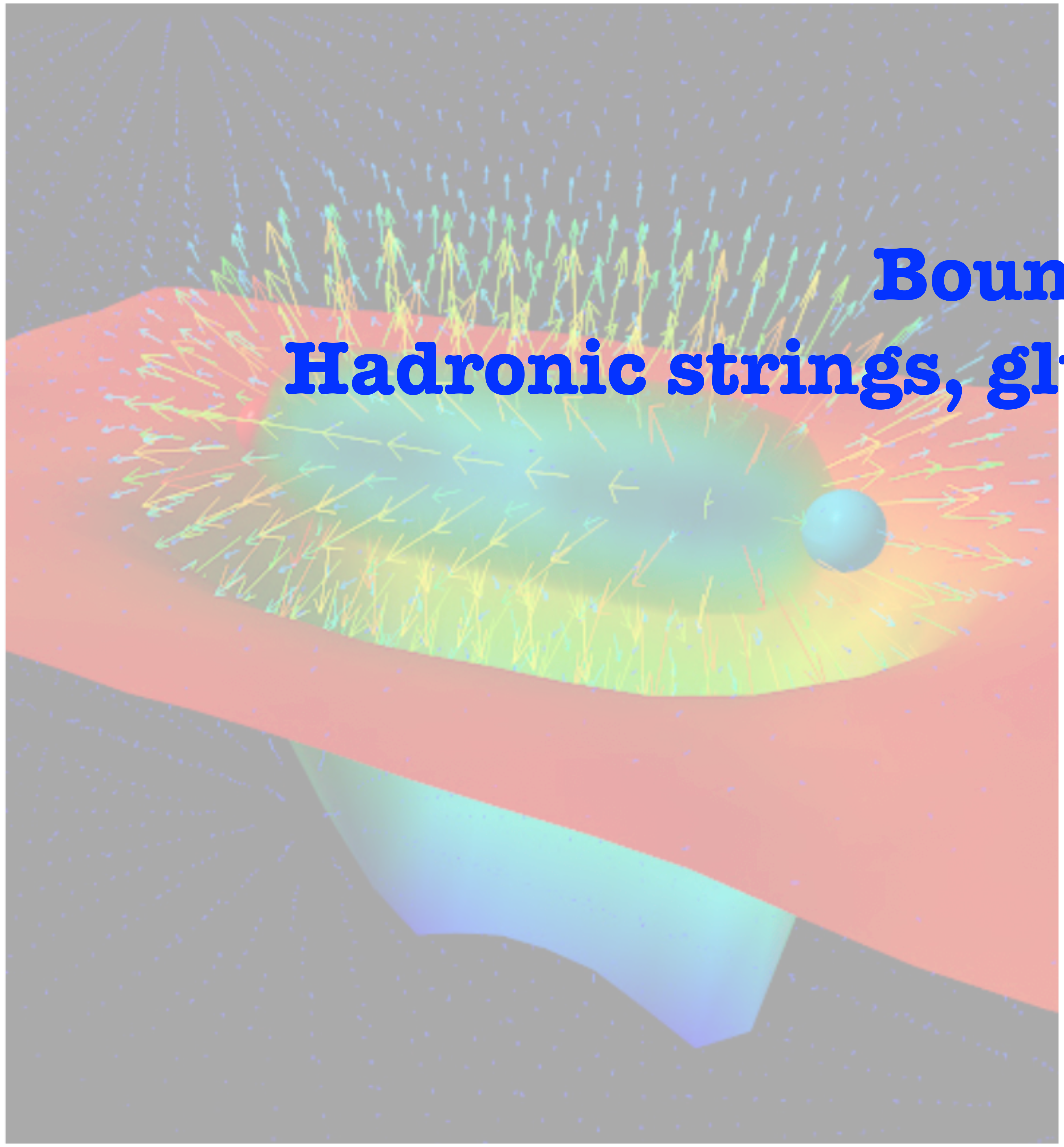
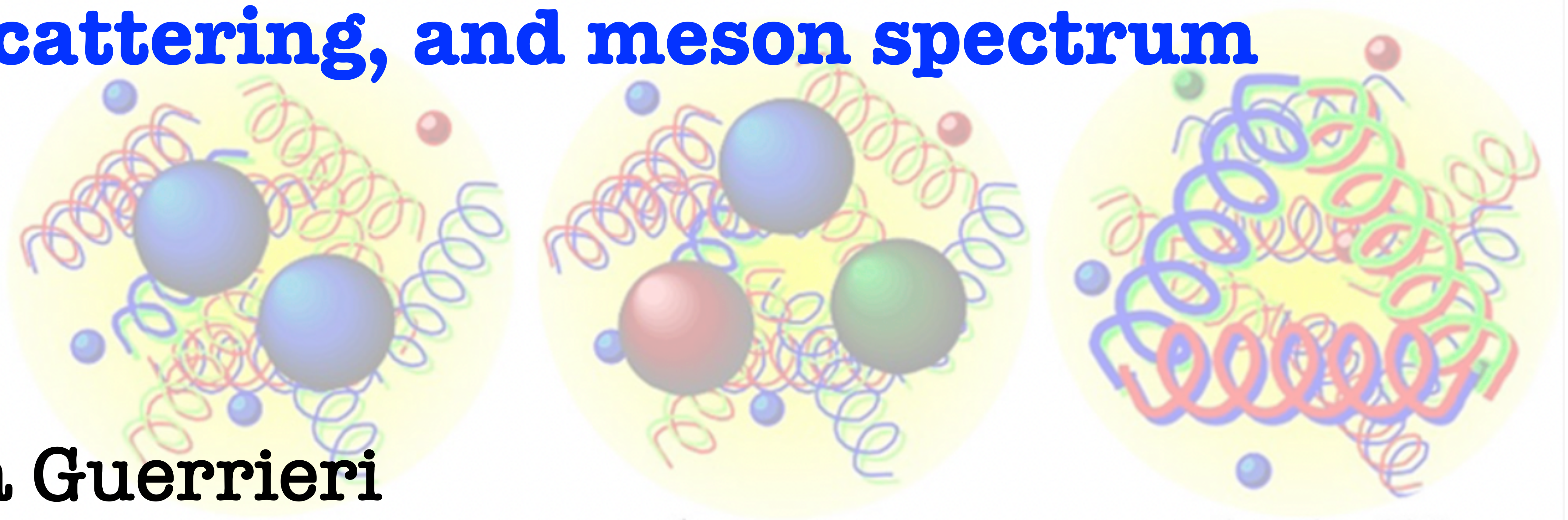


Bounds on QCD Observables: Hadronic strings, glueball scattering, and meson spectrum

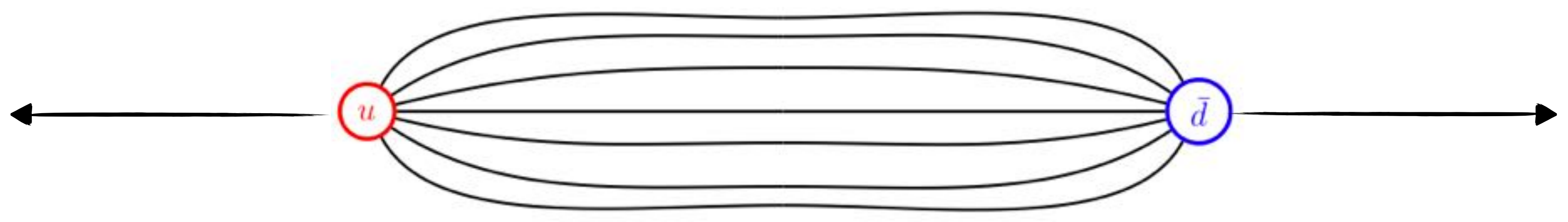


Andrea Guerrieri

February 21, 2024

Plan of the Talk

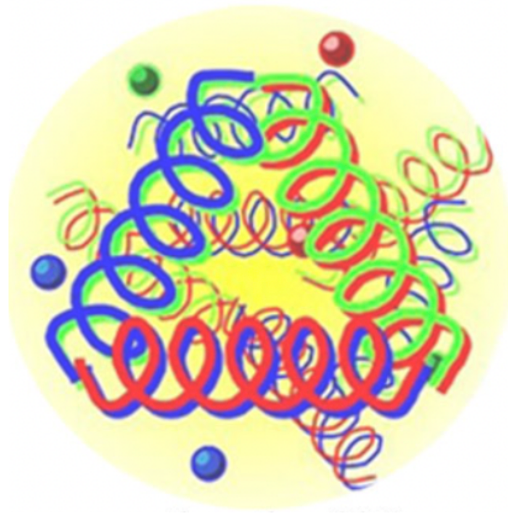
1) The Hadronic String



Bounds on the $q\bar{q}$ potential from Wilson coefficients

Worldsheet QCD axion — Gaikwad, Gorbenko, ALG 2310.20698

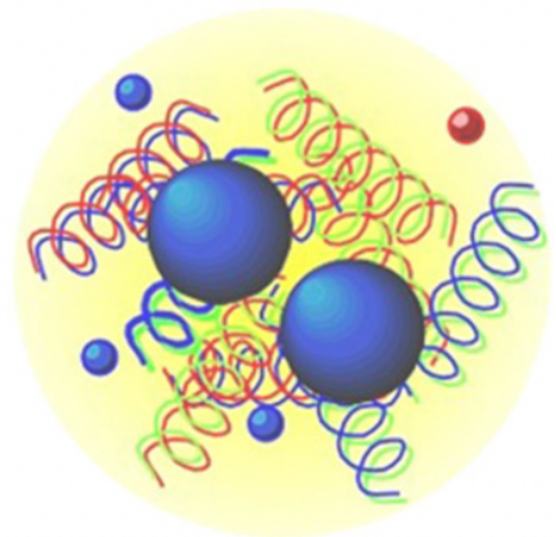
2) Glueballs



Rigorous bounds on SU(3) YM Glueball Scattering

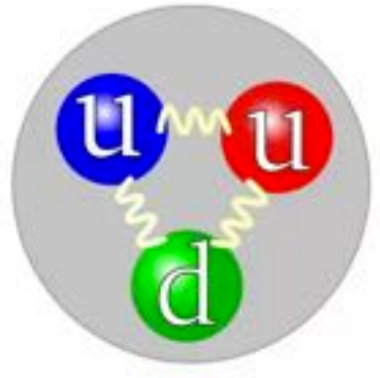
Hebbar, ALG, van Rees 2312.00127

3) Mesons



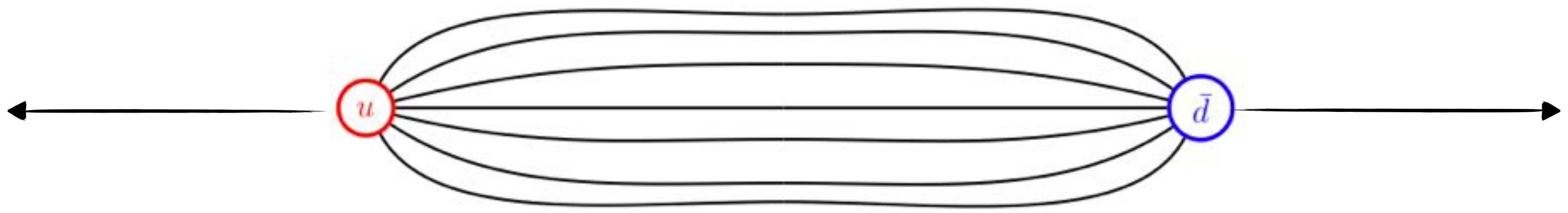
The other side of Ning's story — ALG, Haring, Su work in progress

4) Baryons (for the future)



Bounds on the static $q\bar{q}$ potential

Distance between quarks $R/\ell_s \to \infty$



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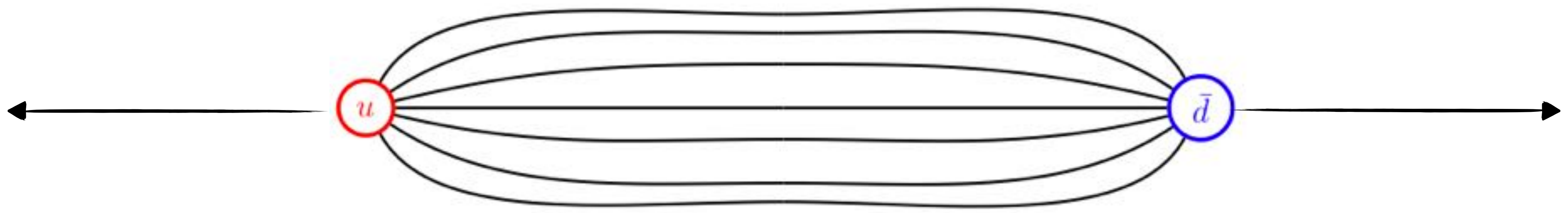

E.g. D=3 Target Space

Universal, consequence of non-linearly realized Lorentz

$$E_0(R) = \frac{R}{\ell_s^2} - \frac{\pi}{6R} - \frac{\pi^2\ell_s^2}{72R^3} - \frac{\pi^3\ell_s^4}{432R^5} + \frac{\Delta_3\ell_s^6}{R^7} + \mathcal{O}\left(\frac{\ell_s^8}{R^9}\right)$$

String tension $\sigma = \ell_s^{-2}$

Lüscher term '80

Lüscher, Weisz, Drummond '04

Aharony, Komargodsky, Dubovsky, Flauger, Gorbenko,...

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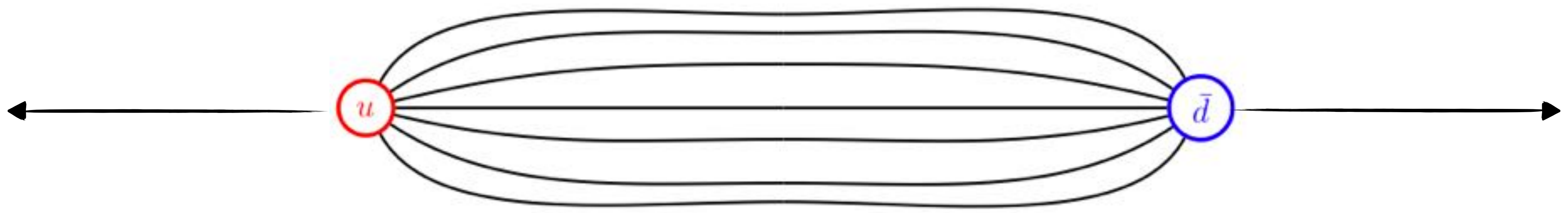

Theory-dependent,
but bounded from first principles

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$$\Delta_3 \leq \frac{\pi^6}{5400} - \frac{5\pi^4}{10368}$$

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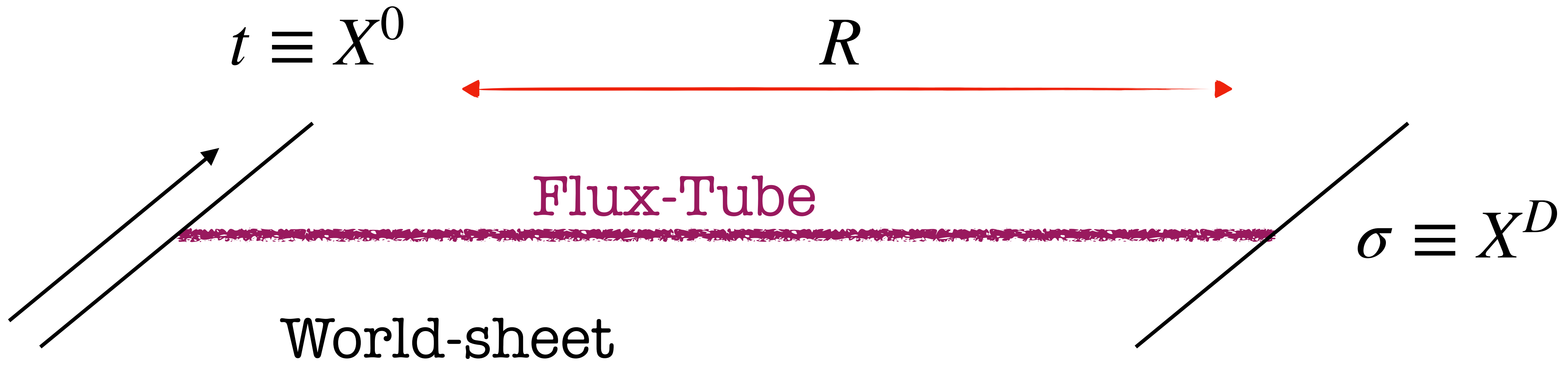
Elias-Miró, ALG, Hebbar, Penedones, and Vieira, PRL 123, no.22 (2019)

Effective String Theory



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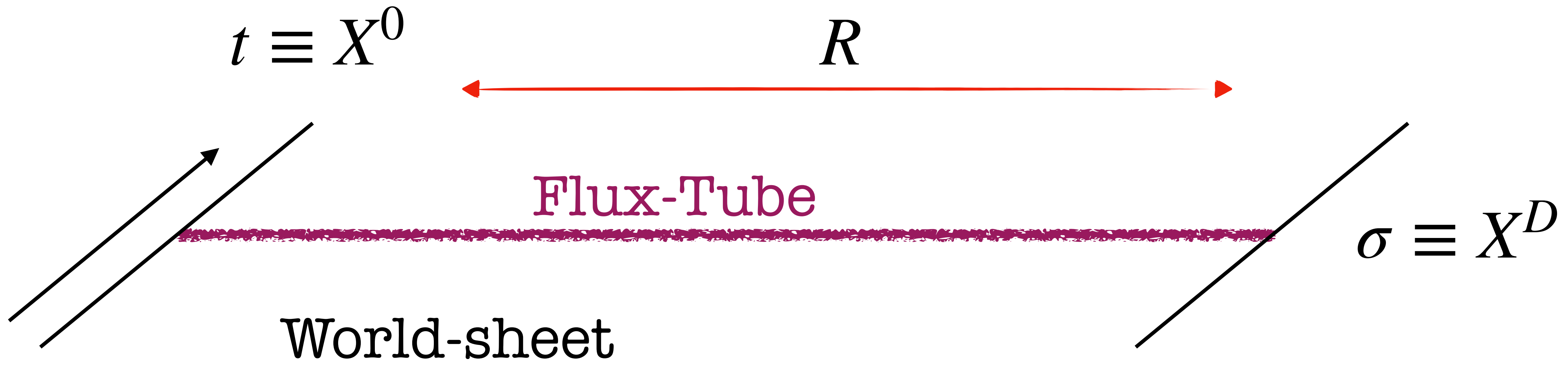

2d gravity theory

$$\mathcal{A}_{EFF} = \int d^2\sigma \sqrt{-h} \left(\frac{1}{\ell_s^2} + \gamma_2 \ell_s^2 R + \gamma_3 \ell_s^4 R^2 + \dots \right)$$

In $D \neq 26$, infinite corrections

Physical Degrees of freedom:

X^i with i=2,...,D massless Goldstones $(SO(1,D-1) \to SO(1,1) \times O(D-2))$

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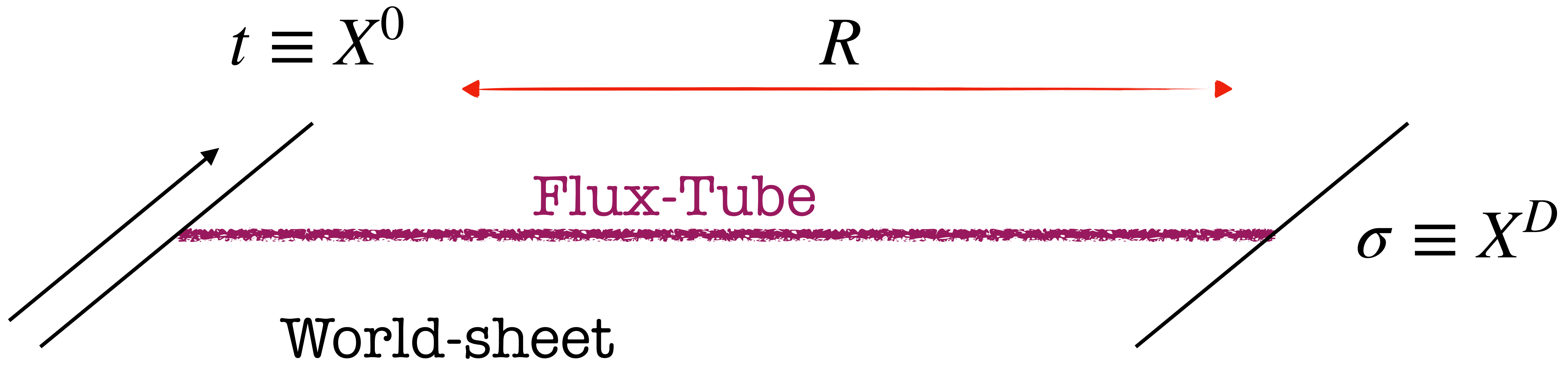

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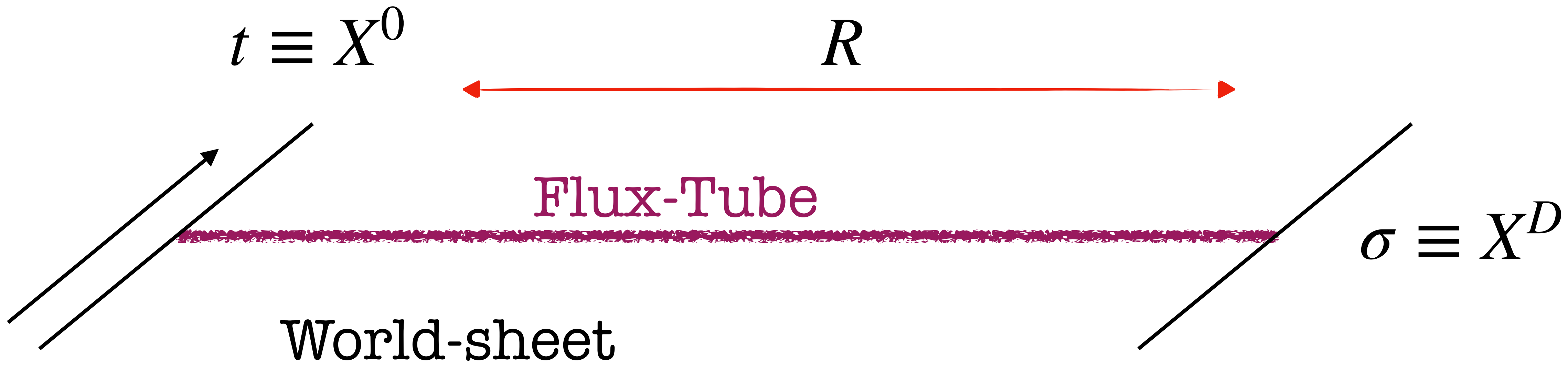

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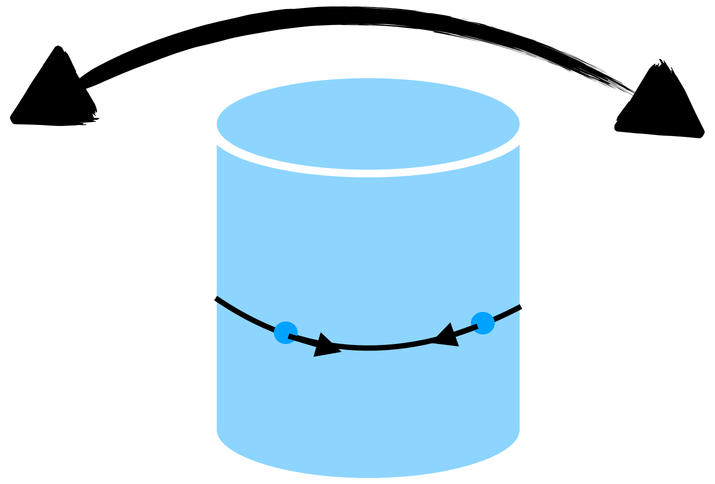
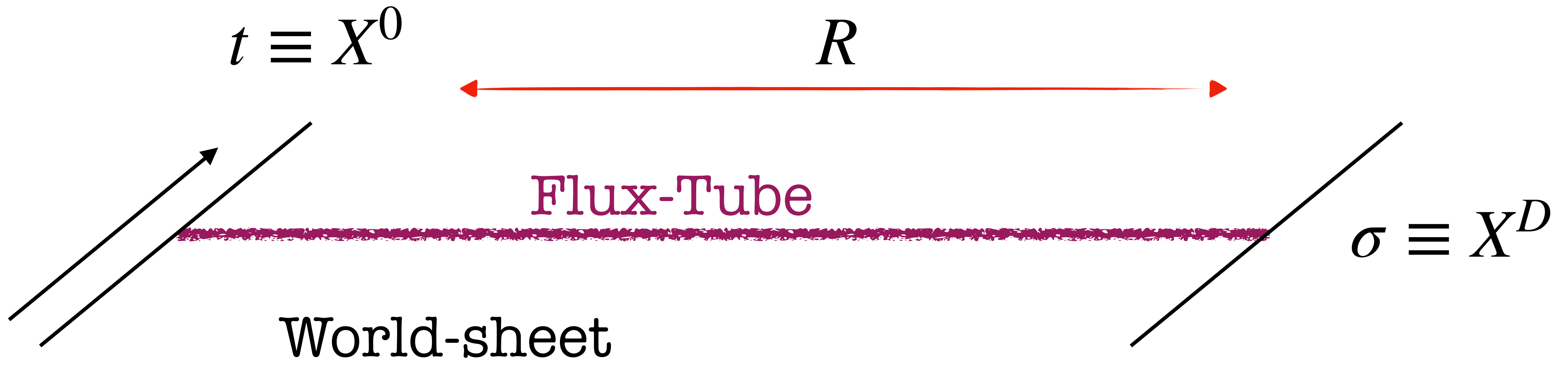


$$\delta(s) = \frac{1}{2i}\log S(s)$$

Elias-Miró, ALG, Hebbar, Penedones, and Vieira, '21

Effective String Theory



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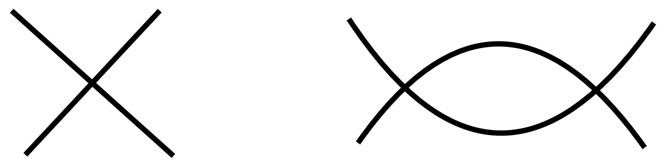

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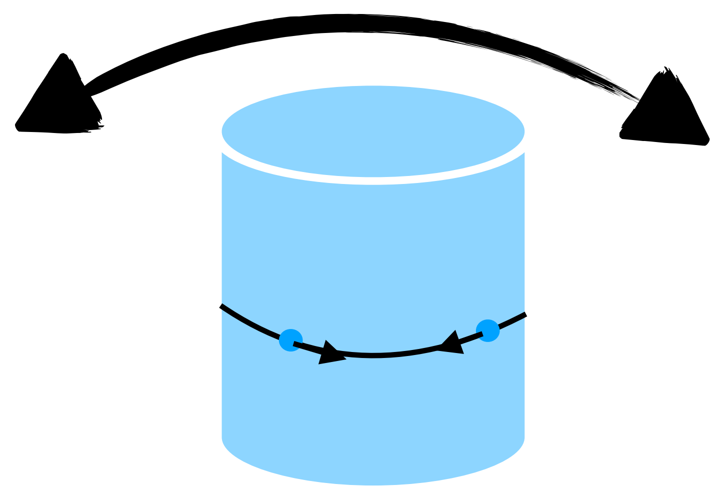

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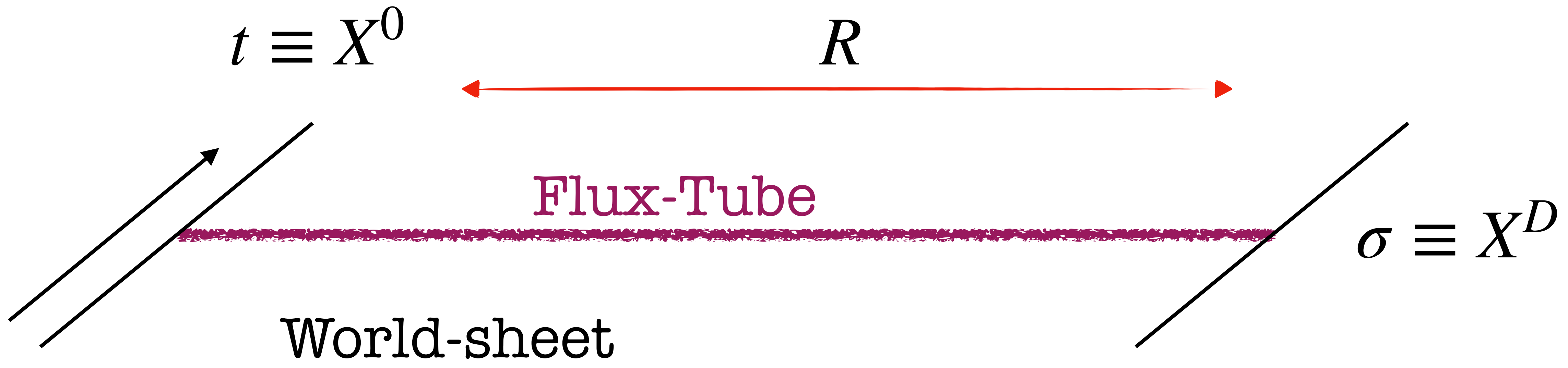

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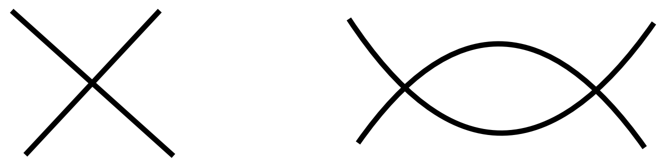

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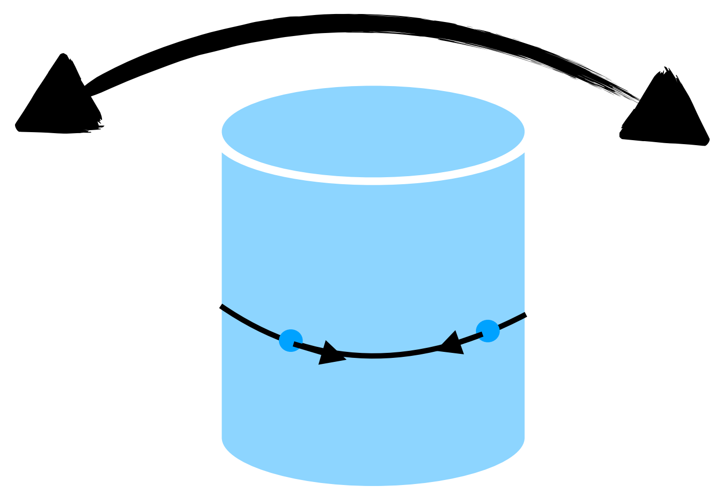

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Bounds on Wilson Coefficients for D=3 flux-tubes

Goal: we bound $\gamma_3 \Longleftrightarrow$ we bound Δ_3

Idea: use the non-perturbative properties of the S-matrix to derive constraints

D=3: 1 Goldstone field, $S_{2\to 2}(s)$ is an analytic function of the $s = 4E^2$ complex variable

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1) Analytic solution: **Schwarz-Pick theorem**

Elias-Miró, ALG, Hebbar, Penedones, and Vieira, '19

$$\gamma_3 \geq -\frac{1}{768}$$

2) Dual functional approach: $\min\limits_{\lambda_2,\lambda_3,\Lambda(s)} \; d = 2\gamma_3 - \frac{1}{192}$

Elias-Miró, ALG '21

```
d[λ2, λ3, Λ] := - λ3/16 - λ2/2 + ∫_0^∞ dz (-2 z Λ - 2 ((1 + z^2 λ2)^2 + z^2 λ3^2)/(π^2 z^9 Λ) + (32 + z^2 (-1 + 32 λ2 + 8 λ3))/(8 π z^4))
```

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Elias-Miró, ALG '21

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Our bound satisfied by lattice simulations!

