Bootstrapping $\mathcal{N} = 4$ SYM for all N and coupling

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Based on 2312.12576 with S. Pufu and R. Dempsey
2310.12322 with L. F. Alday, D. Dorigoni, M. Green and C. Wen

$\mathcal{N} = 4$ Super-Yang-Mills (SYM)

- $\mathcal{N} = 4$ SYM is maximally supersymmetric gauge theory in 4d, defined by gauge group G (e.g. $SU(N)$), coupling g_{YM}, and θ.
  - It is conformal for any complex $\tau \equiv \frac{4\pi i}{g_{\text{YM}}^2} + \frac{\theta}{2\pi}$.
- It's the most well-studied toy model in high energy theory bc e.g.:
  - AdS/CFT: its dual to Type IIB string theory on $AdS_5 \times S^5$, with gravity description for large N and large $\lambda \equiv g_{\text{YM}}^2 N$.
  - Simplest (most symmetric) gauge theory, model for QCD.
- Perturbative approaches: weak coupling for finite N, integrability for $N \to \infty$ and any λ, holography for large N and strong coupling.

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Weak coupling

- When $\lambda \equiv g^2_{\text{YM}} N$ is small, can study SYM with Feynman diagrams for any N like any weakly coupled gauge theory.
- E.g. lowest unprotected singlet (the Konishi) has [Velizhanin '09] :

$$\Delta = 2 + \frac{3\lambda}{4\pi^2} - \frac{3\lambda^2}{16\pi^4} + \frac{21\lambda^3}{256\pi^6}$$
$$+ \frac{\lambda^4 \left(-1440 \left(\frac{12}{N^2} + 1\right)\zeta(5) + 576\zeta(3) - 2496\right)}{65536\pi^8} + O(\lambda^5)$$

  - First non-planar correction only at 4-loops!
- But bulk dual is very stringy in this regime, no gravity approximation, no black holes.

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Holography

- AdS/CFT dictionary for $AdS_5 \times S^5$ string theory with string length ℓ_s and complex string coupling $\tau_s = \chi + i/g_s$:

$$L^4/\ell_s^4 = g_{\rm YM}^2 N \qquad \tau = \tau_s\,,$$

- In principle could study using worldsheet for small g_s, but hard due to RR flux. At finite g_s, no method even in principle.
- At large N, can study $AdS_5 \times S^5$ supergravity, e.g. lowest unprotected singlet is double trace [D'Hoker, Mathur, Matusis, Rastelli '99] :

$$\Delta = 4 - 16/N^2 + O(N^{-7/2})\,,$$

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Planar integrability

- Can compute all scaling dimensions for $N \to \infty$ and finite λ from quantum spectral curve [Gromov, Kazakov, Leurent, Volin '14] .
  - Implemented numerically for entire spectrum just recently [Gromov, Hegedus, Julius, Sokolova '23] .
- At small λ matches weak coupling, at large λ single trace operators like Konishi match stringy prediction:

$$\Delta_{\text{Kon}} = 2\lambda^{1/4} - 2 + 2/\lambda^{1/4} + \dots ,$$

- Higher traces just trivial products of single traces, e.g. lowest double trace has $\Delta = 2 + 2$.
- OPE coefficients not yet computed for generic operators.

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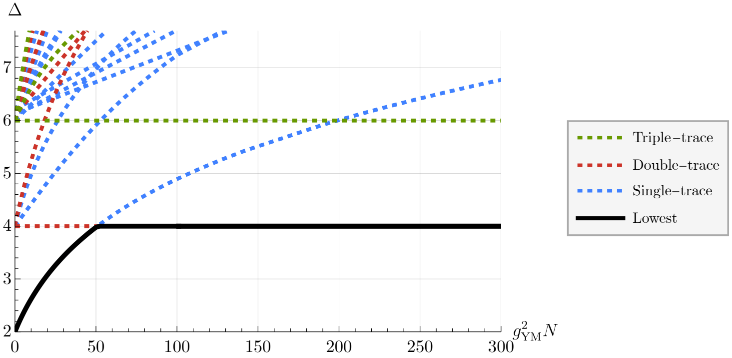

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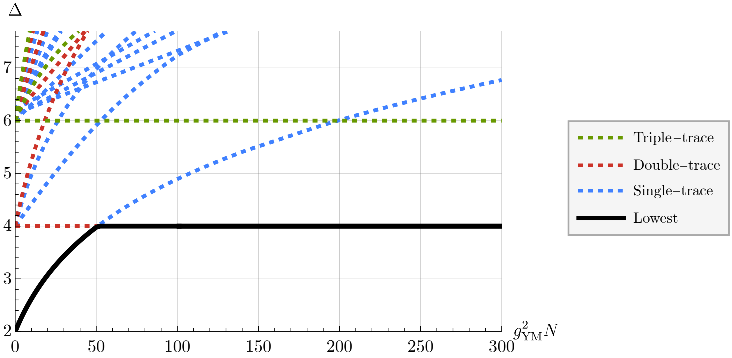

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This talk

Combine non-perturbative methods, bootstrap and supersymmetric localization, to study stress tensor correlator for all N and τ.

