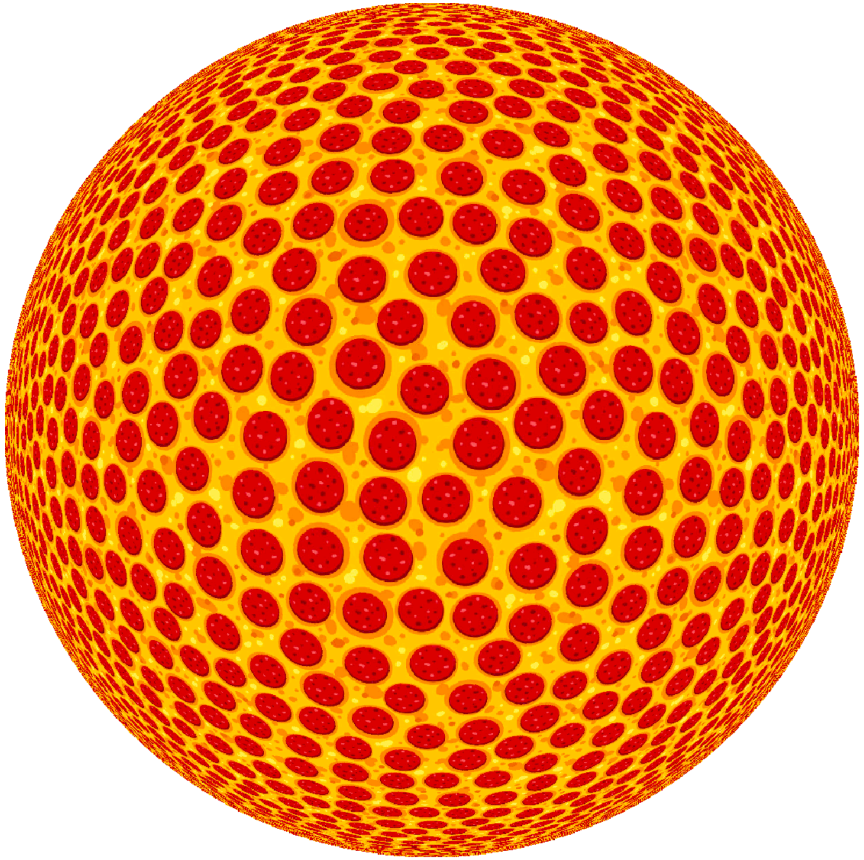
Bootstrability

50 + ε years of the conformal bootstrap
22 Feb 2024
Pisa



European Research Council

EXACTC

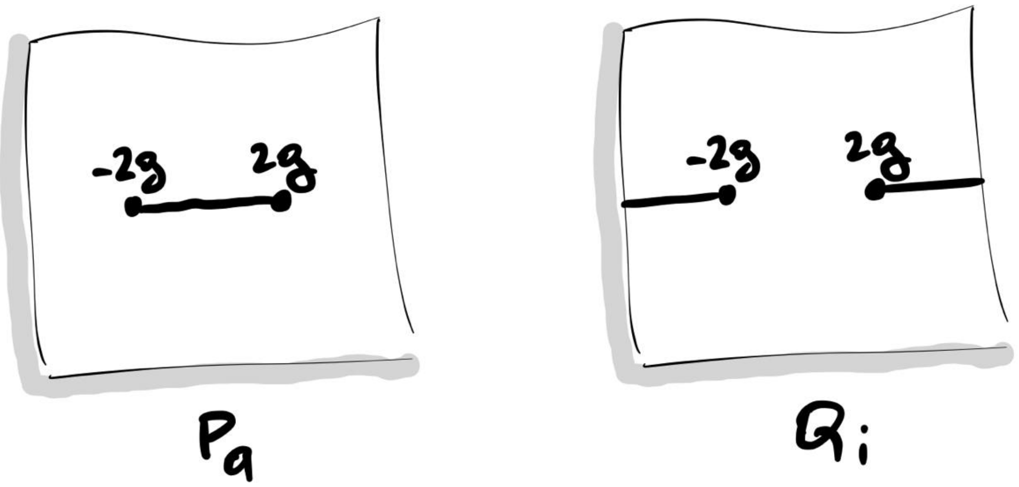
**While bologna actually derives its name from Bologna… The City of Pisa has nothing to do whatsoever with Pizza. Pizza actually is another way of saying pie in Italian…*

ongoing work with Michelangelo, Andrea, Julius and Nika

Integrability: current status and open questions

Quantum Spectral Curve

1) Impose analytic properties on 4+4 functions



2) Impose algebraic constraints

$$P_a(u)\,P^a(u+i) = Q_i(u)\,Q^i(u+i)$$

$$P_a(u)\,P^a(u+2i) = Q_i(u)\,Q^i(u+2i) + Q_i(u+2i)\,Q^i(u+i)\,Q_j(u+i)\,Q_j(u)$$

+ 2 more

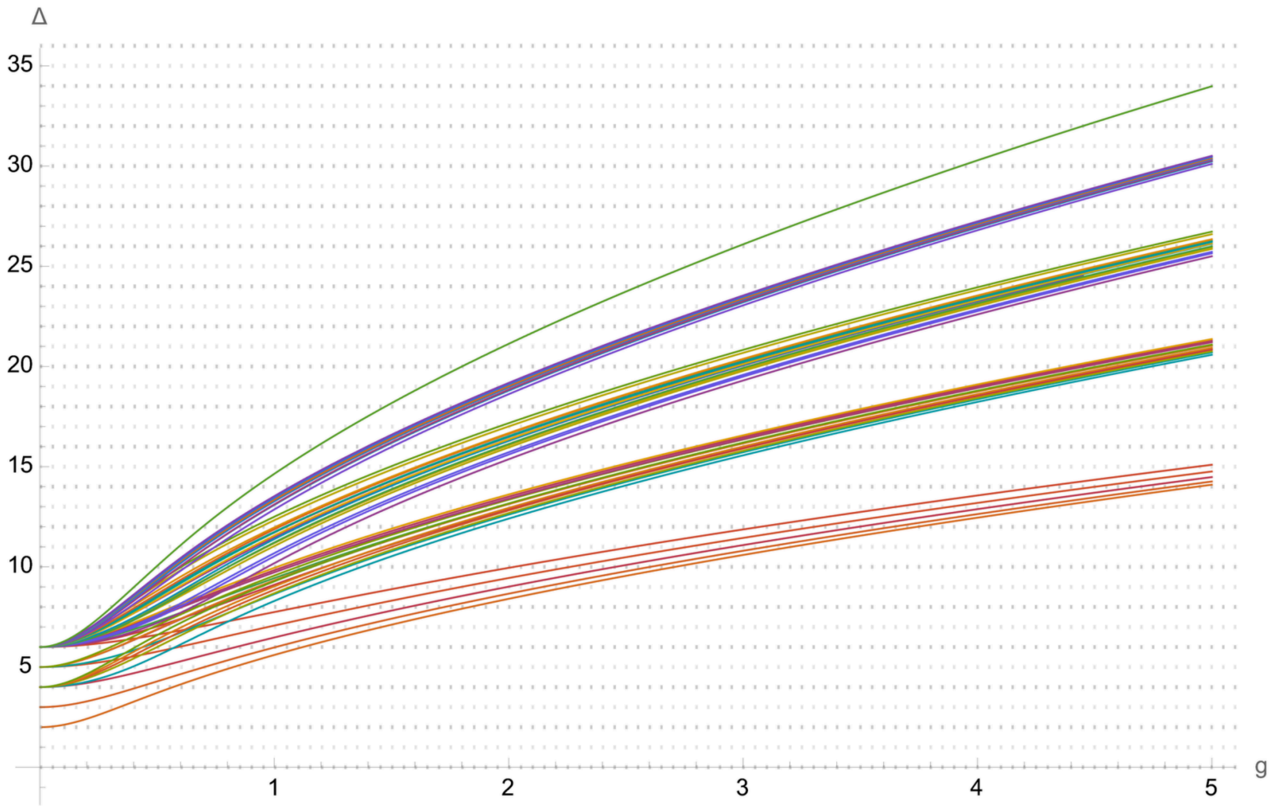
3) Read off charges of the state from asymptotics

$$P_a \sim u^{\text{R-charges}} \qquad Q_i \sim u^{\pm\Delta \pm S}$$

(S ← splus)

Local operators

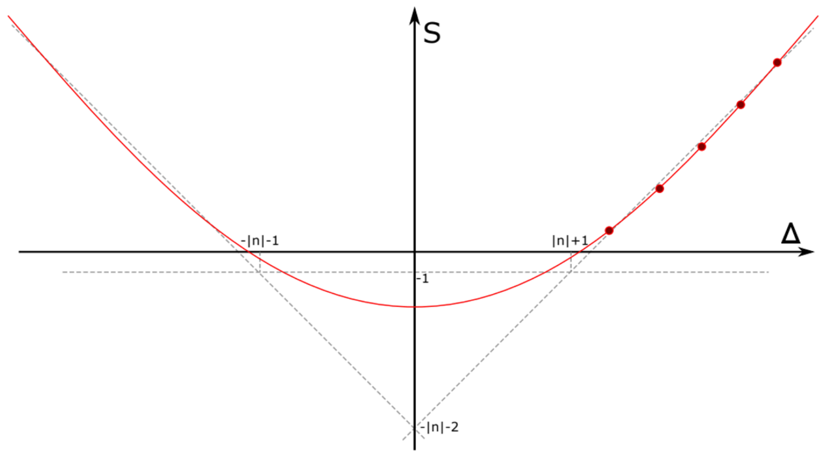
[NG, Julius, N.Sokolova '23]

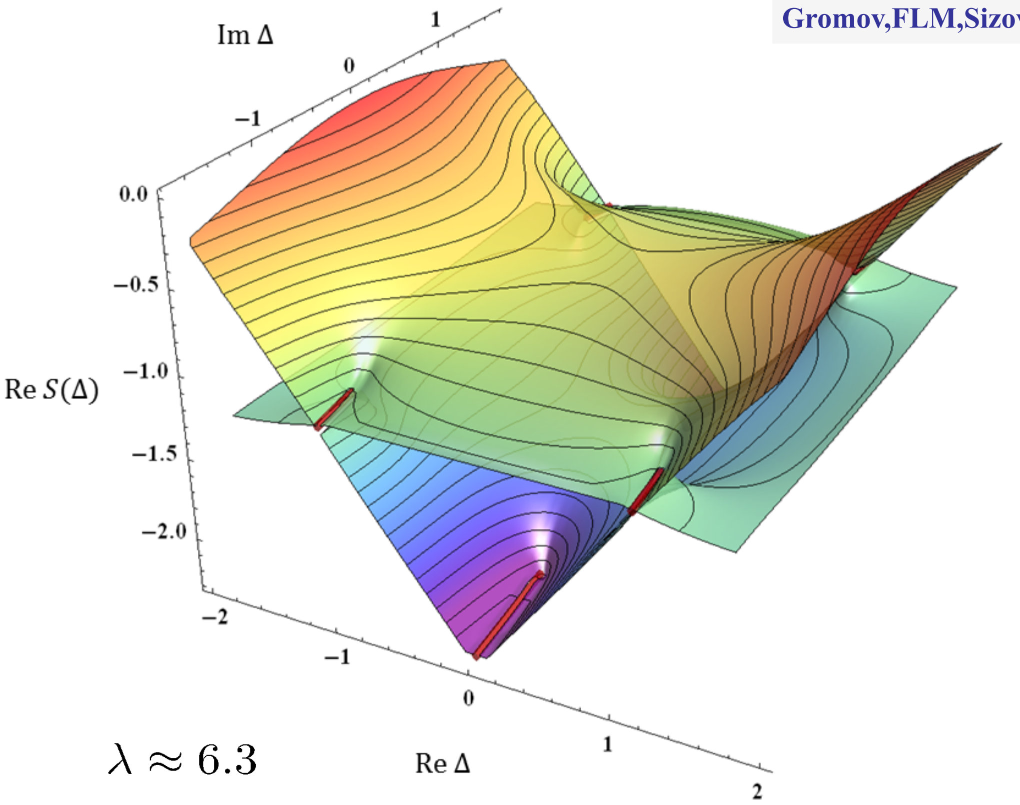


- **All 219 states with bare dimension <= 6**
- **Improved performance at weak coupling, allowing to start numeric from perturbation theory (1 loop could be enough)**
- **C++ code to generate more if needed (https://github.com/julius-julius/qsc)**
- **Merges with analytic bootstrap at strong coupling giving new predictions for structure constants**

[LF Alday, T Hansen, JA Silva '23]
[LF Alday, T Hansen '23]

Light-ray operators?