[4] Baffigo, Caselle '23
[5] Caristo, Caselle, Magnoli, Nada, Panero '21

gauge group	$\mathbb{Z}_2$	$SU(2)$	$SU(6)$	$SU(\infty)$
$\gamma_3 \times 768$	-0.4 [4]	-0.3 [5]	0.2 [1, 6]	0.3

More Bounds in D=3

Elias-Miró, ALG, Hebbar, Penedones, and Vieira, ’21
Elias-Miró, ALG ‘21
Gaikwad, Gorbenko, ALG ’23 (axionic strings in 4D)

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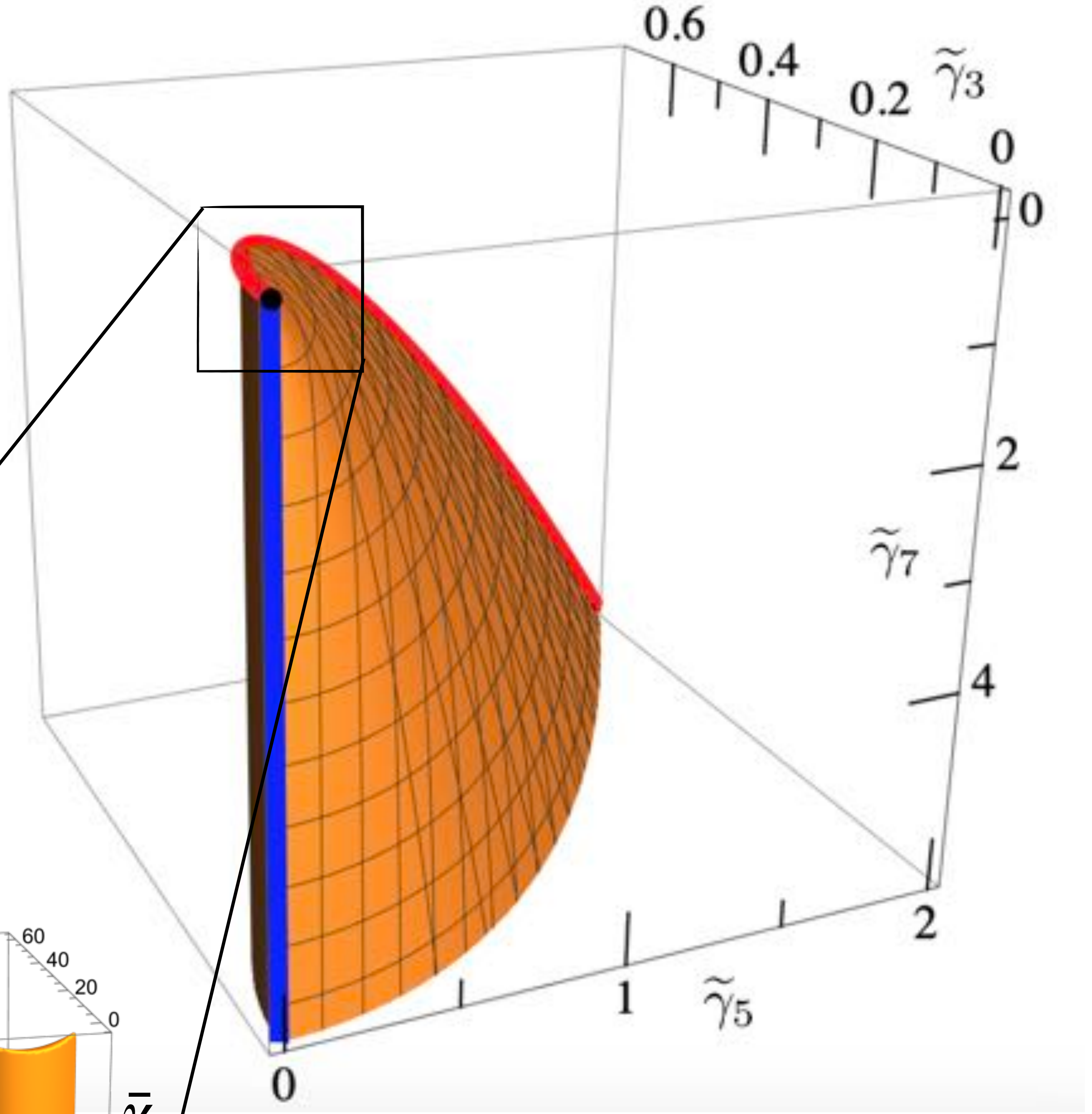
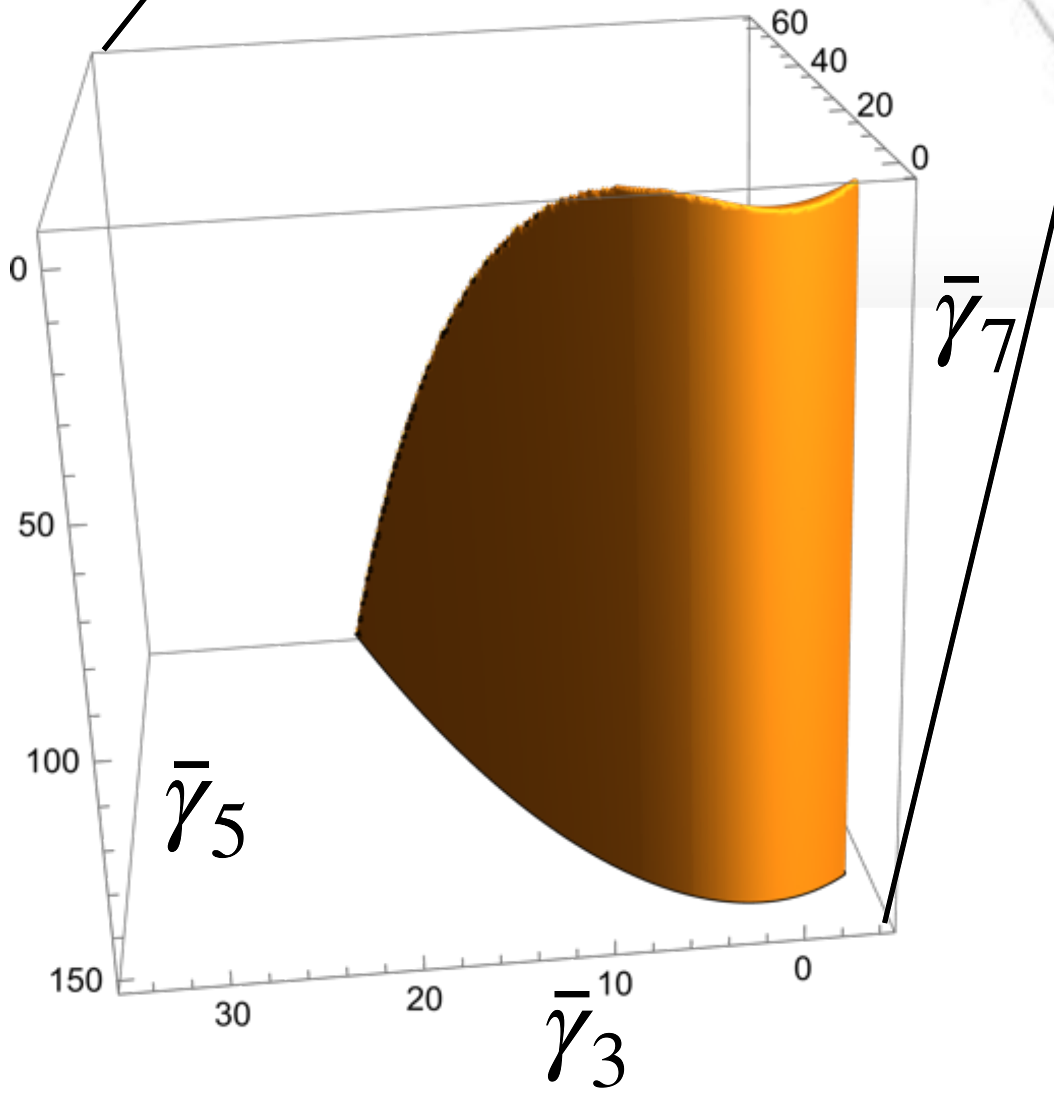
To go beyond we need to include particle production!





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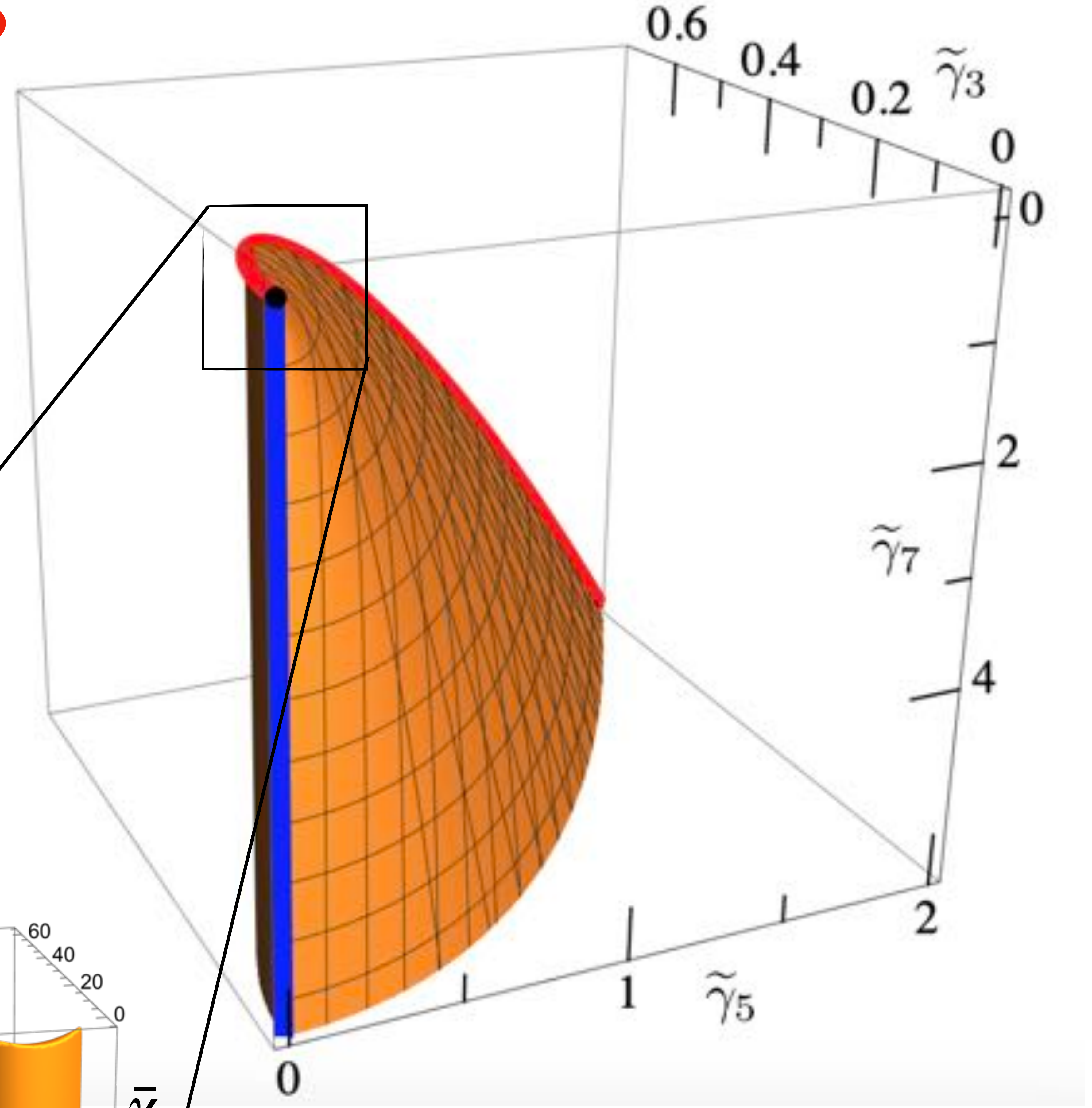
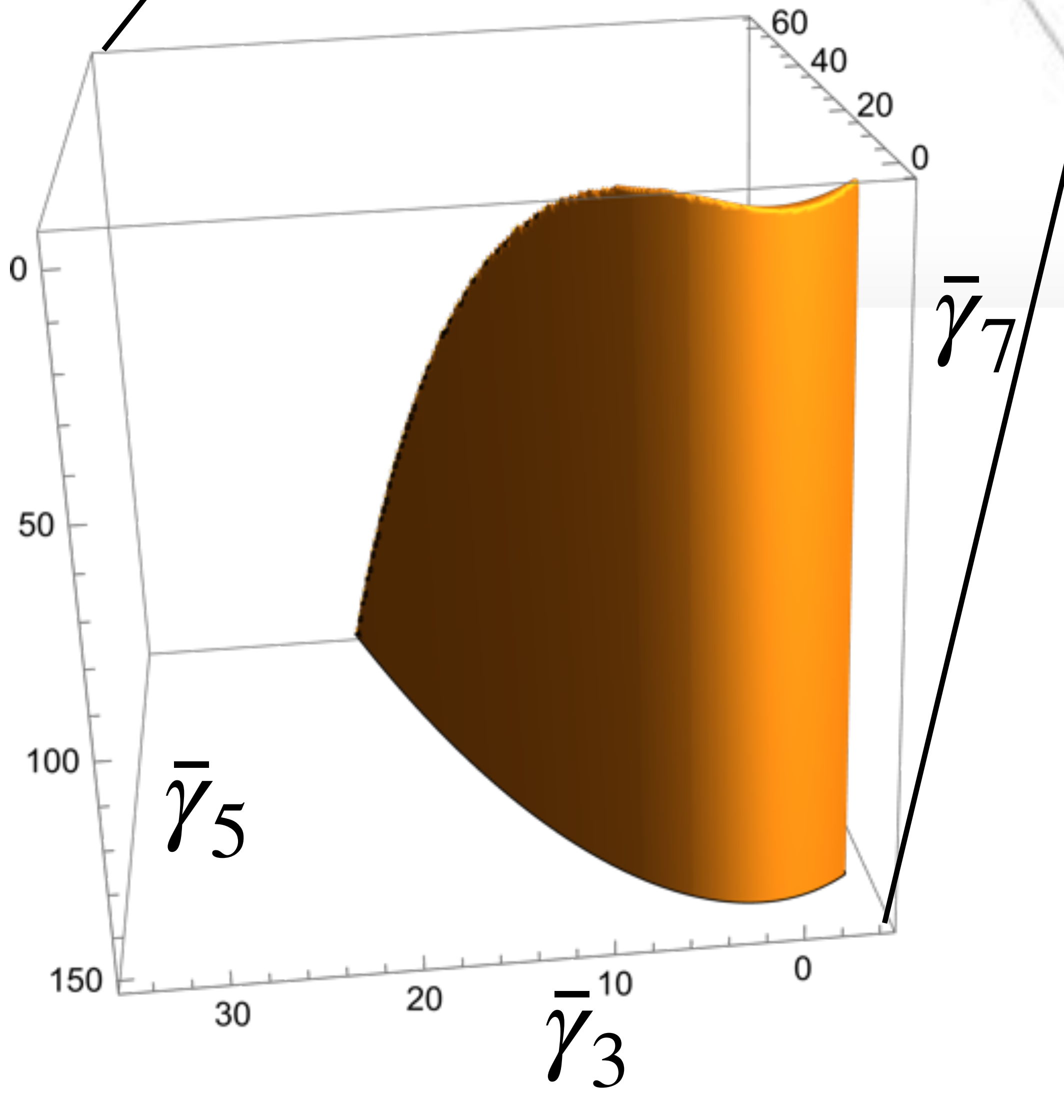
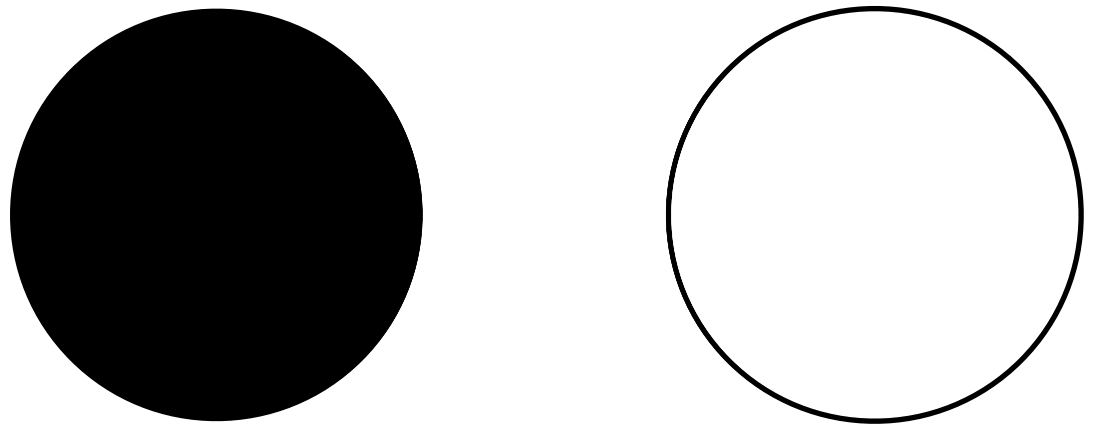




Non-convex!
Elastic unitarity is a non-convex constraint

$$\gamma[7] \geq \frac{\gamma t[5]^2}{\gamma t[3]} + \frac{1}{4096}\gamma t[3] + \frac{1}{64}\gamma t[5] - \frac{1}{16}\gamma t[3]^2 - \frac{1}{7\,340\,032}$$

Convex Non-convex



The Hadronic String in 4D

D=4: X^1, X^2 Goldstones, deviations from Nambu-Goto α_3, β_3

$$\mathscr{A}_{EFF} = \int d^2\sigma\sqrt{-h}\left(\frac{1}{\ell_s^2} + \ldots + \alpha_3 \ell_s^6 K^4 + \beta_3 \ell_s^6 R^2 + \ldots\right)$$

$(\gamma_3 = \alpha_3 - \beta_3)$

New Effect in the amplitude: universal **Polchinski-Strominger** term at 1-loop $\propto \alpha_2 = \frac{D-26}{384\pi}$

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E.g. we can add an axion

$$S_a = \int d^2\sigma\left[-\frac{1}{2}(\partial_\alpha a)^2 - \frac{1}{2}m_a a^2 - \ell_s^2 Q_a a \varepsilon^{ij}\varepsilon^{\alpha\beta}\partial_\alpha\partial_\gamma X^i \partial_\beta \partial^\gamma X^j + \ldots\right].$$

If we tune $Q_a = \frac{\sqrt{22}}{4\sqrt{3\pi}} \simeq 0.378$, and $m_a \to 0$, we can restore integrability

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	$SU(3)$	$SU(5)$	$SU(\infty)$
2^{++}			
$m_a^L \ell_s$	$1.85^{+0.02}_{-0.03}$	$1.64^{+0.04}_{-0.04}$	1.5
Q_a^L	$0.380^{+0.006}_{-0.006}$	$0.389^{+0.008}_{-0.008}$	-
2^{+-}			
$m_a^L \ell_s$	$1.85^{+0.02}_{-0.02}$	$1.64^{+0.04}_{-0.04}$	1.5
Q_a^L	$0.358^{+0.004}_{-0.005}$	$0.358^{+0.009}_{-0.009}$	-

Lattice results show the presence of an axion resonance with the correct coupling, but massive at large N_c

Gaikwad, Gorbenko, ALG '23

Flux-Tube S-matrix Bootstrap in 4D

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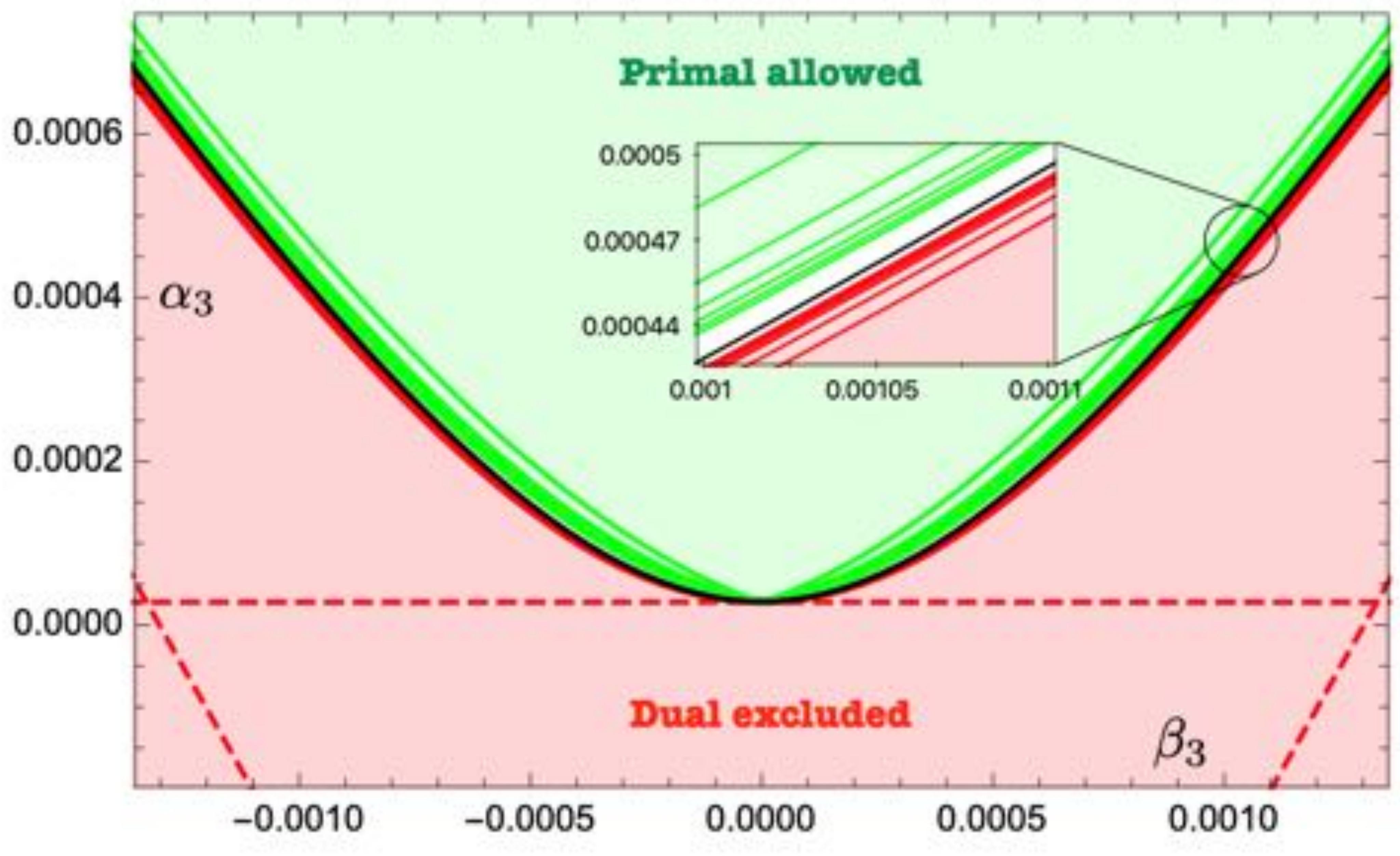

Elias-Miró, ALG, Hebbar, Penedones, Vieira 1906.08098
Elias-Miró, ALG 2106.07957

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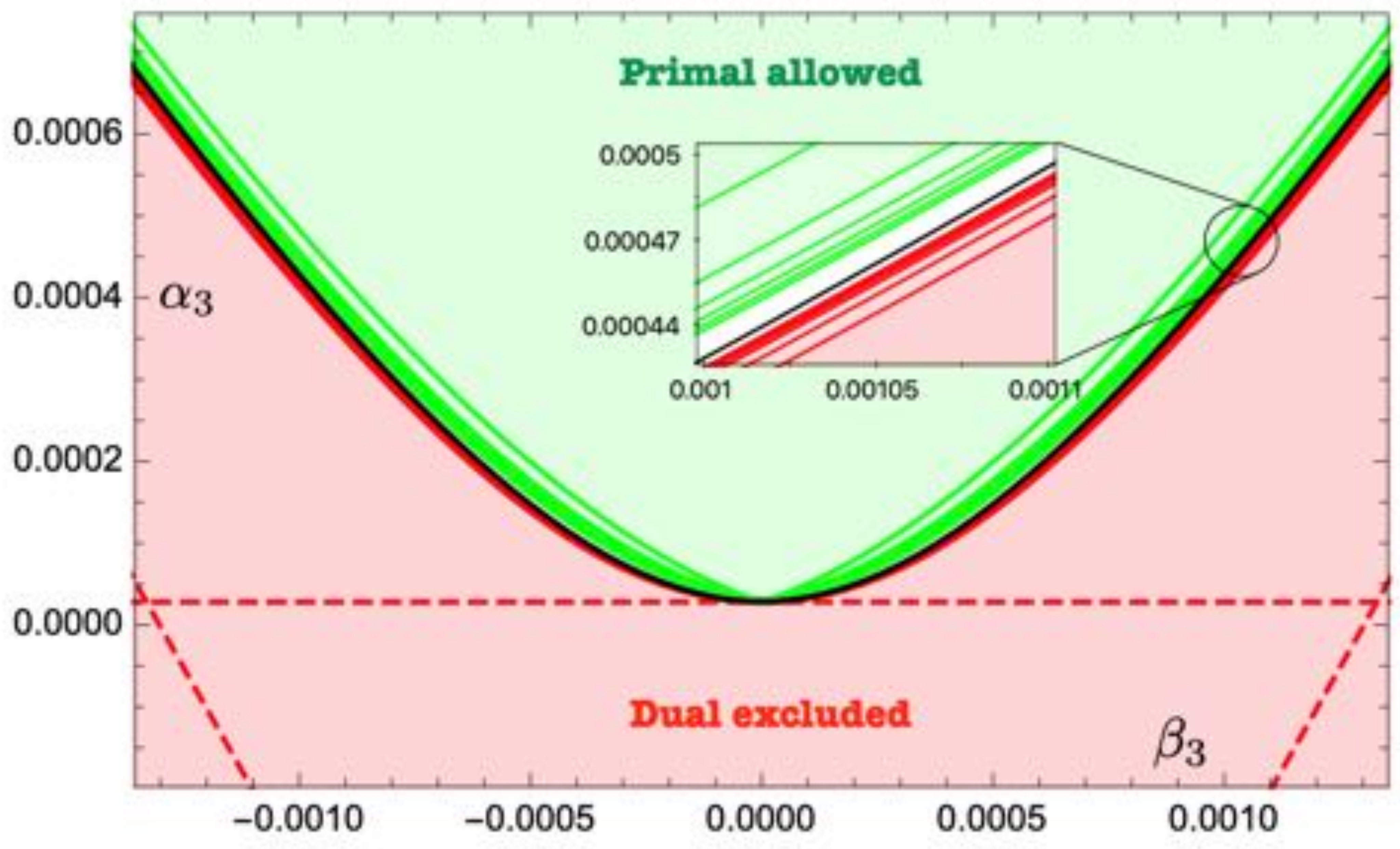

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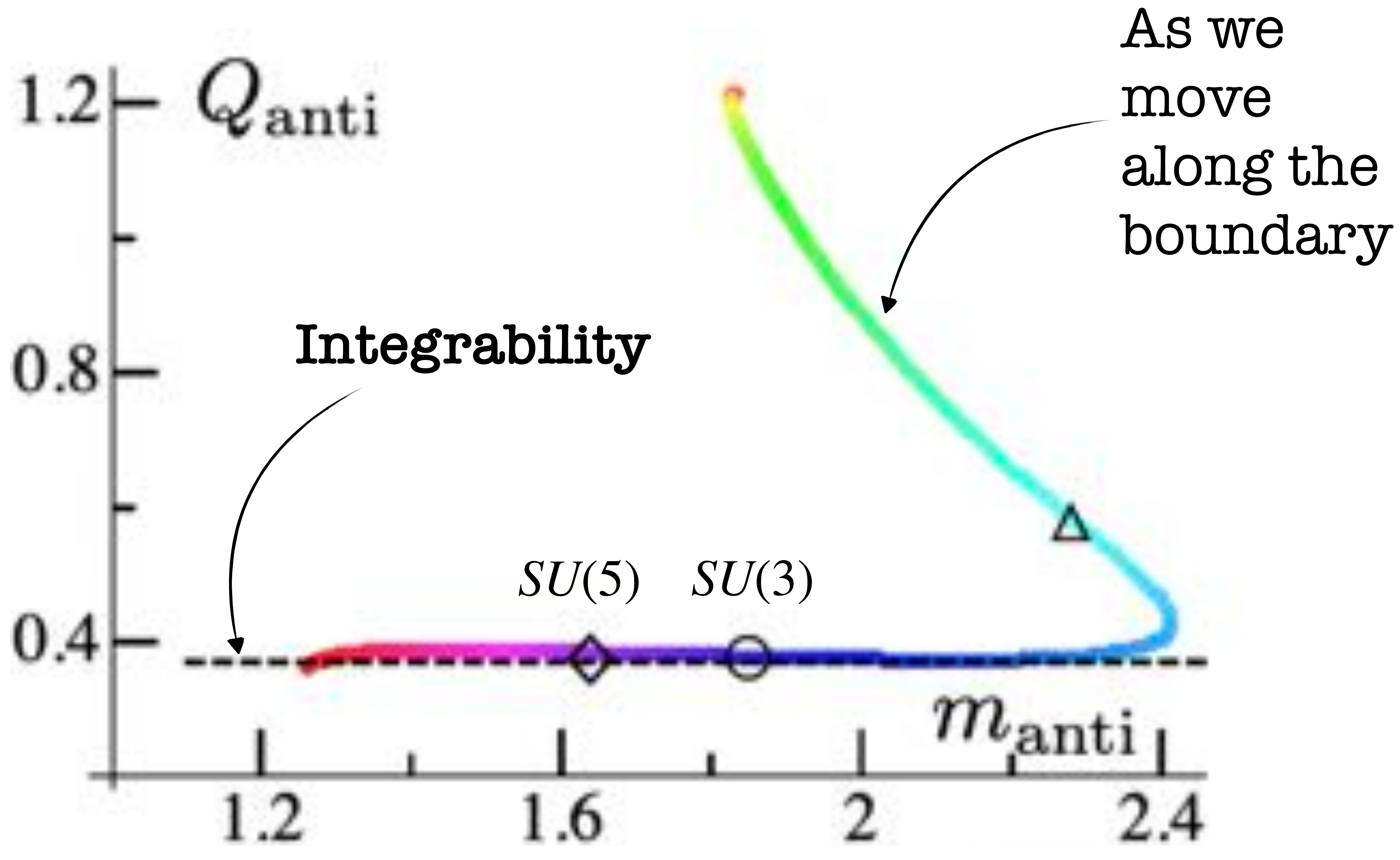
Surprise!
Extremal Bootstrap amplitudes contain an axion with integrable coupling!