Outline:

- Basics of stress tensor correlator.
- Non-perturbative constraints at large or finite N.
- Numerical bootstrap bounds
  - Compare to weak and strong coupling perturbative results.
  - Non-pert improvement to planar integrability spectrum

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$\mathcal{N} = 4$ SYM basics

- All $\mathcal{N} = 4$ CFTs have $SU(4)$ R-symmetry, and are conformal manifolds with one complex parameter τ.
    - Defined by values of central charge $c = \dim(G)/4$ and complex τ.
- $\mathcal{N} = 4$ SYM is gauge theory where matter transform in adjoint of gauge group G, which must be compact classical lie group.
    - For this talk, we take $G = SU(N)$, with $c = \frac{N^2-1}{4}$.
- Duality group of $\mathcal{N} = 4$ $SU(N)$ SYM is $SL(2, \mathbb{Z})$.
    - Self dual points are $\tau = i$ with enhanced $\mathbb{Z}_2$, and $\tau = e^{\frac{i\pi}{3}}$ with $\mathbb{Z}_3$.

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Stress tensor correlator

- 4-point function of stress-tensor superprimary S^a with $\mathbf{20}'$ index a:

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- $\langle SSSS\rangle$ Ward identity has formal solution [Dolan, Osborn '02] :

$$G^{abcd}(U,V) = G^{abcd}(U,V)_{\text{short}} + \Theta^{abcd}(U,V)\mathcal{T}(U,V)\,.$$

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- 4-point function of stress-tensor superprimary S^a with $\mathbf{20}'$ index a:

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- Expand $\mathcal{T}(U, V)$ in even spin ℓ 4d conformal blocks $g_{\Delta,\ell}(U, V)$:

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- Impose that $\langle S^a(x_1)S^b(x_2)S^c(x_3)S^d(x_4)\rangle$ is permutation invariant.
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$$\mathcal{T} = \frac{\mathcal{T}_R}{c} + b_1 \frac{\mathcal{T}_{R^4}}{c^{7/4}} + \frac{\mathcal{T}_{R|R} + b_2 \mathcal{T}_{R^4}}{c^2} + \frac{b_3 \mathcal{T}^1_{D^4R^4} + b_4 \mathcal{T}^2_{D^4R^4}}{c^{9/4}} + \dots .$$

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- Derivatives of free energy $F(m)$ deformed by hyper mass relate to S^4 integrals of correlator [Binder, SMC, Pufu, Wang '19; SMC, Pufu '20] :

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Mass deformed sphere partition function

- Computed using localization in terms of rank(G) dimensional matrix model integral for gauge group G [Pestun '08] .
- For $SU(N)$ we have explicitly (with $a_{ij} \equiv a_i - a_j$):

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  - $H(z)$ is product of Barnes G-functions.
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- Eisenstein series diverges for weak coupling due to $1/g_{\rm YM}$ terms:

$$E(\tfrac{3}{2};\tau) = \frac{16\pi^{3/2}\zeta(3)}{g_{\rm YM}^3} + \frac{1}{3}\pi^{3/2} g_{\rm YM} + \sum_{k=1}^{\infty} \frac{32\cos(\theta)\pi^{3/2} k\sigma_{-2}(k) K_1\left(\frac{8k\pi^2}{g_{\rm YM}^2}\right)}{g_{\rm YM}}$$

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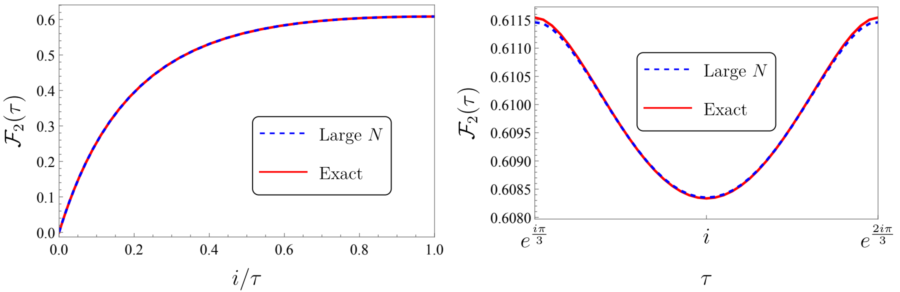

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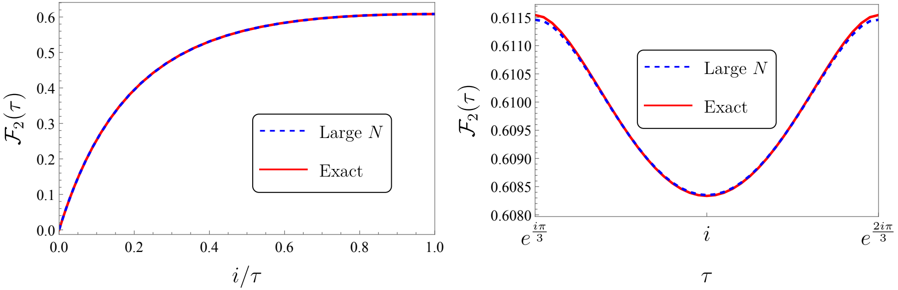

