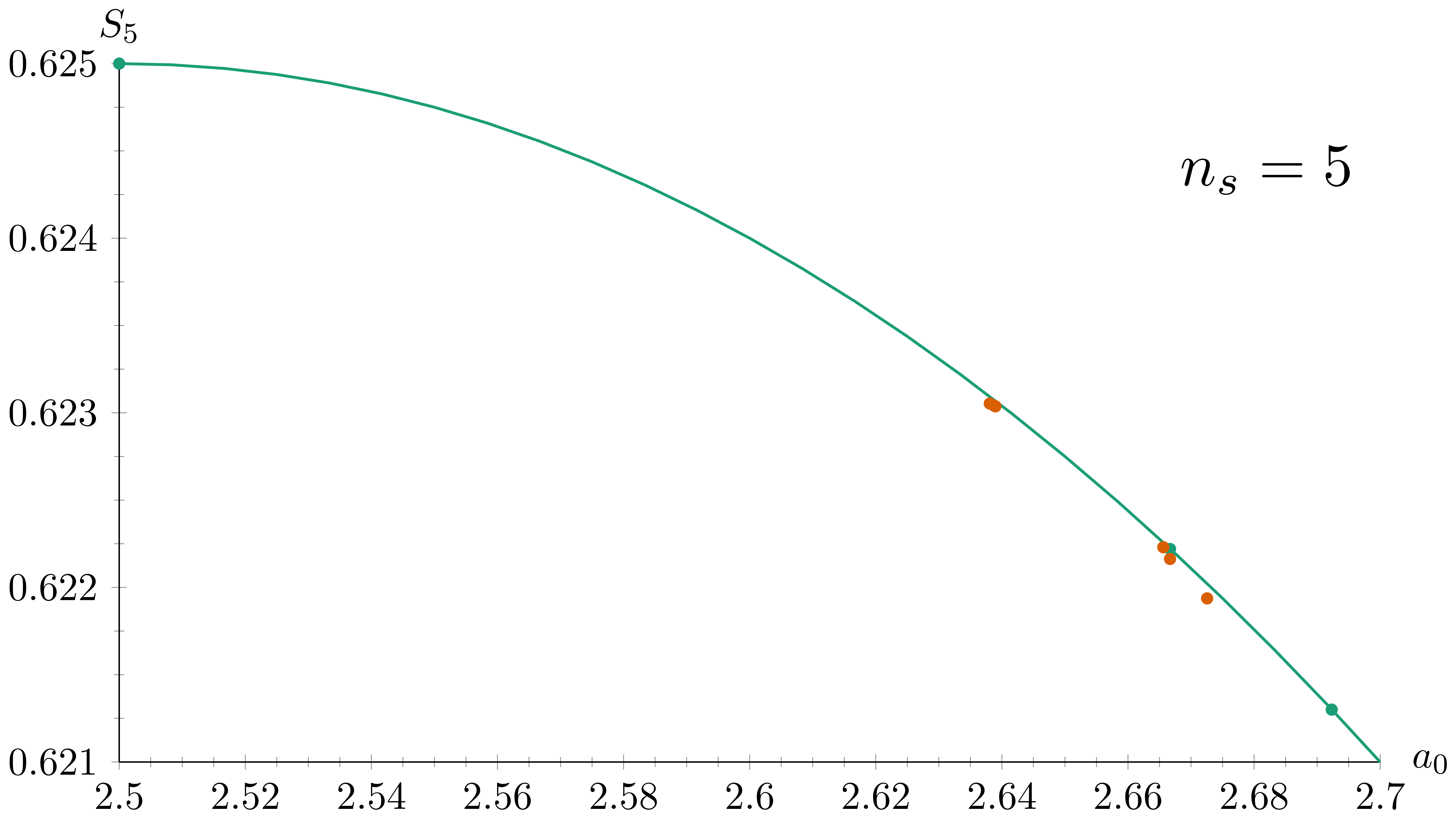
The bound is saturated for $n_1 = n_2 = 2,\ n_3 = 34$

This appears to be an isolated case and can be obtained from a bifundamental theory with $U(2) \times O(34)$ symmetry