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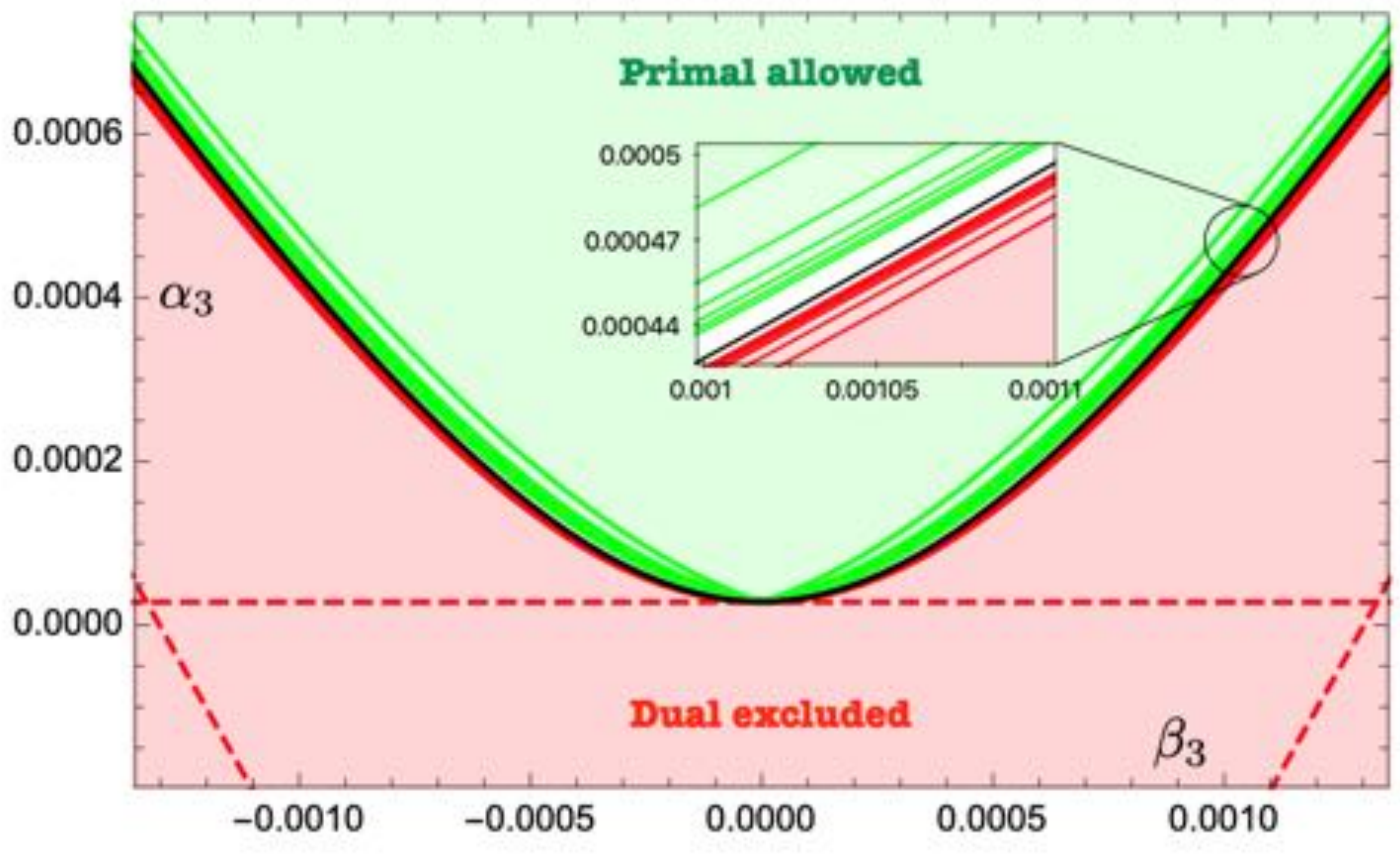
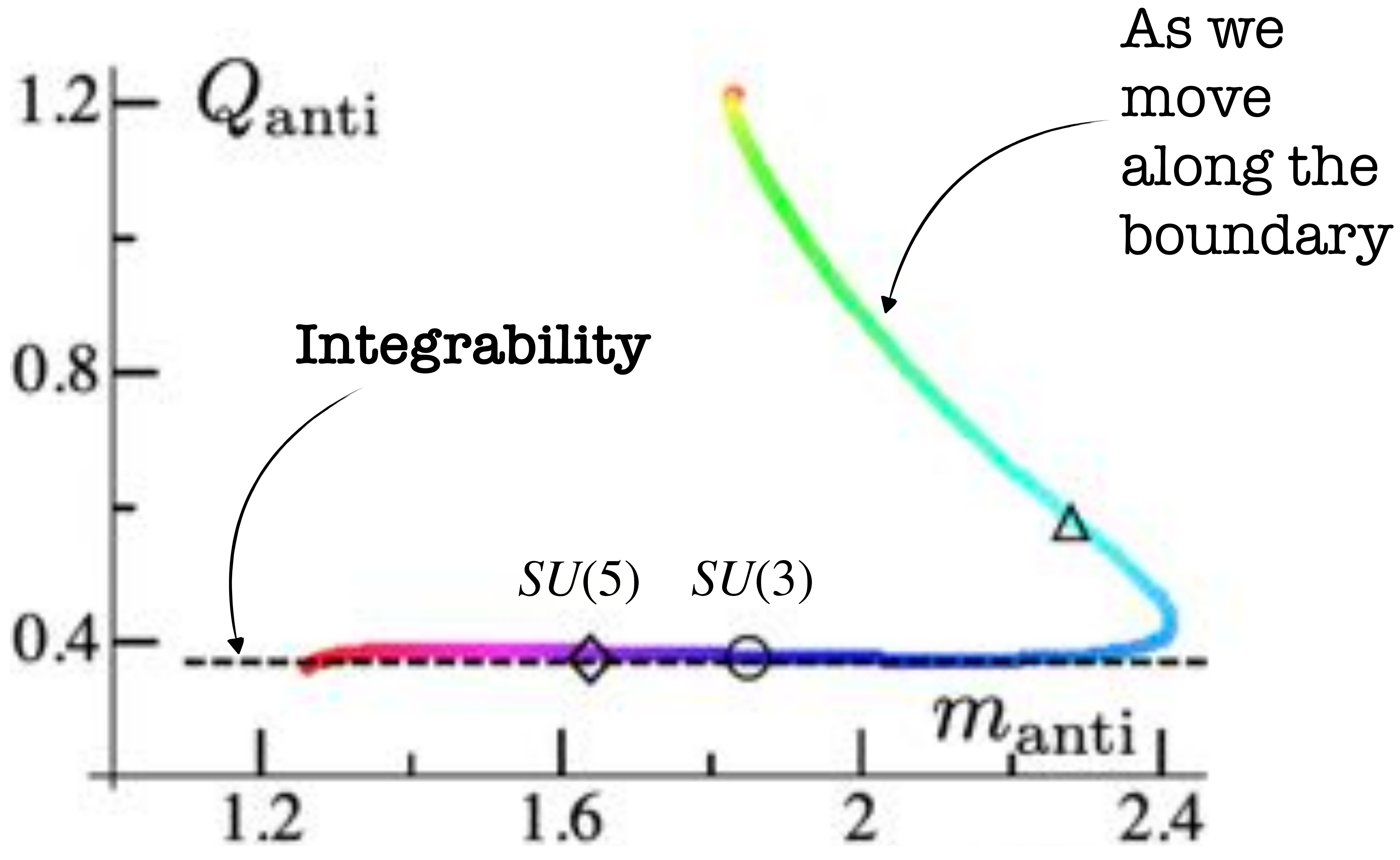


Elias-Miró, ALG, Hebbar, Penedones, Vieira 1906.08098
Elias-Miró, ALG 2106.07957

Surprise!
Extremal Bootstrap amplitudes contain an axion with integrable coupling!



$$Q_a^L \approx Q_a^c \approx Q_a^b$$

Can we explain this **triple** coincidence?

An EFT for the Bootstrap extremal amplitudes

Can we develop an EFT for the Bootstrap amplitudes?

Gaikwad, Gorbenko, ALG 2310.20698

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Weakly-coupled EFT of branons interacting through an axion

$$\mathcal{L}_a = -\frac{1}{2}(\partial X^i)^2 - \frac{1}{2}(\partial a)^2 - \frac{1}{2}m_a^2 a^2 - g_a a \varepsilon_{ij}\varepsilon^{\alpha\beta}\partial_\alpha\partial_\gamma X^i \partial_\beta \partial^\gamma X^j + \dots$$

$$M_{\text{sing}} = -\frac{g_a^2}{4}\frac{s^4}{(s+m_a^2)},$$

$$M_{\text{anti}} = -\frac{g_a^2}{4}\frac{s^4(s+3m_a^2)}{(s-m_a^2)(s+m_a^2)},$$

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We match the EFT and the Bootstrap low energy expansion and express $\{m_a, g_a\}$ as a function of $\{\ell_s, \beta_3\}$ for $\beta_3 \to \infty$

$$\mathscr{A}_{EFF} = \int d^2\sigma \sqrt{-h}\left(\frac{1}{\ell_s^2} + \dots + \alpha_3 \ell_s^6 K^4 + \beta_3 \ell_s^6 R^2 + \dots\right)$$

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$$\alpha_3 = \beta_3$$

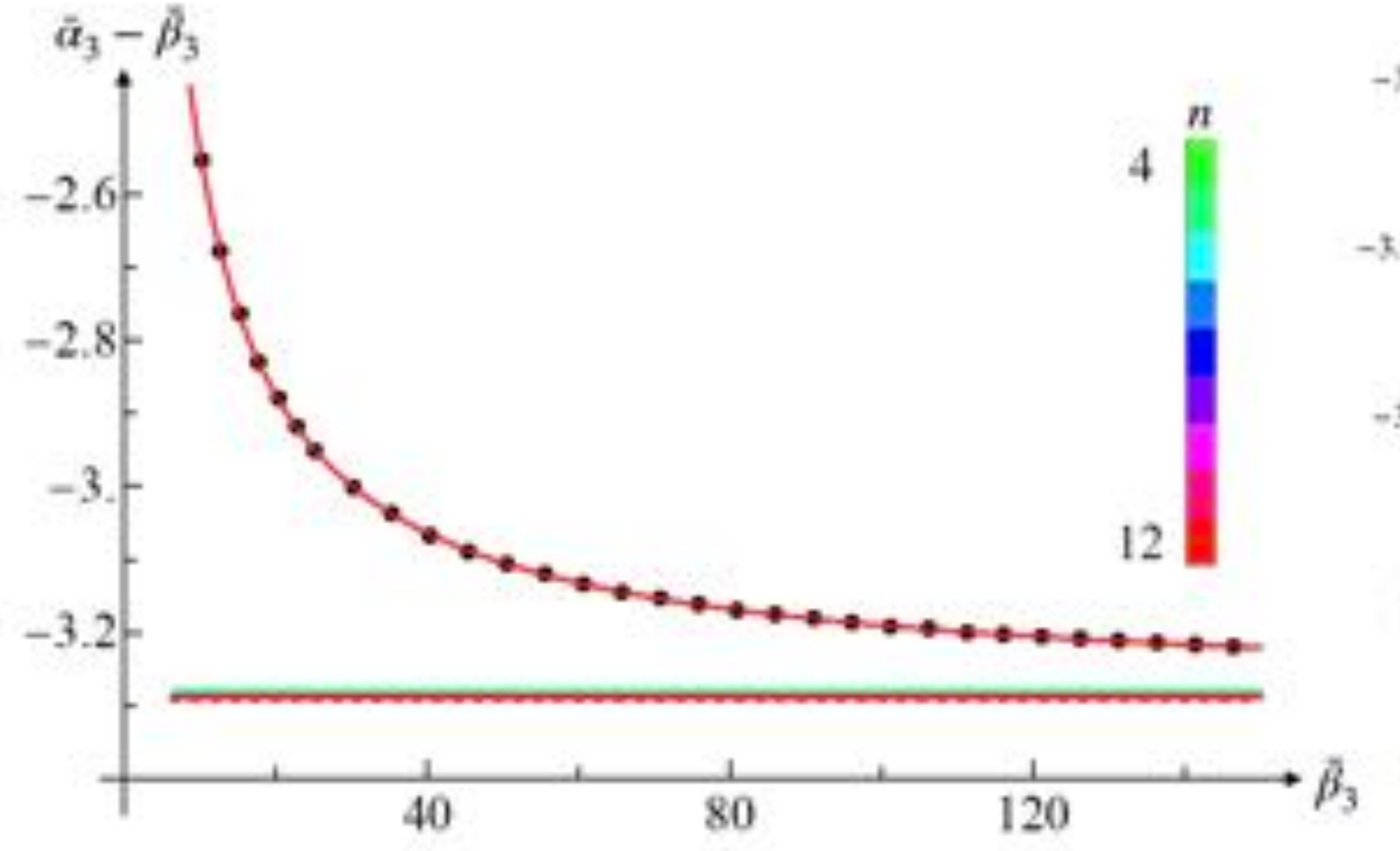
$$m_a \ell_s = \sqrt{\frac{|\alpha_2|}{\beta_3} + i\frac{|\alpha_2|^{7/2}}{2\beta_3^{5/2}} + \dots}$$

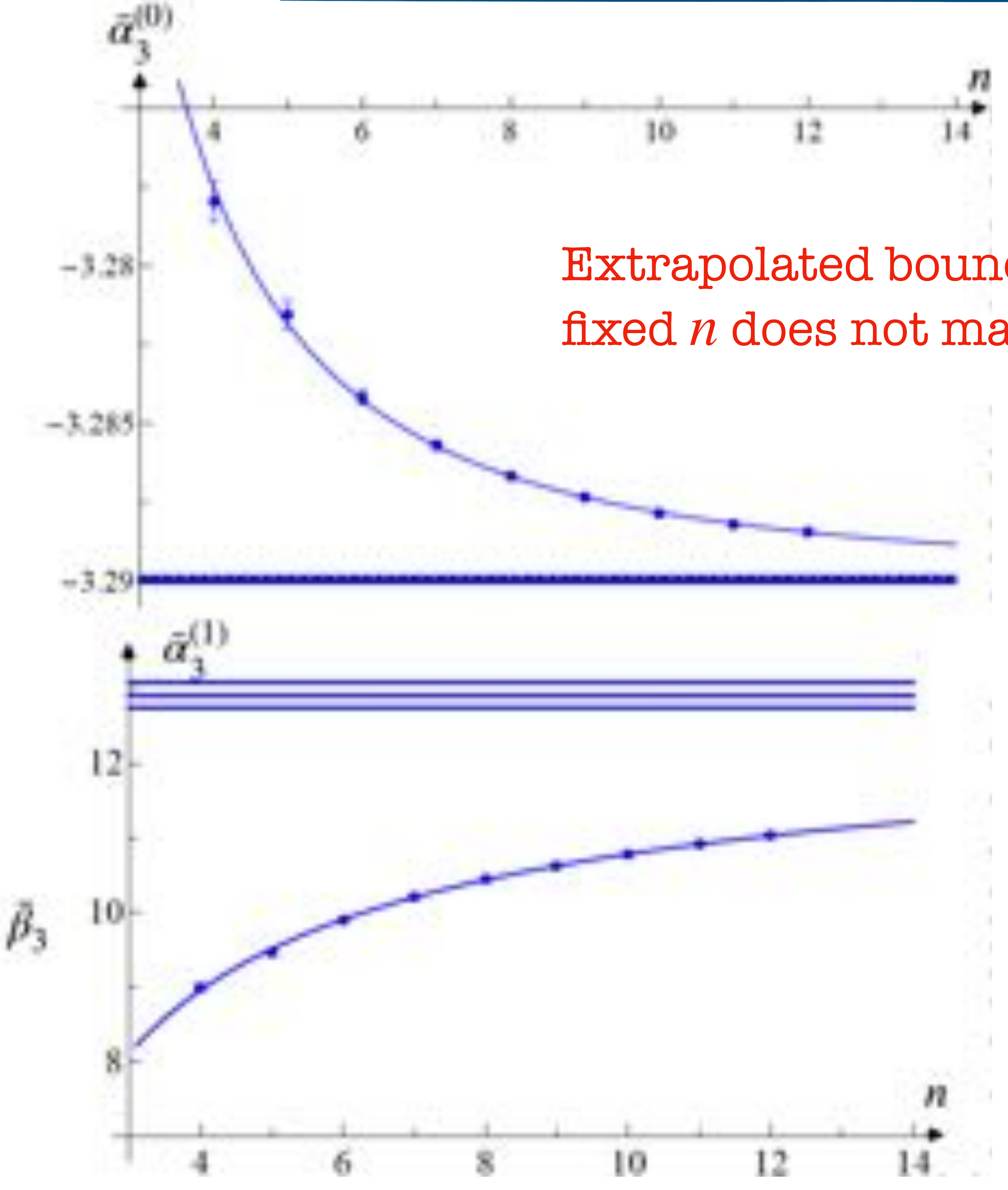
$$Q_a = \frac{\sqrt{8\Gamma_a}}{m_a^{5/2}\ell_s^2} = 2\sqrt{2|\alpha_2|} + \dots$$

Bootstrap Results: $\lim_{\beta_3\to\infty} \frac{\alpha_3}{\beta_3} = const$

EFT inspired asymptotic expansion $\bar{\alpha}_3(\bar{\beta}_3) - \bar{\beta}_3 = \sum_{k=0}^{n} \frac{\bar{\alpha}_3^{(k)}}{\bar{\beta}_3^k}.$

$$\mathscr{A}_{EFF} = \int d^2\sigma\sqrt{-h}\left(\frac{1}{\ell_s^2} + \ldots + \alpha_3\ell_s^6 K^4 + \beta_3 \ell_s^6 R^2 + \ldots\right)$$

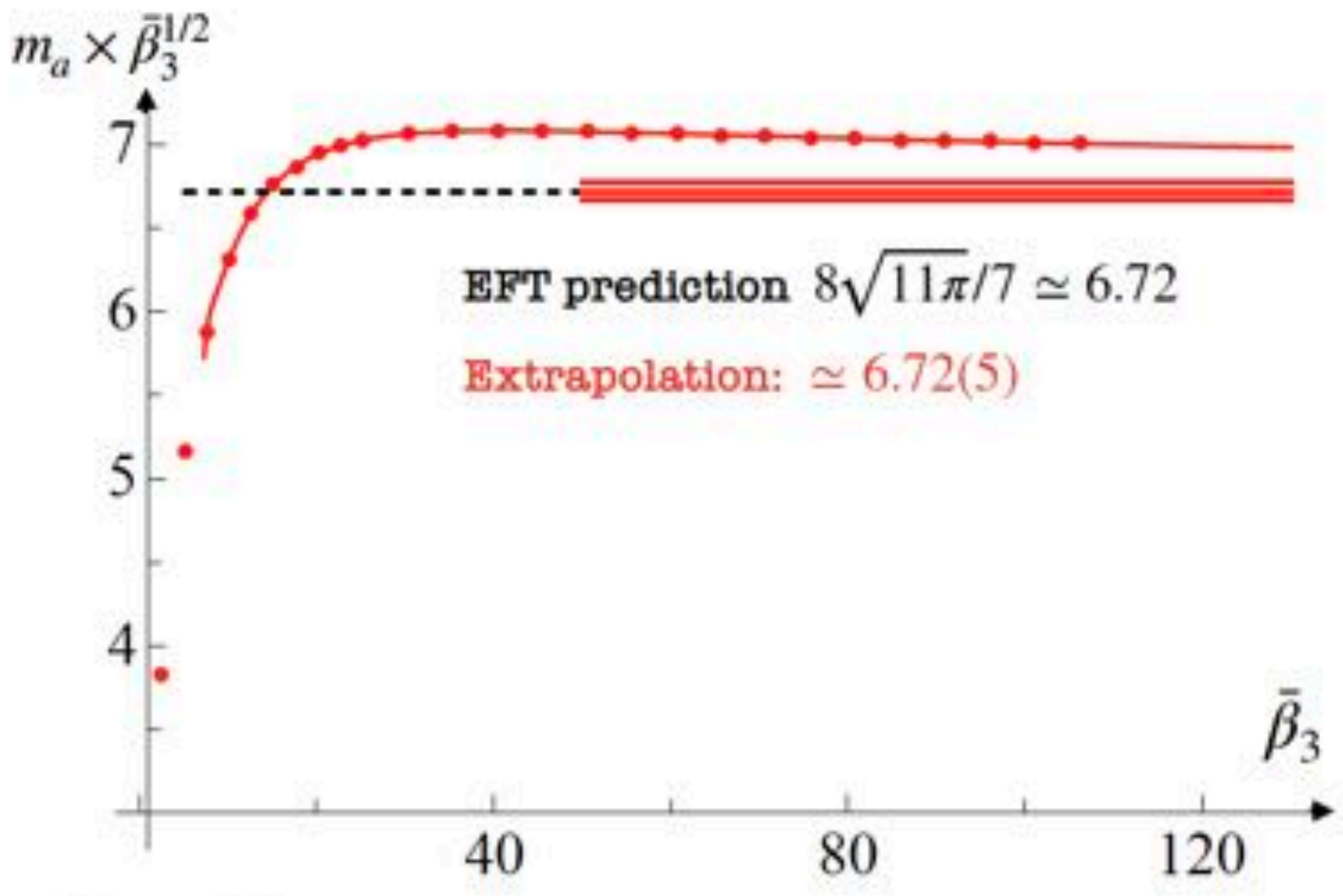


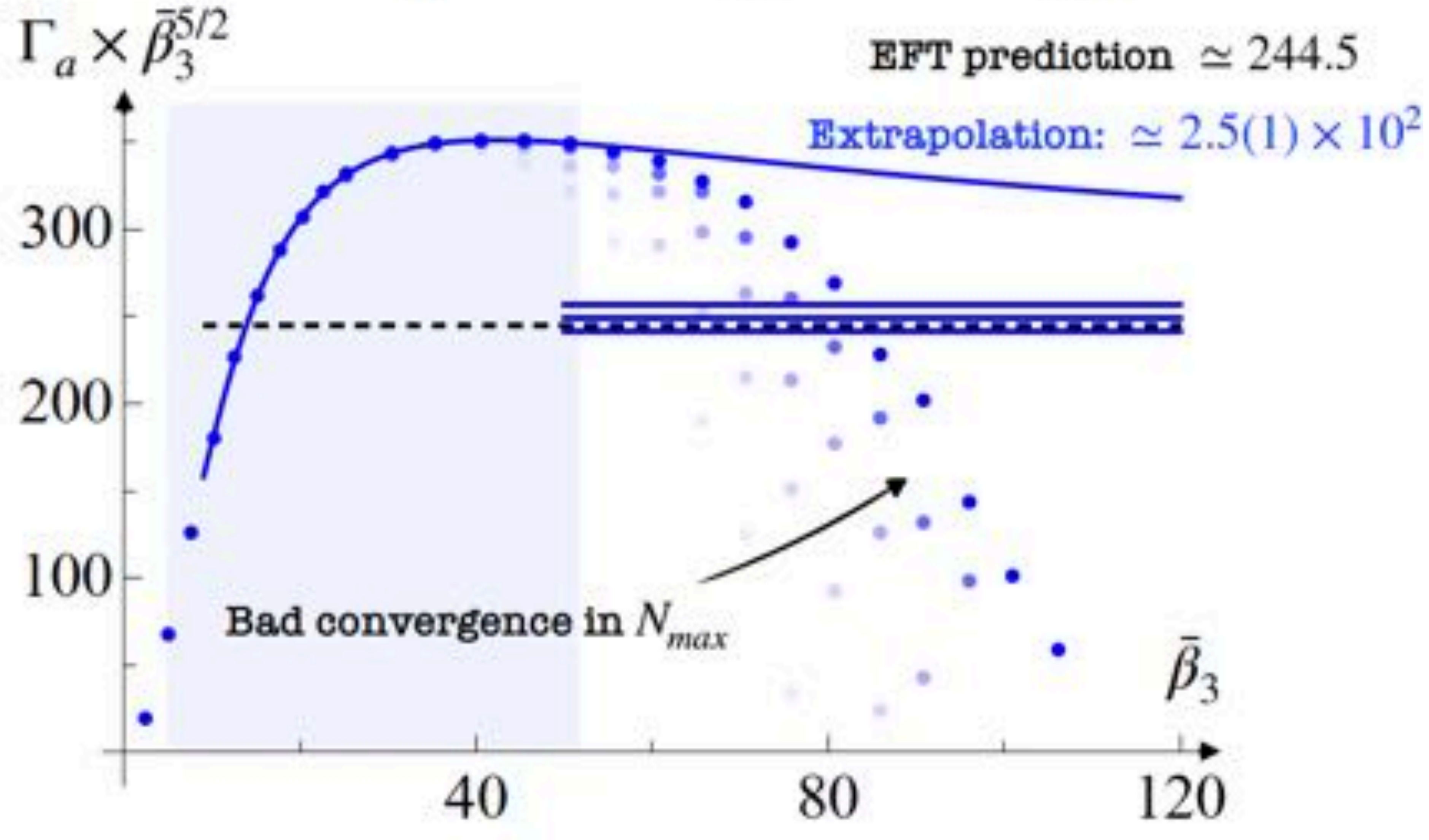


Extrapolated bound for fixed n does not match

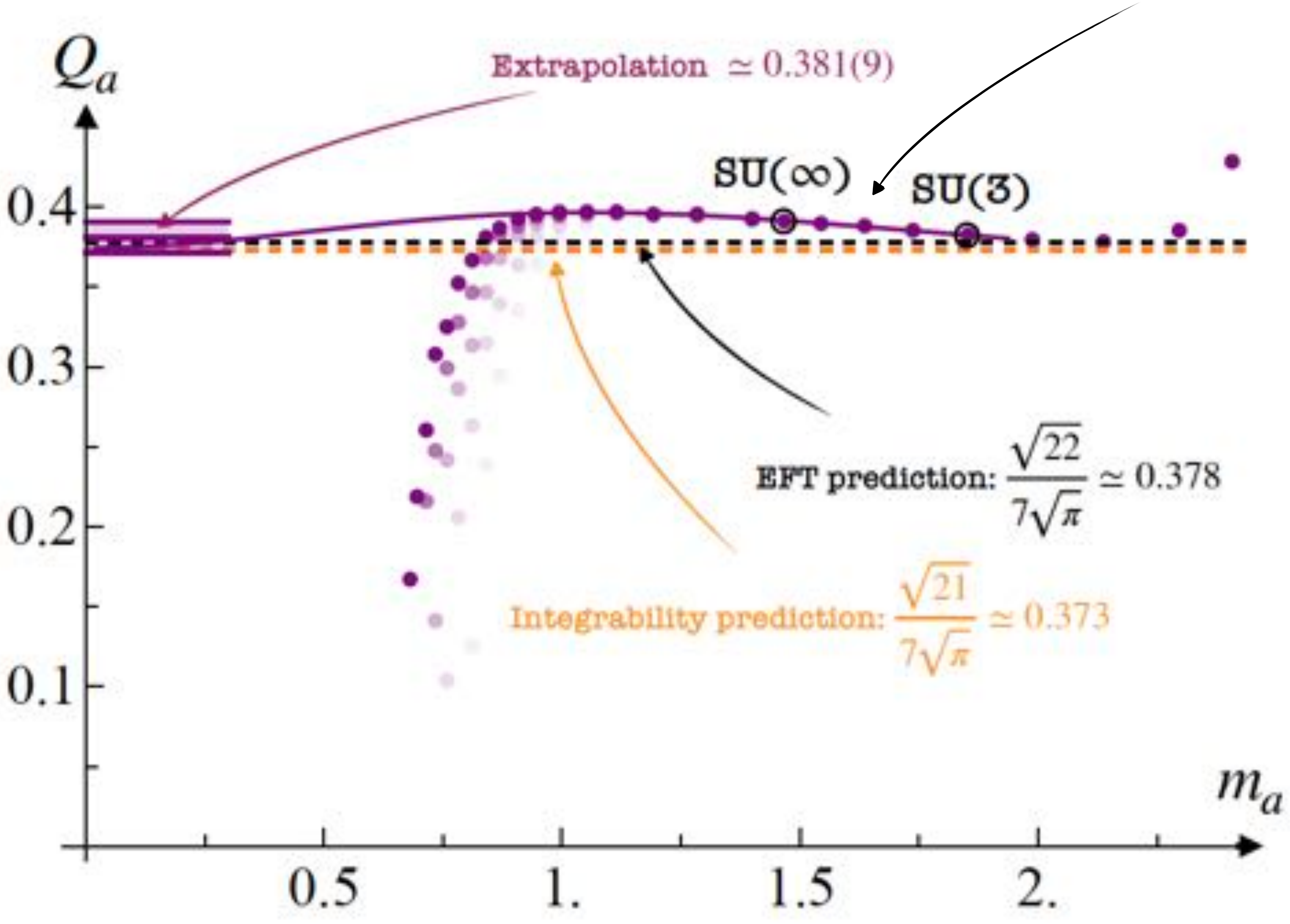
Careful with extrapolations taking into account "global significance"

Bootstrap Results: $m_a(\beta_3), \Gamma_a(\beta_3)$





EFT breaks down for SU(N) axion masses



Bootstrap bounds compatible with the EFT explanation

The coincidence remains…but…

Axion dominance and approximate integrability

For $m_a^2 \ll \hat{s} \ll \ell_s^{-2}$, analyticity locks the axion coupling to cancel the Weyl anomaly α_2 in the EFT.
We expect smaller particle production in the UV (1%)

$$2\delta_{\text{anti}}(\hat{s}) = \frac{\hat{s}}{4} - \alpha_2 \hat{s}^2 + \alpha_2 \hat{s}^3 \frac{\hat{s} - \frac{3\alpha_2}{\beta_3}}{\left(\hat{s} + \frac{\alpha_2}{\beta_3}\right)\left(\hat{s} - \frac{\alpha_2}{\beta_3}\right)} \leq c\hat{s}$$

What if $m_a \simeq \ell_s^{-1}$, when the EFT breaks down?