$\lambda \approx 6.3$

Gromov,FLM,Sizov 2015

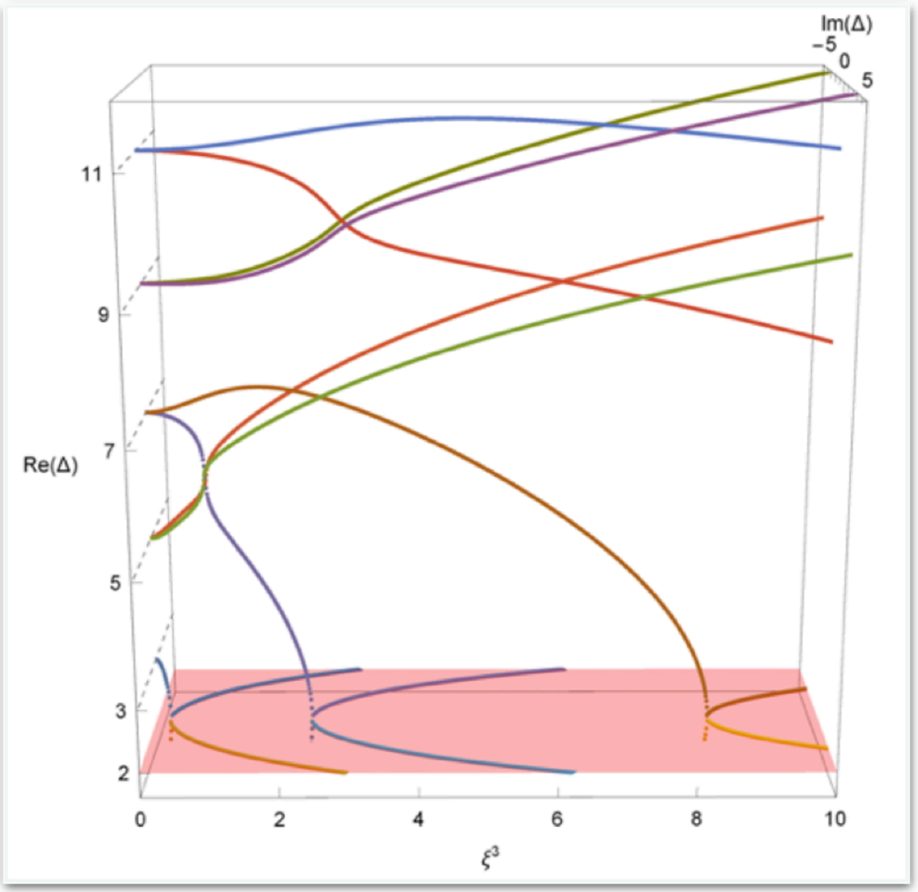
SUSY?

β, γ – deformations, orbifolds etc

Extreme case – fishnet:

$$\mathcal{L}_{4d} = N\,\mathrm{tr}\left(|\partial\phi_1|^2 + |\partial\phi_2|^2 + (4\pi)^2\xi^2\phi_1^\dagger\phi_2^\dagger\phi_1\phi_2\right)\,,$$

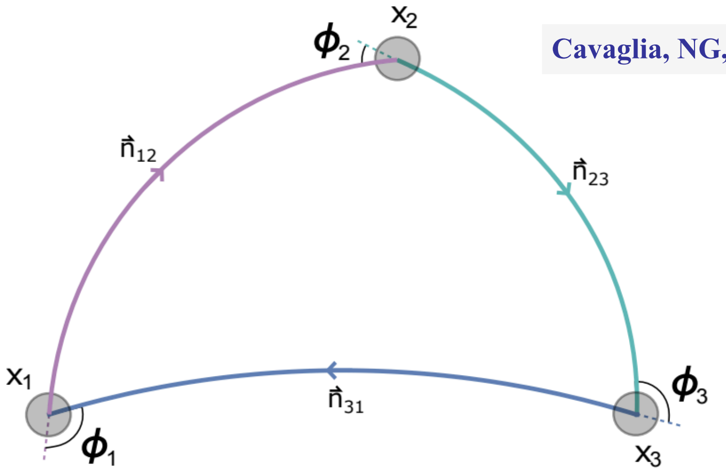
RG-flow for dbl trace terms at 7 loops:



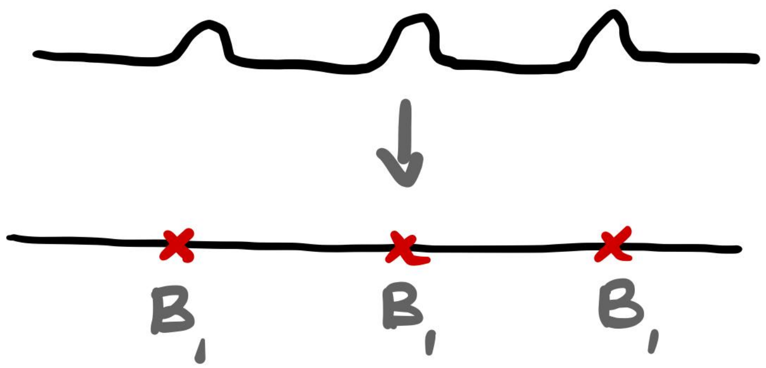
Going beyond the spectrum

- In a planar theory n-point correlators of local reduce to a product of 2-point functions
- As it is a gauge theory there are **non-local** observables such as Wilson-Loops

Cavaglia, NG, Fedor LM `18



Strategy 1: approximate by cusps SoV



Strategy 2: expand around straight line (**this talk**)

- First non-trivial correction to 3pt is very interesting: before wrapping at weak coupling integrability, Bethe ansatz and more general Hexagon works well till the wrapping order
- Beyond wrapping there are signs that Hexagon+QSC can re-sum all orders for HLL at least

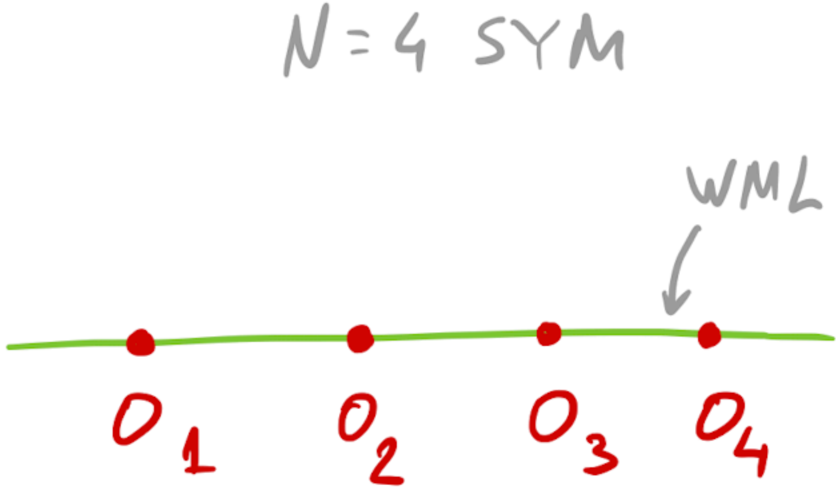
Tools to go beyond spectrum

Integrability: SoV, Hexagons

Bootstrability = Integrability + Bootstrap

Is the spectrum of N=4 and its deformations sufficient to solve planar theory?

Concrete set-up



P. Liendo, C. Meneghelli, and V. Mitev `18
P. Ferrero and C. Meneghelli `21
P. Liendo and C. Meneghelli `16
L. Bianchi, G. Bliard, V. Forini, L. Griguolo, and D. Seminara `20

N. Drukker `12
D. Correa, J. Maldacena, and A. Sever `12
N. NG and F. Levkovich-Maslyuk `15

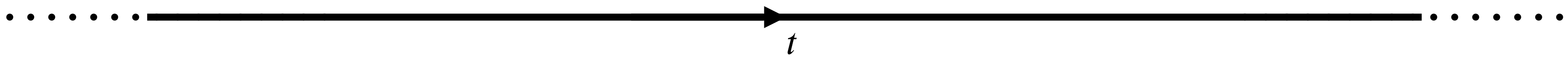
Know the spectrum of the defect CFT

All correlators are O(N^0)

No “double traces” problem

Price to pay: less symmetry, same number of operators

1/2 BPS Wilson line



$$W = \mathrm{TrP}e^{\int dt\left(iA_t + \Phi_{||}\right)}$$

[Maldacena, '98]

- 16 preserved supercharges

  Bosonic subgroup $SO(1,2) \times SO(3) \times SO(5)$ ⇒ $OSp(2,2|4)$

- Superconformal defect where local operators reorganise in representations of the preserved superalgebra.

Protected 1/2 BPS multiplets

$$\mathscr{B}_1 = \Phi^i_\perp,$$

$$\mathscr{B}_2 = \Phi^{(i}_\perp \Phi^{j)}_\perp,$$

Non-protected long operators

$$O_{\Delta_1} = \Phi_{||},$$

$$\Phi^2_{||}, \quad \Phi^i_\perp \Phi^i_\perp, \quad \dots$$

- OPE e.g.

$$\mathscr{B}_1 \times \mathscr{B}_1 = \mathscr{I} + \mathscr{B}_2 + \sum_\Delta \mathscr{L}^\Delta_{0,[0,0]}$$

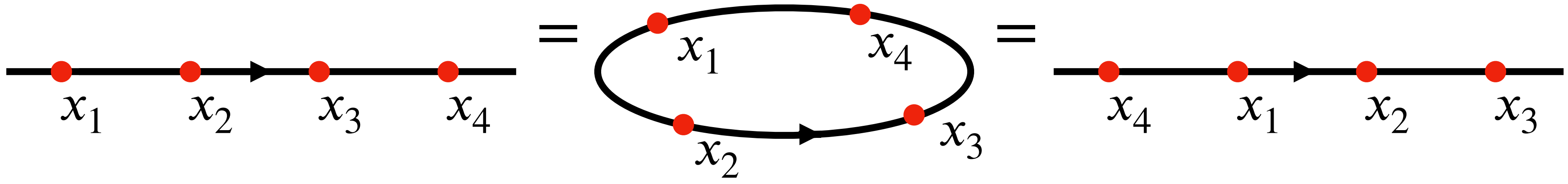
Simplest 4-point function

The simplest operator in $\mathscr{B}_1$:

$$\left\langle\left\langle \Phi_\perp(x_1)\Phi_\perp(x_2)\Phi_\perp(x_3)\Phi_\perp(x_4) \right\rangle\right\rangle = \frac{1}{x_{12}^2}\frac{1}{x_{34}^2}\mathscr{A}(\chi)$$

With cross ratio $\chi = \dfrac{x_{12}x_{34}}{x_{13}x_{24}}$ and $x_{ij} = x_i - x_j$

Crossing equation:



$$\frac{1}{x_{12}^2 x_{34}^2}\mathscr{A}(\chi) = \frac{1}{x_{14}^2 x_{23}^2}\mathscr{A}(1-\chi)$$

Constraints of superconformal symmetry

Studying the related topological sector

$$\left\langle W_{\text{circle}} \right\rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$$

$$\mathbb{F} = 1 + C_{BPS}^2 = \frac{3 W_{\text{circle}} W''_{\text{circle}}}{(W'_{\text{circle}})^2}$$

reproduced by integrability

[Cavaglià, NG, Julius, Preti '22]

Known differential operator

$$\mathcal{A}(\chi) = \mathbb{F}\chi^2 + \mathcal{D}_\chi \circ f(\chi)$$

[Liendo Meneghelli Mitev '17]

Operator Product Expansion

$$f(\chi) = f_I(\chi) + C_{BPS}^2(\lambda) f_{\mathcal{B}_2}(\chi) + \sum\nolimits_\Delta C_{1,1,\Delta}^2 f_\Delta(\chi)$$

Superconformal blocks

$$f_I(\chi) = \chi$$

$$f_{\mathcal{B}_2}(\chi) = \chi\big(1 - {}_2F_1(1,2,4;\chi)\big)$$

$$f_\Delta(\chi) = \frac{1}{1-\Delta}\left[\chi^{\Delta+1}\, {}_2F_1(\Delta+1, \Delta+2, 2(\Delta+2); \chi)\right]$$

singlets of $SO(5) \times SO(3)$,
nontrivial dimensions

Constraints of conformal symmetry

The reduced correlator $f(\chi)$ still obeys to the crossing equation

$$\mathcal{G}(\chi) \equiv (1-\chi)^2 f(\chi) + \chi^2 f(1-\chi) = 0$$

and rewriting it in terms of the OPE expansion we obtain

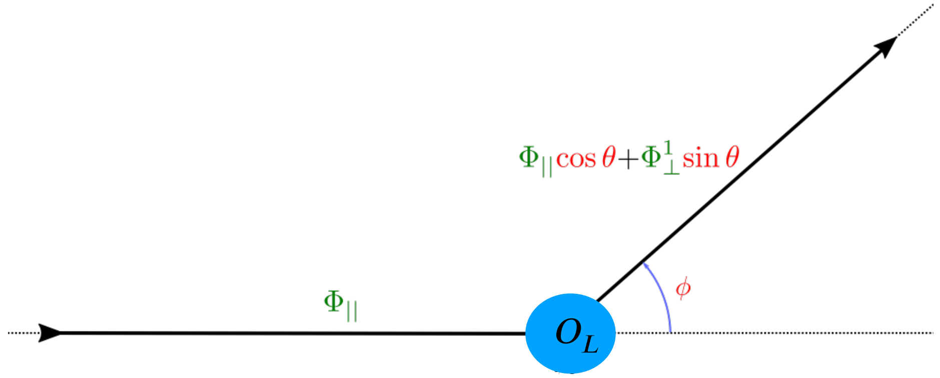
$$\underbrace{\mathcal{G}_I(\chi) + C^2_{BPS}(\lambda)\mathcal{G}_{\mathcal{B}_2}(\chi)} + \sum_{\Delta} C^2_{1,1,\Delta}\mathcal{G}_{\Delta}(\chi) = 0$$

Existing results

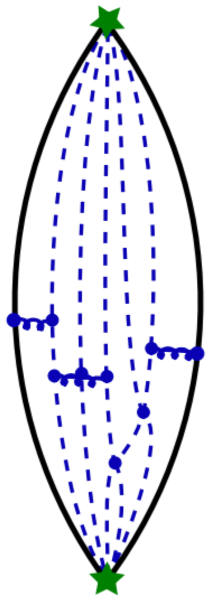
- Numerical bootstrap [Liendo, Meneghelli, Mitev '17]
- Functional bootstrap at strong coupling [Meneghelli Ferrero '21, +more recently]