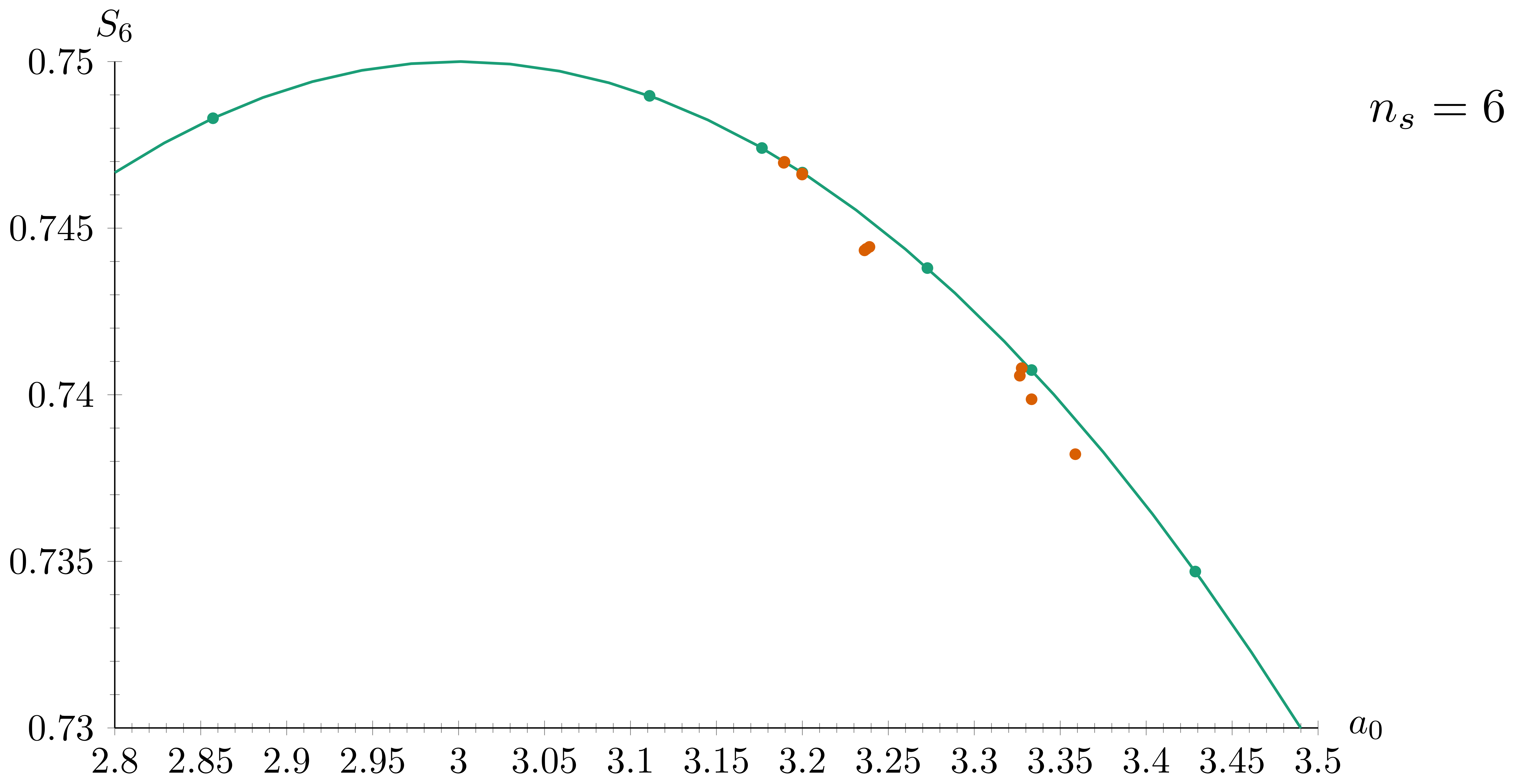
In the spirit of Rychkov Stergiou
I will off a bottle of Trinity College vintage port to anyone finding another
Infinite sequence saturating the bound.