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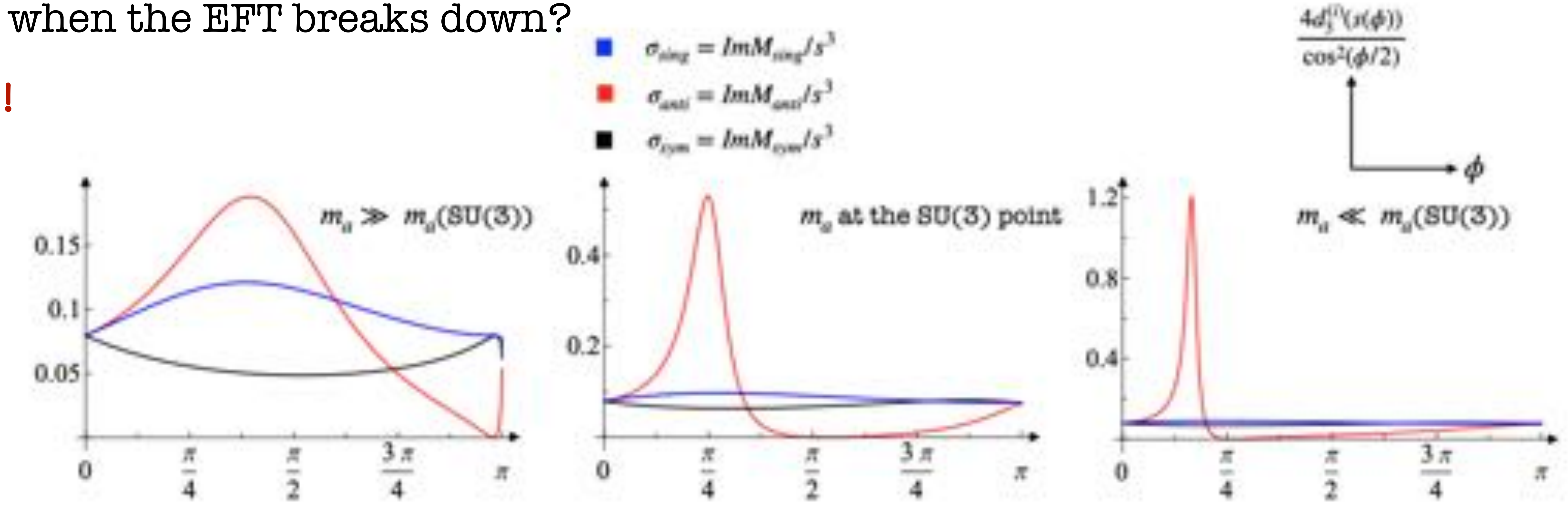

Generic Nonperturbative | QCD regime | EFT Regime

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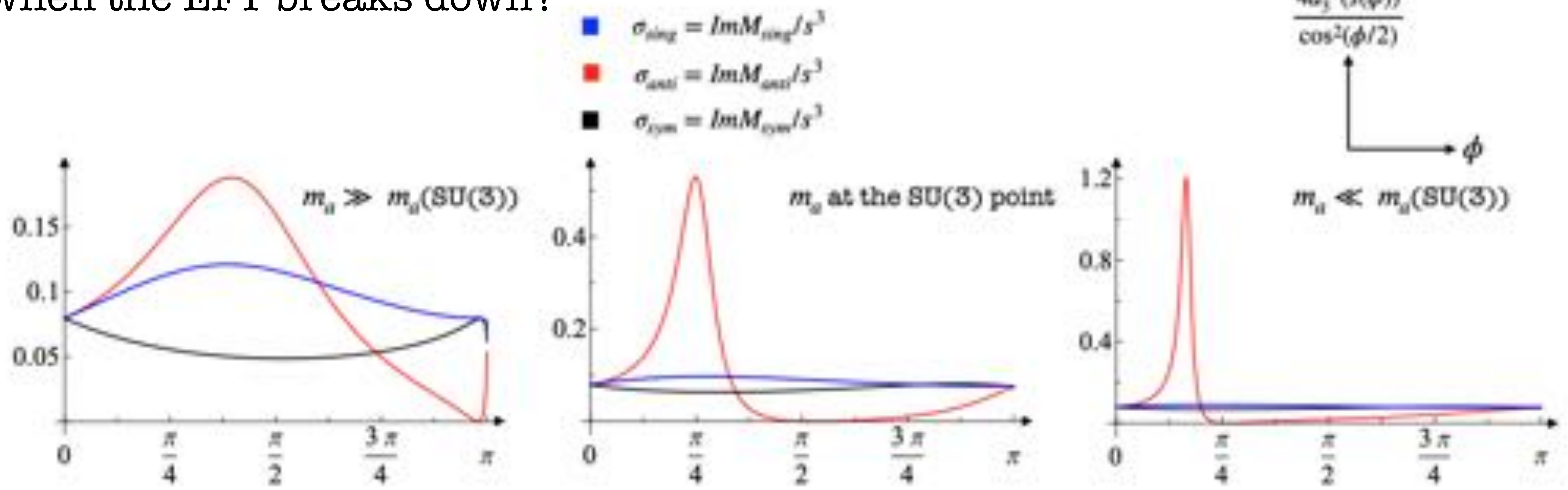


Generic Nonperturbative — QCD regime — EFT Regime

Nonperturbative relation between the charge and the anomaly

$$\frac{\Gamma_a}{m_a^5} = -\alpha_2 - \frac{1}{4\pi m_a^2}\int_0^\infty \frac{ds \log|S_f|^2}{s^2}$$

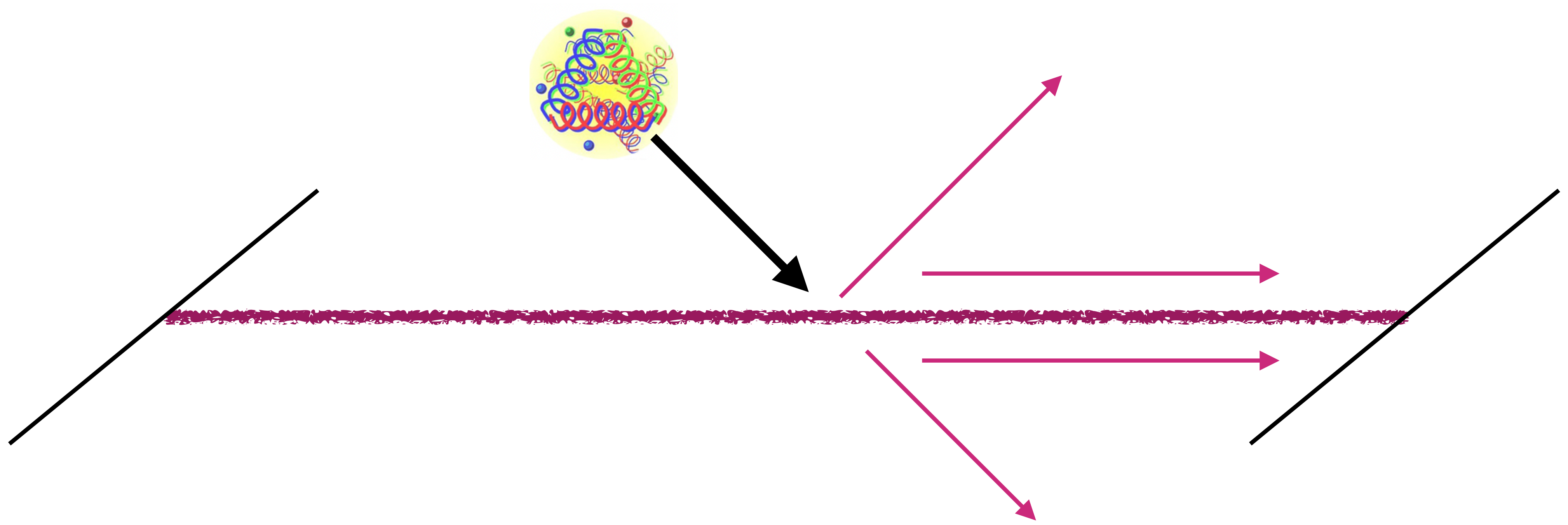
5% correction at SU(3) point proportional to Integrability breaking

Flux-Tube Bootstrap: What's next?

Our argument relies on the assumption QCD flux-tube S-matrix "close" to the real-world one

Emission/Absorption of Glueballs

Hebbar, ALG (to appear)



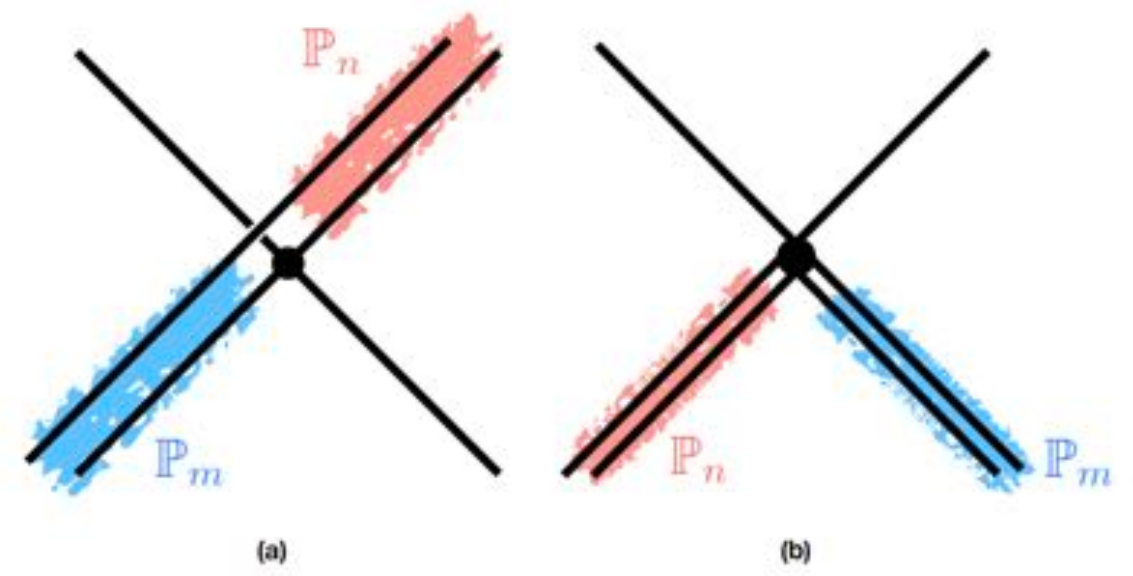
$$\begin{pmatrix} < bb \,|\, bb > & < bb \,|\, G > \\ < G \,|\, bb > & < G \,|\, G > \end{pmatrix}$$

World-sheet particle production: multi-particle Bootstrap

Naively: Stronger constraints!

$$\sum_n P_{2\to n} = 1 \implies P_{2\to 2} + P_{2\to 4} + \ldots \leq 1$$

Homrich, ALG, Penedones, Vieira (to appear this year perhaps)



$3 \to 3$ $\quad$ $2 \to 4$

2-particle Jet States

Glueballs in SU(3) pure YM

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Regime in which the S-matrix Bootstrap shows its power: cutoff $\Lambda = 2m$, no small parameters.

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Stable Glueballs spectrum

	J^{PC}	Mass
G	0^{++}	1
H	2^{++}	1.437 ± 0.006
G^*	0^{++}	1.72 ± 0.01
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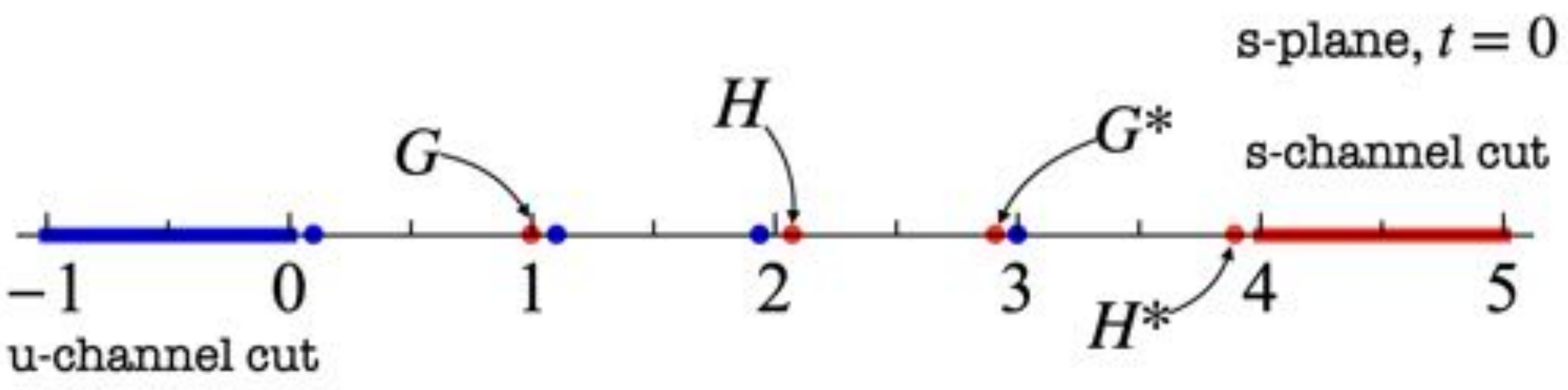
Athenodorou, Teper 2007.06422, 2106.00364

Pole Structure in GG->GG scattering



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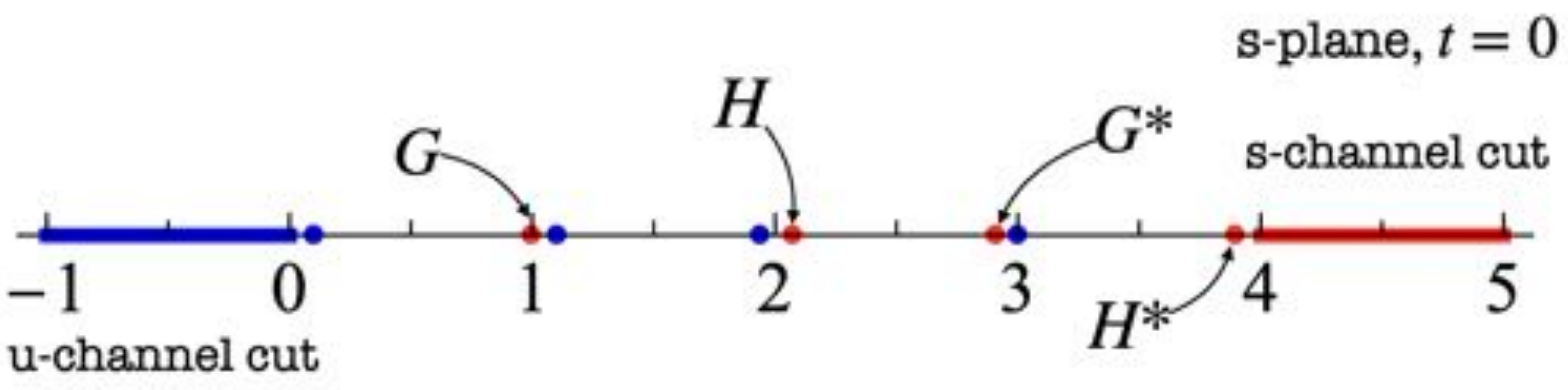
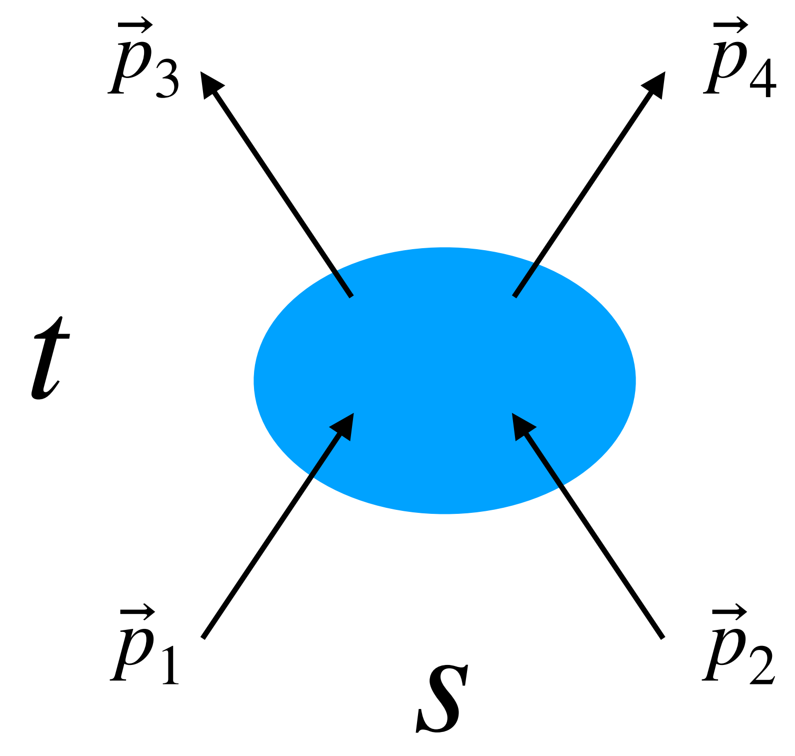


Can we bound these couplings using only general principles?

$$M \supset -g_G^2 \frac{1}{s - m_G^2} - g_{G*}^2 \frac{1}{s - m_{G*}^2} - g_H^2 \frac{t^2 + \ldots}{s - m_H^2} - g_{H*}^2 \frac{t^2 + \ldots}{s - m_{H*}^2} + \ldots$$

AG, Hebbar, van Rees 2312.00127

Amplitudes in 3+1 D: general properties



Crossing: $M(s,t,u)$ symmetric in the three variables

$$s = (p_1 + p_2)^2, t = (p_1 - p_3)^2, u = (p_1 - p_4)^2$$

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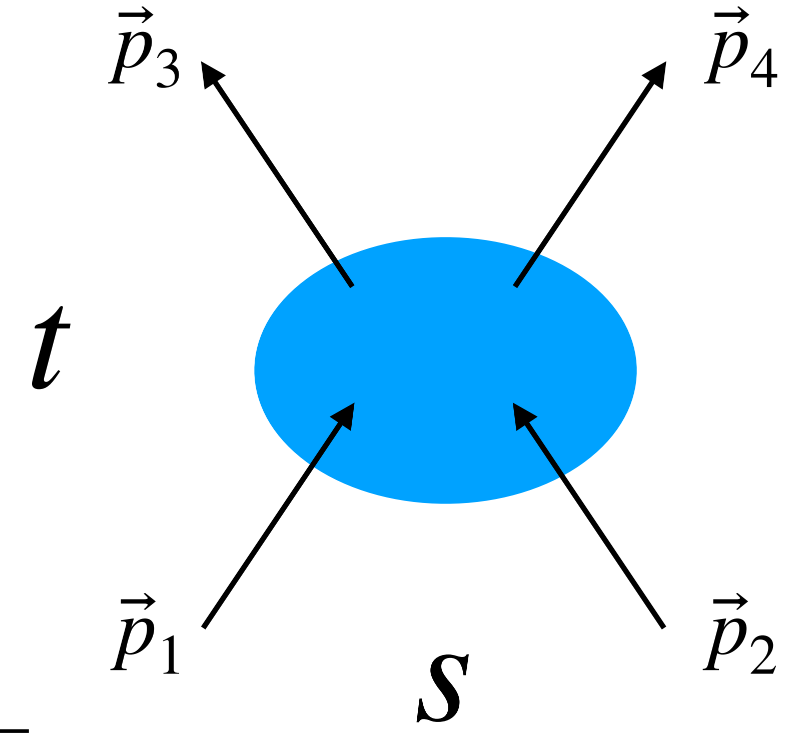

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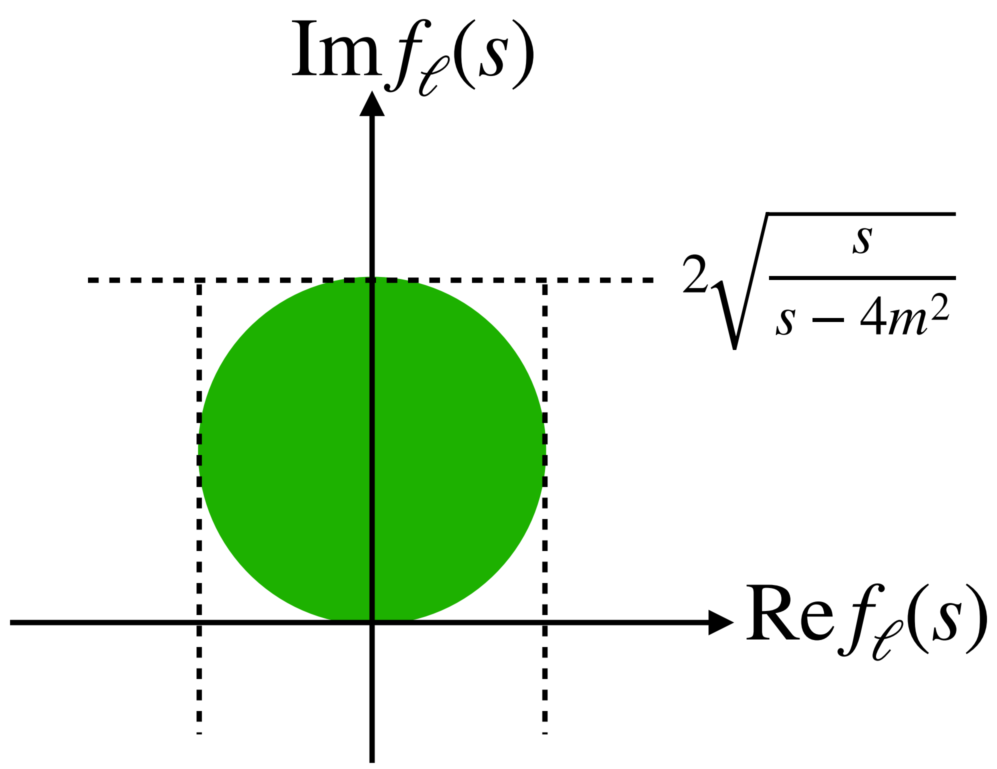

Unitarity: $2Imf_\ell \geq \sqrt{\frac{s-4m^2}{s}}|f_\ell|^2$

$$f_\ell = \frac{1}{32\pi}\int_{-1}^{1} dx P_\ell(x) M(s,t(s,x))$$

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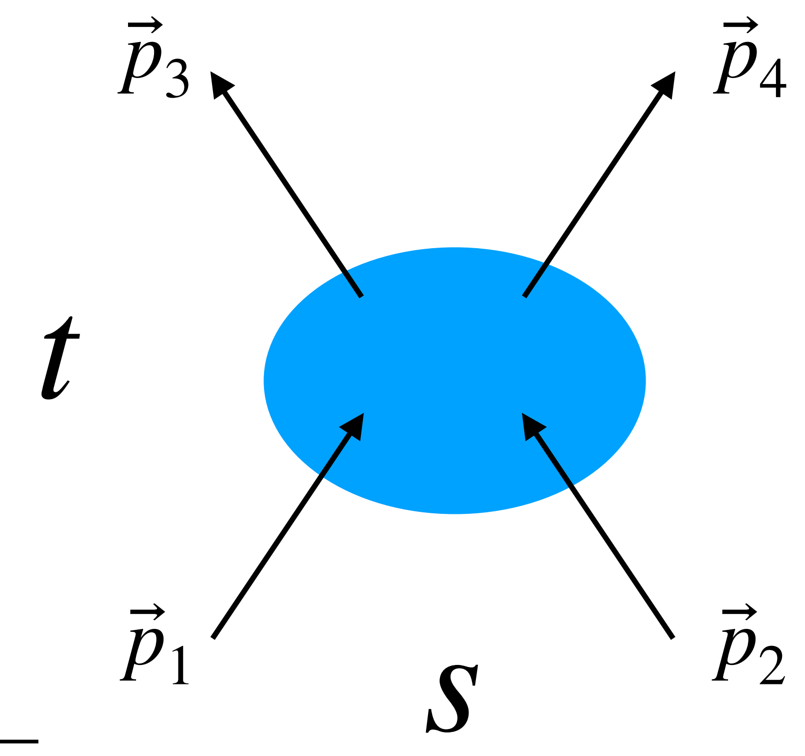

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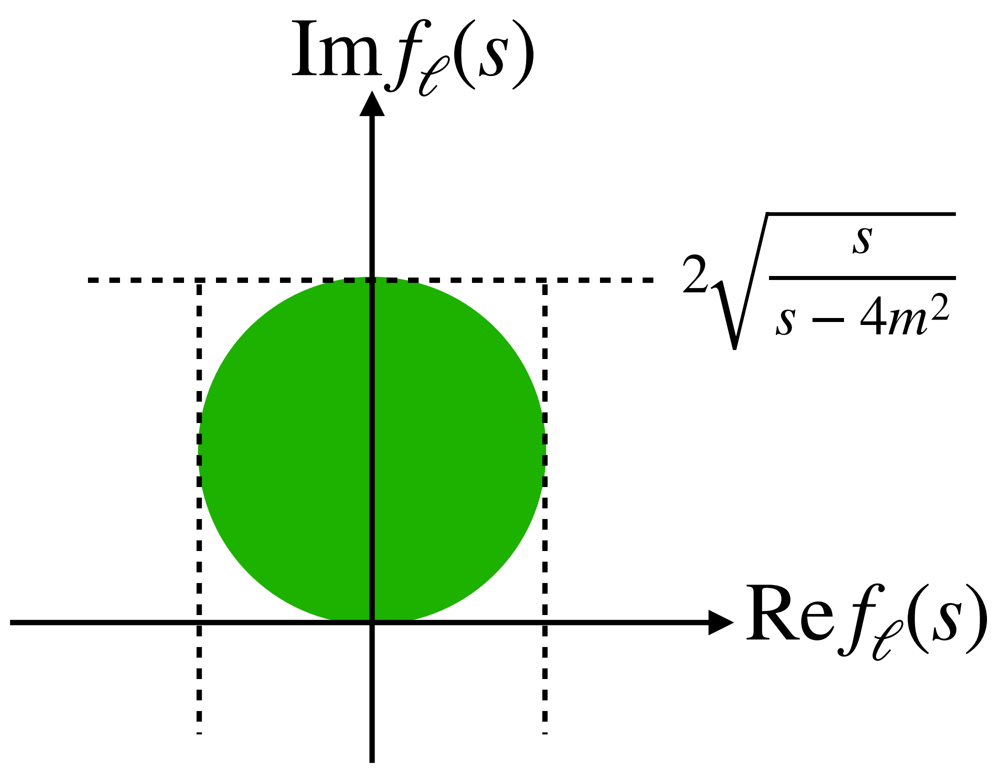
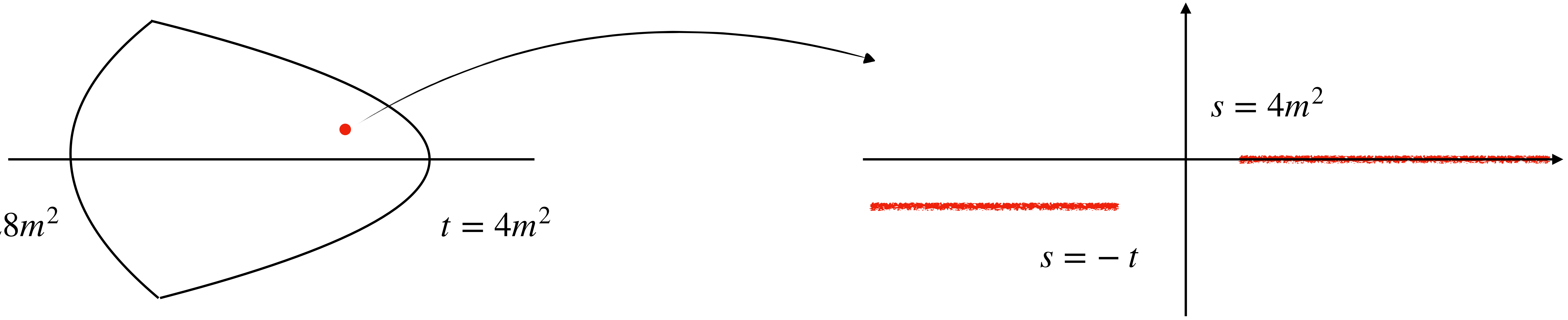


Analyticity:

Martin '66



For any fixed-t, analytic in the cut plane

Amplitudes in 3+1 D: constraints

s-u crossing + analyticity

Doubly-subtracted fixed-t dispersion relations

$$M(s,t) - M(s_0,t_0) = \int_{Disc} \frac{dv}{\pi}(M_v(v,t)K(v,s,t:t_0) + M_v(v,t_0)K(v,t,t_0,s_0)),$$

with $K(v,s,t;t_0) = \frac{1}{v-s} + \frac{1}{v-u} - \frac{1}{v-t_0} - \frac{1}{v-4+t+t_0}$

Froissart bound

$$\lim_{s\to\infty} \frac{M(s,t<t_0)}{|s|^2} = 0$$

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$s < 60$ according to Martin, but in practice $s < 12$

Projecting into partial waves w.r.t to t: Roy Equations '73

Unitarity + J Roy Equations $\Longrightarrow$ Rigorous Bounds

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Equation that generates the “null constraints” used in the positivity literature

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Unitarity

$$\begin{pmatrix} 1+\mathrm{Re}[S_\ell] & \mathrm{Im}[S_\ell] \\ \mathrm{Im}[S_\ell] & 1-\mathrm{Re}[S_\ell] \end{pmatrix} = \begin{pmatrix} 2-\tilde{\rho}_s \mathrm{Im}[f_\ell] & \tilde{\rho}_s \mathrm{Re}[f_\ell] \\ \tilde{\rho}_s \mathrm{Re}[f_\ell] & \tilde{\rho}_s \mathrm{Im}[f_\ell] \end{pmatrix} \succeq 0$$

We can use SDPB!