Integrability for the spectrum of insertions

Integrability for the cusped Wilson line



$$W = \mathrm{tr}\,\mathrm{P}\exp\left(\int_{-\infty}^{0} dt(iA\cdot\dot{x}+\vec{\Phi}\cdot\vec{n}|\dot{x}|)\right)\times O_L \times \mathrm{P}\exp\left(\int_{0}^{\infty} dt(iA\cdot\dot{x}_\phi+\vec{\Phi}\cdot\vec{n_\theta}|\dot{x}_\phi|)\right)$$



Weak coupling

For "orthogonal" insertions: [Correa, Maldacena, Sever '12] [Drukker '12] [Bonini, Griguolo, Preti, Seminara '15] (wrapping terms)

For "parallel" insertions: [Correa, Leoni, Luque '18] (1-loop, one sector)

Non-perturbative

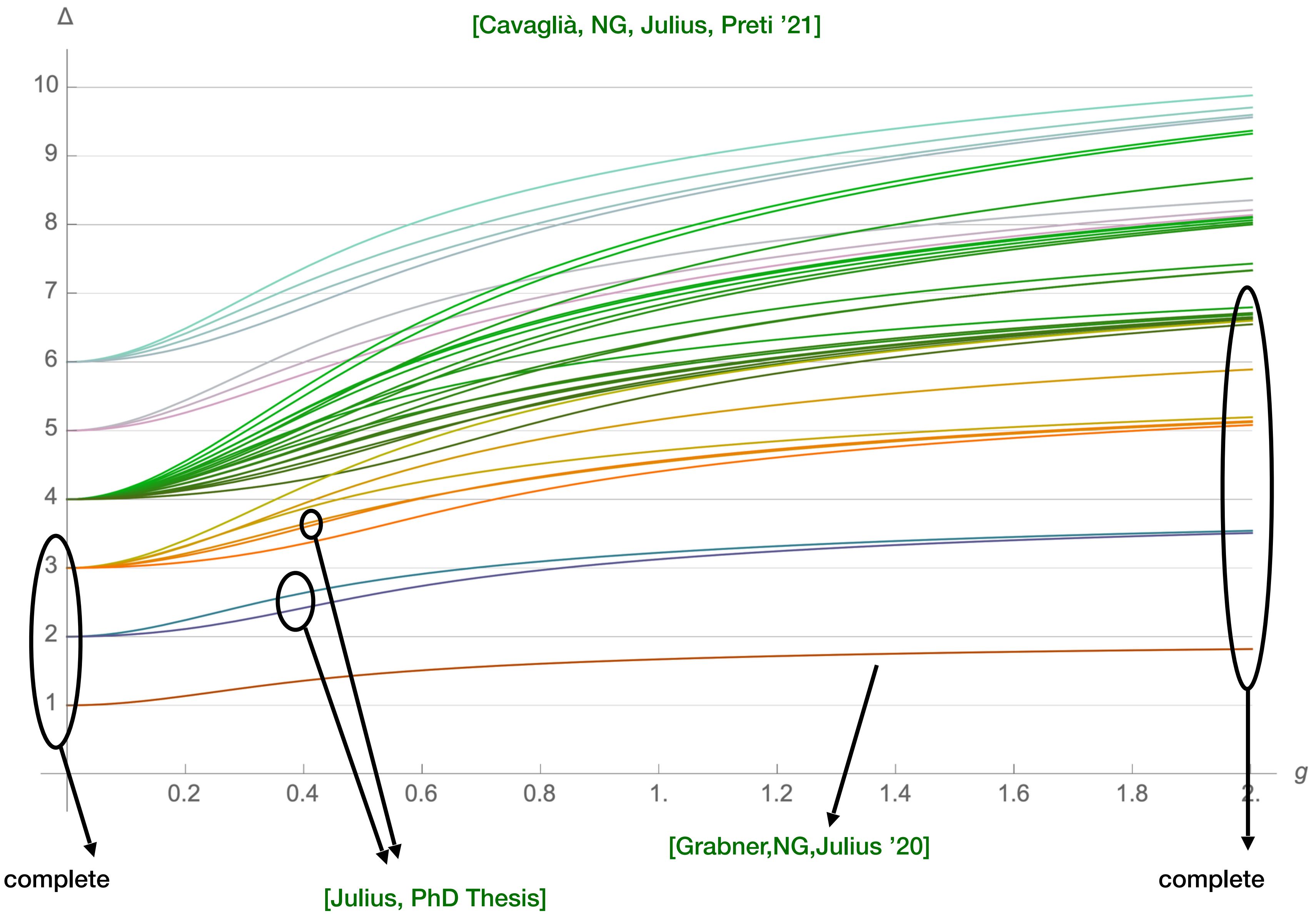
For "orthogonal" insertions: [Correa Maldacena Sever '12] [Drukker '12] (TBA)

[NG,Levkovich-Maslyuk'15] (QSC)

For "parallel" insertions: [Cavaglià, NG, Levkovich-Maslyuk '15] (ladder)

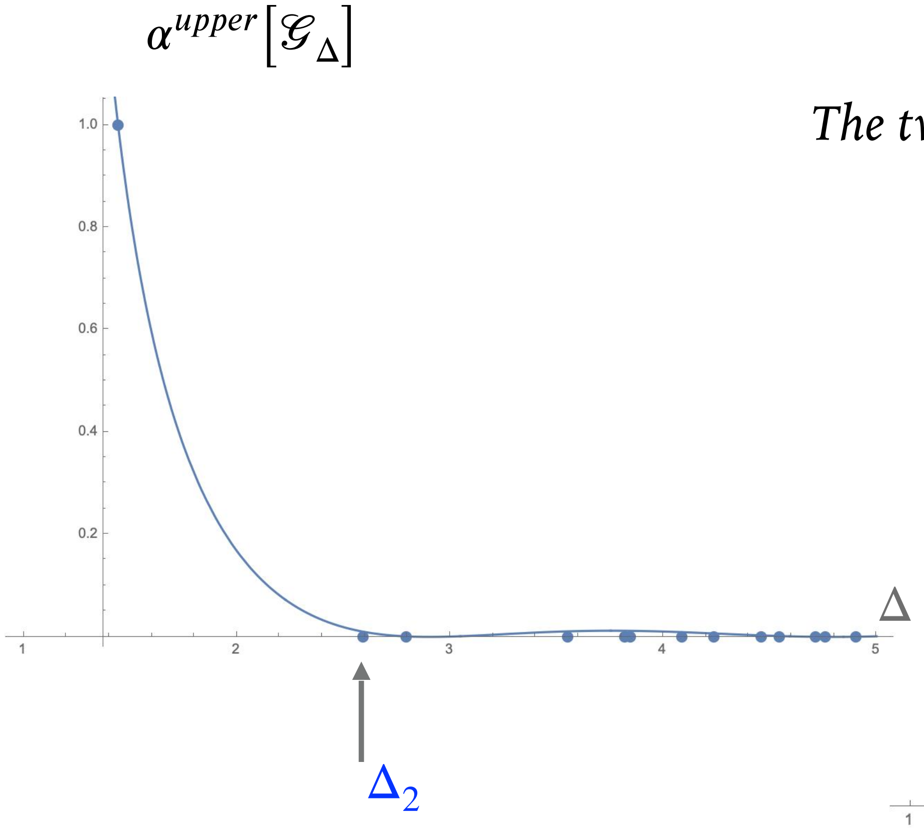
[Grabner, NG, Julius '20] (general θ, ϕ)