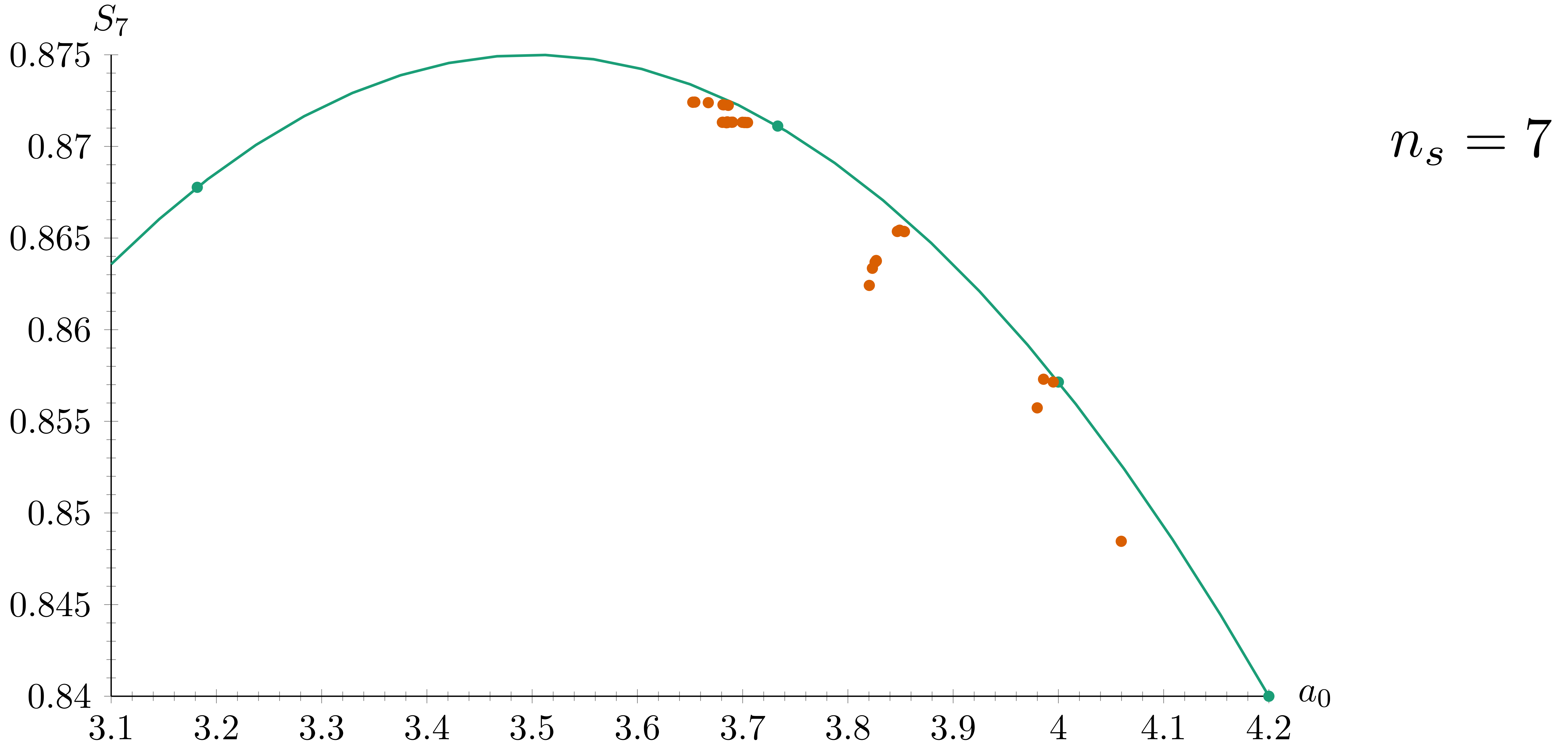
S_4
0.5
0.4998
0.4996
0.4994
0.4992
0.499
0.4988
1.9
1.91
1.92
1.93
1.94
1.95
1.96
1.97
1.98
1.99
2
a_0
$n_s = 4$



S_5
0.625
0.624
0.623
0.622
0.621
$n_s = 5$
a_0
2.5
2.52
2.54
2.56
2.58
2.6
2.62
2.64
2.66
2.68
2.7

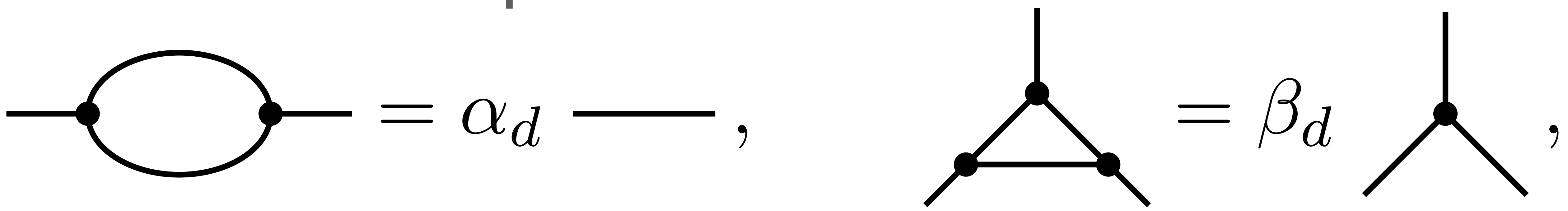


S_6
0.75
0.745
0.74
0.735
0.73
2.8
2.85
2.9
2.95
3
3.05
3.1
3.15
3.2
3.25
3.3
3.35
3.4
3.45
3.5
a_0
$n_s = 6$



S_7
0.875
0.87
0.865
0.86
0.855
0.85
0.845
0.84
3.1
3.2
3.3
3.4
3.5
3.6
3.7
3.8
3.9
4
4.1
4.2
a_0
$n_s = 7$