The Glue-Hedron

In the $GG \to GG$ scattering we measure the coupling $g_X X G^2$

max $\|g_G\|$	max $\|g_H\|$	max $\|g_{G^*}\|$	max $\|g_{H^*}\|$
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206	156	217	–

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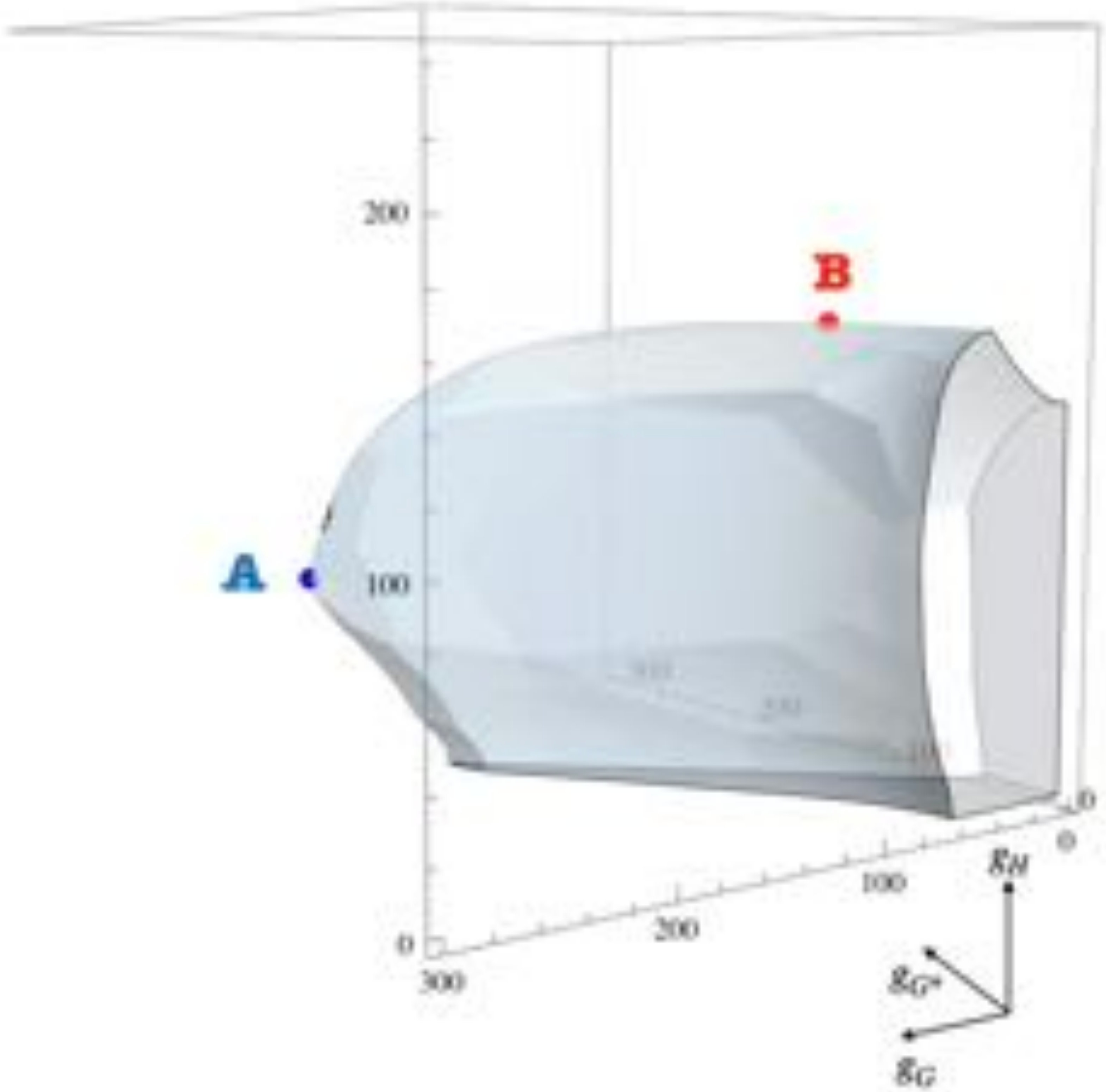

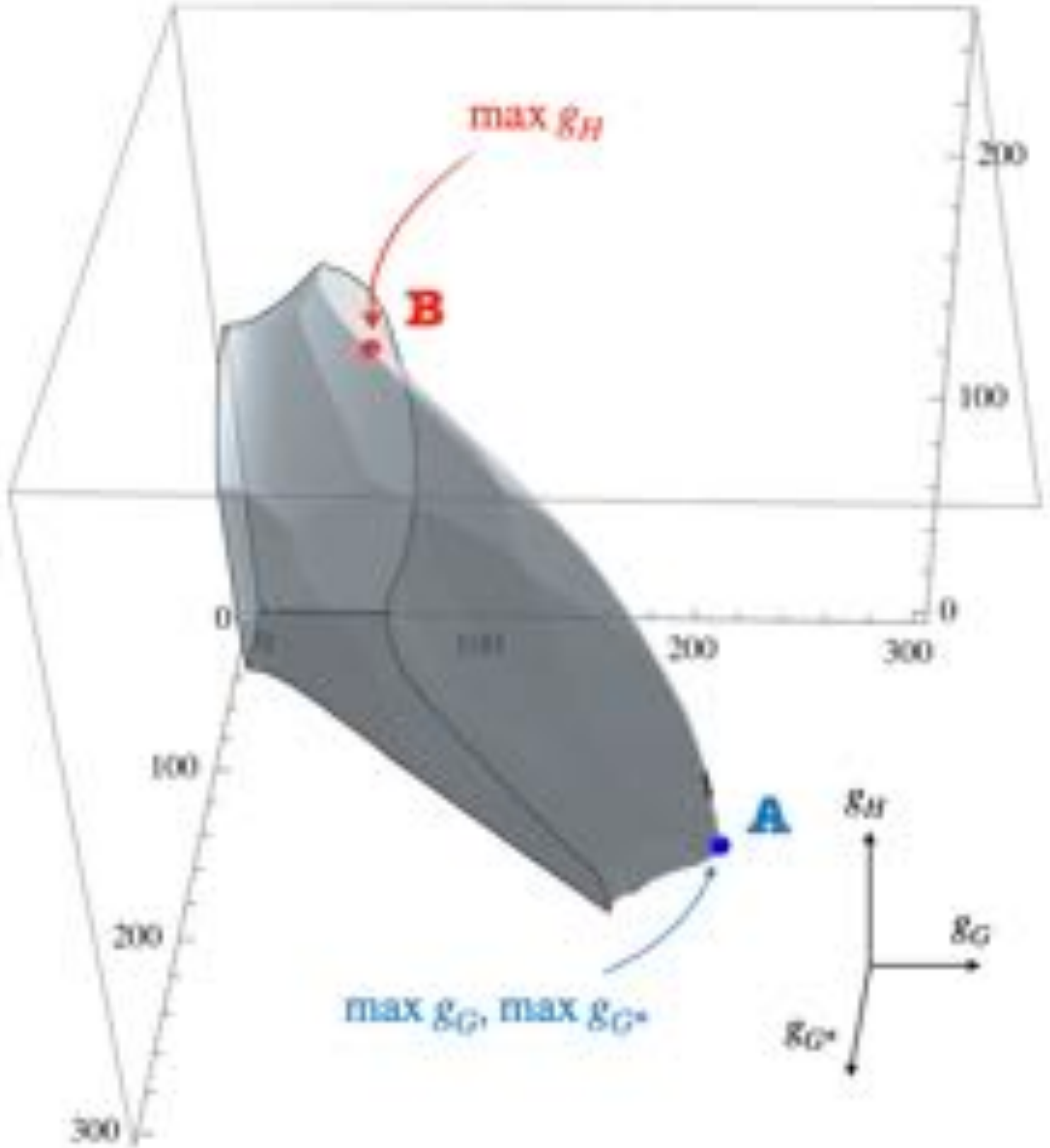

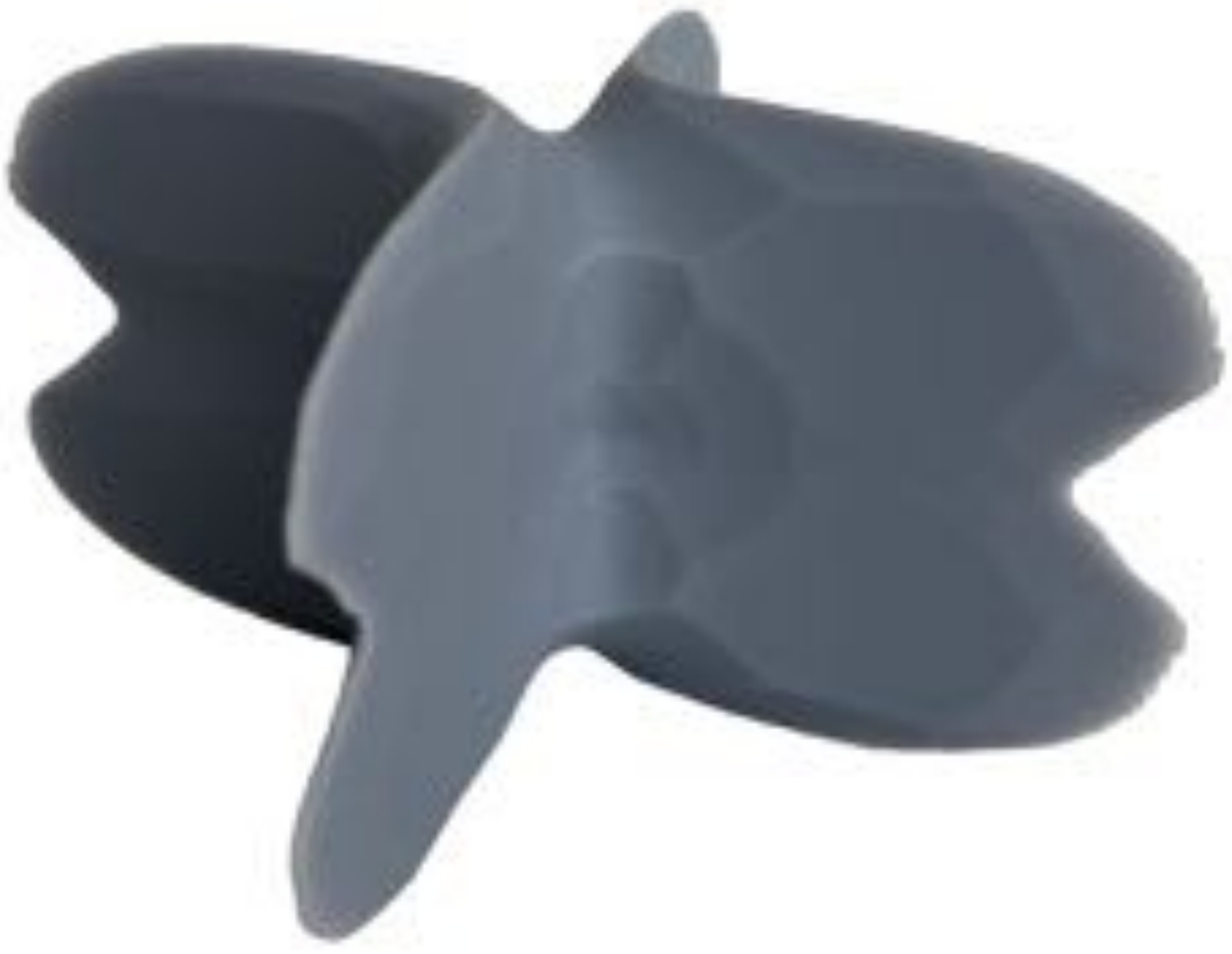


Glueballs: What's next?

Fixed-t dual Bootstrap: amplitude can be reconstructed up to $s = 12$!

1) We need better dispersion relations

2) We need to include other processes G*G*->G*G*, GG*->GG*, but anomalous thresholds!

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Elias-Miro', Gumus, ALG, Zahed (to appear)

(Auberson-Khuri), more recently Gopakumar, Sinha et al

$y = ax$

Use $x = (st + su + tu)$, $y = stu$, crossing symmetric variables

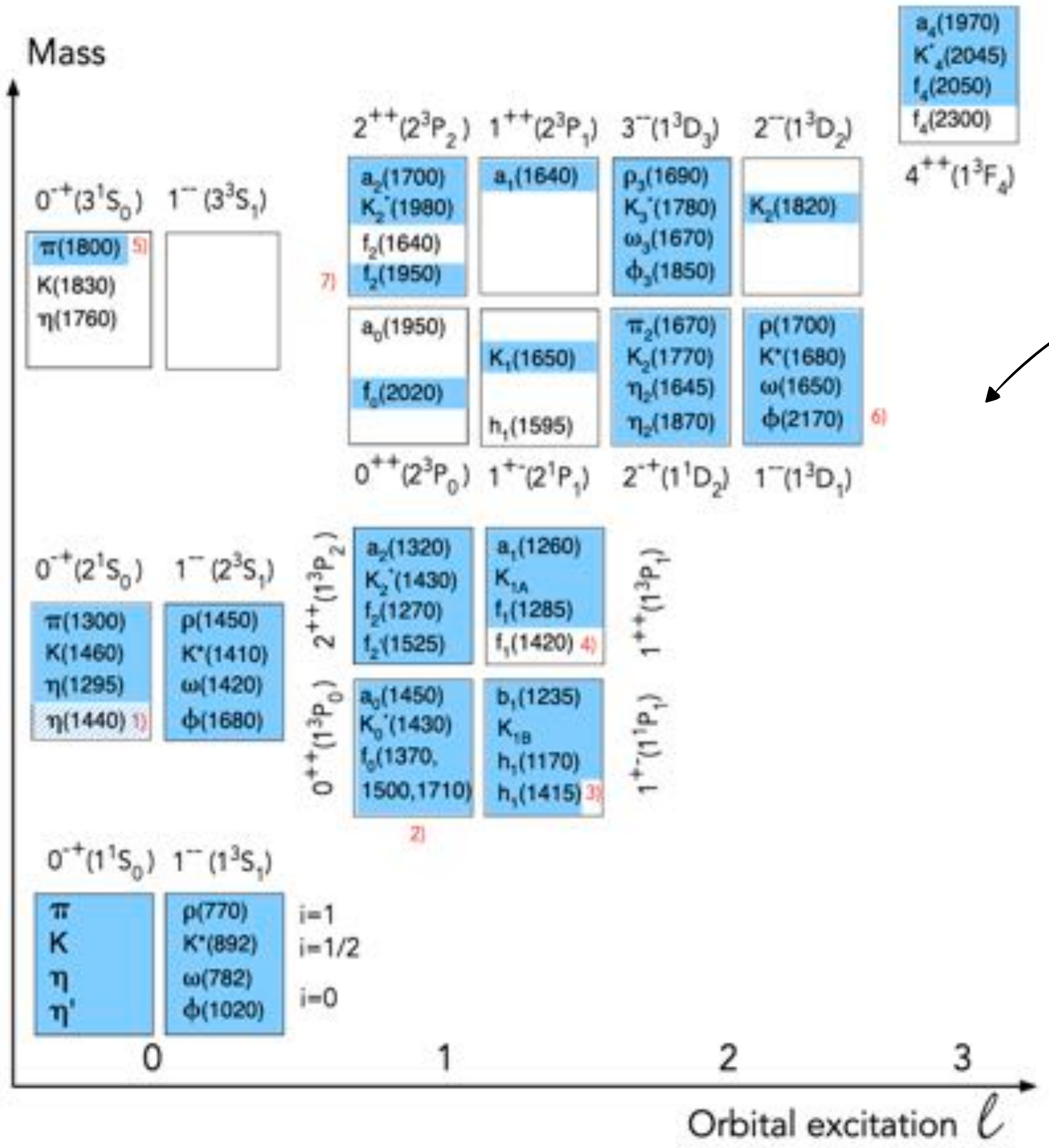
Roy-Wanders-Mahoux dispersion relations: $y = a(x - x_0)$

We are on a quest to find which function can get us to $s \to \infty$

Elias-Miro', Gumus, ALG, Zahed (work in progress)

Precision physics from Bootstrap: QCD Spectroscopy?

LIGHT UNFLAVORED MESONS



Unitarized χPT
Dispersive Roy Equations analysis

Easy to work with physical pion masses
Hard to control systematics

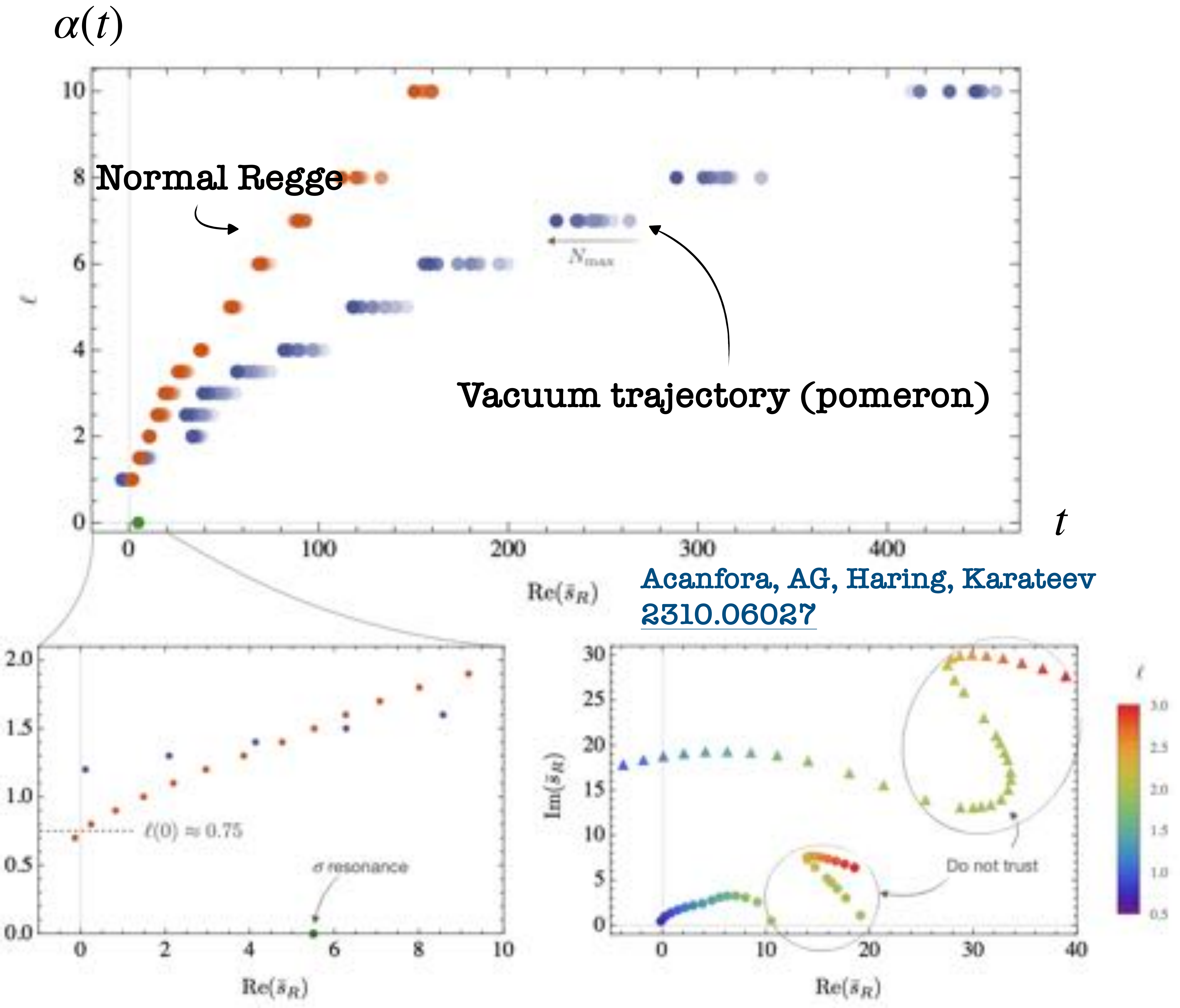
Lattice QCD

Hard to study physical pion masses
Clean Systematics

Bootstrap as a rigorous tool to predict the physics and extrapolate the spectrum?

https://pdglive.lbl.gov/ParticleGroup.action?init=0&node=MXXX005

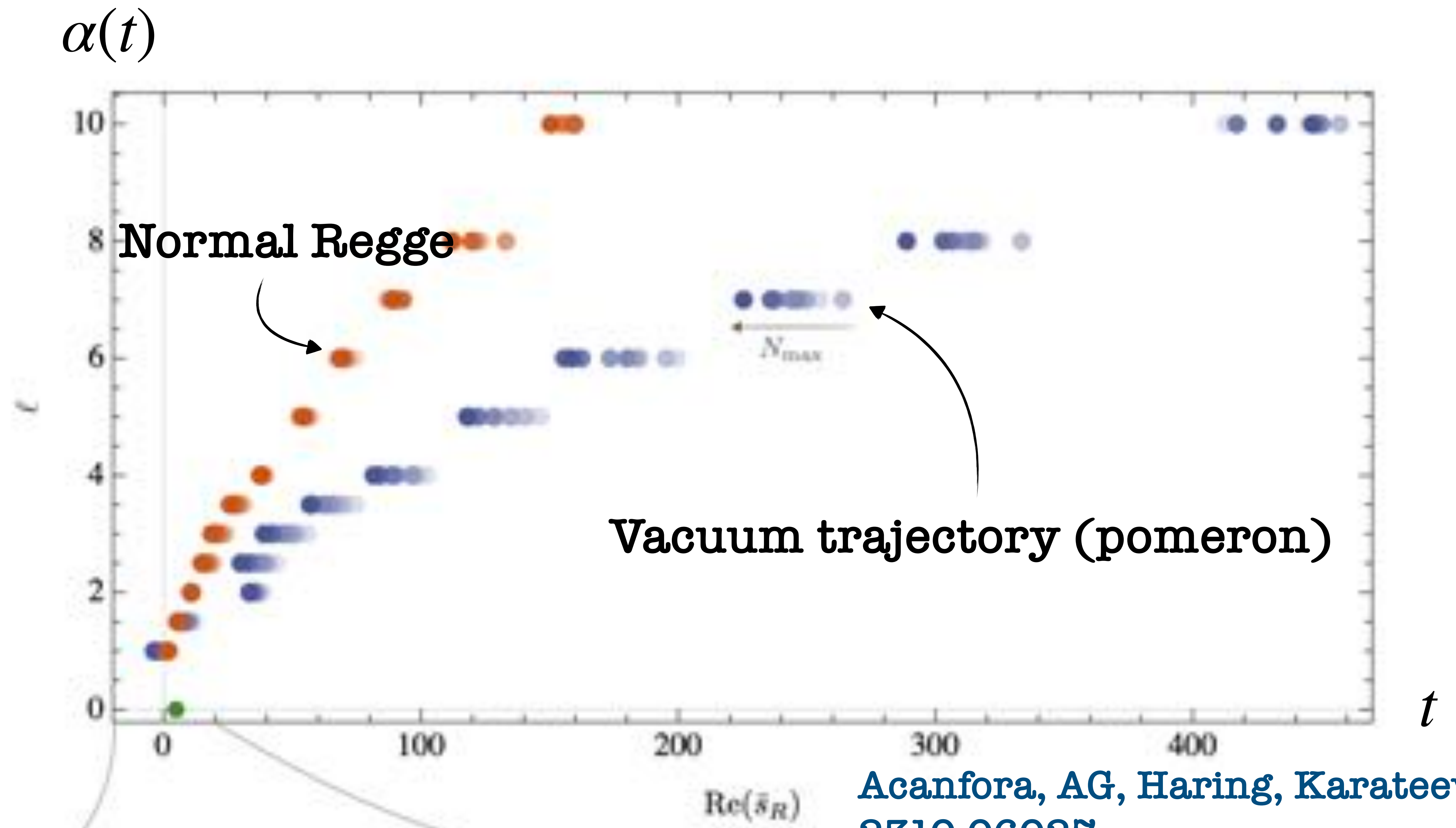
Non-perturbative properties of amplitudes



Acanfora, AG, Haring, Karateev
2310.06027

2->2 Amplitude of U(1) Goldstones in 4 dimensions

Non-perturbative properties of amplitudes



Acanfora, AG, Haring, Karateev
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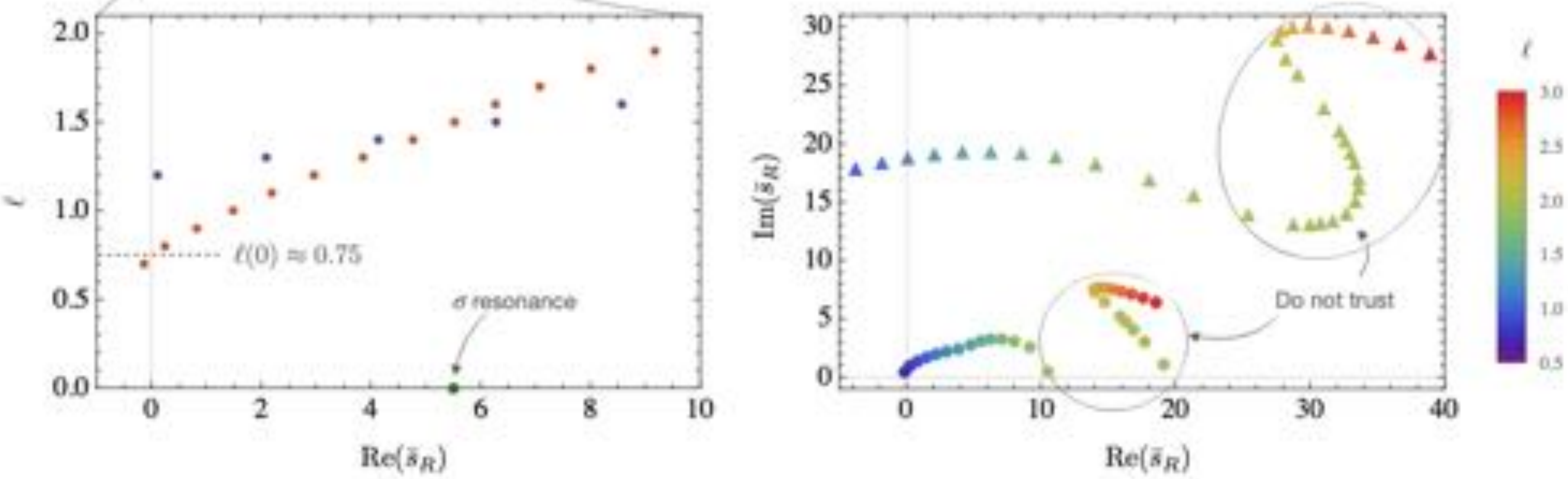
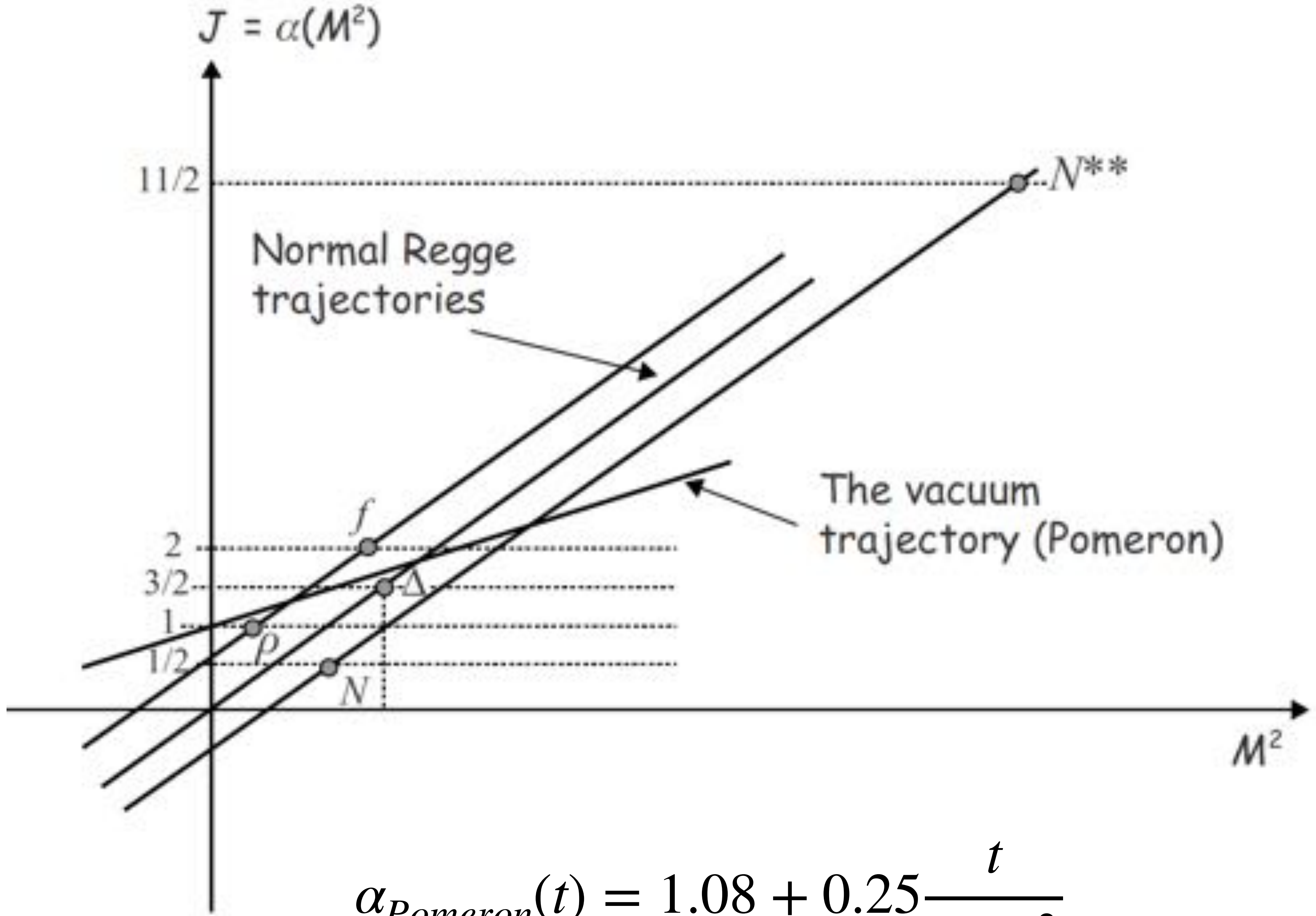
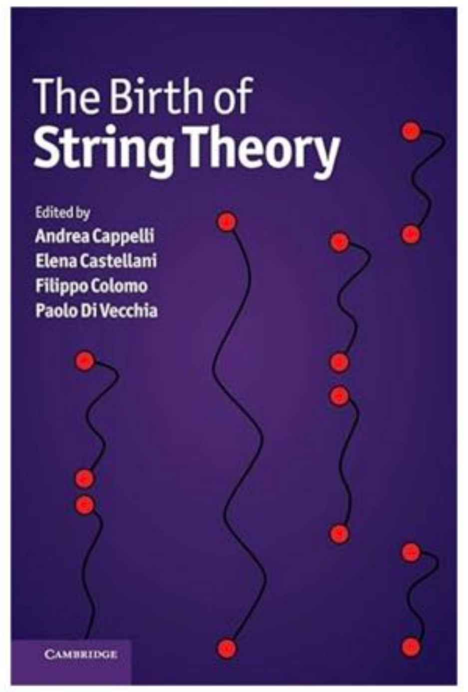


2->2 Amplitude of U(1) Goldstones in 4 dimensions



$$\alpha_{Pomeron}(t) = 1.08 + 0.25\frac{t}{GeV^2}$$

$$\alpha_{\rho}(t) = 0.52 + 0.9\frac{t}{GeV^2}$$



Advancements for Primal I: L_{max} convergence

Primal Bootstrap: at the moment more flexible, powerful, simpler to code

1) Ansatz for M(s,t,u)

2) Numerically project $f_\ell = \frac{1}{32\pi}\int_{-1}^{1} dx P_\ell(x) M(s, t(s,x))$, for $\ell \leq L_{max}$

3) Impose $2Imf_\ell \geq \sqrt{\frac{s-4m^2}{s}}|f_\ell|^2$, for $\ell \leq L_{max}$

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Improved Positivity constraints

$$Im\,M(s,0 \leq t < 4) = 16\pi \sum_\ell (2\ell+1) P_\ell\left(1+\frac{2t}{s-4}\right) Im f_\ell(s) \geq 0$$

$$Im\,M(s,0 \leq t < 4) - \sum_{\ell \leq L_{max}} = 16\pi \sum_{\ell > L_{max}} (2\ell+1) P_\ell\left(1+\frac{2t}{s-4}\right) Im f_\ell(s) \geq 0$$

For any t necessary positivity constraints on the tail of higher spins!