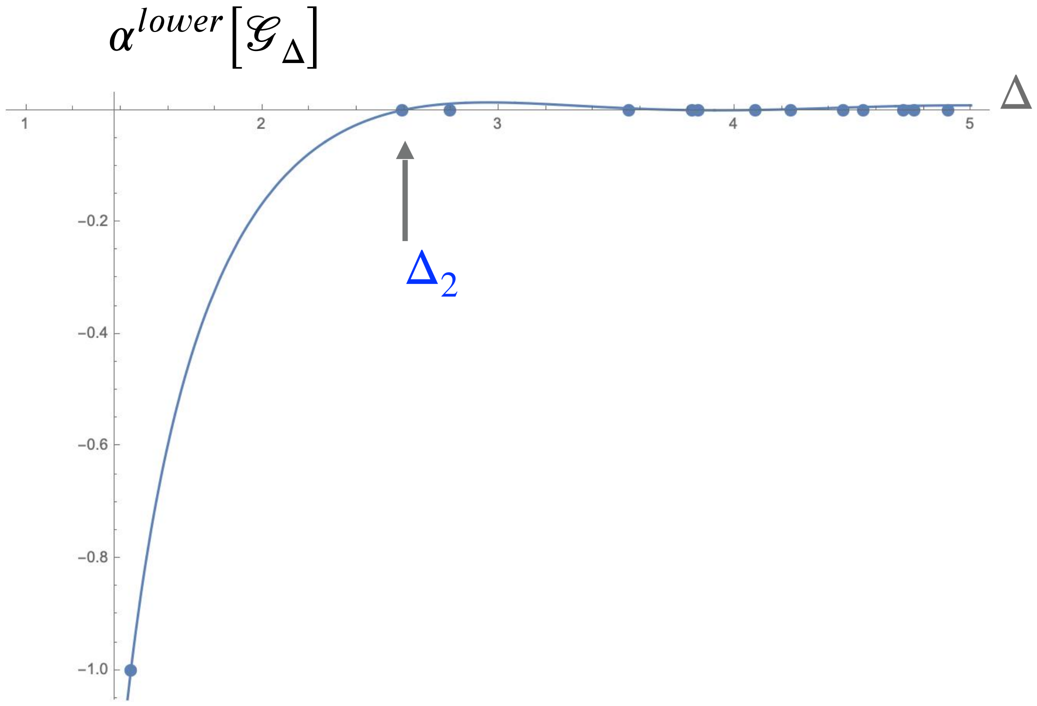
The spectrum, straight WL+2 insertions



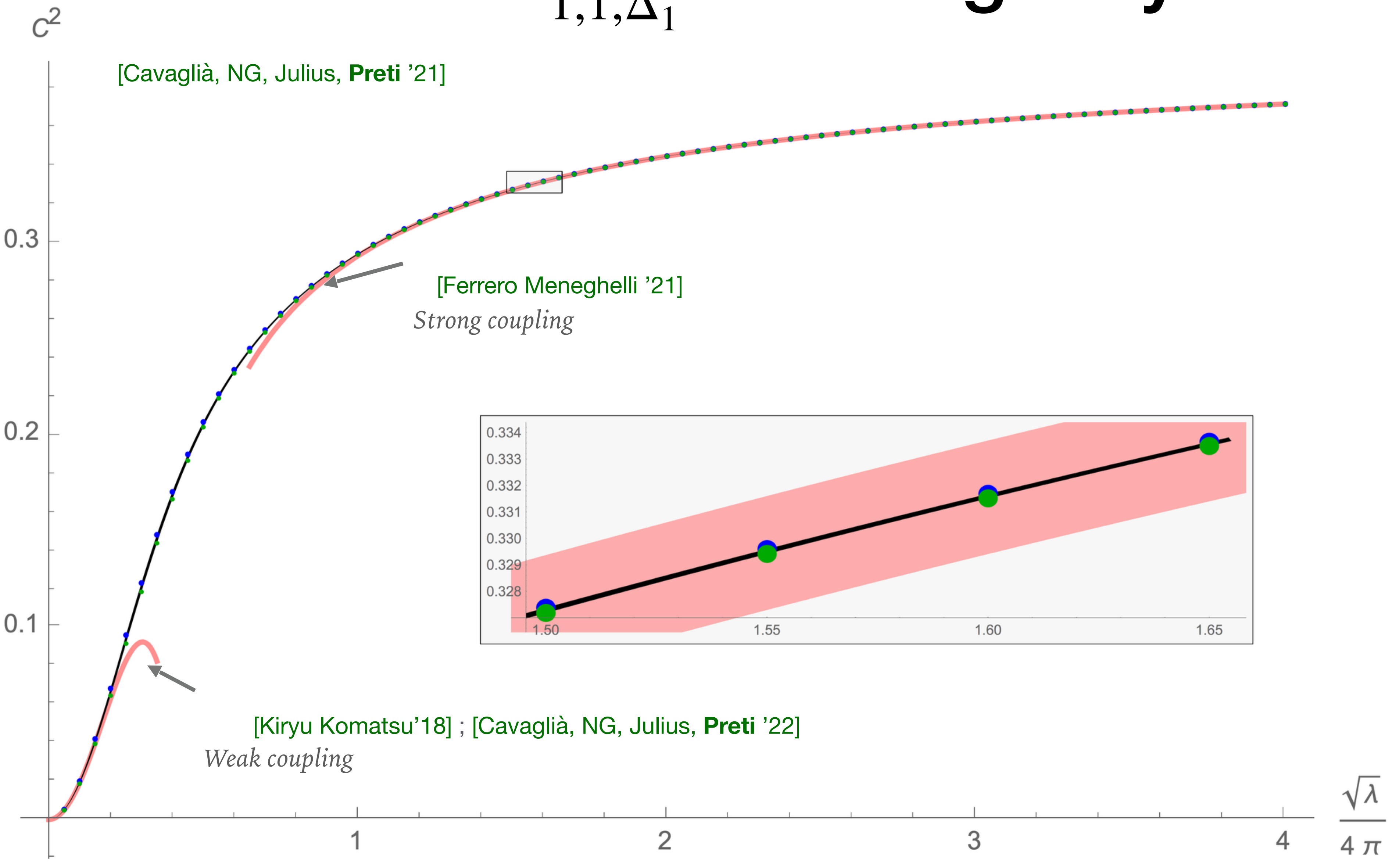
Bootstrapping OPE coefficients

The two functionals for coupling $g=1/2$, $N_{der}=20$



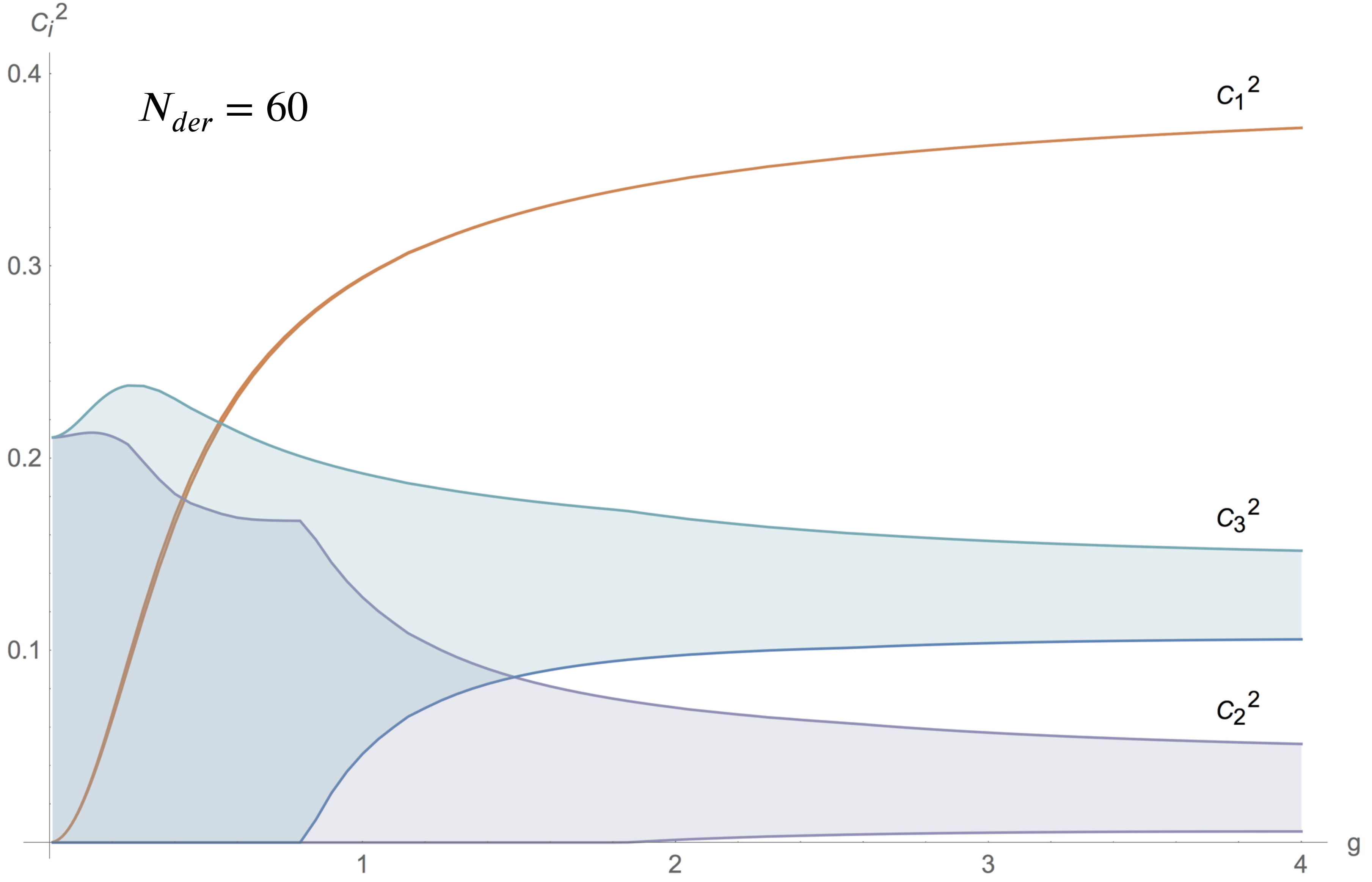


OPE coefficient $C^2_{1,1,\Delta_1}$ including only 2 states



The error is computed measuring the thickness of the region, namely $1/2(C^2_{\text{upper}} - C^2_{\text{lower}})$

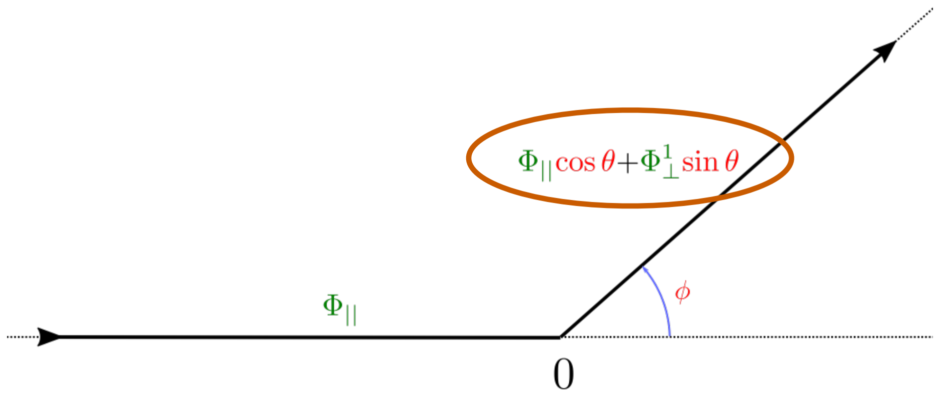
First 3 OPE coefficients including the first 10 states



Can we do better?

Integrated correlators

Line deformations



The cusp anomalous dimension at order $\sin^2\theta$ is computed exactly

$$\Gamma_{\text{cusp}} \sim \mathbb{B}\sin^2\theta \qquad \mathbb{B} = \frac{1}{2\pi^2}\lambda\partial_\lambda \log\left\langle W_{\text{circle}}\right\rangle$$

Bremsstrahlung function

[Correa, Henn, Maldacena, Sever '12]

$$\int dt \left\langle\left\langle \Phi_\perp(t)\Phi_\perp(0)\right\rangle\right\rangle = -2\mathbb{B}$$

Can we exploit the line deformations to obtain constraints for our four-point function?

New integrated correlators

We can follow the same logic but expanding the cusp anomaly to the next order

$$\Gamma_{\text{cusp}} = \mathbb{B}\sin^2\theta + \frac{1}{4}(\mathbb{B} + \mathbb{C})\sin^4\theta + \mathcal{O}(\sin^6\theta)$$

[NG,Levkovich-Maslyuk'15]

Curvature function

$$\mathbb{C}(g) = -4\mathbb{B}^2 - \frac{1}{2}\oint\frac{du_x}{2\pi i}\oint\frac{du_y}{2\pi i}S[u_x, u_y]$$

Analytic expansion at weak/strong coupling

$$\mathbb{C} = 4g^4 - \left(24\zeta_3 + \frac{16\pi^2}{3}\right)g^6 + \left(\frac{64\pi^2\zeta_3}{3} + 360\zeta_5 + \frac{64\pi^4}{9}\right)g^8 - \left(\frac{112\pi^4\zeta_3}{5} + 272\pi^2\zeta_5 + 4816\zeta_7 + \frac{416\pi^6}{45}\right)g^{10}$$
$$+\left(\frac{3488\pi^6\zeta_3}{135} + \frac{2192\pi^4\zeta_5}{9} + \frac{9184\pi^2\zeta_7}{3} + 63504\zeta_9 + \frac{176\pi^8}{15}\right)g^{12} + O\left(g^{14}\right)$$

[Cavaglià, NG, Julius, **Preti** '22]

$$\mathbb{C} = \frac{\left(2\pi^2-3\right)g}{6\pi^3} + \frac{-24\zeta_3+5-4\pi^2}{32\pi^4} + \frac{11+2\pi^2}{256\pi^5 g} + \frac{96\zeta_3+75+8\pi^2}{4096\pi^6 g^2} + \frac{3\left(408\zeta_3-240\zeta_5+213+14\pi^2\right)}{65536\pi^7 g^3} + \frac{3\left(315\zeta_3-240\zeta_5+149+6\pi^2\right)}{65536\pi^8 g^4} + O\left(\frac{1}{g^5}\right)$$

New integrated correlators

The four-point function $\mathscr{A}$ and the reduced correlator f obey the following independent constraints

Constraint 1

$$\int_0^1 \delta\mathscr{A}(\chi)\frac{1+\log\chi}{\chi^2}d\chi = \frac{3\mathbb{C}-\mathbb{B}}{8\mathbb{B}^2}$$

Constraint 2

$$\int_0^1 \frac{\delta f(\chi)}{\chi}d\chi = \frac{\mathbb{C}}{4\mathbb{B}^2} + \mathbb{F} - 3$$

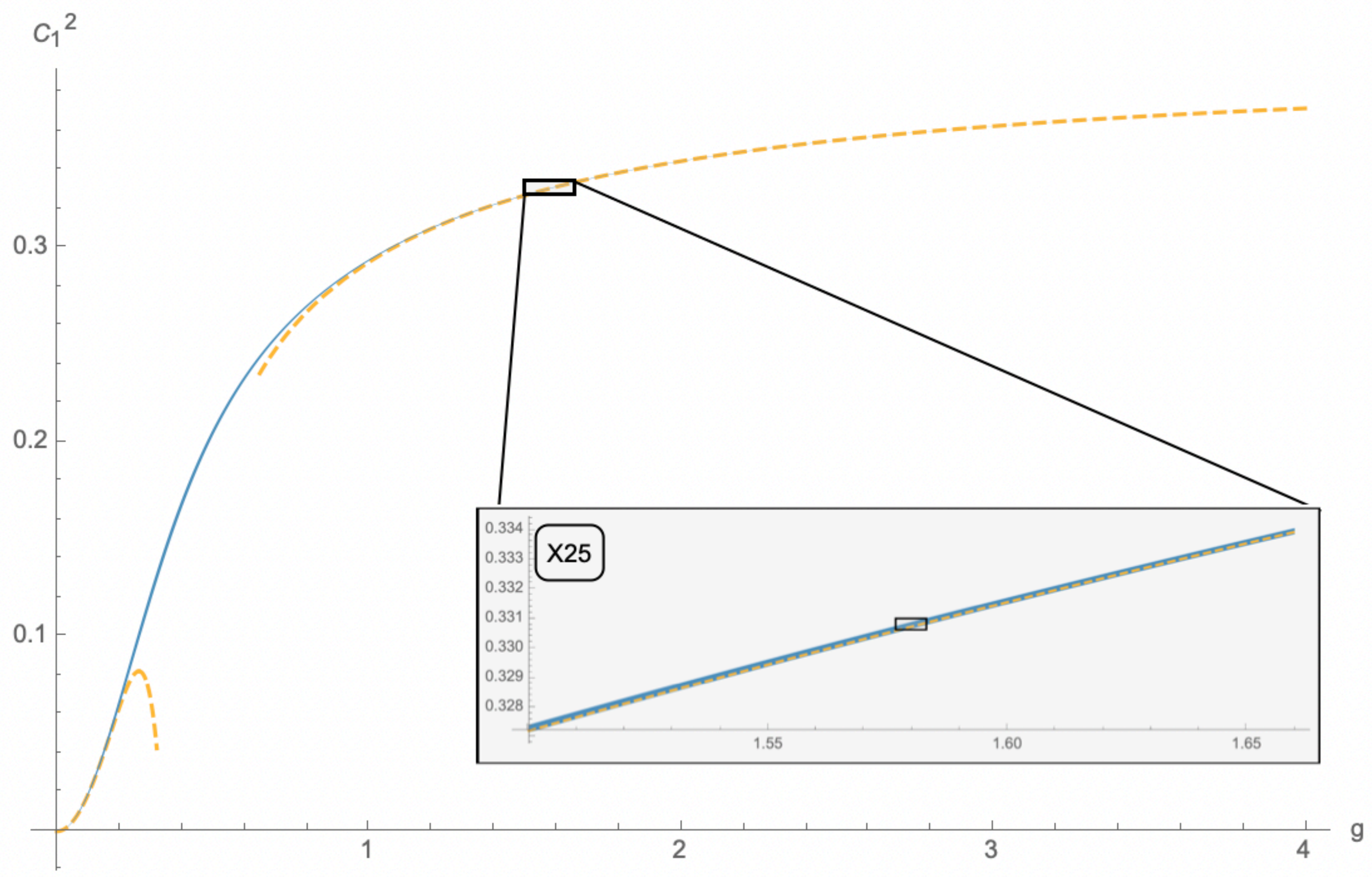
- We tested both at weak and strong coupling.
- Then a linear combination of them has been derived [Drukker, Kong, Sakkas '22]
- A complete proof [Cavaglià, NG, Julius, `22]