Numerical bootstrap+localization

- Combine all non-perturbative constraints (unitarity, crossing, localization) to bootstrap CFT data [SMC, Dempsey, Pufu '21] .
  - Input N via c in short contributions.
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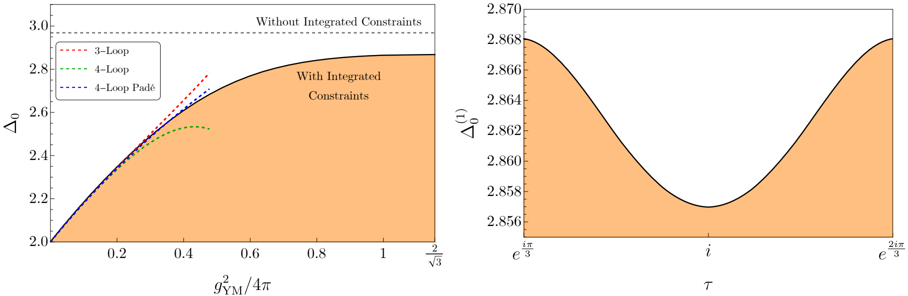

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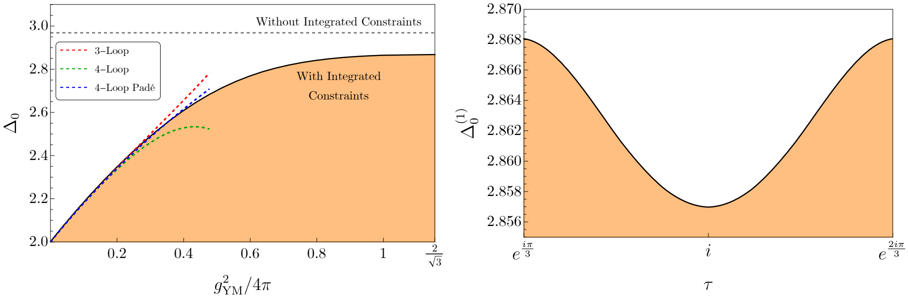

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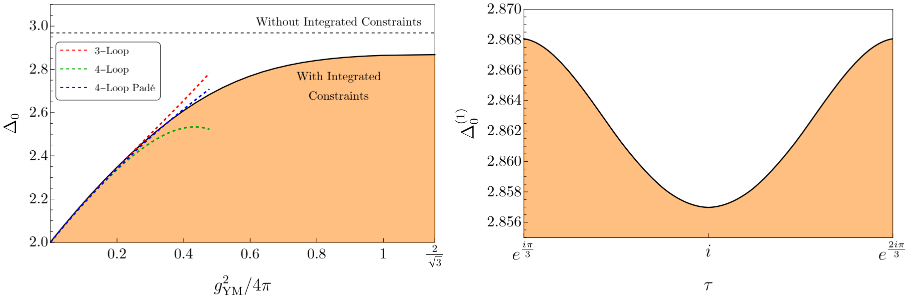

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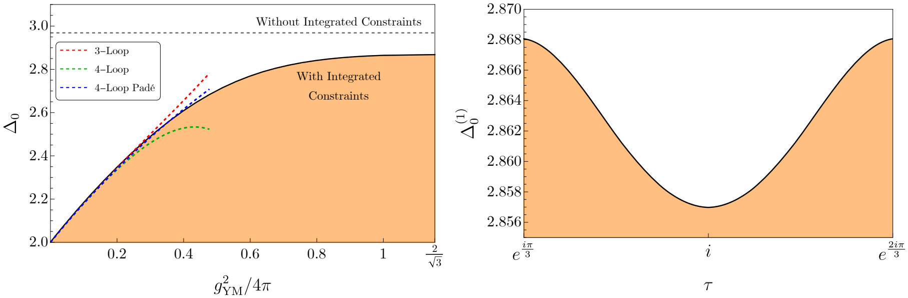

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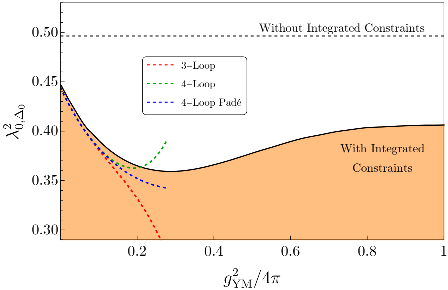

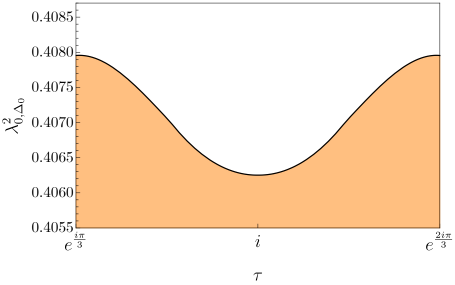

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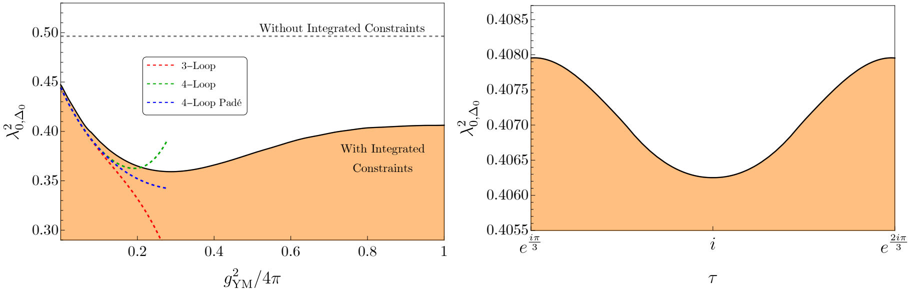