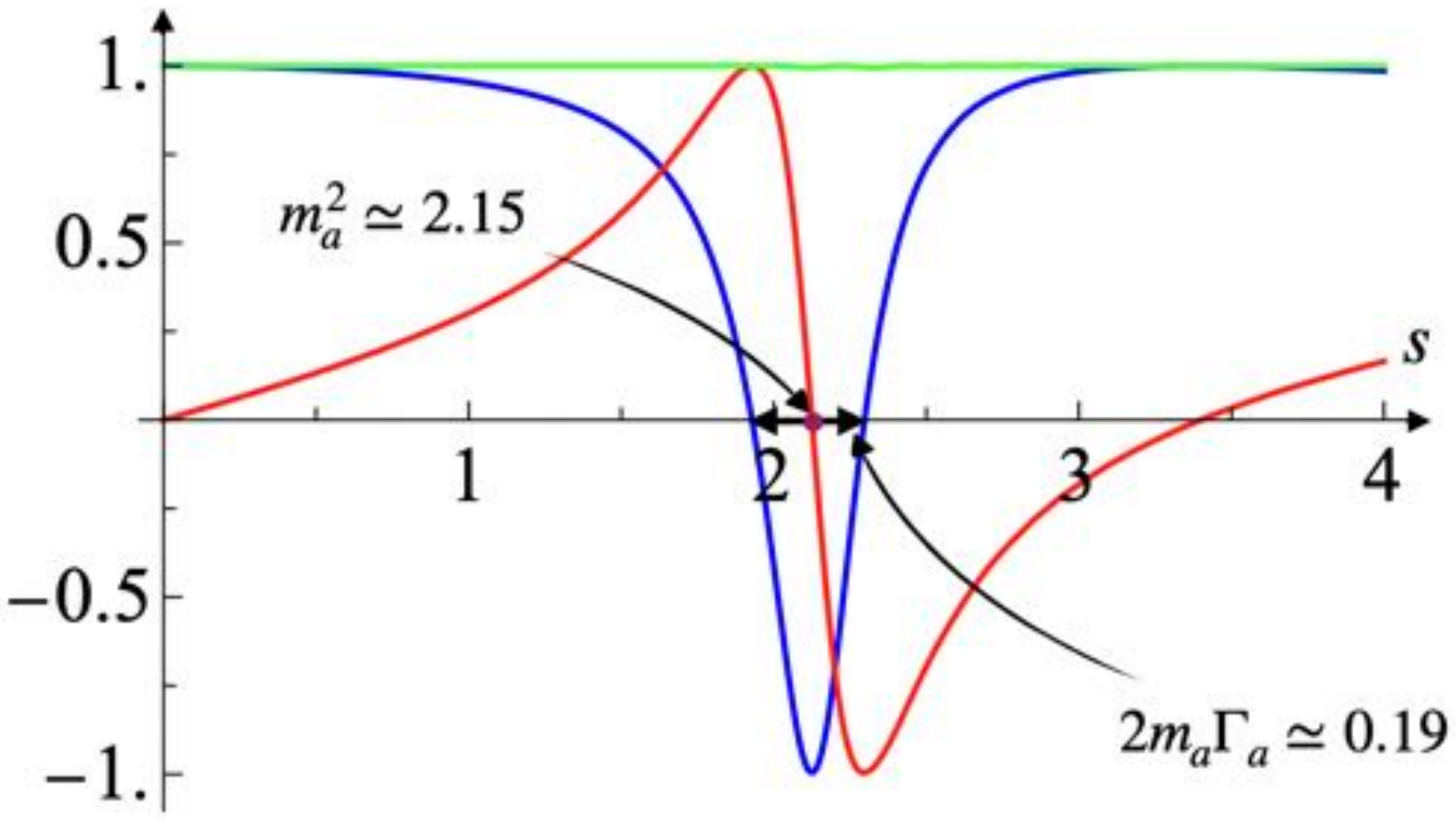
Advancements for Primal II: N_{max} convergence

1) Ansatz for M(s,t,u): powerful enough to describe weakly coupled resonances

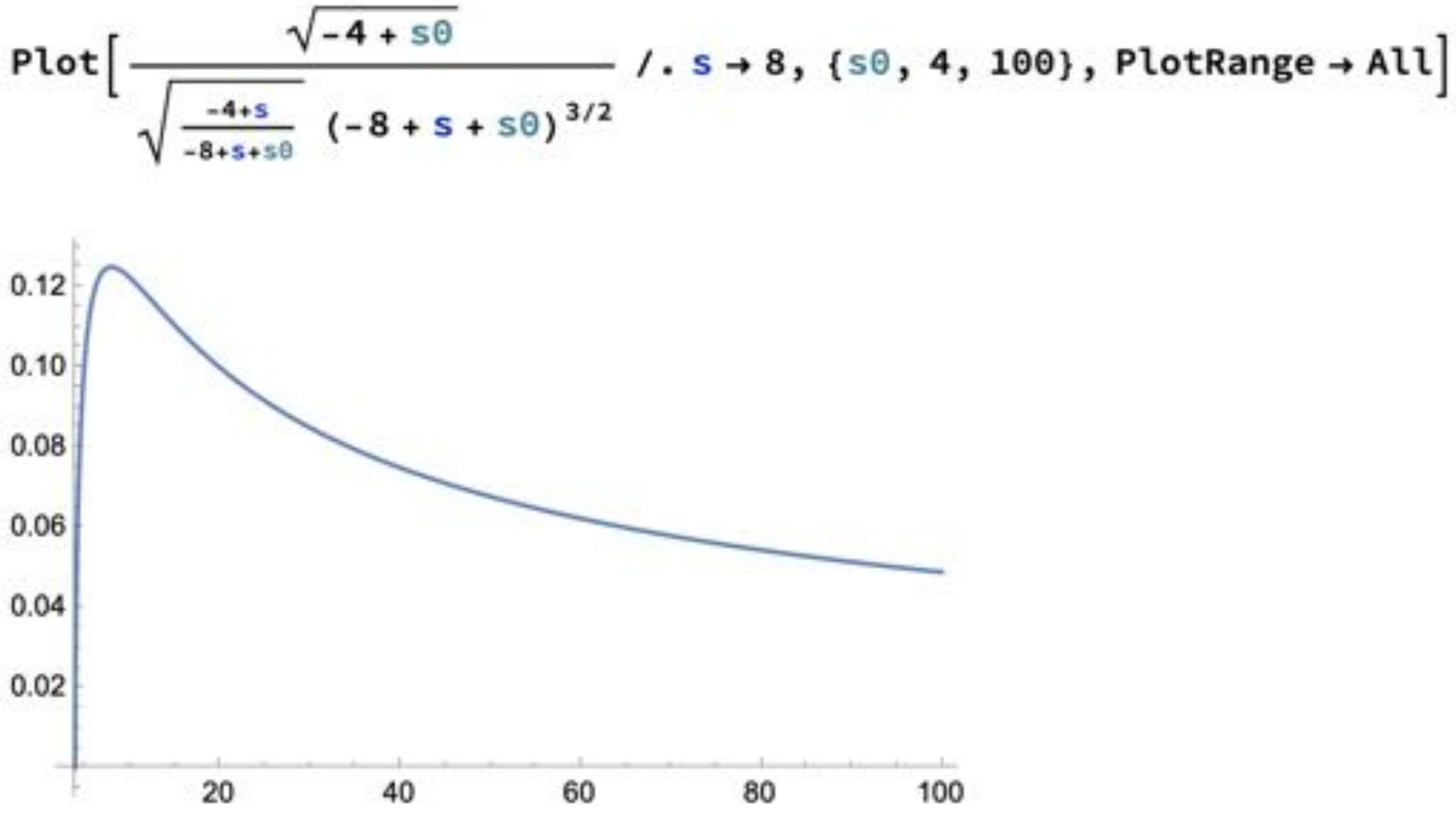
$$\rho(s, s_0) = \frac{\sqrt{s_0 - 4} - \sqrt{4 - s}}{\sqrt{s_0 - 4} + \sqrt{4 - s}}$$

Real $s \to e^{i\phi}$

$$\Delta\phi = Jac \times \Delta s = Jac \times 2m\Gamma$$



Resonant structure, coupling $\propto \frac{\Gamma}{m}$

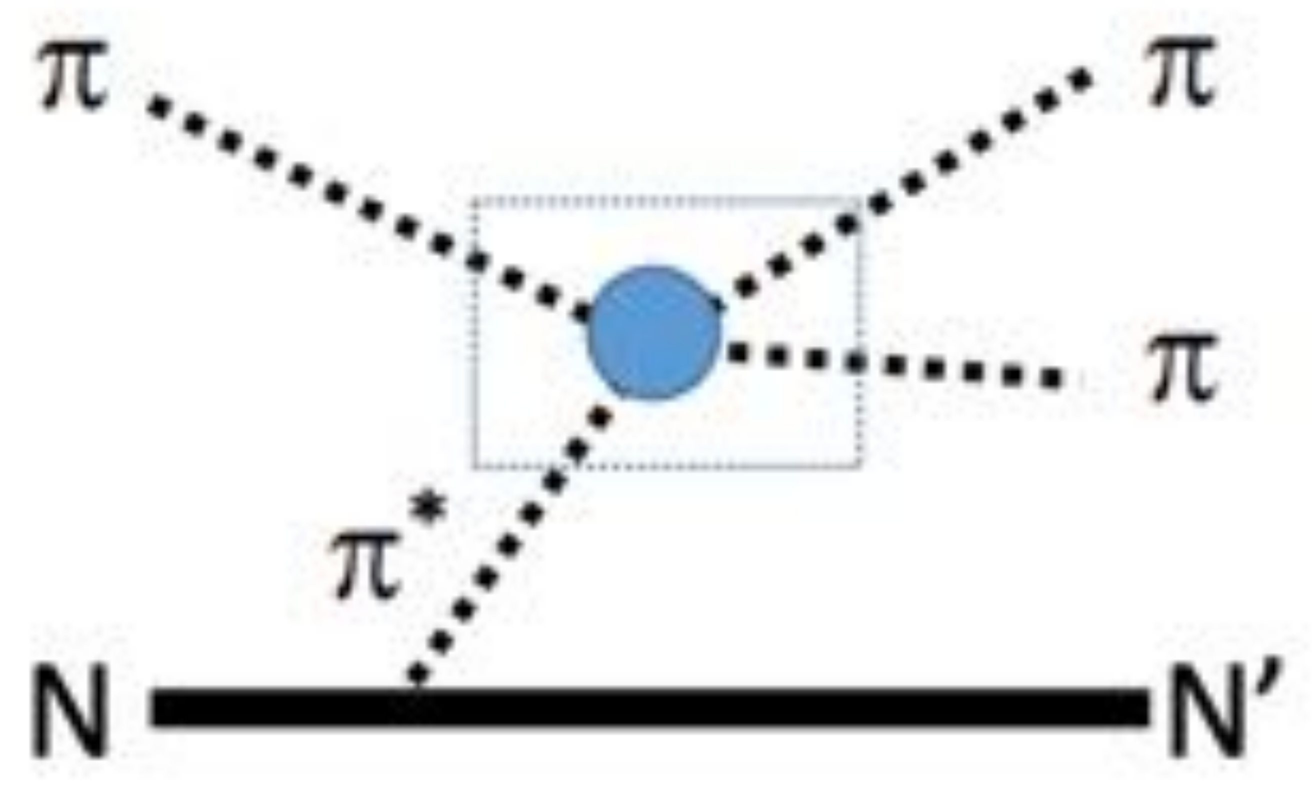


max Jac for $s_0 = m$

We choose different foliations $\{s_0 = 6.67{,}30{,}50{,}80{,}\ldots\}$

Real world QCD spectroscopy (work in progress)

Goal: Use the Bootstrap to "**Fit**" Experimental Data AG, Haring, Su (work in progress)



Experimental situation incredibly messy!

Pions are unstable, and we don't detect the scattering directly.

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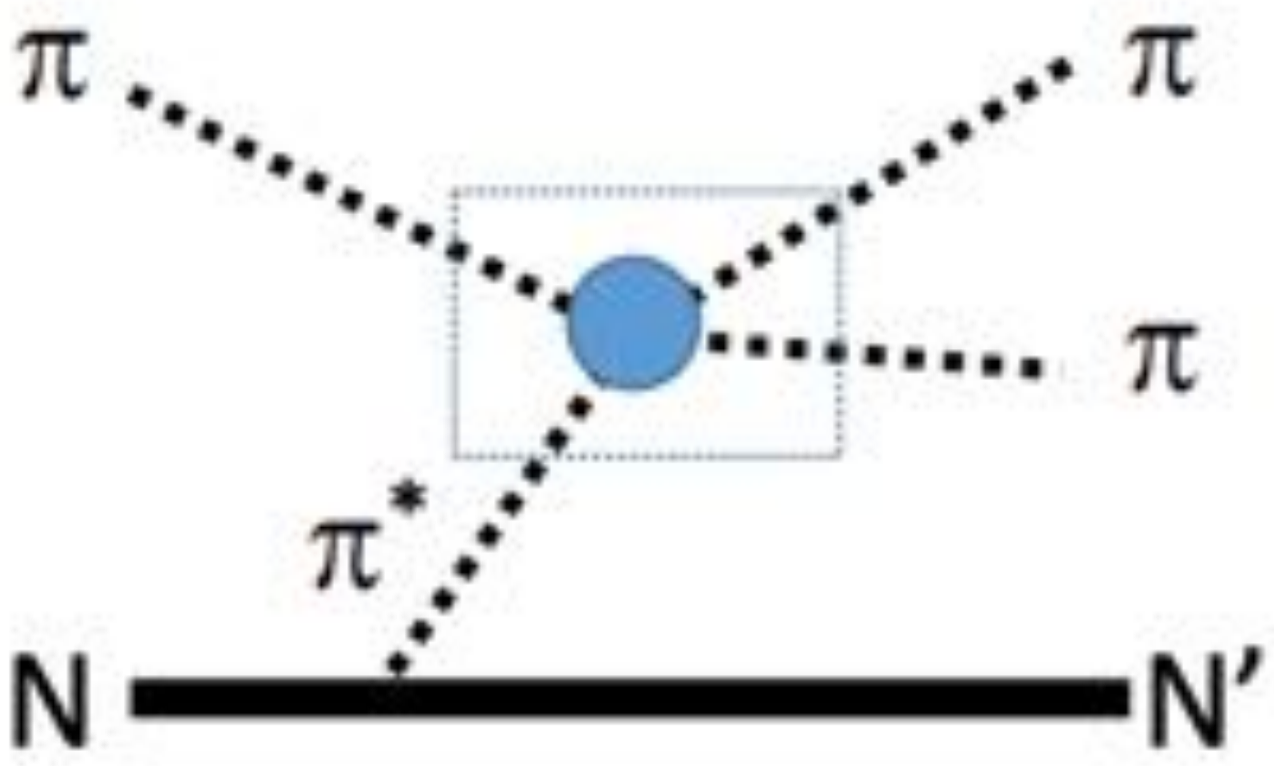
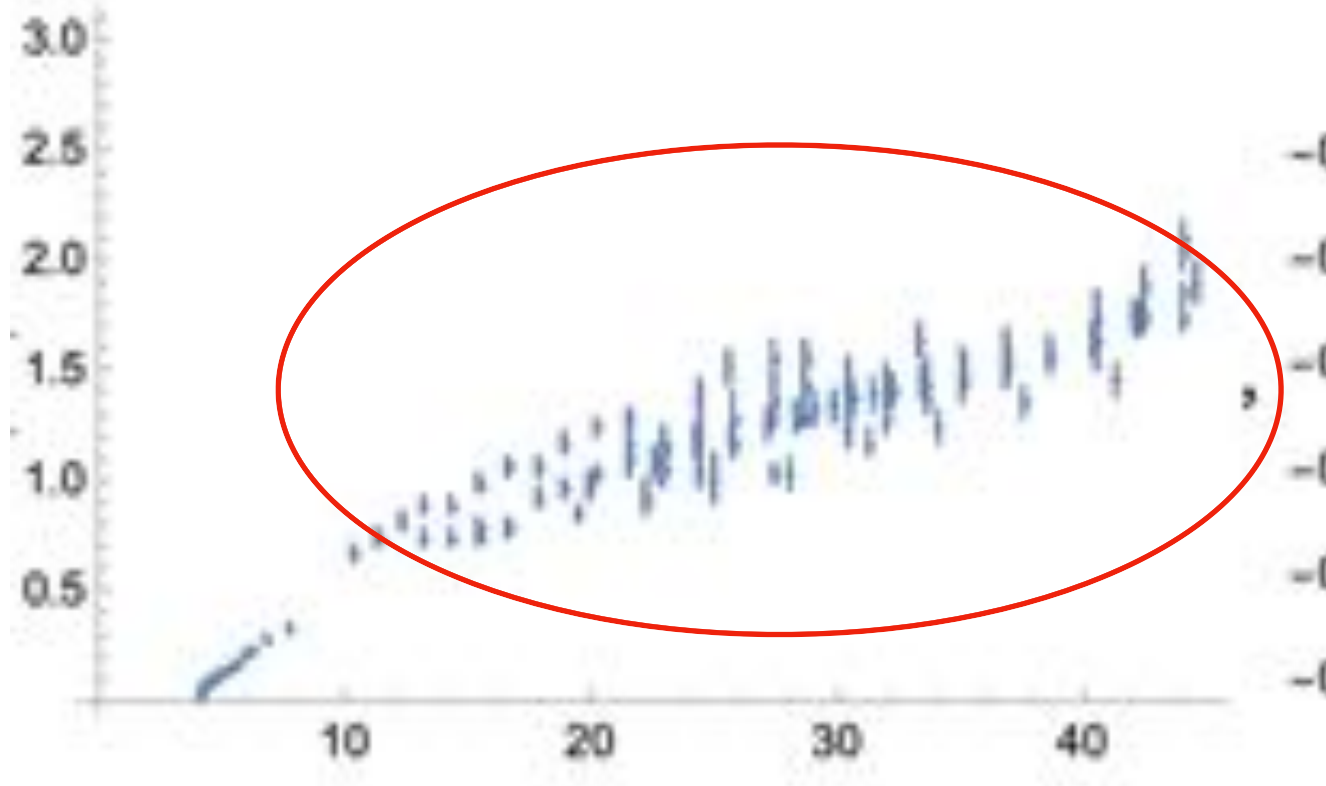


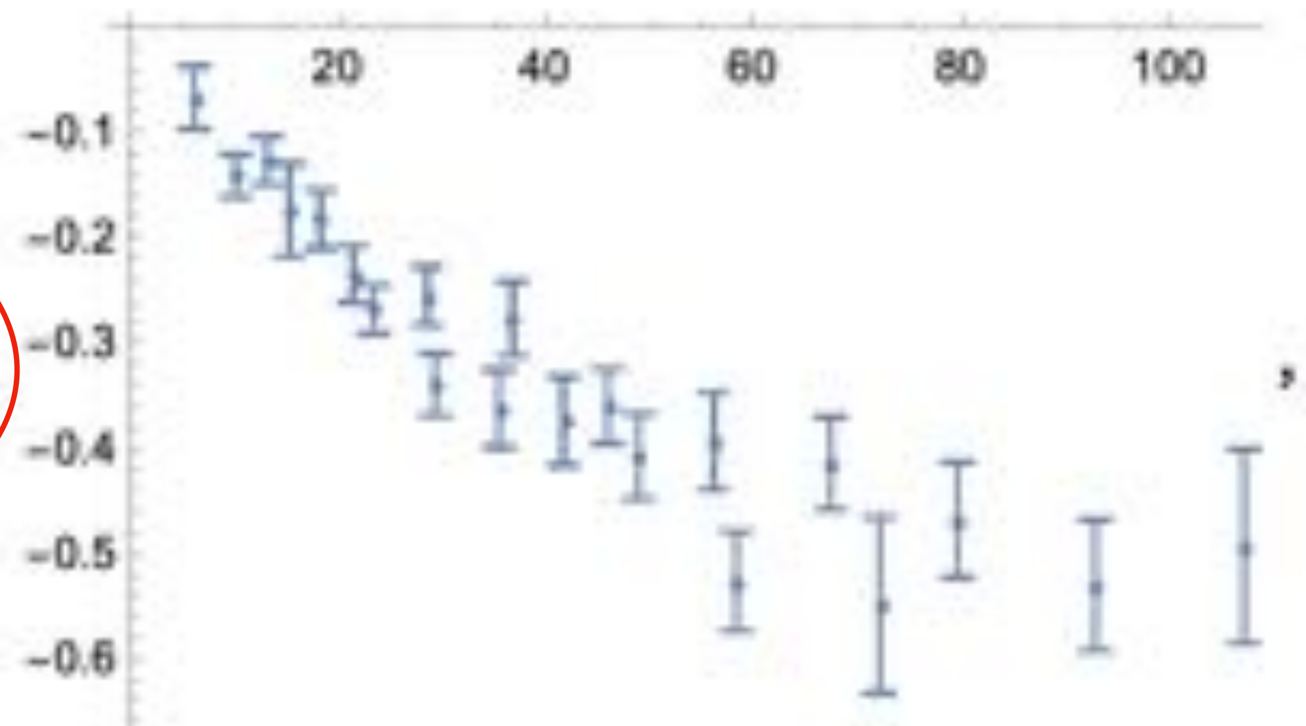
Experimental situation incredibly messy!

Pions are unstable, and we don't detect the scattering directly.

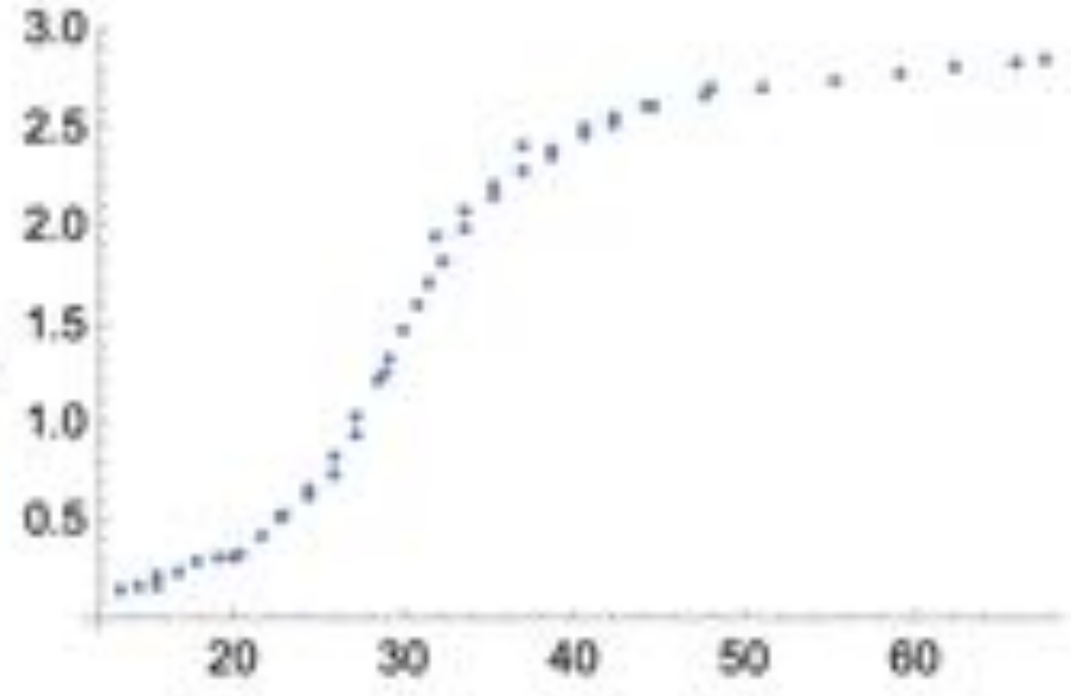
In the $\ell = 0, I = 0$ channel data coming from different experiments are incompatible!



$I = 0, \ell = 0$



$I = 2, \ell = 0$



$I = 1, \ell = 1$

Before applying the Bootstrap to phenomenology is important we carefully choose the data to use!

The Pion Kink

Idea: construct a class of crossing symmetric, analytic and unitary non-perturbative amplitudes depending on few parameters to
1) Fit Data
2) Extrapolate

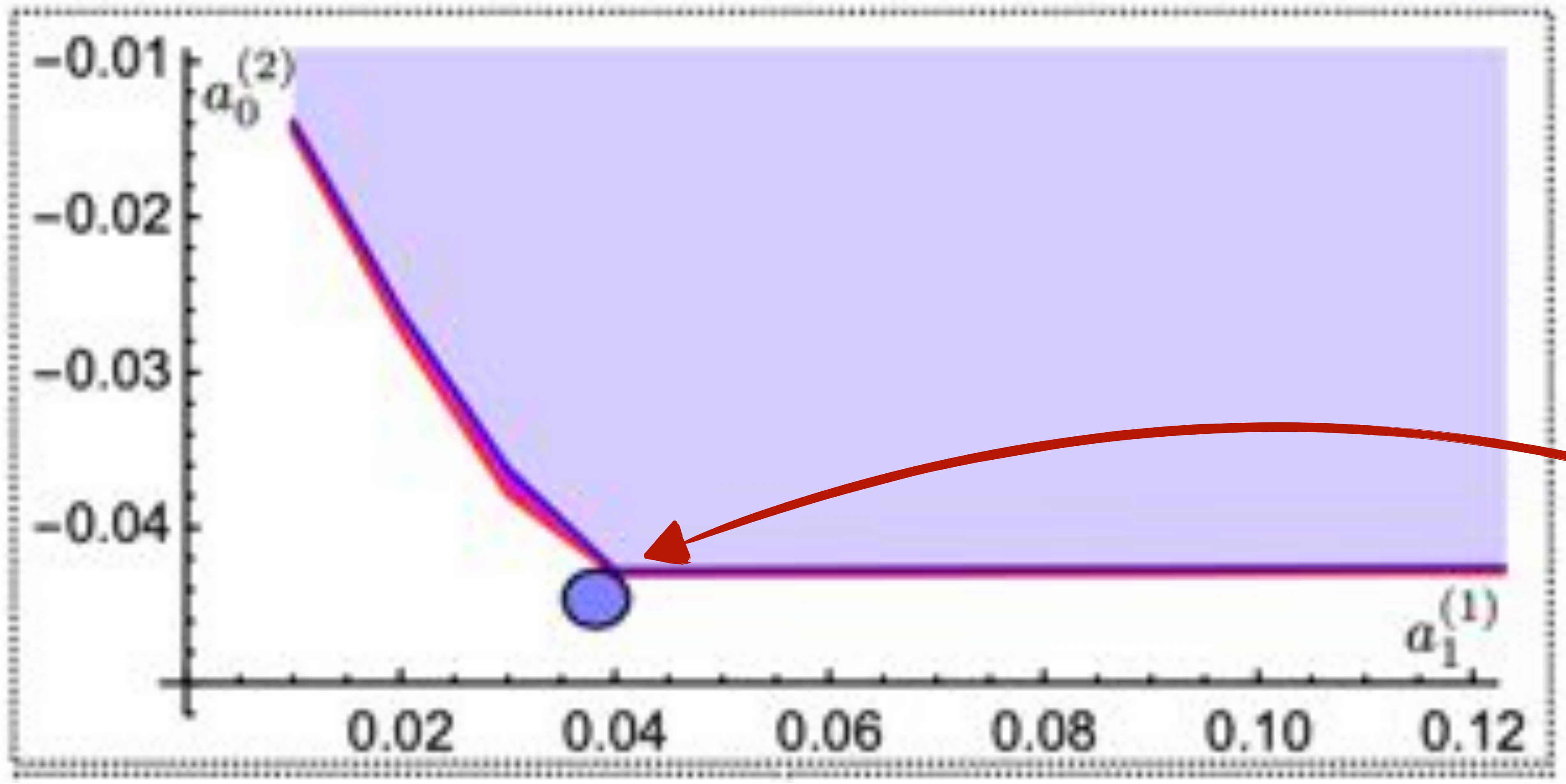
How can we construct such an amplitude?

AG, Penedones, Vieira '18

Moving these three inputs the kink moves

$$a_0^{(0)} = 0.22$$
$$z_0 = 0.36$$
$$z_2 = 2.04$$

Low Energy constants in χPT



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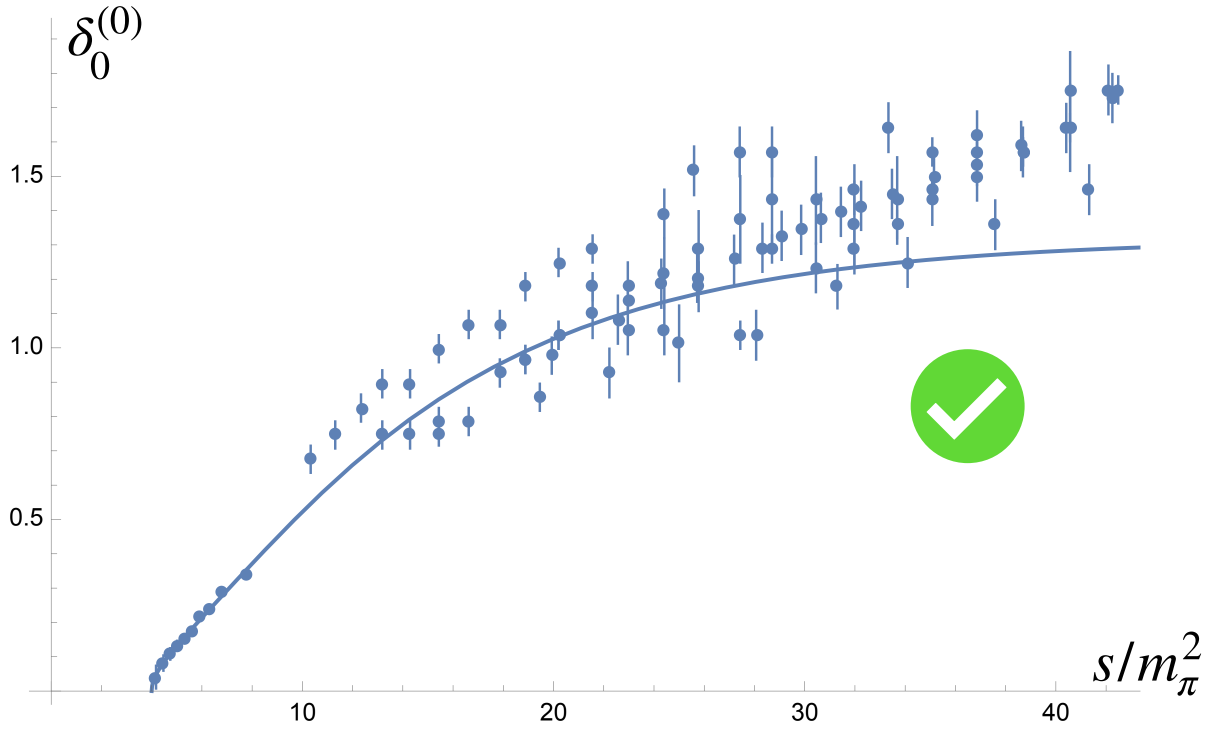
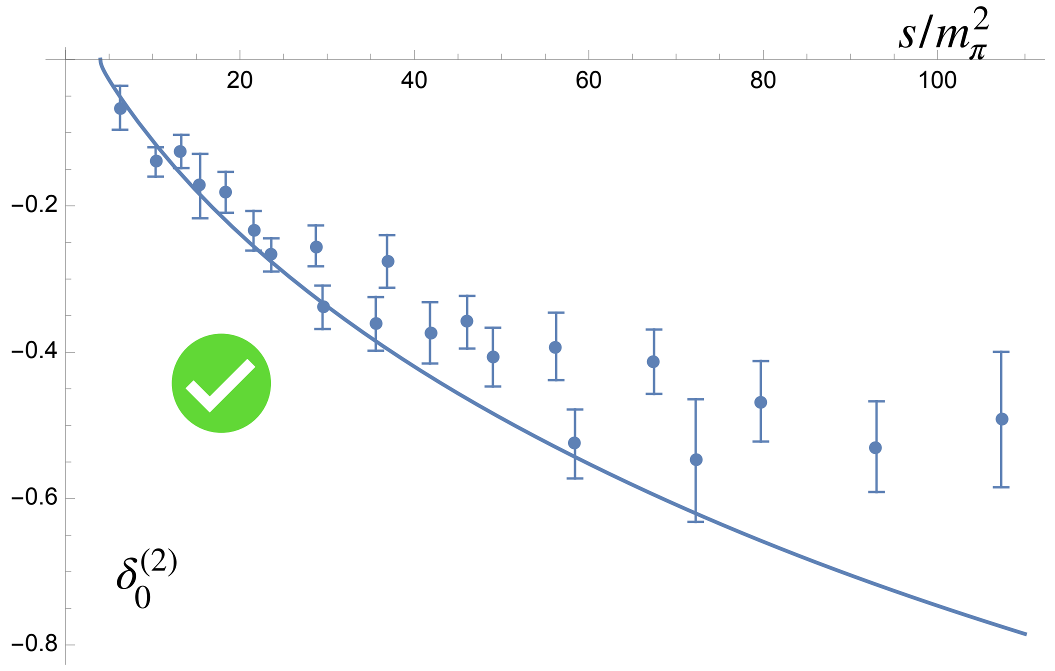
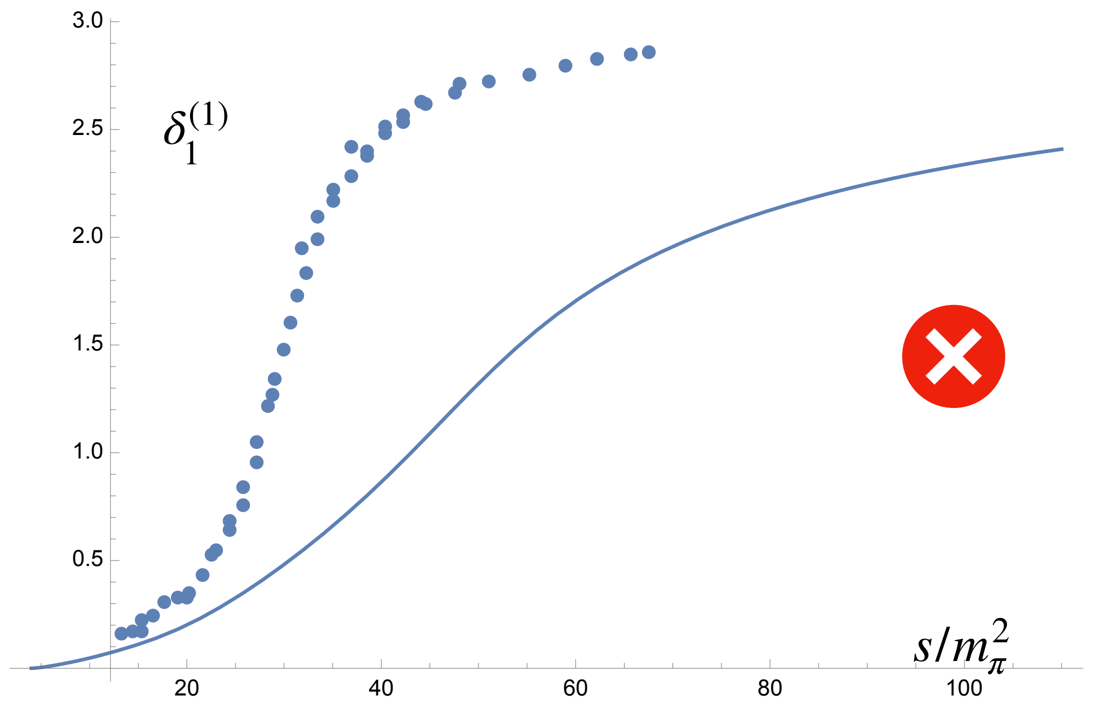