↓

Big impact on the Bootstrability output

Super tight bounds for structure constant

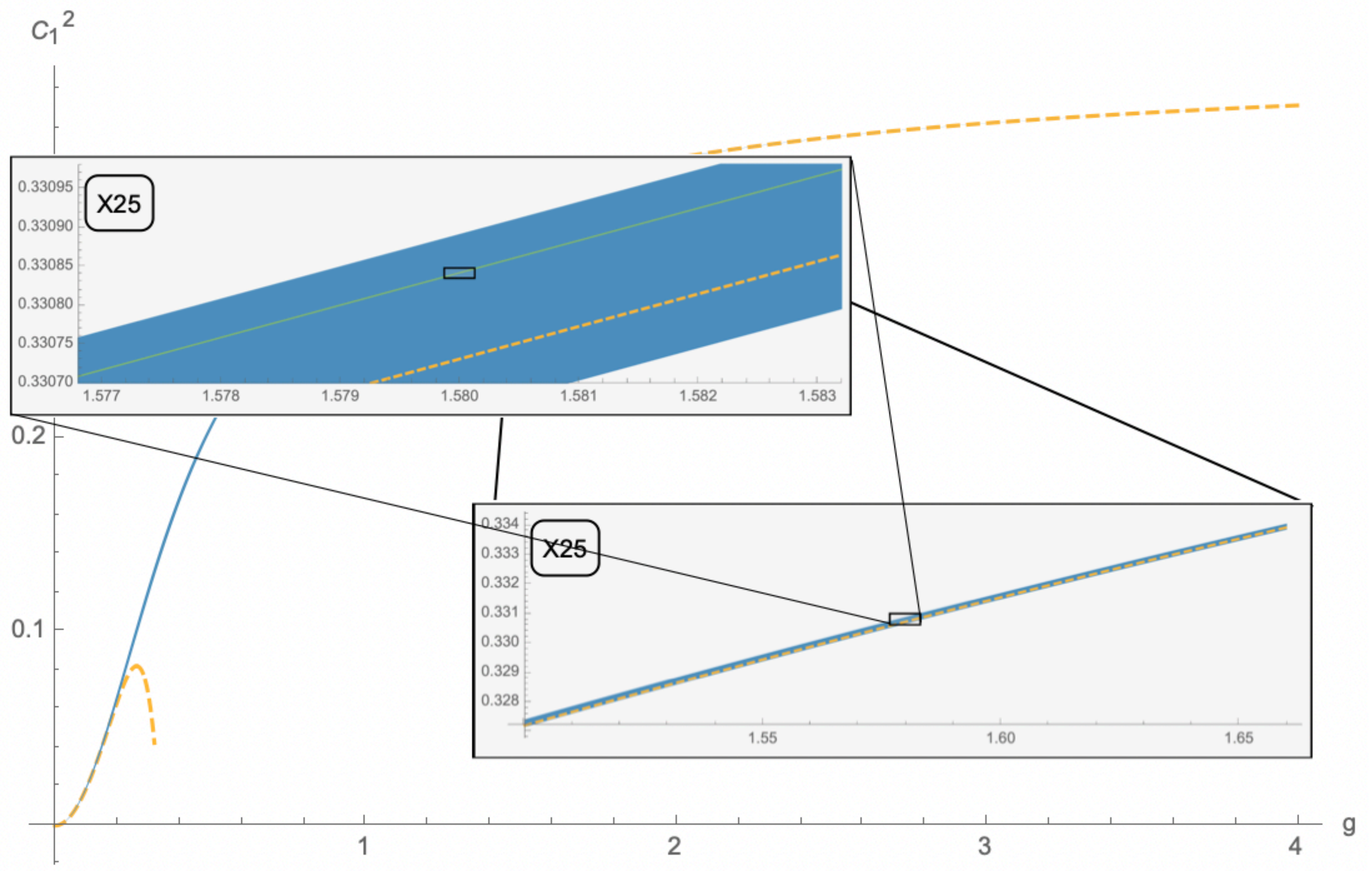
2 states from QSC



A.Cavaglià, NG, J.Julius, M.Preti, N.Sokolova `21

Super tight bounds for structure constant

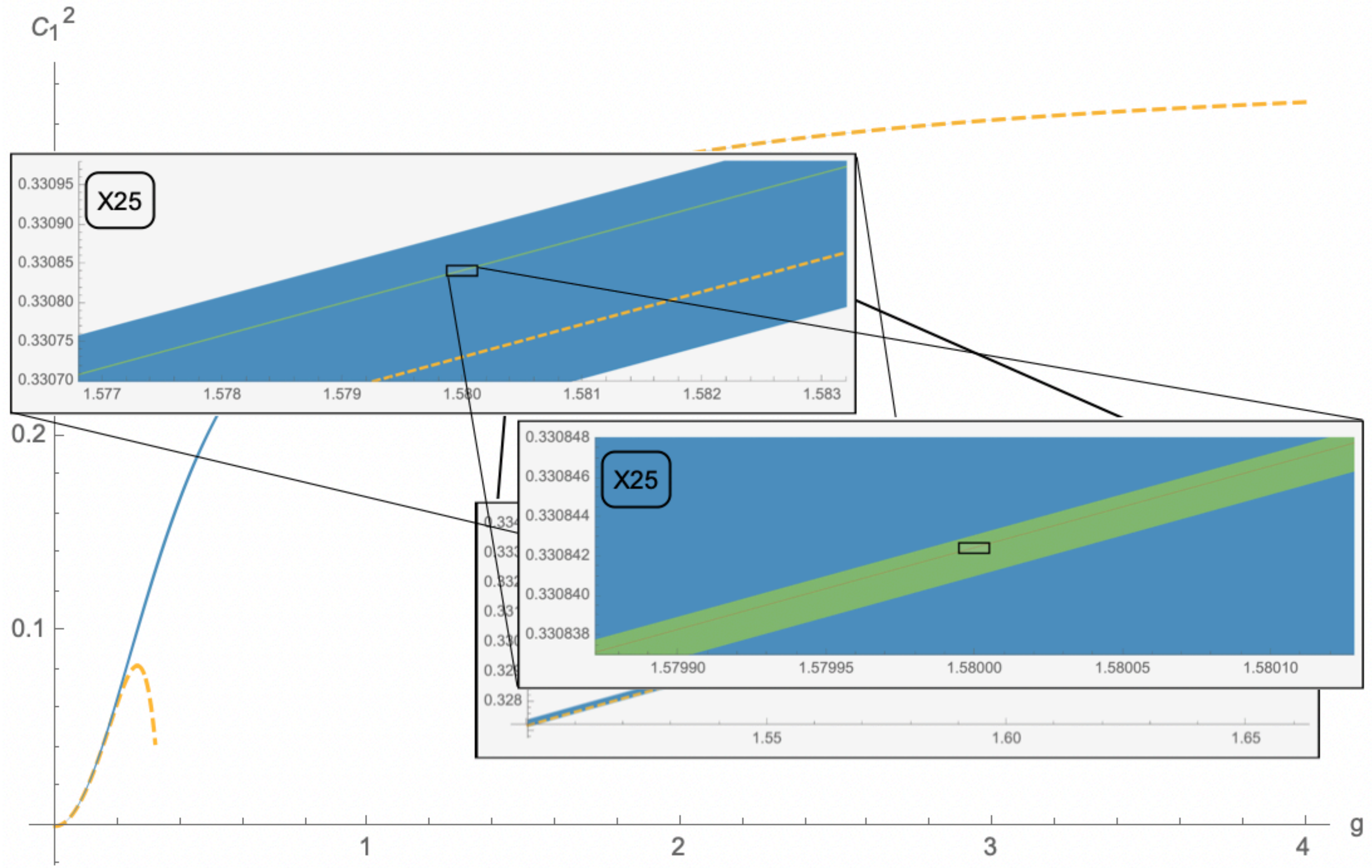
10 states,
+curvature function



A.Cavaglià, NG, J.Julius, M.Preti, N.Sokolova `22

Super tight bounds for structure constant

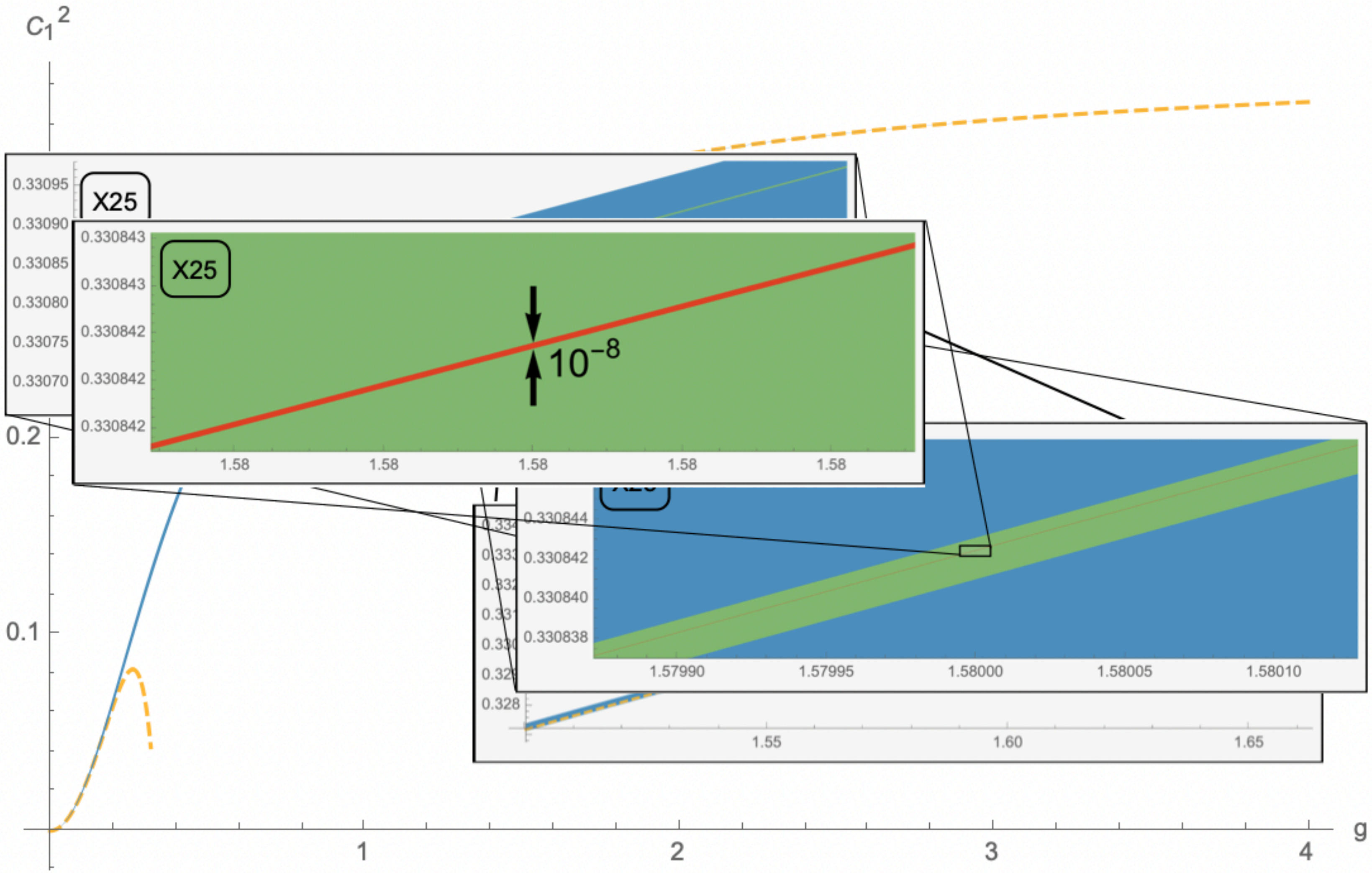
10 states,
+curvature function



A.Cavaglià, NG, J.Julius, M.Preti, N.Sokolova `22

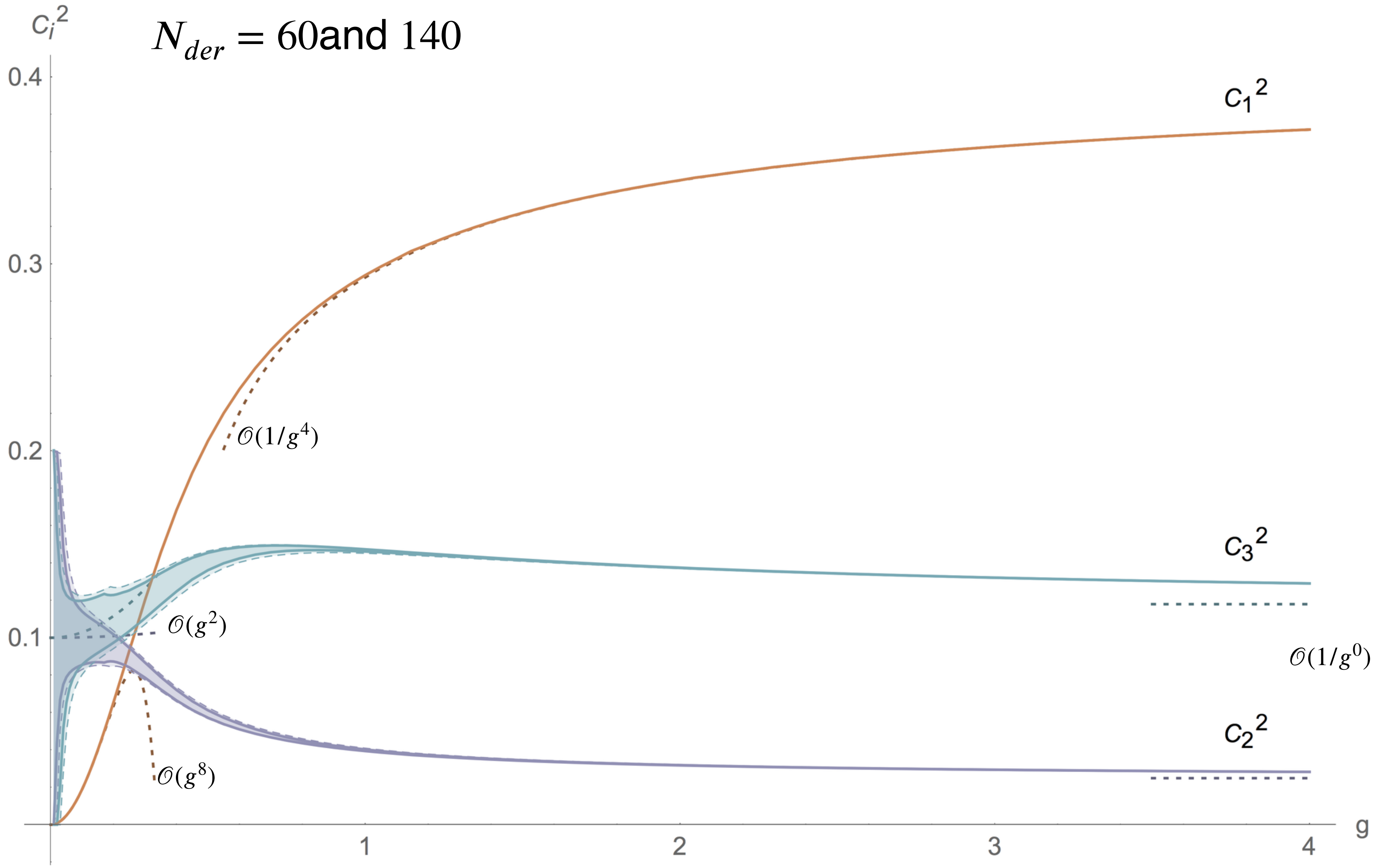
Super tight bounds for structure constant