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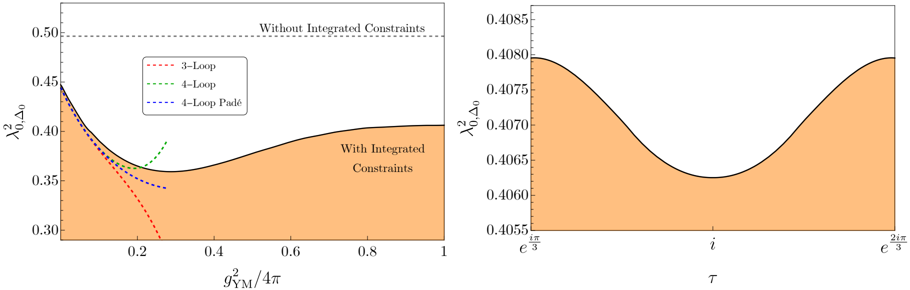

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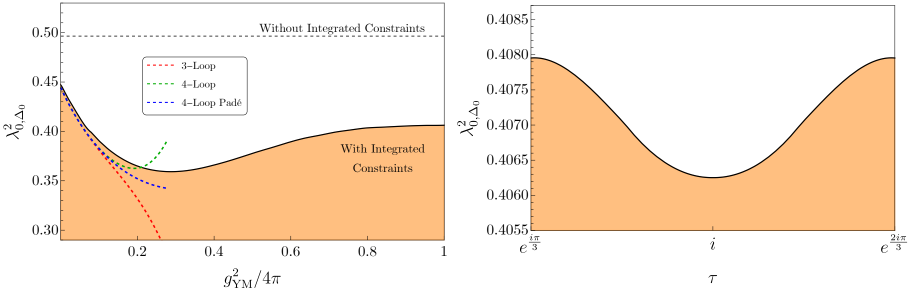

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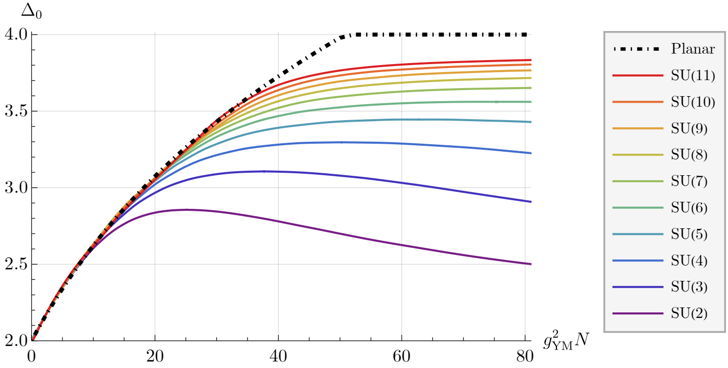

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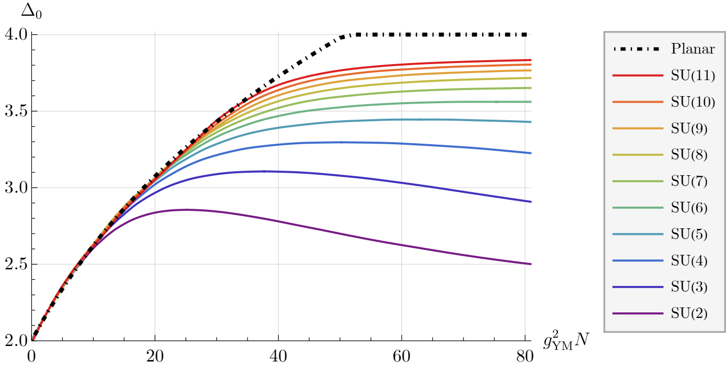

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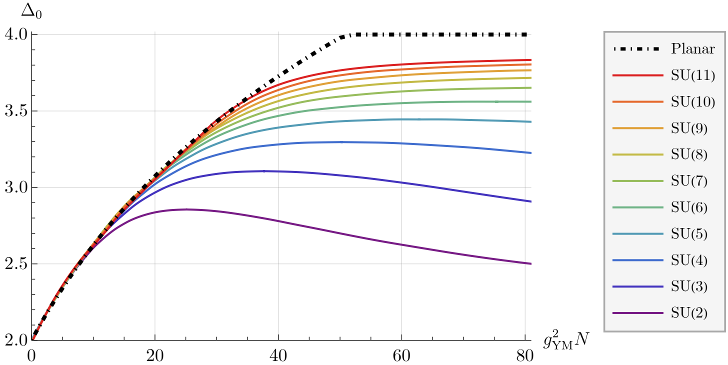