Good candidate for a fitting function, but bad χ^2

Navigating towards the kink

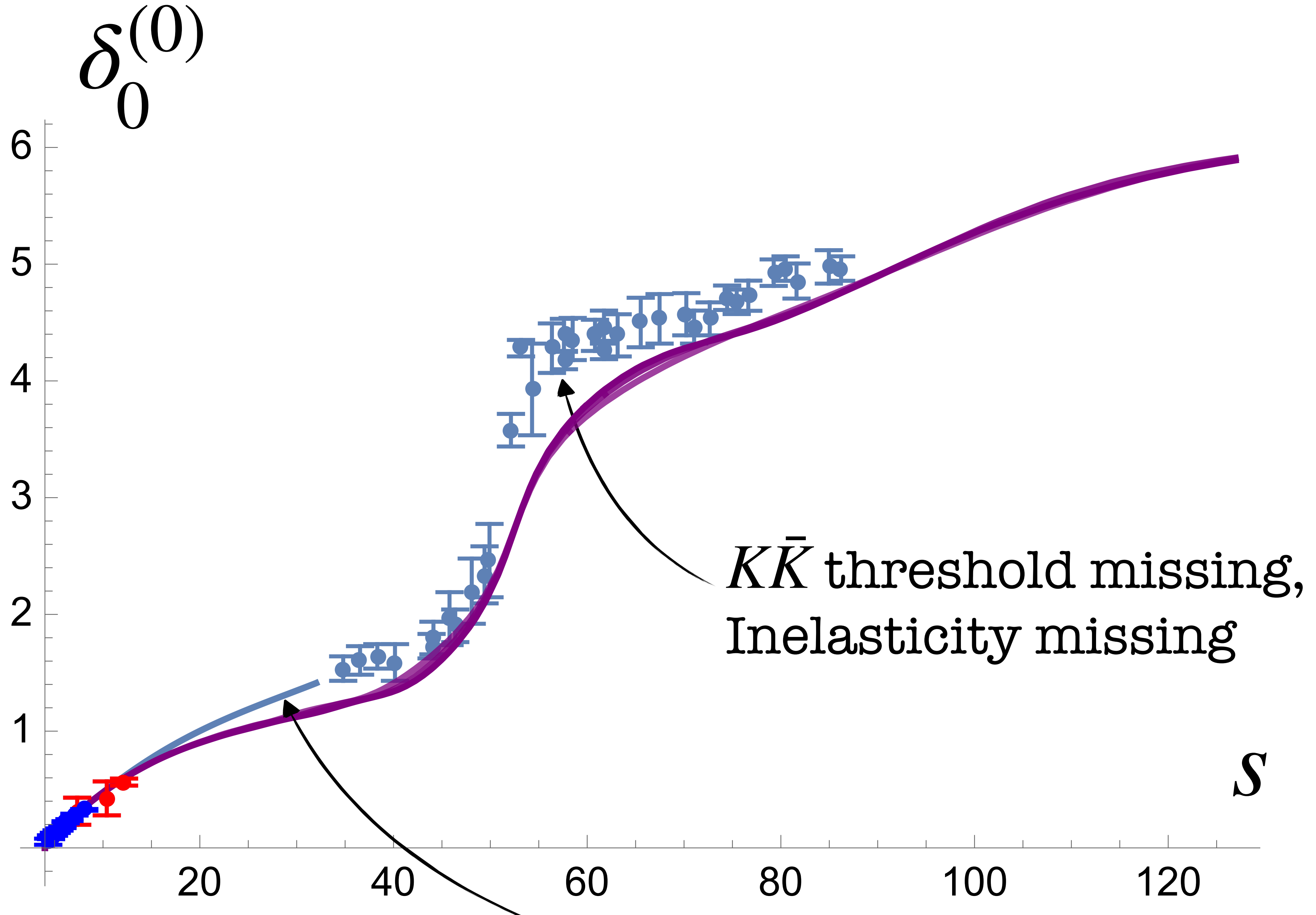
AG, Haring, Su (work in progress)

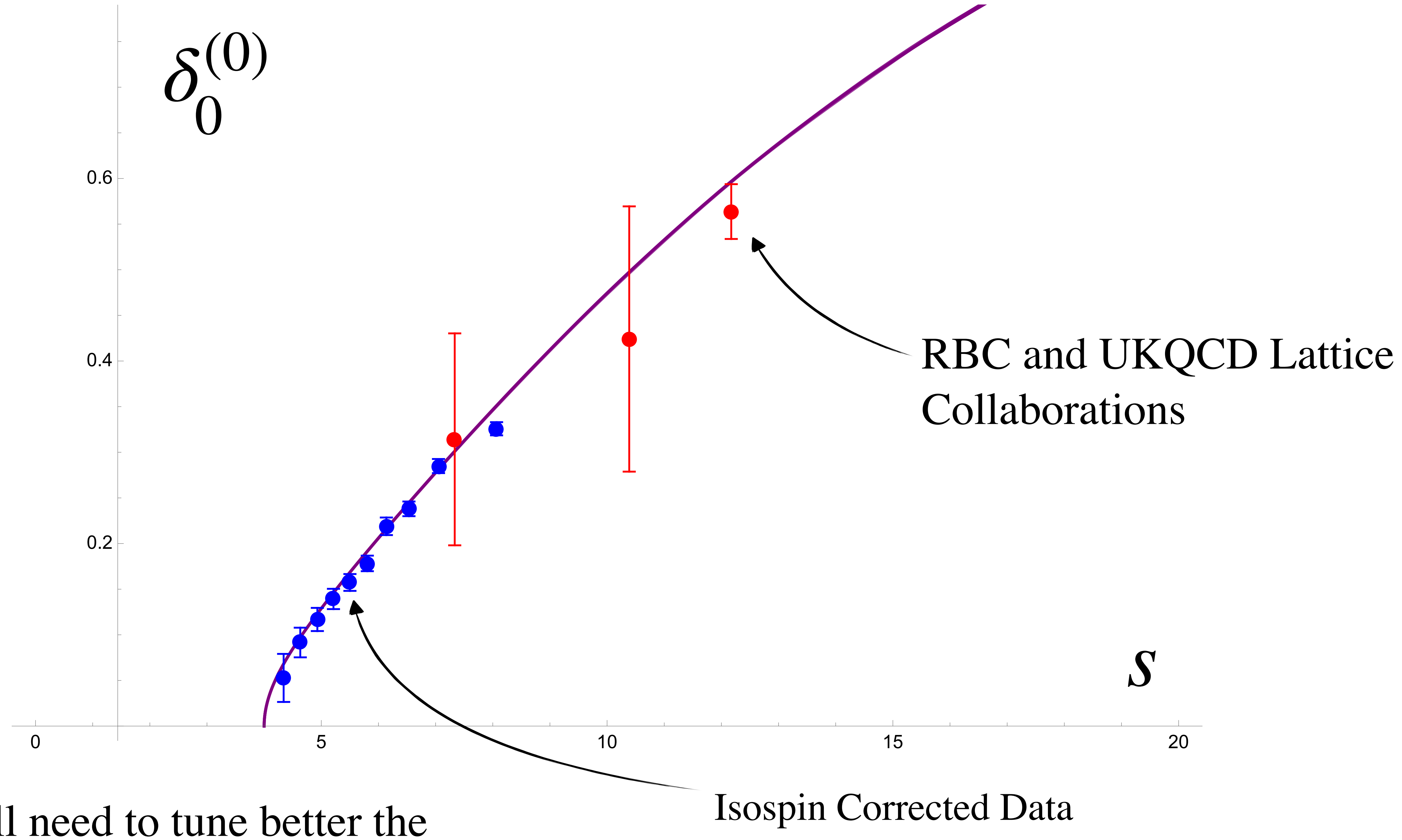
Chiral zeros $\{f_0^{(0)}(z_0) = 0, f_0^{(2)}(z_2) = 0\}$,

We navigate in the 11 red parameters to find the best values!

1 scattering length $f_0^{(0)}(4) = 2a_0^{(0)}$

$$\{S_2^{(0)}(m^2_{f_2(1270)}) = 0, S_0^{(0)}(m^2_{f_0(980)}) = 0, S_0^{(0)}(m^2_{f_0(1350)}) = 0, S_1^{(1)}(m^2_{\rho(770)}) = 0\}$$

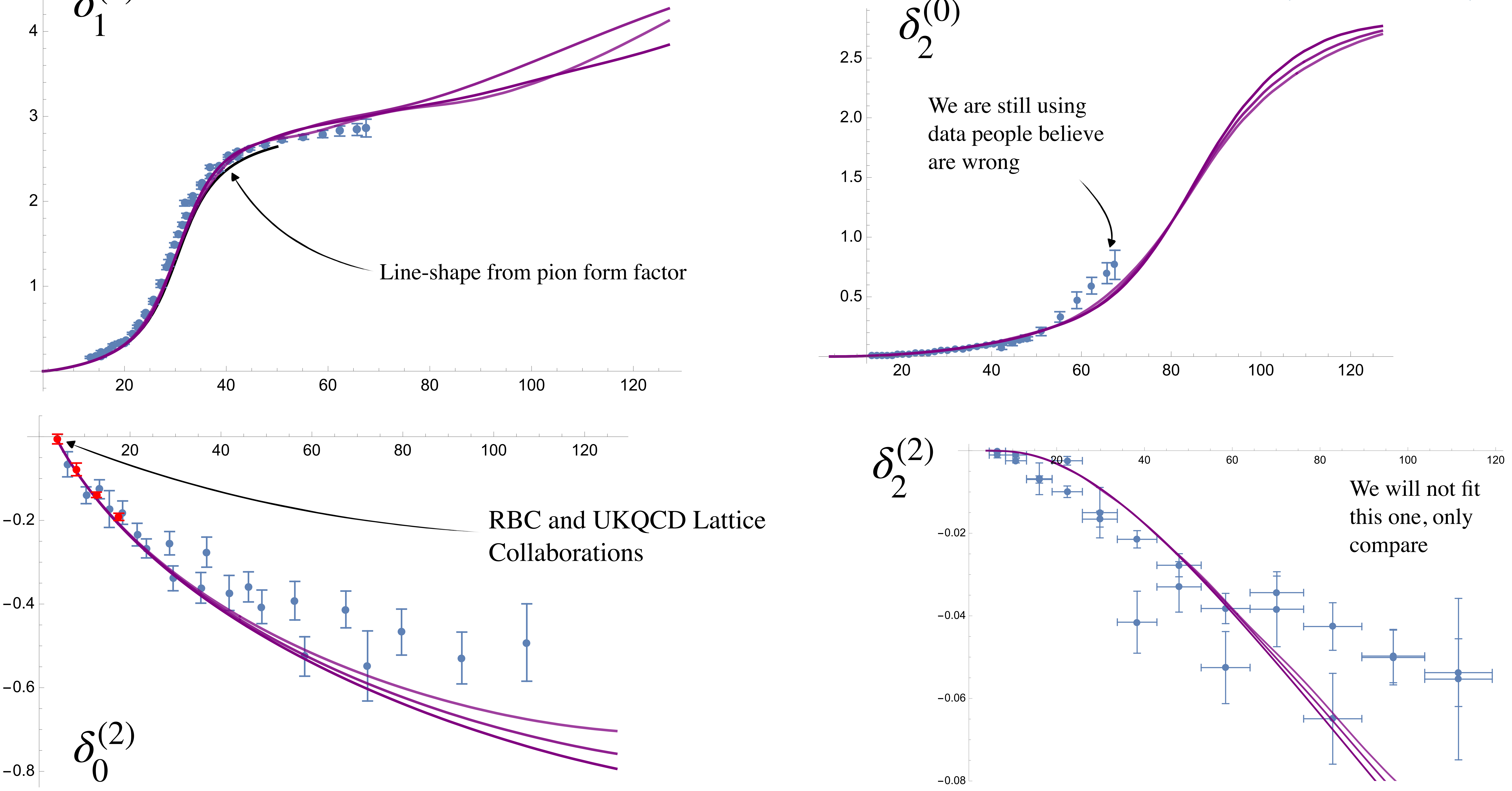




Discrepancy with Colangelo: we still need to tune better the parameters!

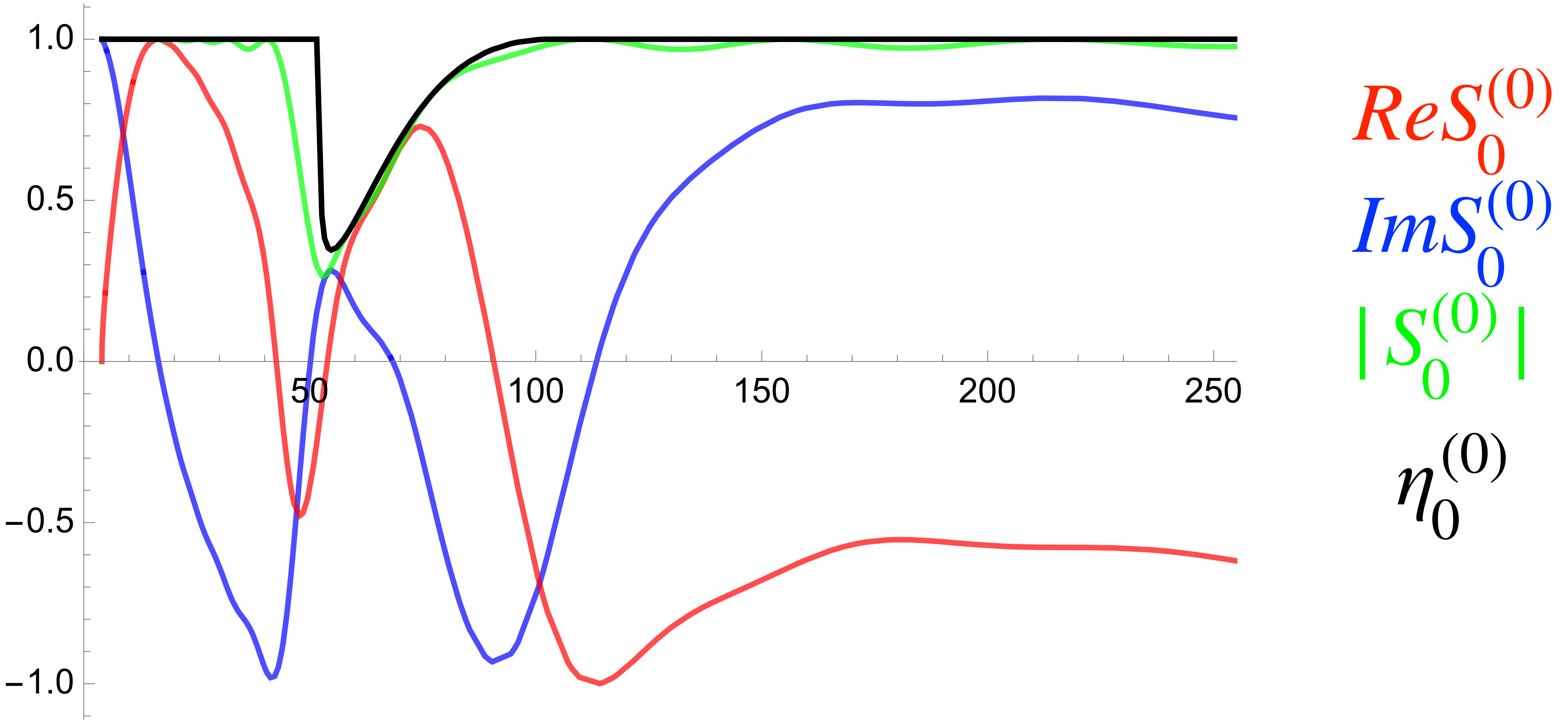
Step 1 of the navigator algorithm II

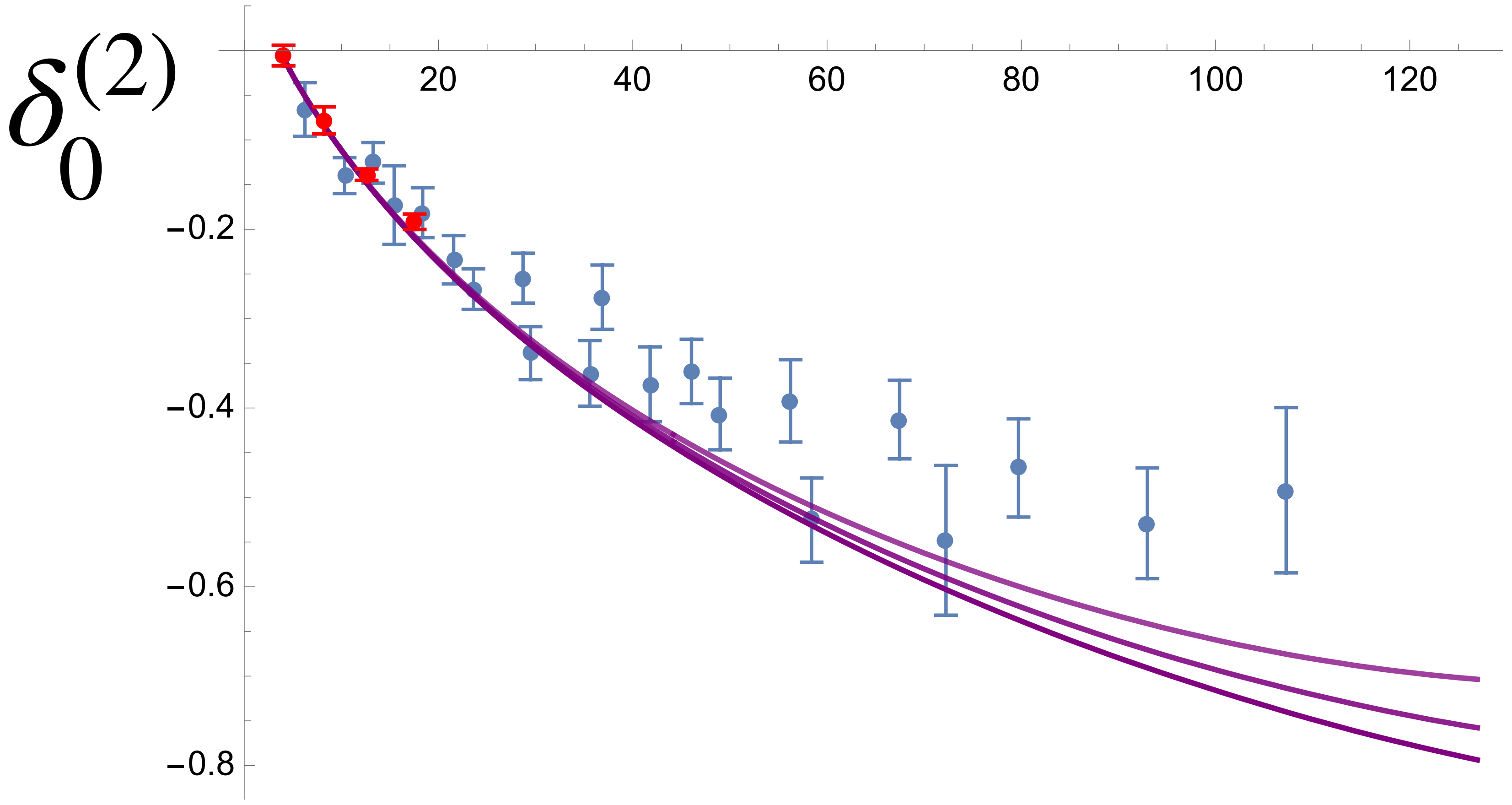
AG, Haring, Su (work in progress)



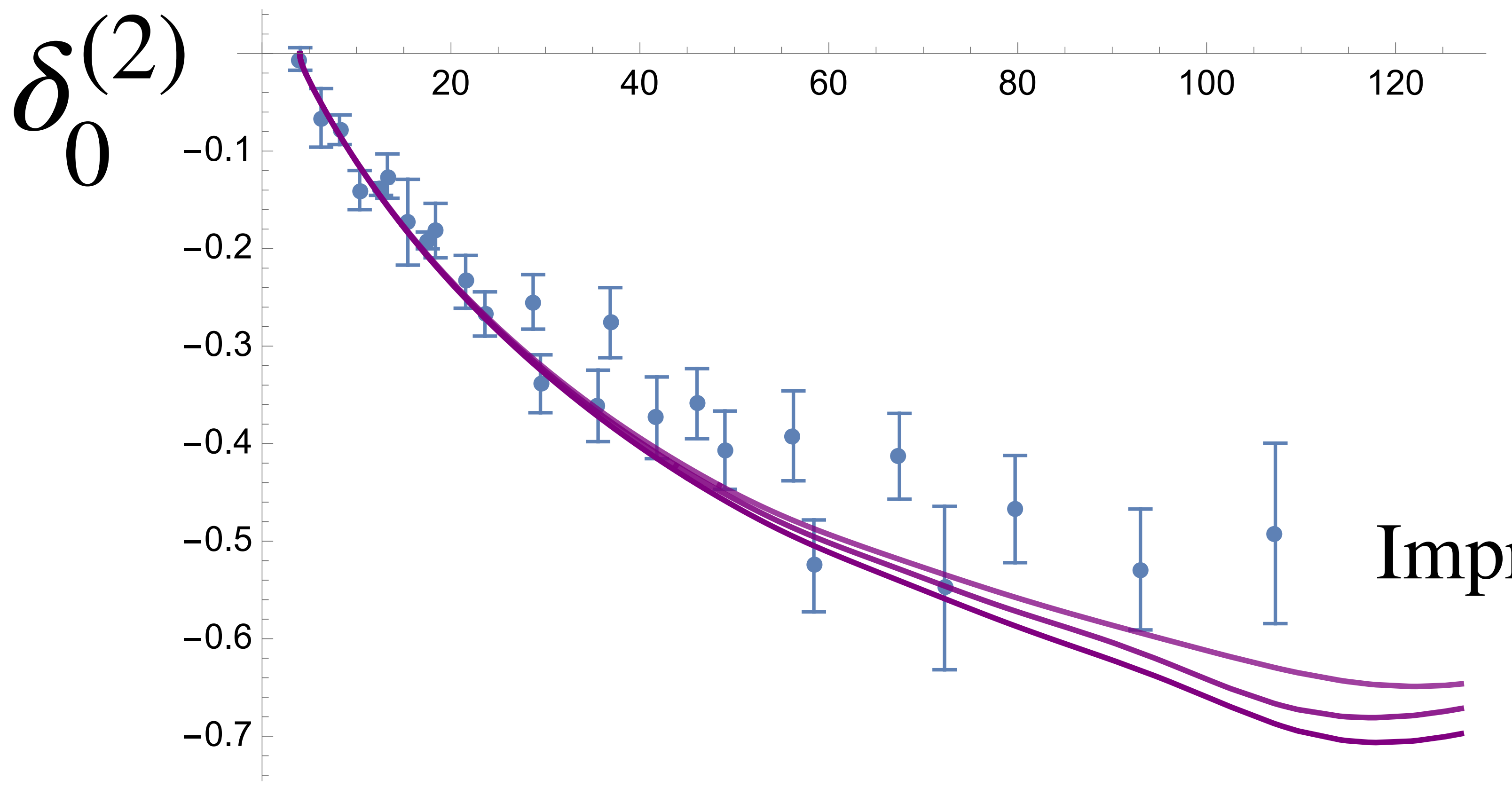
Including Inelasticity

$$S_0^{(0)} \equiv \eta_0^{(0)} e^{2i\delta_0^{(0)}}$$





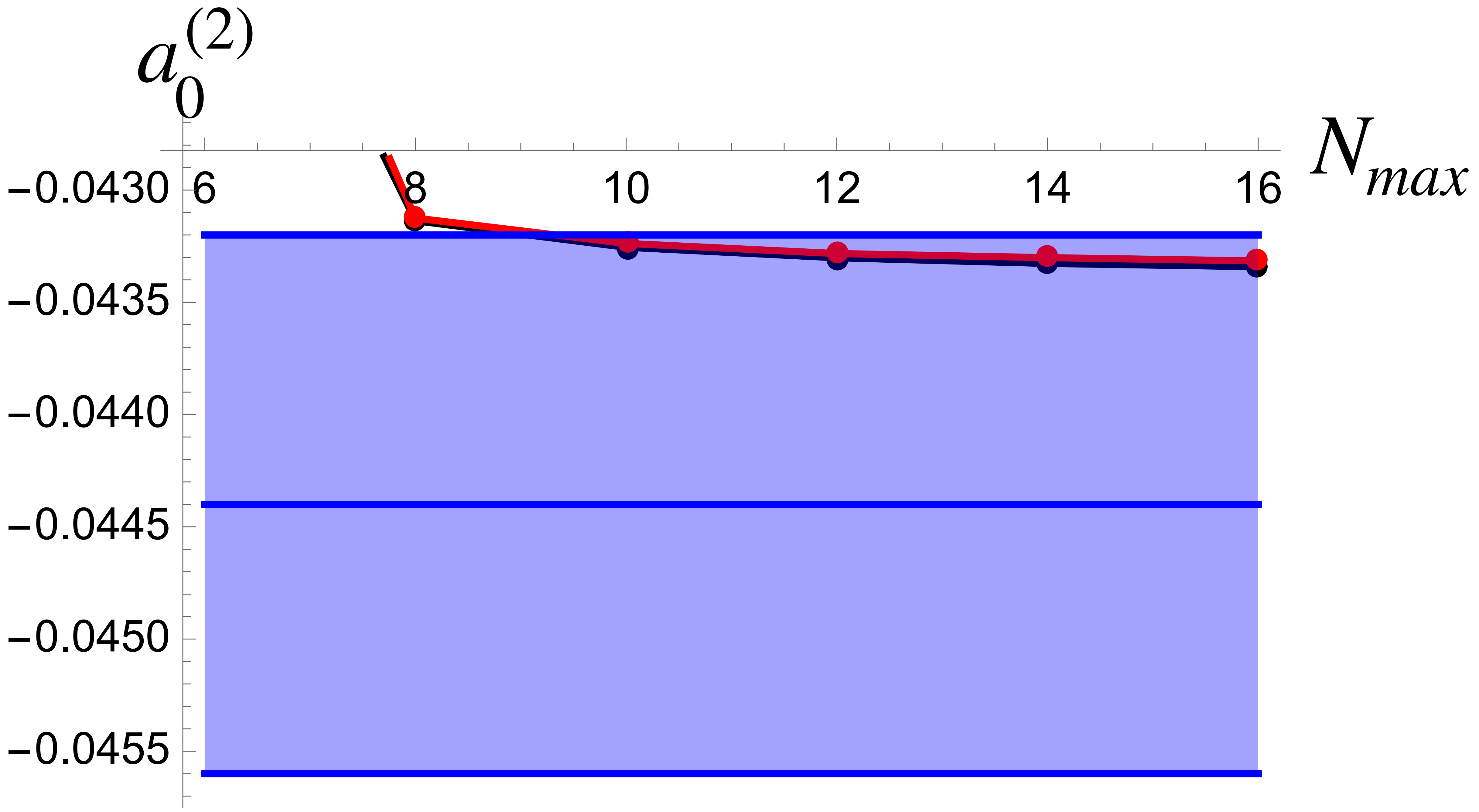
Without Inelasticity

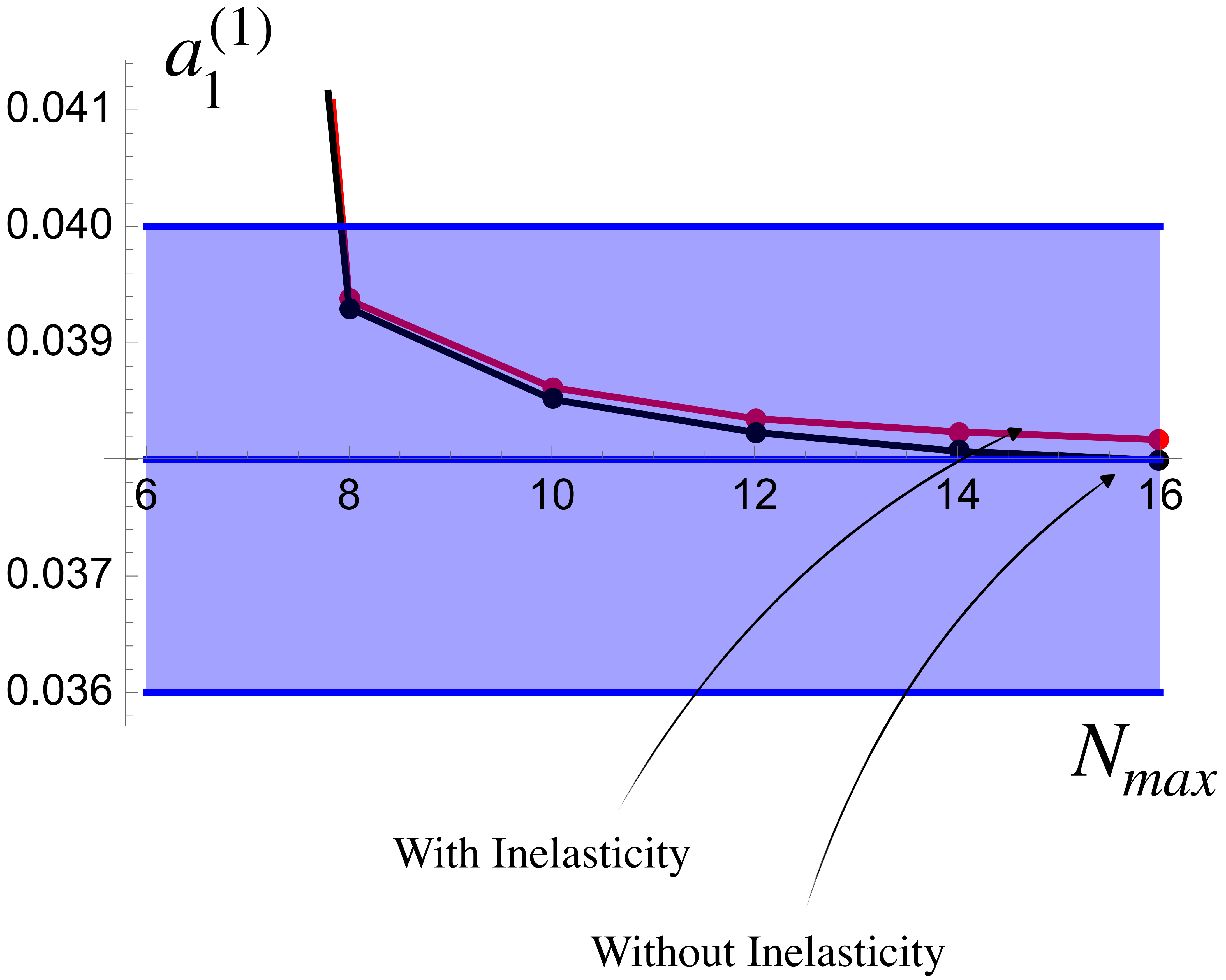


Improvement!