Approaching precision
of the anomalous magnetic
dipole momentum of electron!
(only 2 digits away)



Only spectrum goes in, no extra **localization** data, this includes spectrum of deformed theory, which gives integrated correlators

[Cavaglià, NG, Julius, **Preti** ’22]



Analytic Bootstrability

Strategy at weak coupling

The four-point function at weak coupling is known only up to order g^2 [Kiryu Komatsu'18]

Our strategy to compute higher orders is to formulate an ansatz in terms of Harmonic Polylogarithms (HPL) and then fix the coefficients using

- Crossing equation
- Conformal data matching
- Integral relations
- Uniform transcendentally

At order g^4 we obtain [Cavaglià, NG, Julius, **Preti** '22]

$$G^{(2)}_{\text{weak}}(x) = \frac{4(1-2x)}{(1-x)^2}\Big[\frac{\pi^2}{3}(H_2 - H_{1,0} + H_{1,1}) - 3\zeta_3 H_1 + 2(H_{1,2,1} - H_{1,1,2} - H_{2,0,0} - H_{1,1,0,0})$$
$$+\frac{(x-1)x+1}{2x-1}\left(2H_{2,1} - 2H_{1,0,0} + (x-1)H_{1,1,0} - (x-2)H_{2,0} + xH_3 + \frac{\pi^2}{3}(H_1 - xH_0)\right)$$
$$+\frac{(x^3+1)}{1-2x}H_{1,2} + H_{1,3} - H_{2,2} + 2H_{3,0} + 2H_{3,1} + H_{1,2,0} - H_{2,1,0} + x\frac{2\pi^4 x - 45((x-1)x+1)\zeta_3}{15(2x-1)}\Big]$$

OPE coefficients at weak coupling

As a bi-product we obtain several results for the structure constants of which the most interesting ones are the following

Classical dimension = 1

$$C_1^2(g) = 2g^2 - \left(24 - \frac{4\pi^2}{3}\right)g^4 + \left(320 - 16\pi^2 + 48\zeta_3 - \frac{76\pi^4}{45}\right)g^6$$
$$-\left(4480 - \frac{832\pi^2}{3} + 256\zeta_3 - \frac{224\pi^4}{15} + 880\zeta_5 - \frac{64\pi^6}{45}\right)g^8 + O(g^{10})$$

Classical dimension = 2

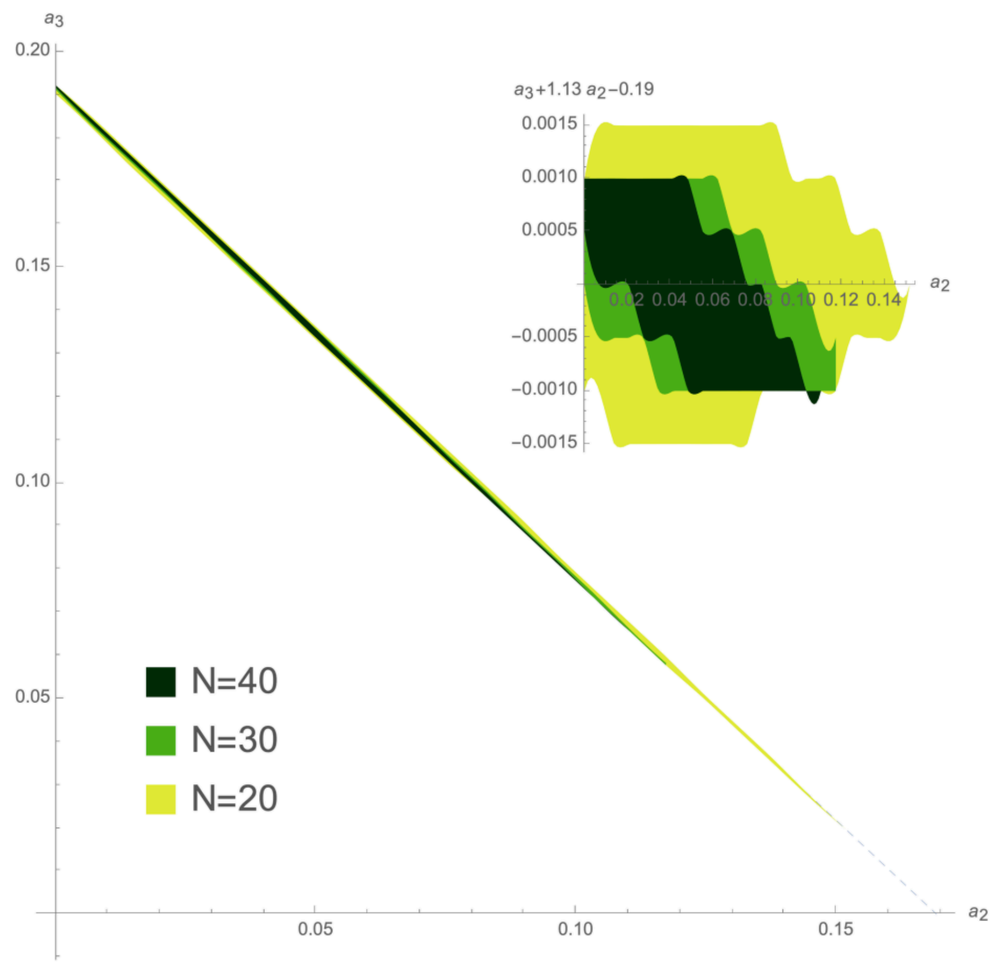
$$C_2^2 = \frac{1}{10} + \frac{1}{150}\left(10\pi^2 - 75 - 9\sqrt{5}\right)g^2 + O(g^4)$$
$$C_3^2 = \frac{1}{10} + \frac{1}{150}\left(10\pi^2 - 75 + 9\sqrt{5}\right)g^2 + O(g^4)$$

Classical dimension = 3

$$C_6^2 = \frac{1}{14} + \frac{2}{7\sqrt{37}} + O(g^2) \qquad C_8^2 = \frac{1}{14} - \frac{2}{7\sqrt{37}} + O(g^2)$$

[Cavaglià, NG, Julius, **Preti** ’22]

4-point function



a_3
0.20
0.15
0.10
0.05
0.05
0.10
0.15
a_2
$a_3+1.13\,a_2-0.19$
0.0015
0.0010
0.0005
−0.0005
−0.0010
−0.0015
0.02 0.04 0.06 0.08 0.10 0.12 0.14
a_2
N=40
N=30
N=20

How do we bound a linear combination with generic weights H_Δ?

$$T \equiv \sum_{n=1}^{\infty} C_n^2 H_{\Delta_n}$$

e.g. $H_\Delta = f_\Delta(x_0)$ → reduced correlator at cross ratio x_0

$H_\Delta = \mathscr{D}_x f_\Delta(x_0)$ → full 4-point function $G(x_0)$

Idea: rewrite the Bootstrap equation….

$$\mathscr{G}_{1+\mathscr{B}_2}(\lambda, x) + \sum_{n\geq 1} C_n^2 \mathscr{G}_{\Delta_n}(x) = 0$$

… so that T comes in front of the Δ_1 block ….

$$\underbrace{\frac{\sum_{n=1}^{\infty} C_n^2 H_{\Delta_n}}{H_{\Delta_1}}} + \sum_{n\geq 2} C_n^2 \left(\frac{\mathscr{G}_{\Delta_n}(x)}{\mathscr{G}_{\Delta_1}(x)} - \frac{H_{\Delta_n}}{H_{\Delta_1}} \right) + \frac{\mathscr{G}_{1+\mathscr{B}_2}(\lambda, x)}{\mathscr{G}_{\Delta_1}(x)} = 0$$

Quantity we want to bound

n=1 term is cancelled

new functions playing the role of “blocks”

We go around this by dividing the tail in two regions

In the finite interval: precise polynomial interpolation of new blocks

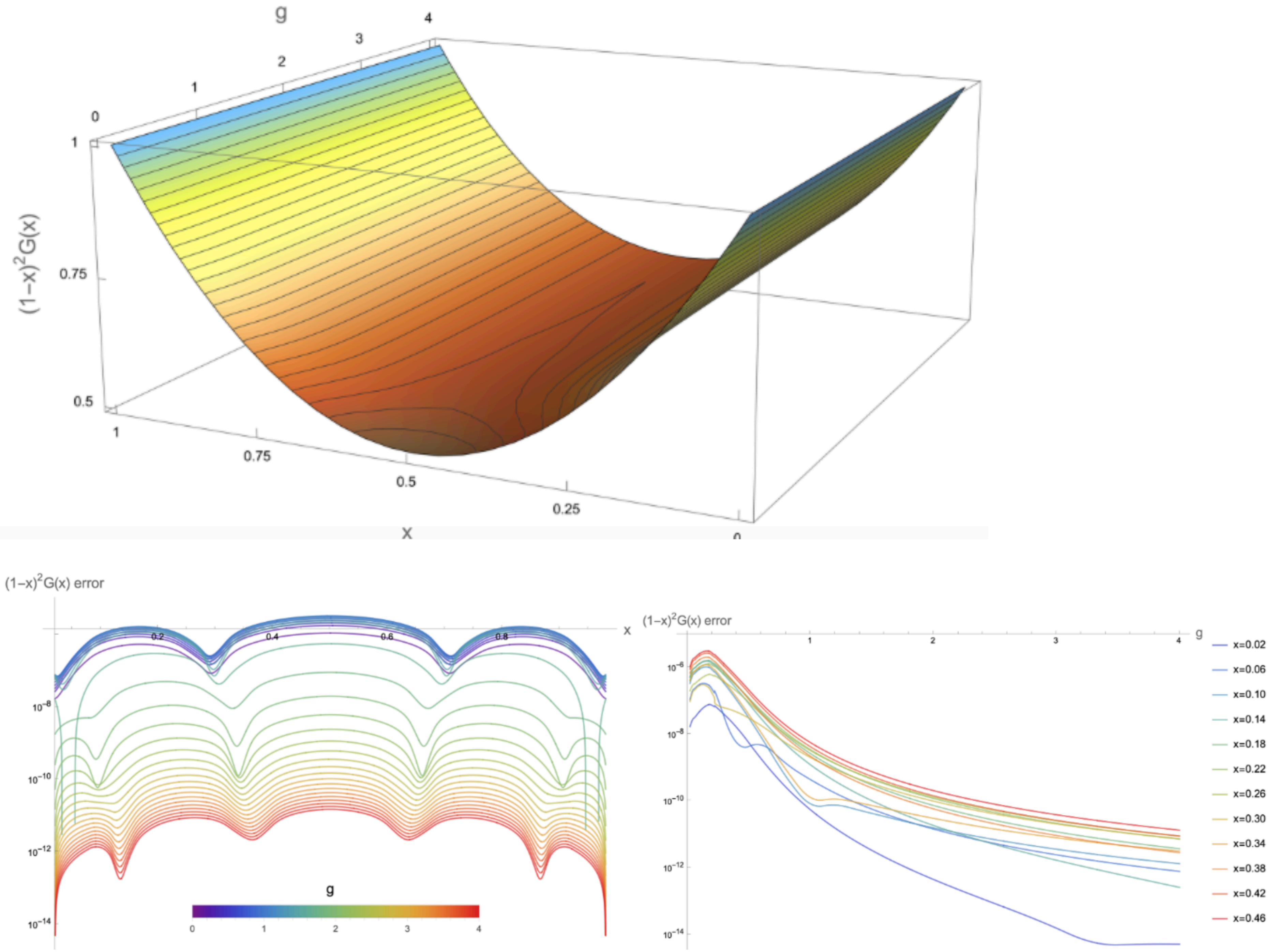
$$\Delta - \Delta_{\rm gap} = \frac{\Delta_{\rm up} - \Delta_{\rm gap}}{y+1}\,, \qquad y \in [0,\infty]$$

$$\hat{P}_m(\Delta - \Delta_{\rm gap}) = \frac{1}{(y+1)^M} P_m^{\rm extra}(y)$$

$$\frac{\mathcal{G}_{\Delta}(1/2)}{\mathcal{G}_{\Delta_1}(1/2)} - \frac{H_{\Delta}}{H_{\Delta_1}}$$

Above very large cutoff: just neglect subleading exponential

(two conditions for SDPB - plus positivity for discrete states below gap)



g
0
1
2
3
4
(1−x)²G(x)
1
0.75
0.5
x
1
0.75
0.5
0.25
0
(1−x)²G(x) error
x
0.2
0.4
0.6
0.8
10⁻⁸
10⁻¹⁰
10⁻¹²
10⁻¹⁴
g
0
1
2
3
4
(1−x)²G(x) error
g
1
2
3
4
10⁻⁶
10⁻⁸
10⁻¹⁰
10⁻¹²
10⁻¹⁴
x=0.02
x=0.06
x=0.10
x=0.14
x=0.18
x=0.22
x=0.26
x=0.30
x=0.34
x=0.38
x=0.42
x=0.46