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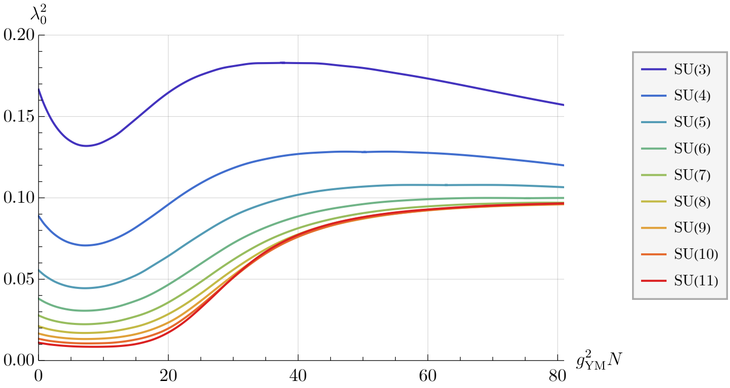

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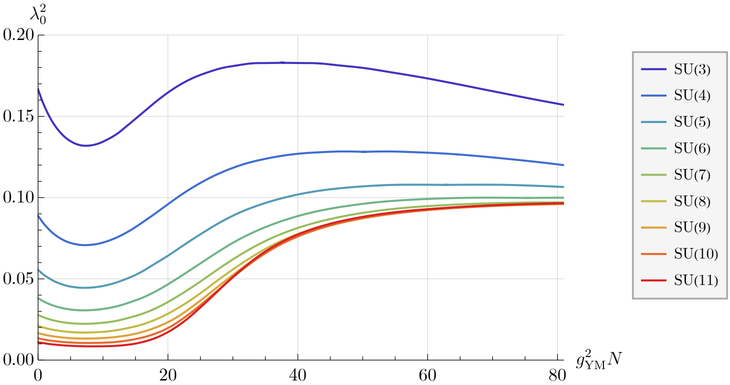

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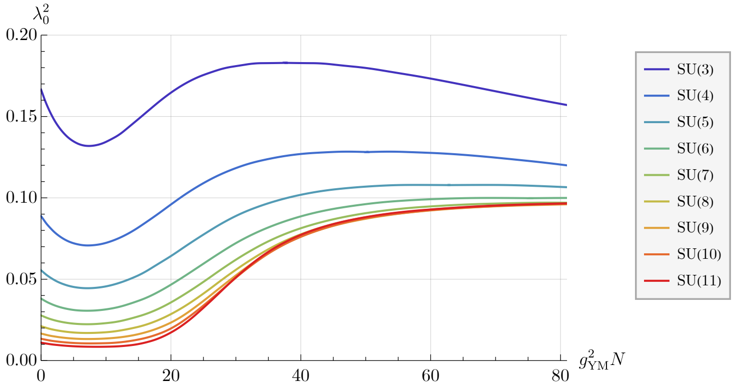

- For large N, convergence gets worse, we computed many Λ and extrapolated to $\Lambda \to \infty$ (see next slides for more details).
- No planar integrability results to compare to now.

Analytic bootstrap+localization

- Recall that analytic bootstrap (i.e. crossing, pole structure of Witten diagrams, and flat space limit) fixes correlator to:

$$\mathcal{T} = \frac{\mathcal{T}_R}{c} + b_1 \frac{\mathcal{T}_{R^4}}{c^{7/4}} + \frac{\mathcal{T}_{R|R} + b_2 \mathcal{T}_{R^4}}{c^2} + \frac{b_3 \mathcal{T}^1_{D^4 R^4} + b_4 \mathcal{T}^2_{D^4 R^4}}{c^{9/4}} + \dots .$$

- 2 localization constraints fix b_i in terms of Eisensteins and generalized eisensteins that appear in localization inputs.
    - Matches type IIB S-matrix in flat space limit at finite τ [SMC, Green, Pufu, Wang, Wen '19] . 1-loop b_2 fixed in [SMC '19] .
- Extract CFT data of double trace operators, e.g. lowest Δ:

$$\Delta_{4,0} = 4 - \frac{4}{c} + \frac{135}{7\sqrt{2}\pi^{3/2}c^{7/4}} E(\tfrac{3}{2},\tau) + \frac{1199}{42c^2} - \frac{3825}{32\sqrt{2}\pi^{5/2}c^{9/4}} E(\tfrac{5}{2},\tau) + \dots$$

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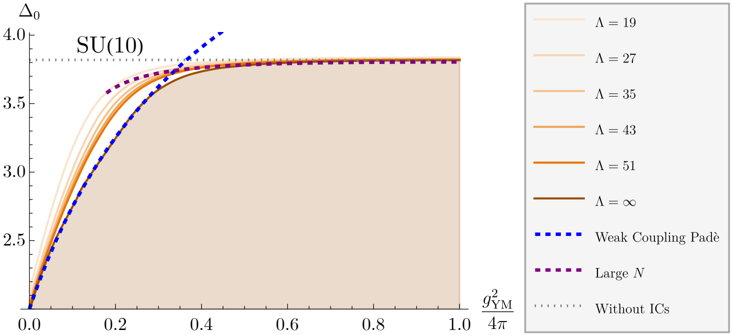

- Use extrapolation to overcome slow convergence (next slides).
- Matches BOTH weak coupling and strong coupling expansions!
- Observe non-pert **level repulsion**, in between weak coupling for single trace and strong coupling for double trace.