There is related problem for three index traceless tensor d_{ijk}

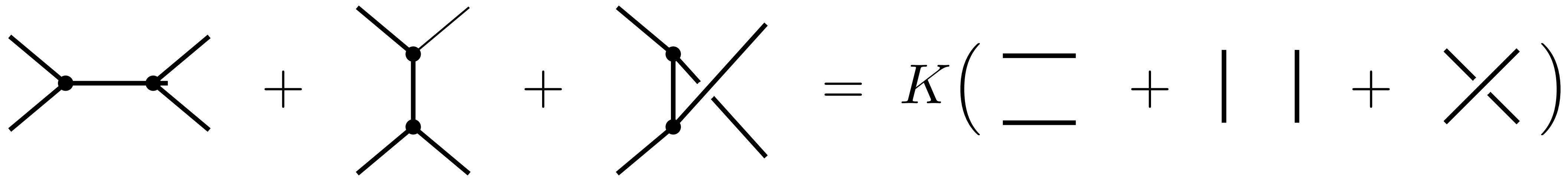


Relevant for $\mathcal{N}=1$ supersymmetric theories in three dimensions

In general there are bounds

$$-\frac{n_s-2}{2(n_s+2)} \leq \beta_d/\alpha_d \leq \frac{n_s-2}{n_s-1}$$

The upper bound requires



There are four solutions $n_s = 3 \times (1,2,4,8)+2$

3×3 Hermitian traceless matrices e_i satisfying $\frac{1}{2}(e_i e_j + e_j e_i) = \frac{1}{3}\delta_{ij}\, I_3 + d_{ijk}\, e_k$

Then if $x = x_i e_i$ the Cayley Hamilton theorem $x^3 - \frac{1}{2}\,\mathrm{tr}(x^2)\, x - \frac{1}{3}\,\mathrm{tr}(x^3)\, \mathbb{1}_3 = 0\,.$

Is equivalent to the above with $K=\frac{1}{6}$

This theorem holds for real, complex, quaternionic and octonionic matrices

Some comments on the A theorem and the epsilon expansion

In four dimensions one can demonstrate perturbatively

$$\mathrm{d}A(g) = T_{IJ}(g)\,\mathrm{d}g^I \beta^J(g)\,, \qquad T_{[IJ]} = \partial_I W_J - \partial_J W_I\,.$$

These equations are invariant under

$$\delta A = g_{IJ}\,\beta^I \beta^J\,, \quad \delta\, T_{IJ} = \mathcal{L}_\beta\, g_{IJ} + \partial_I\big(g_{JK}\beta^K\big) - \partial_J\big(g_{IK}\beta^K\big)\,, \quad \delta W_J = g_{JK}\beta^K\,,$$

for any symmetric g_{IJ}

In some cases this can be use to set the antisymmetric part of T_{IJ} to zero

In $4-\varepsilon$ dimensions, with minimal subtraction,

$$\beta^I(g) \to \hat\beta^I(g) = -\varepsilon\, g^I + \beta^I(g)$$ This fixes the scheme

Suppose the A equation is valid with $\beta^I \to \hat\beta^I\,,\ A \to \hat A\,,\ T_{IJ} \to \hat T_{IJ}$
then one can use $\hat\beta^2$ variations to make $\hat A$ linear in ε
and $\hat T_{IJ}$ independent

For the theorem to be valid away from four dimensions it is necessary that

$$T_{IJ}g^J = \partial_I A'$$ This is not automatic as it requires an integrability condition

In ϕ^4 theory at L loops there are N_L vacuum graphs, N_V inequivalent vertices, N_S graphs with one inequivalent vertex, N_T contributions to T_{IJ} and N_G contributions to symmetric G_{IJ}

L	$N(L)$	$N_V(L)$	$N_S(L)$	$N_T(L)$	$N_G(L)$
3	1	1	1	1	1
4	1	1	1	1	1
5	3	3	3	7	7
6	5	10	2	26	18
7	17	36	5	142	97
8	42	164	2		453
9	177	819	9		

N_V Is the number of terms you get on amputating one vertex from a vacuum graph

L loop A is relevant for $L-3$ loop beta's and you need $L-1$ loop T
the number of antisymmetric contributions to T is $N_V - N$
the β^2 freedom for L loop A involves $L-1$ loop G

Maybe there are deeper principles at work. It is a surprise to me that there is a consistent framework for the six loop beta functions with additional consistency conditions

It hasn’t fallen yet!

Thank you for your attention

And to the organisers for a great meeting and the invitation to come