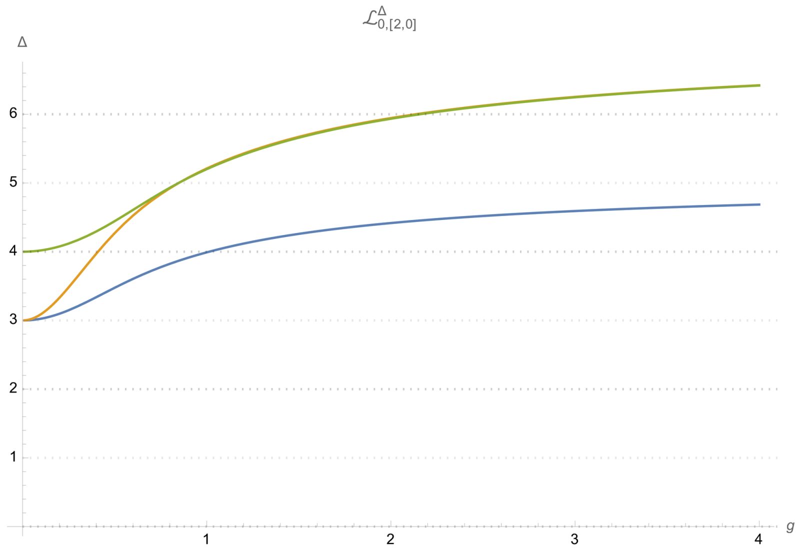
Multi-correlators

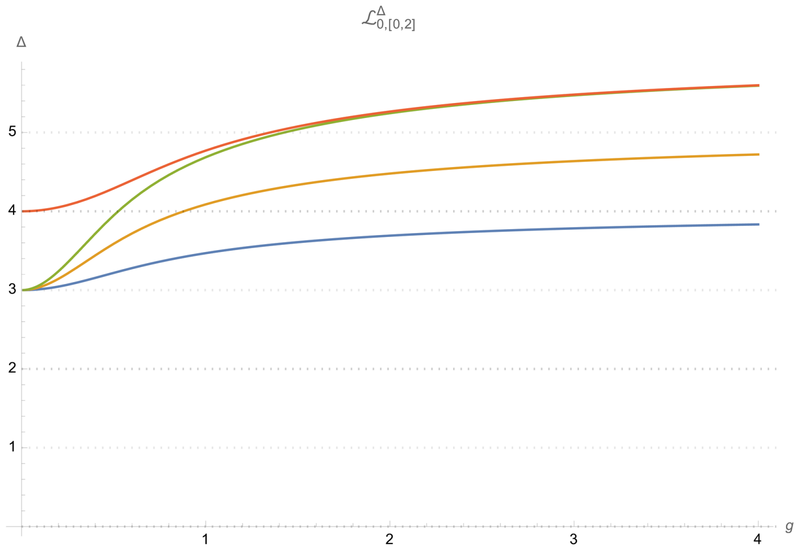
4 types of correlators

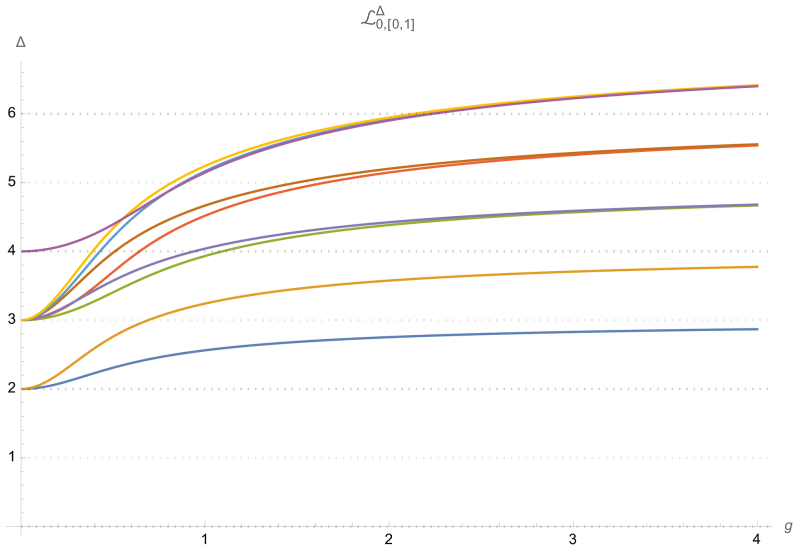
$$\mathcal{A}_{\{1,1,1,1\}}(\chi,\zeta_1,\zeta_2)\,, \qquad \mathcal{A}_{\{1,2,1,2\}}(\chi,\zeta_1,\zeta_2)\,, \qquad \mathcal{A}_{\{1,2,2,1\}}(\chi,\zeta_1,\zeta_2)\,, \qquad \mathcal{A}_{\{2,2,2,2\}}(\chi,\zeta_1,\zeta_2)$$

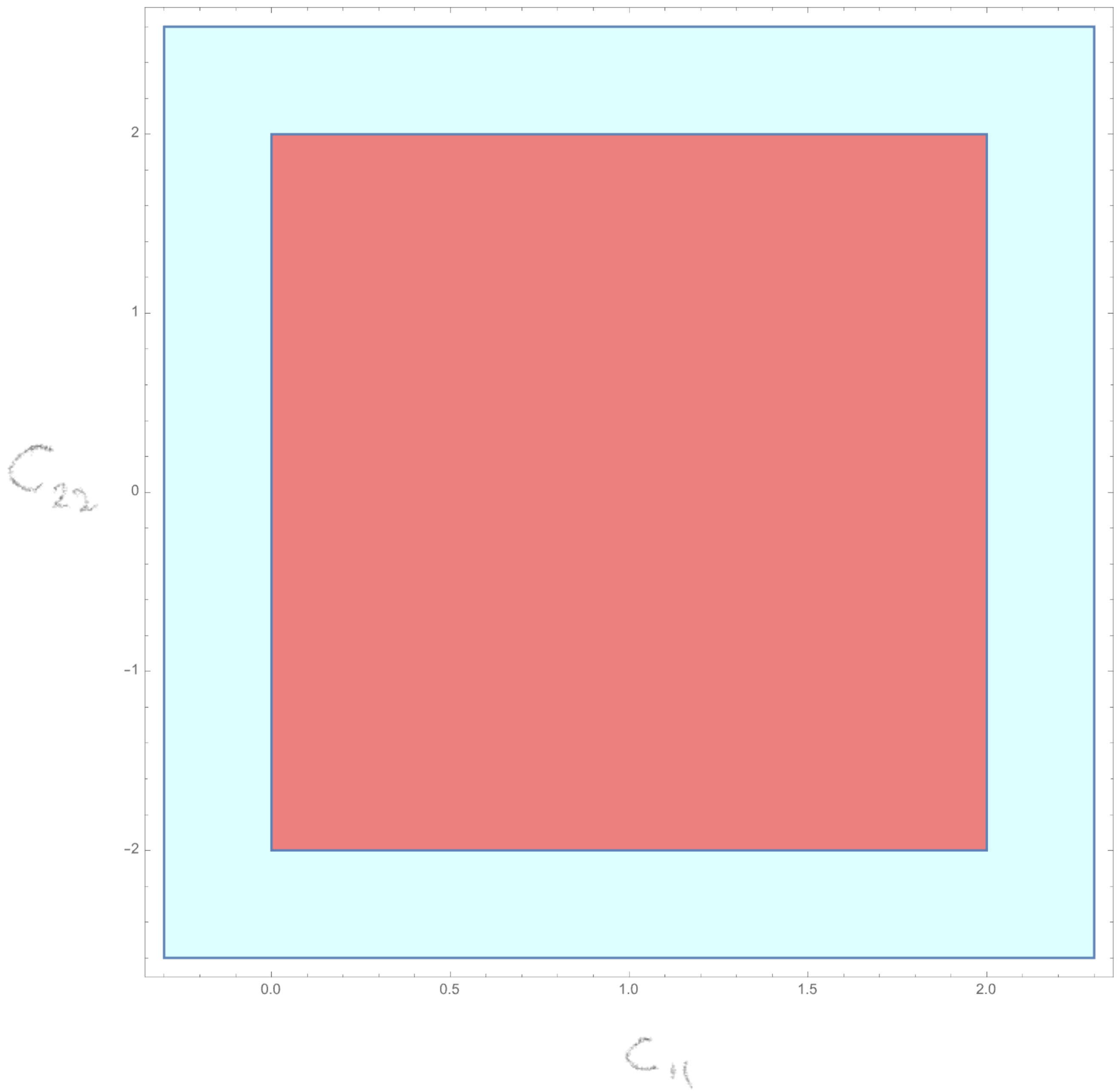
$$\begin{aligned}
\mathcal{B}_1 \times \mathcal{B}_1 &= \mathcal{I} + \mathcal{B}_2 + \sum_{\Delta\geq 1} \mathcal{L}^{\Delta}_{[0,0]}\,, \\
\mathcal{B}_1 \times \mathcal{B}_2 &= \mathcal{B}_1 + \mathcal{B}_3 + \sum_{\Delta\geq 2} \mathcal{L}^{\Delta}_{[0,1]}\,, \\
\mathcal{B}_2 \times \mathcal{B}_2 &= \mathcal{I} + \mathcal{B}_2 + \mathcal{B}_4 + \sum_{\Delta\geq 1} \mathcal{L}^{\Delta}_{[0,0]} + \sum_{\Delta\geq 3} \left(\mathcal{L}^{\Delta}_{[2,0]} + \mathcal{L}^{\Delta}_{[0,2]}\right)
\end{aligned}$$