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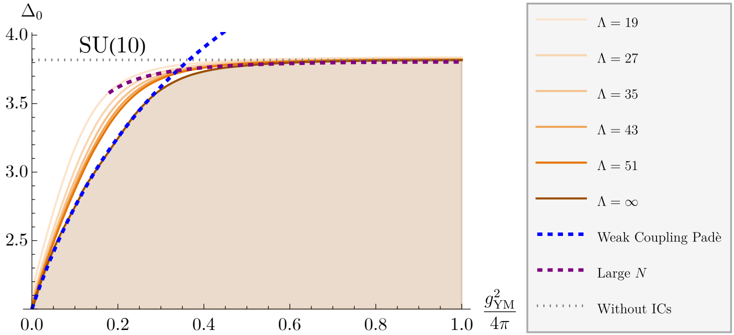

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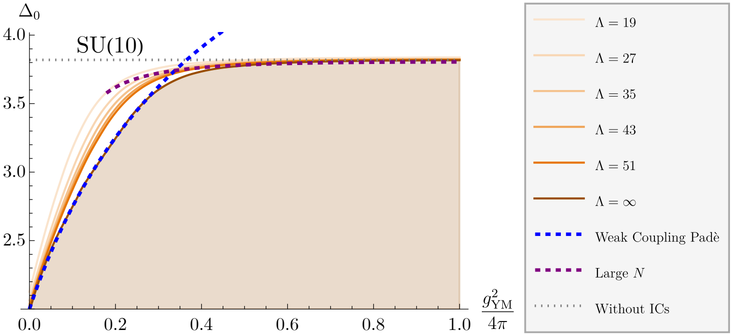

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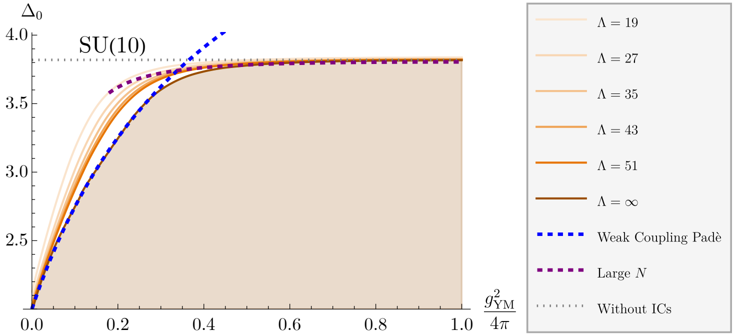

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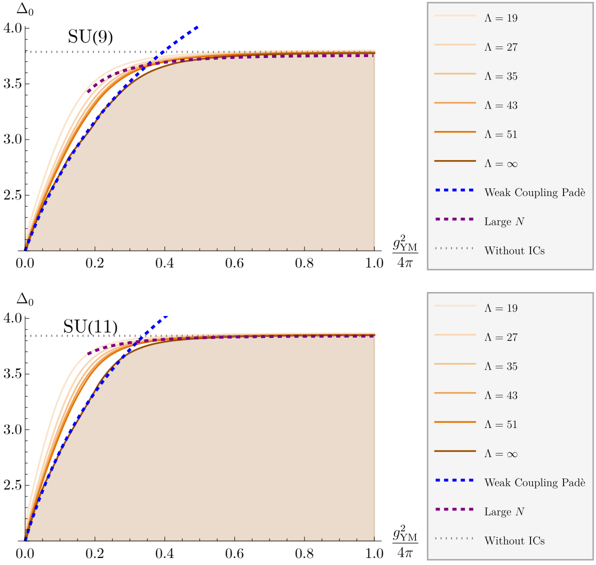
Bounds: Lowest Δ for $SU(9)$ and $SU(11)$



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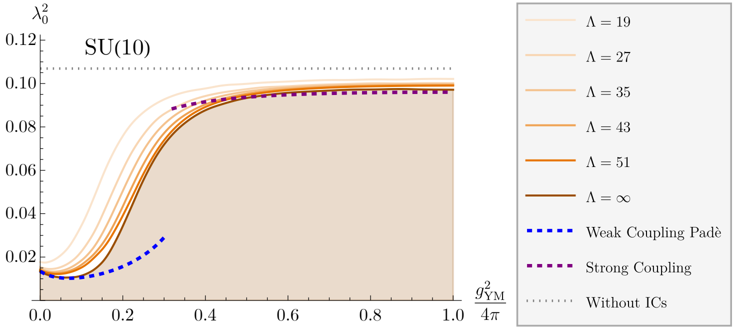

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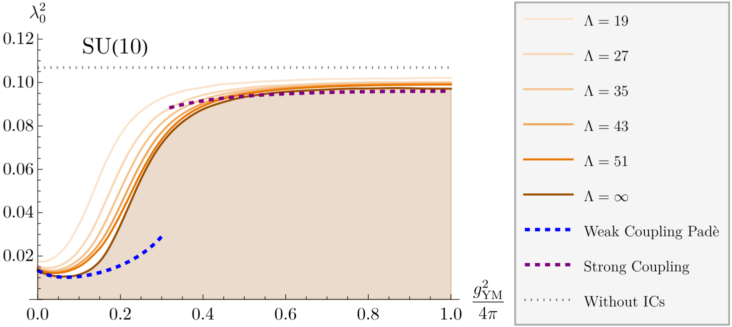

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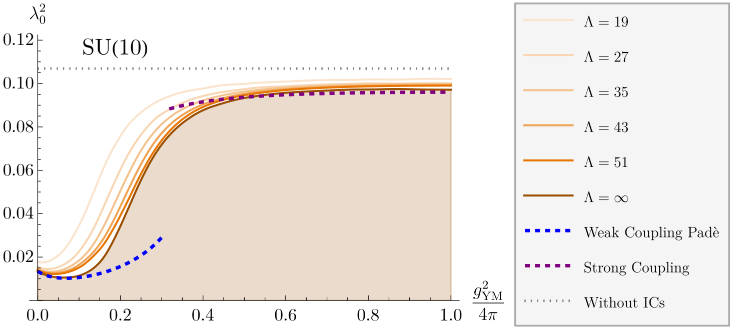

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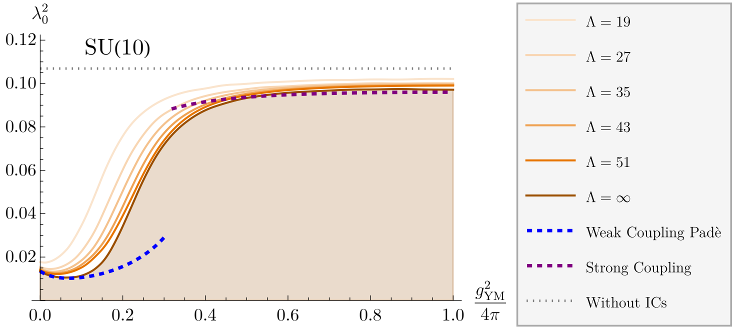

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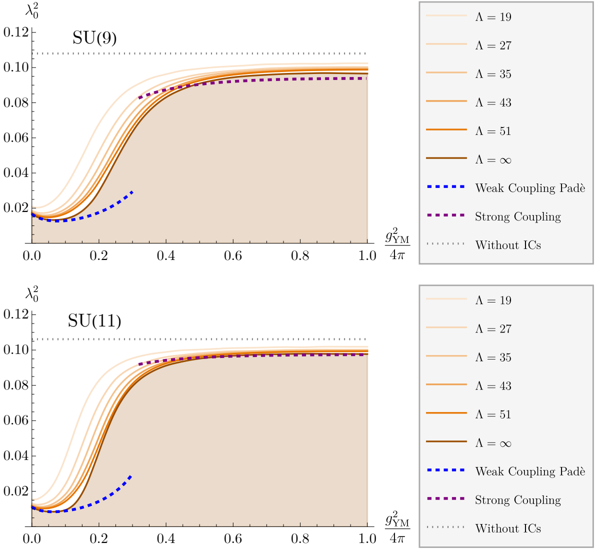
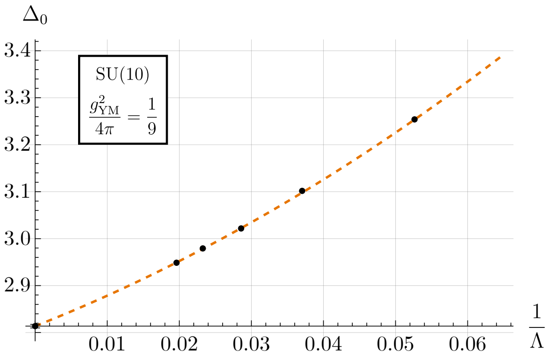


Extrapolation in Λ



- We use simple polynomial ansatz for extrapolation: $\Delta = \Delta_0 + \Delta_1/\Lambda + \Delta_2/\Lambda^2$.
- Similar ansatz used in original $\mathcal{N} = 4$ bootstrap [Rastelli, van Rees '13] .
- Extrapolation gives results that match perturbative data for all N.

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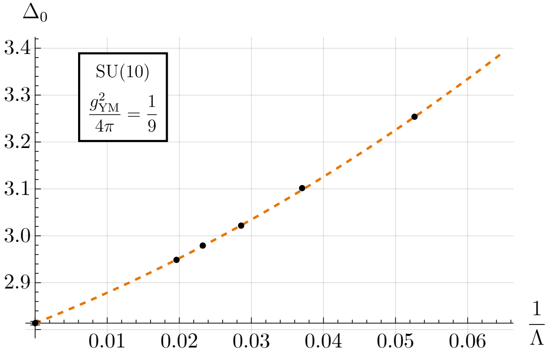

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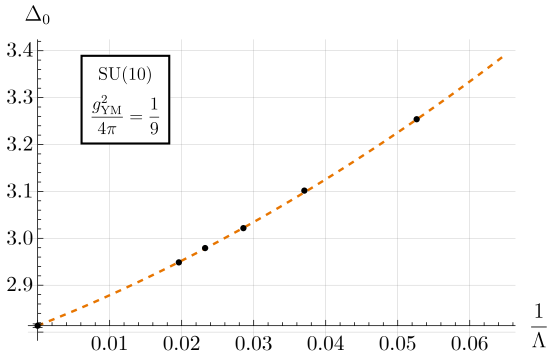