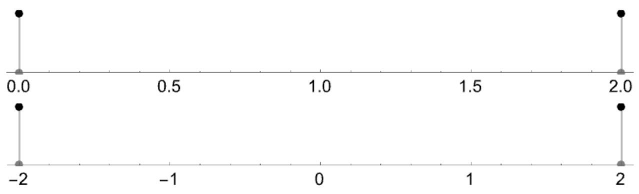
Extra spectra

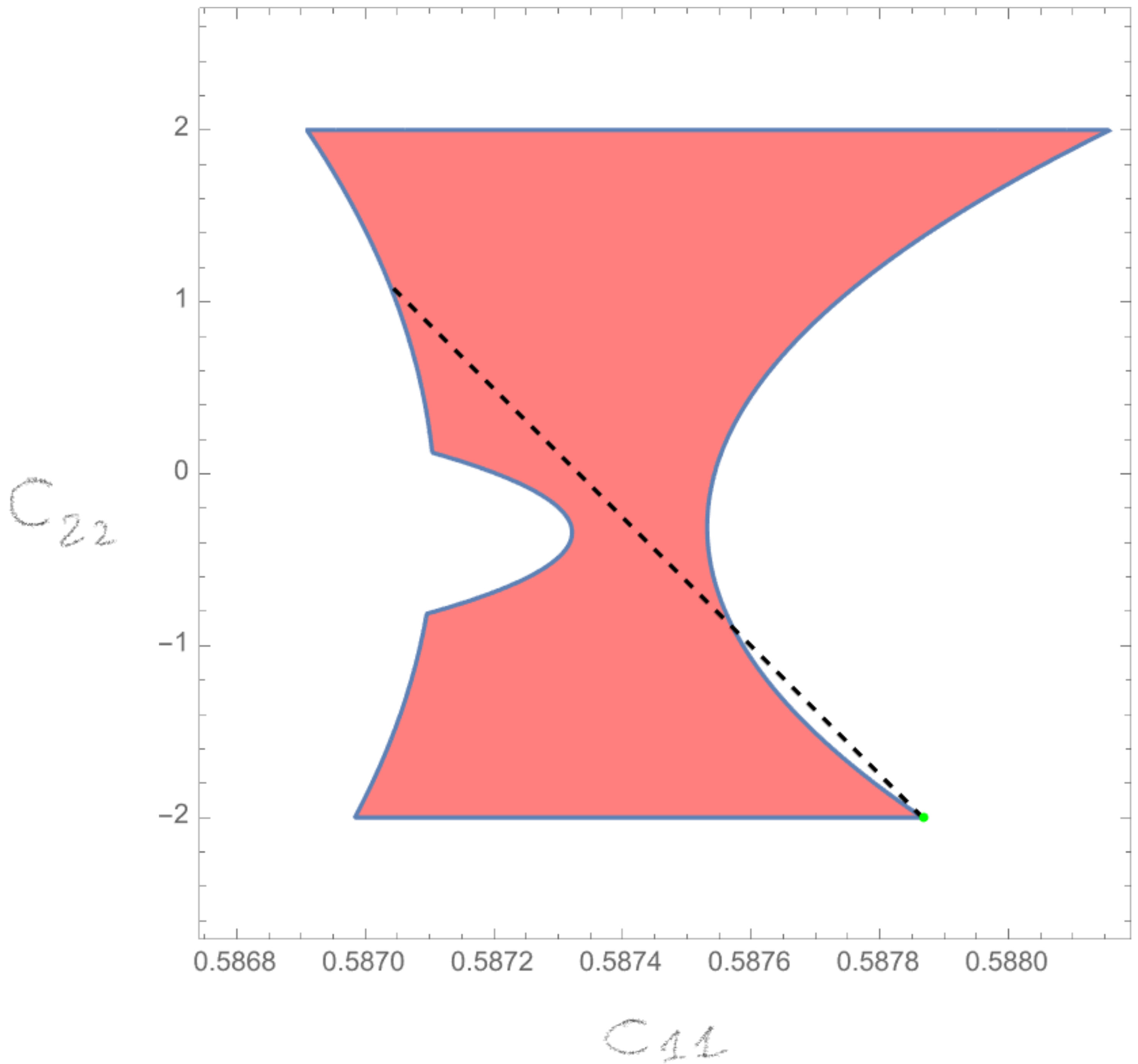




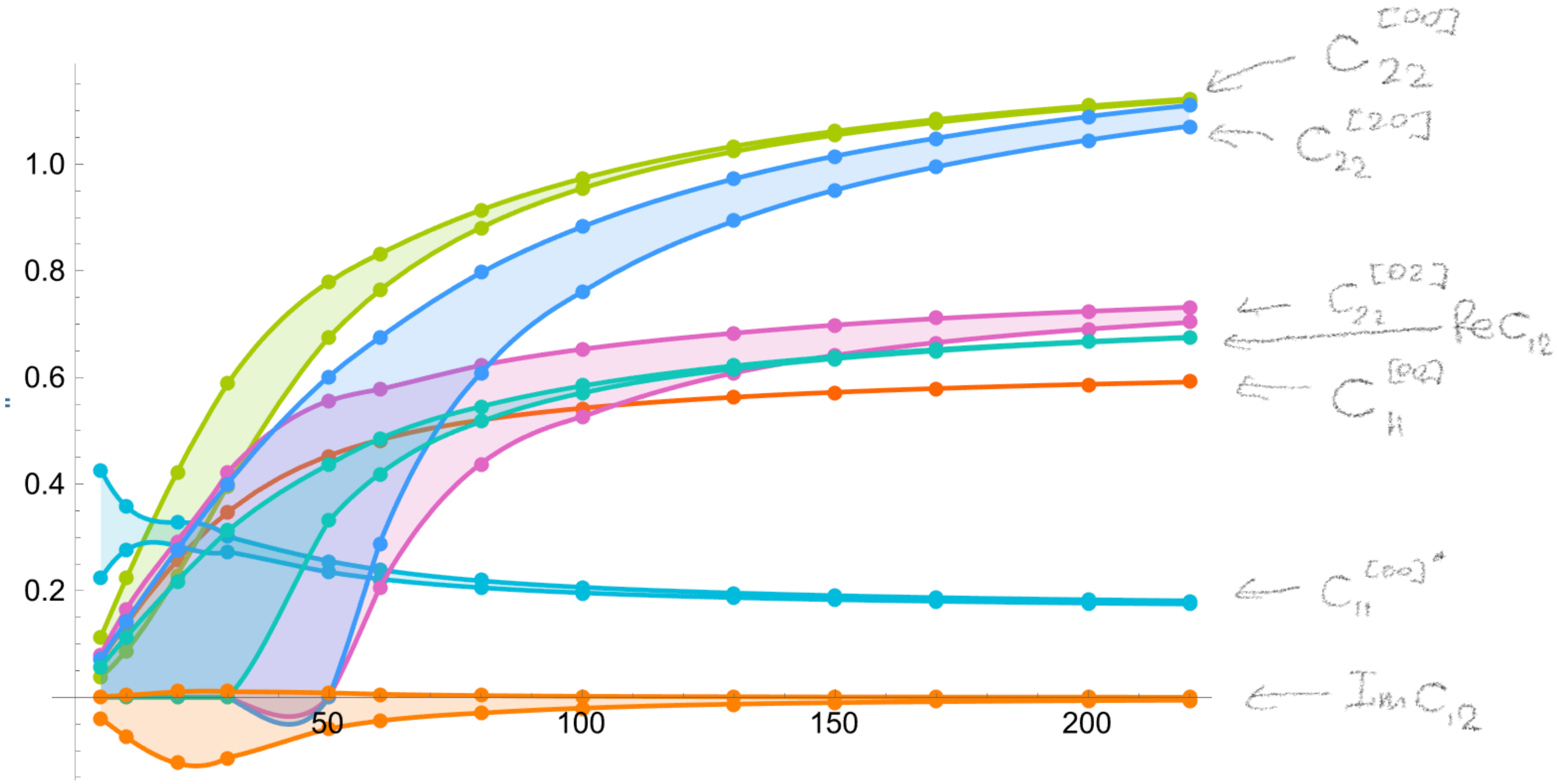



2
1
0
-1
-2
0.0
0.5
1.0
1.5
2.0


0.0
0.5
1.0
1.5
2.0
−2
−1
0
1
2



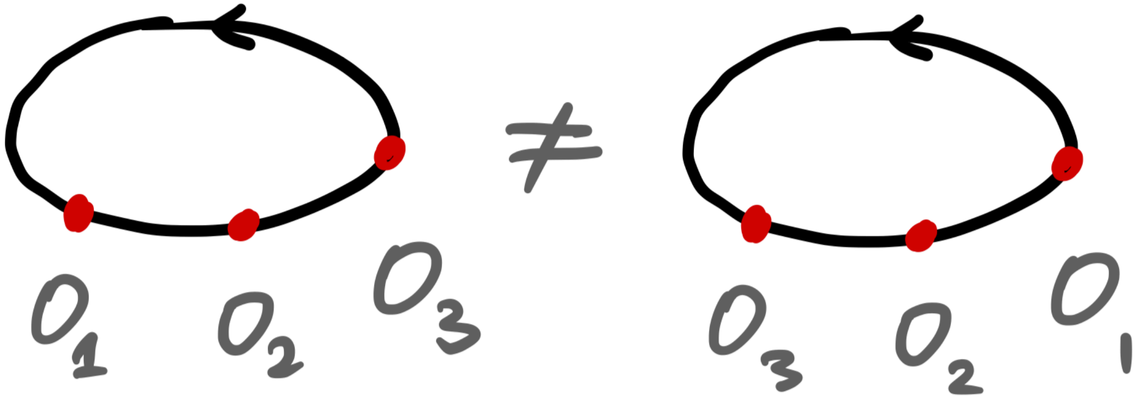
2
1
0
−1
−2
0.5868
0.5870
0.5872
0.5874
0.5876
0.5878
0.5880
c_{22}
c_{11}



preliminary: 12D, 30 derivatives

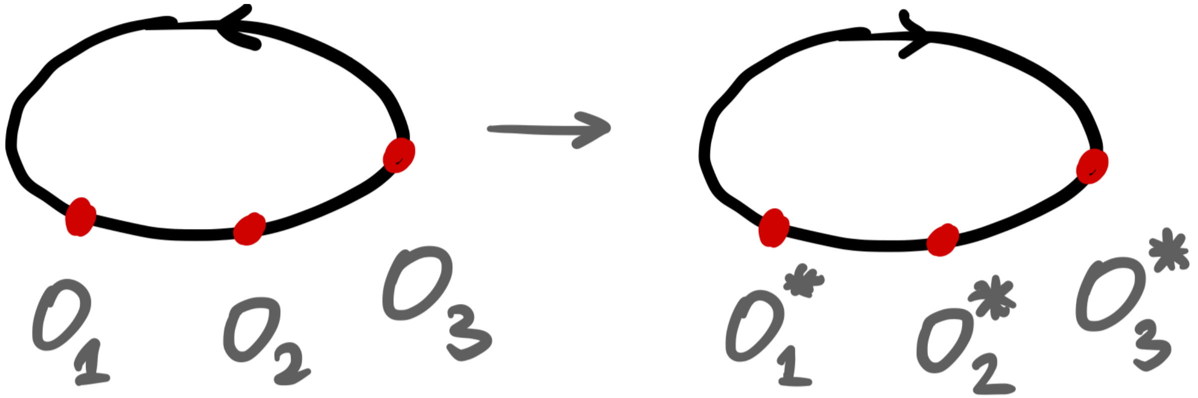
Why some parameters are zero?

$C_{123} \neq C_{321}$



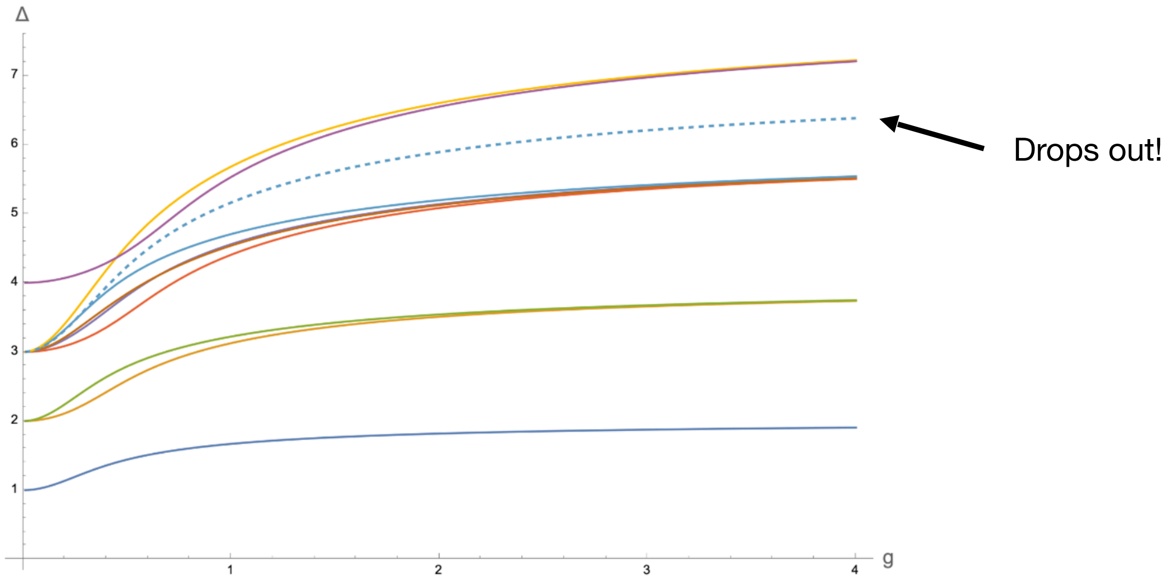
$F \to -F^T$

“Charge conjugation” on $N \times N$ SYM fields, leaves action invariant



$C_{O_1O_2O_3} = C_{O_3O_2O_1}\ \mathbb{P}_1\mathbb{P}_2\mathbb{P}_3$

$$C_{O_1 O_1 O_{\mathbb{P}-odd}} = 0$$

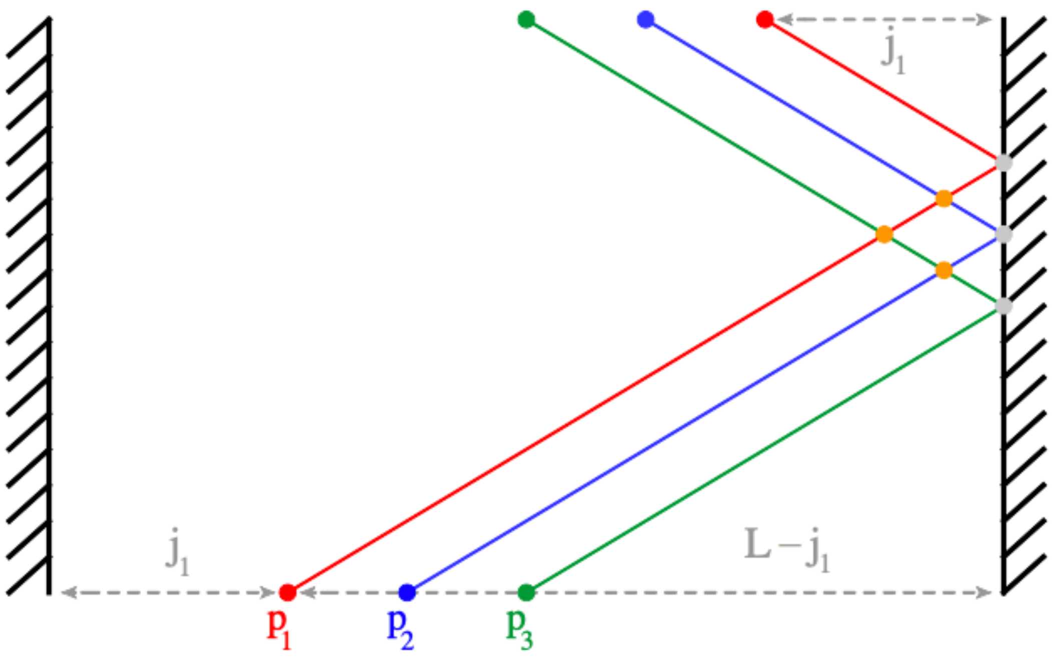


Gain factor of ~2 of the precision

Complex conjugation:

$$\bar{C}_{O_1 O_2 O_3} = \mathbb{P}_1 \mathbb{P}_2 \mathbb{P}_3 C_{\bar{O}_1 \bar{O}_2 \bar{O}_3}$$

Parity from integrability data:



Scanning through the states we know:

$$\mathbb{P} = (-1)^{\Delta_{\lambda=\infty}+R_1+R_2}$$

$$\mathbb{P} = \prod_{k=1}^{M_4/2}\left[-\left(\frac{u_{4,k}-i/2}{u_{4,k}+i/2}\right)^L \prod_{j>k}^{M_4/2}\frac{u_{4,k}+u_{4,j}+i}{u_{4,k}+u_{4,j}-i}\prod_{j=1}^{M_3}\frac{u_{4,k}-u_{3,i}-i/2}{u_{4,k}-u_{3,i}+i/2}\right]$$

What's next?

- Non-protected **multi-correlators** With ~10 states, there are a lot of correlators!
- 6-point integrated correlator, or integrated correlators with non-protected states
- Local operators - we gain a lot more symmetry (4D CFT) with a similar (smaller?) number of states.
- We can also add data from deformations of N=4 in the form of integrated correlators (additional to those from localization)