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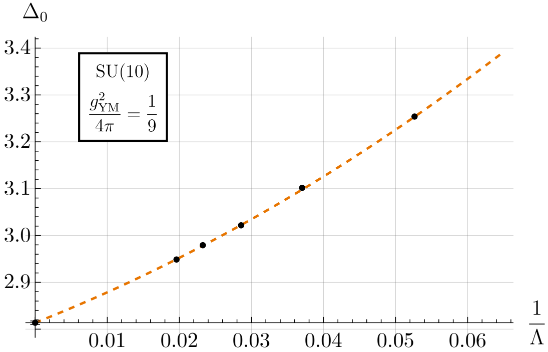

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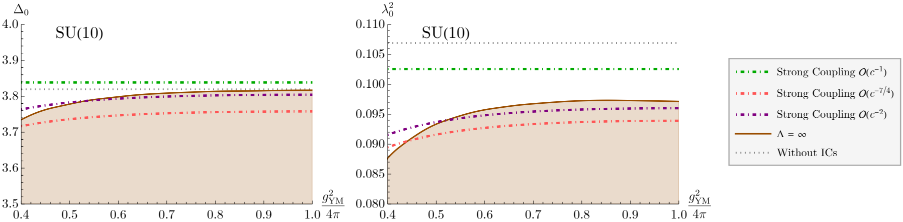

- For largish N (e.g. $SU(10)$), we see that analytic bootstrap result gets closer to bound as we include more $1/c$ corrections.
- $1/c$ is supergravity, $1/c^{7/4}$ is R^4 correction [SMC, Green, Pufu, Wang, Wen '19] , $1/c^2$ is 1-loop correction [Alday, Bissi '17; Aprile, Drummond, Heslop, Paul '17] (which included contact term fixed from localization [SMC '19]).
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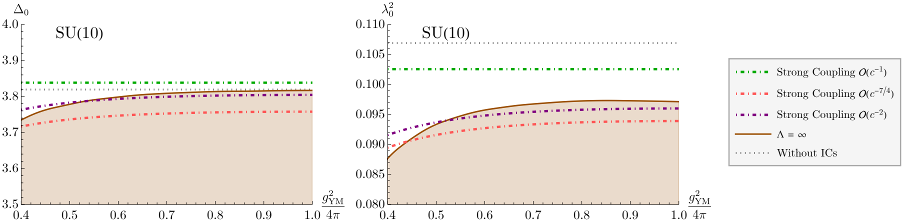

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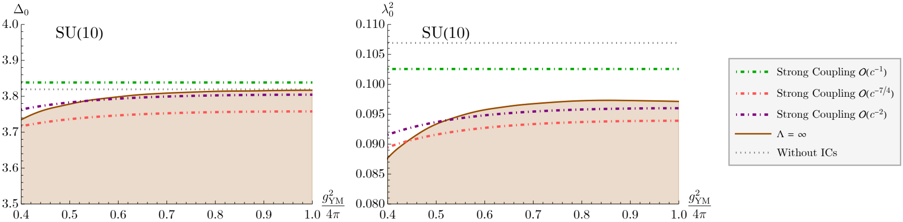

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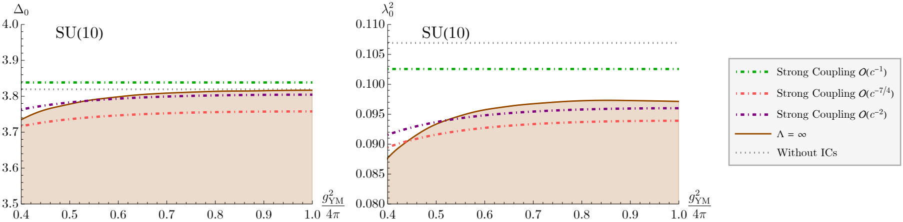

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- For smallish g_{YM}, bounds saturated by weak coupling (indistinguishable from integrability in this regime) for single trace.
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Near future directions

- More accurate bounds, sensitive to higher twist or spin operators.
  - In bootstrap without localization, higher spins usually easier to access than higher twists, but with localization both equally hard.
- Get greater accuracy from imposing more localization constraints (e.g. from the squashed sphere), or from mixing with lowest dimension long operator (which is also relevant).
- If we are sensitive to second lowest twist, then impose ≤ 2 relevant operators to get islands for each τ, rigorously solve SYM!
- Combine localization + bootstrap to numerically solve ANY 3d $\mathcal{N}=2$, 4d $\mathcal{N}=2$, or 5d $\mathcal{N}=1$ Lagrangian CFT, e.g.:
  - 4d $\mathcal{N}=2$ dual to open strings [SMC '22; Behan, SMC, Ferrero '23] .
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- Imagine bootstrap sensitive to higher twist operators with $\Delta \sim c \sim N^2$.
- These operators are dual to black-hole states for largish N.
  - First steps to computing 1/16-BPS black-hole states for low N in [Chang, Lin '22] , but not for unprotected black hole states.
- Can study statistics of black-hole states, i.e. how many states appear in given window of Δ.
- Can see how these statistics change as function of τ and N, i.e. as we go from weak to strong coupling.

See you in Kyoto!

- Bootstrap, Localization, and Holography, May 20-24
- Some funding for students, poster session!