With Inelasticity

Outputs: Scattering Lengths

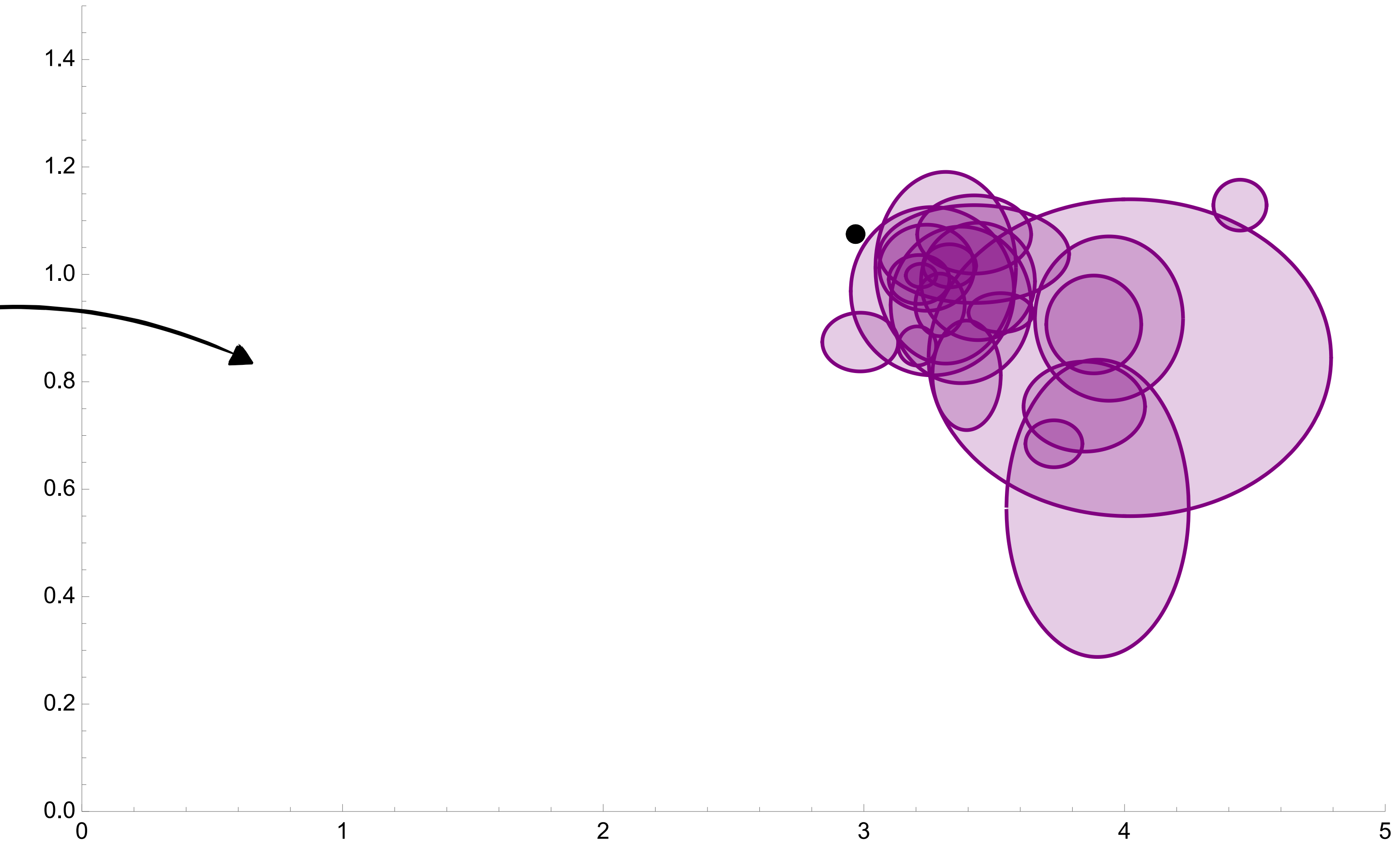




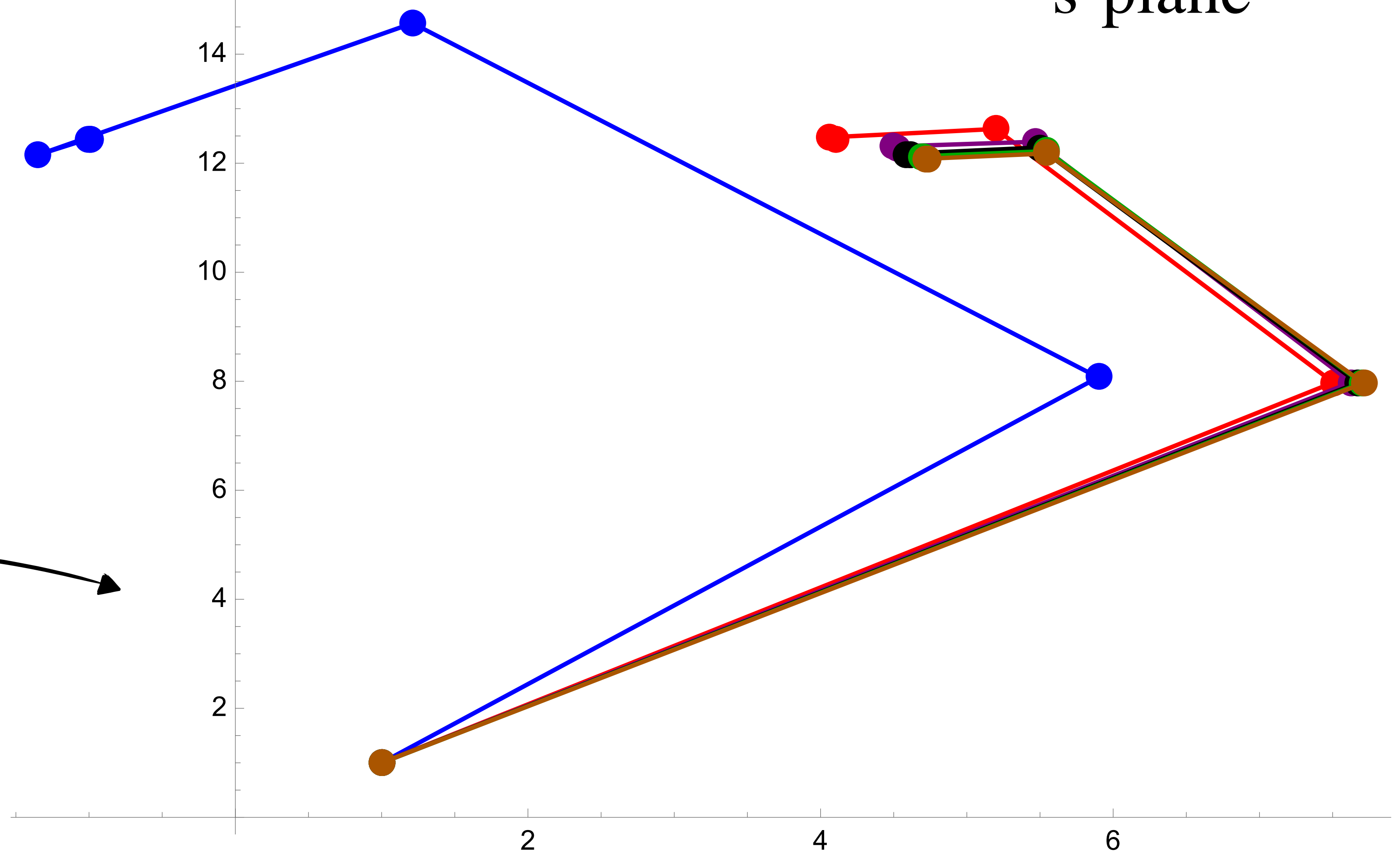
Outputs: Spectrum

Different determinations of the σ since 2001

$(410 \pm 20) - i(240 \pm 15)$	SARANTSEV	2021	RVUE	$J/\psi(1S) \to \gamma\{\pi\pi, K\bar{K}, \eta\eta, \omega\phi\}$
$(512 \pm 15) - i(188 \pm 12)$	[1] ABLIKIM	2017	BES3	$J/\psi \to \gamma 3\pi$
$(440 \pm 10) - i(238 \pm 10)$	[2] ALBALADEJO	2012	RVUE	Compilation
$(445 \pm 25) - i(278^{+22}_{-18})$	[3,4] GARCIA-MARTIN	2011	RVUE	Compilation
$(457^{+14}_{-13}) - i(279^{+11}_{-7})$	[5,3] GARCIA-MARTIN	2011	RVUE	Compilation
$(442^{+5}_{-8}) - i(274^{+6}_{-5})$	[6] MOUSSALLAM	2011	RVUE	Compilation
$(452 \pm 13) - i(259 \pm 16)$	[7] MENNESSIER	2010	RVUE	Compilation
$(448 \pm 43) - i(266 \pm 43)$	[8] MENNESSIER	2010	RVUE	Compilation
$(455 \pm 6^{+31}_{-13}) - i(278 \pm 6^{+34}_{-43})$	[9] CAPRINI	2008	RVUE	Compilation
$(463 \pm 6^{+31}_{-17}) - i(259 \pm 6^{+33}_{-34})$	[10] CAPRINI	2008	RVUE	Compilation
$(552^{+84}_{-106}) - i(232^{+81}_{-72})$	[11] ABLIKIM	2007A	BES2	$\psi(2S) \to \pi^+\pi^- J/\psi$
$(466 \pm 18) - i(223 \pm 28)$	[12] BONVICINI	2007	CLEO	$D^+ \to \pi^-\pi^+\pi^+$
$(472 \pm 30) - i(271 \pm 30)$	[13] BUGG	2007A	RVUE	Compilation
$(484 \pm 17) - i(255 \pm 10)$	GARCIA-MARTIN	2007	RVUE	Compilation
$(430) - i(325)$	[14] ANISOVICH	2006	RVUE	Compilation
$(441^{+16}_{-8}) - i(272^{+9}_{-12.5})$	[15] CAPRINI	2006	RVUE	$\pi\pi \to \pi\pi$
$(470 \pm 50) - i(285 \pm 25)$	[16] ZHOU	2005	RVUE	
$(541 \pm 39) - i(252 \pm 42)$	[17] ABLIKIM	2004A	BES2	$J/\psi \to \omega\pi^+\pi^-$
$(528 \pm 32) - i(207 \pm 23)$	[18] GALLEGOS	2004	RVUE	Compilation
$(533 \pm 25) - i(249 \pm 25)$	[19] BUGG	2003	RVUE	
$517 - i\,240$	BLACK	2001	RVUE	$\pi\pi \to \pi\pi$
$(470 \pm 30) - i(295 \pm 20)$	[13] COLANGELO	2001	RVUE	$\pi\pi \to \pi\pi$
$(535^{+48}_{-50}) - i(155^{+76}_{-53})$	[20] ISHIDA	2001		$\Upsilon(3S) \to \Upsilon\pi\pi$
$610 \pm 14 - i(310 \pm 13)$	[21] SUROVTSEV	2001	RVUE	$\pi\pi \to \pi\pi, K\bar{K}$



s-plane



How we extract the σ

Different paths of the Newton Method to find the zero in $S_0^{(0)}$, for different N_{max} (super stable Bootstrap solution)

Plan to finish the paper (s)

1) Not fixing the ρ, but navigate to a better kink (we know it is possible, example in backup slides)

2) Propagate errors

3) Check different data sets

4) Bonus: extract higher spin resonances

Backup Slides

An analytic bound on scattering

Goal: we bound $c_4 \Longleftrightarrow$ we bound Δ_3

What are the non-perturbative properties of the branons scattering amplitude?

Unitarity: define $S(s) = 1 + \frac{i}{2s} T_{2\to 2}(s)$ then $|S(s)|^2 \leq 1$ for s>0

S-matrix measures probabilities

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Crossing: $S(s) = S(u)$ where t=0, u=-s

In 2d there is no scattering angle

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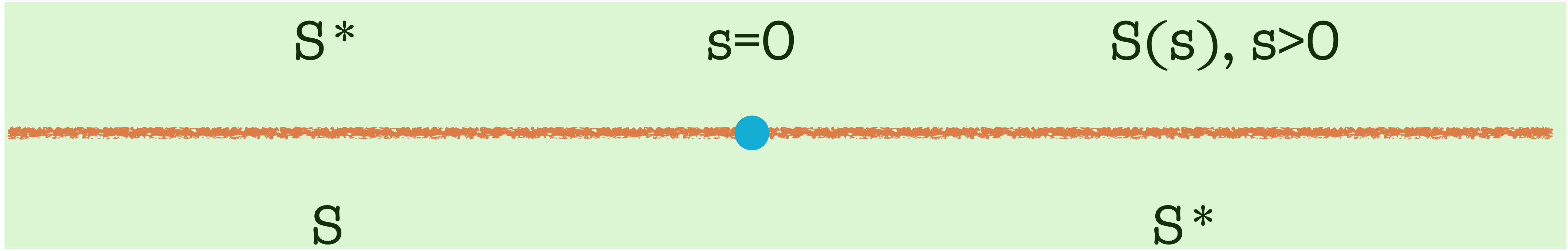
Analytic away from the real axis

Analyticity



Low Energy Constraints: $S(s) = 1 + i\frac{s}{4} - \frac{s^2}{32} + i(\gamma_3 - \frac{1}{384})s^3 + \ldots$

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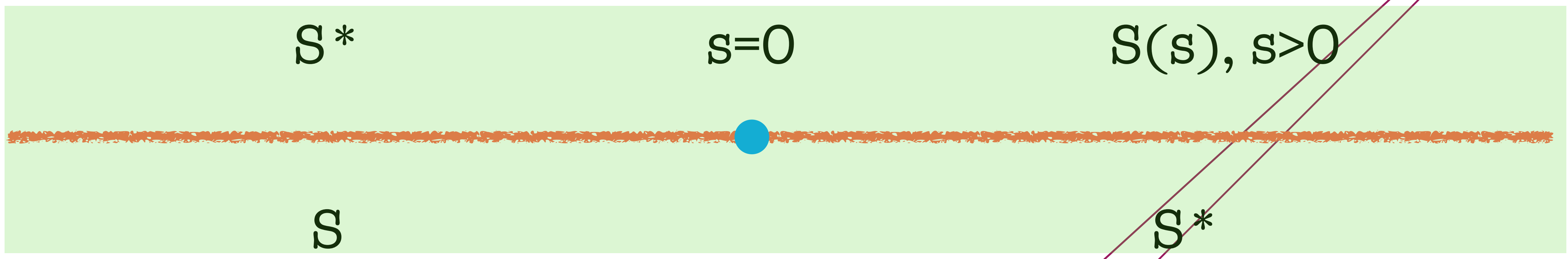

Low Energy Constraints: $S(s) = 1 + i\frac{s}{4} - \frac{s^2}{32} + i(\gamma_3 - \frac{1}{384})s^3 + \ldots$

Solution (Schwarz-Pick Theorem)

$$\gamma_3 \geq -\frac{1}{768}$$

$$S(s) = \frac{8i - s}{8i + s}$$

Bootstrap Prediction $\gamma_3 > -\frac{1}{768} \sim -0.0013$

Lattice for SU(2) YM $\gamma_3 = -0.00034(6)$

Caristo, Caselle, Magnoli, Nada, Panero '21

Lattice for $\mathbb{Z}_2$ gauge theory $\gamma_3 = -0.00048(4)$

Baffigo, Caselle '23

A numerical bound

What if we were not good enough to find an analytic solution?

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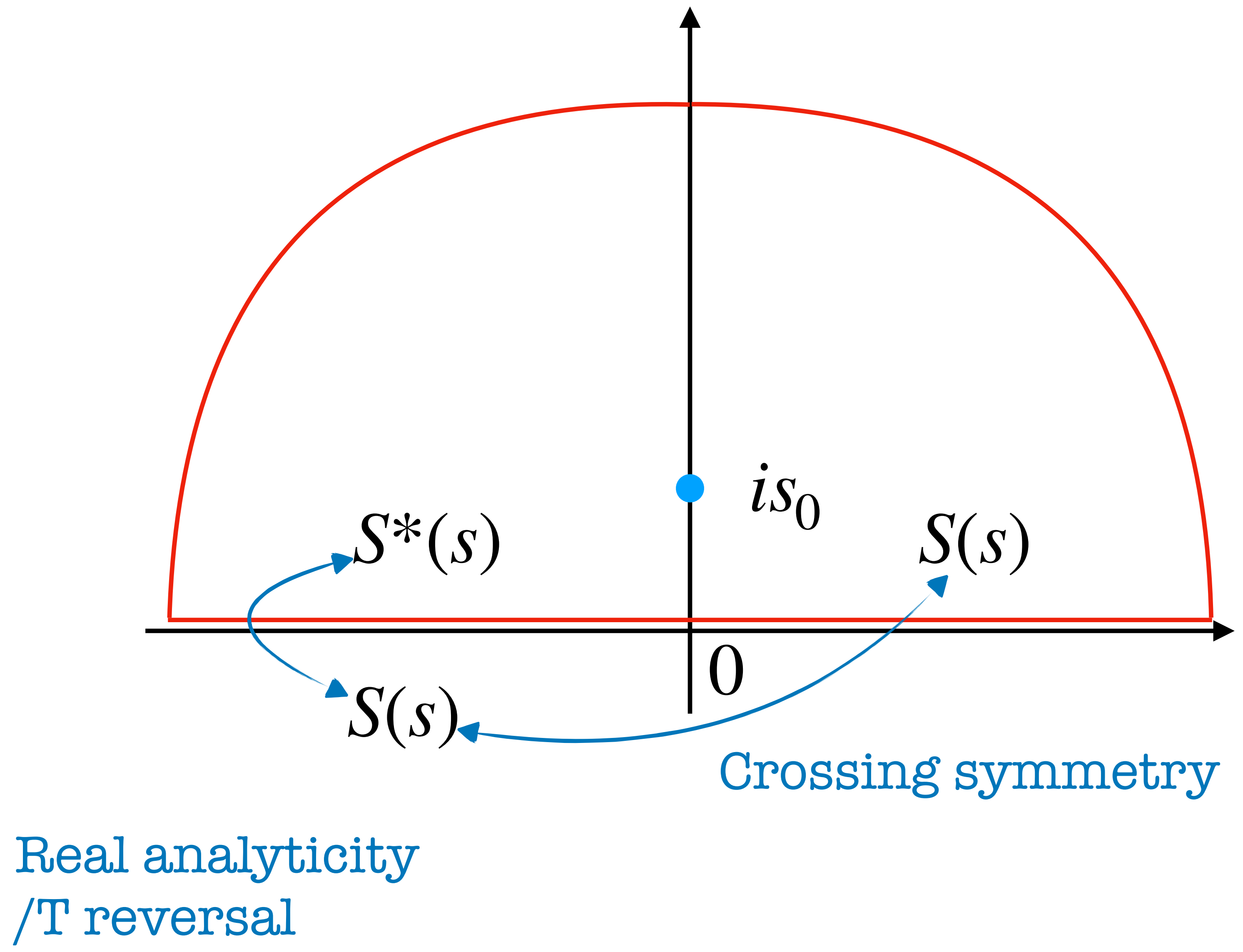
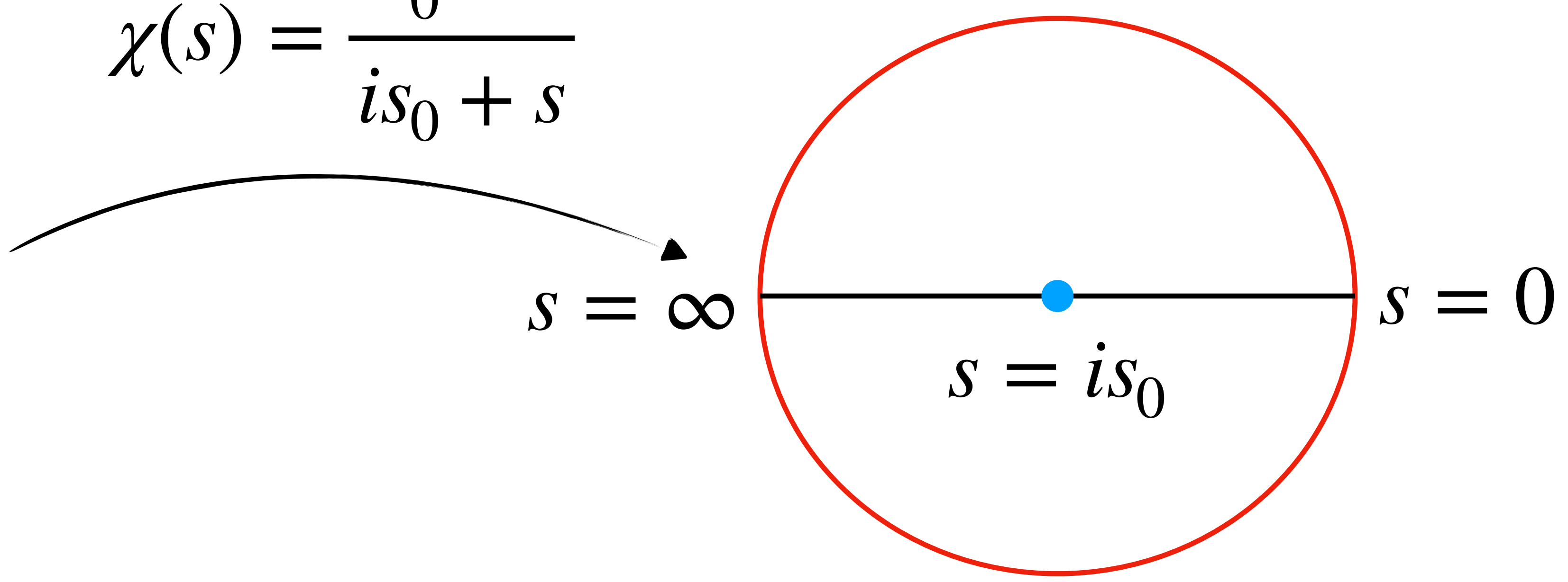


$$\chi(s) = \frac{is_0 - s}{is_0 + s}$$



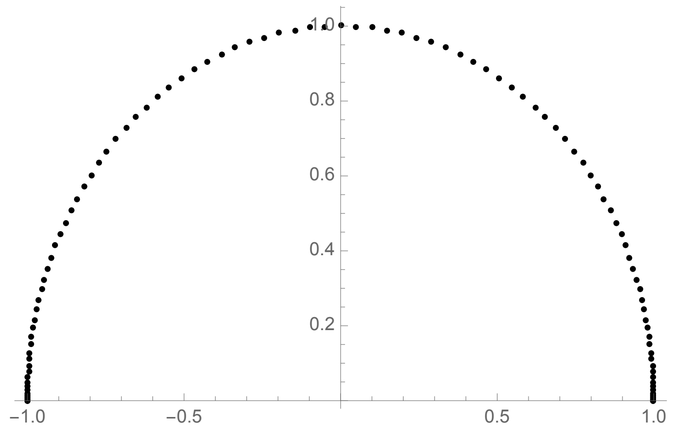
Ansatz manifestly **Analytic** and **crossing symmetric**:

$$S(s) = \sum_n a_n \chi(s)^n$$

We check unitarity numerically:

$$|S(s)|^2 \leq 1, \quad S(s) = 1 + \frac{i}{2s} T(s)$$

Unitarity imposed on a grid of **M** points



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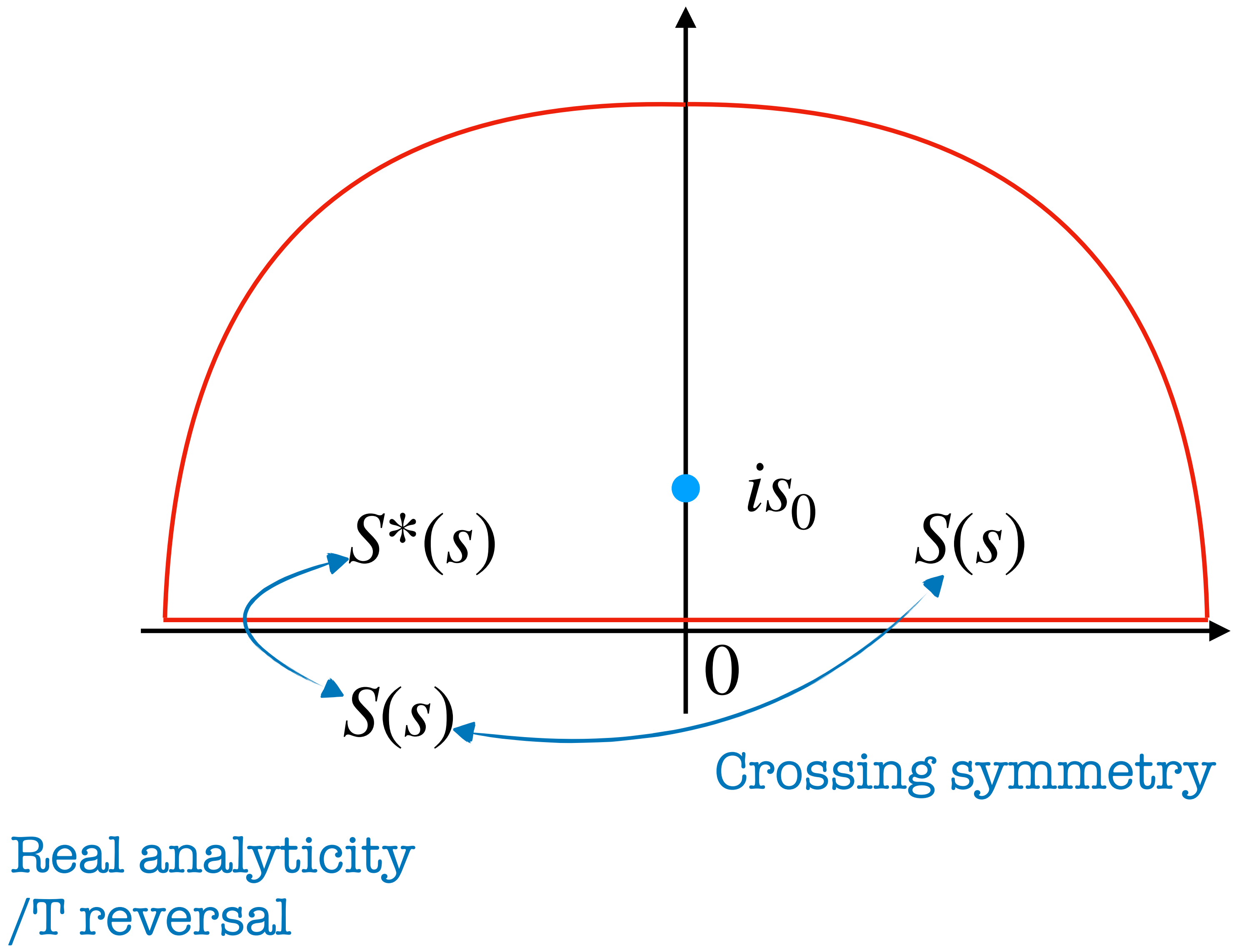
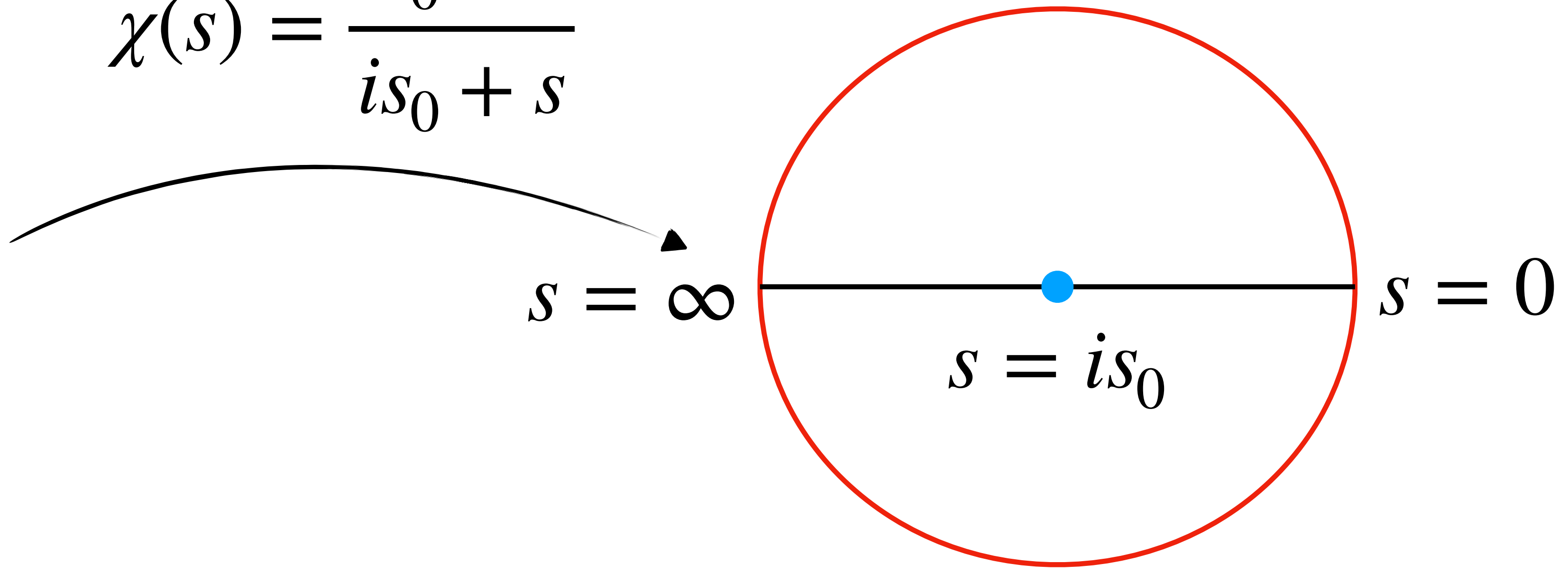


$$\chi(s) = \frac{is_0 - s}{is_0 + s}$$



FindMinimum $\gamma_3(a_n)$ with $T(s) = \sum_n^N a_n \chi(s)^n$ and $|S(s)|^2 \leq 1$

Order of limits for convergence:

1) number of constraint large $M \to \infty$
2) number of terms large $N \to \infty$

Ansatz manifestly **Analytic** and **crossing symmetric**:

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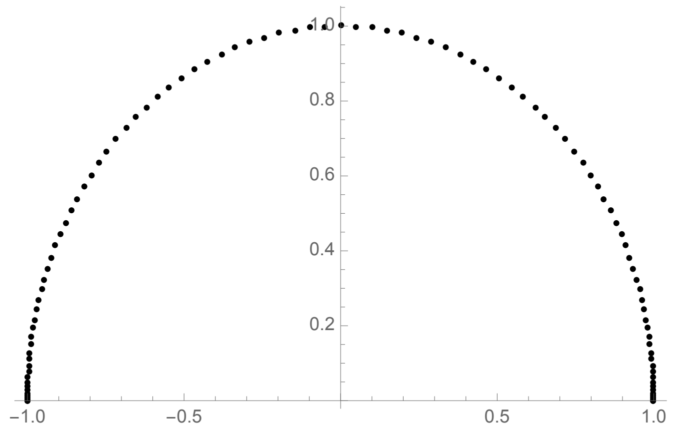
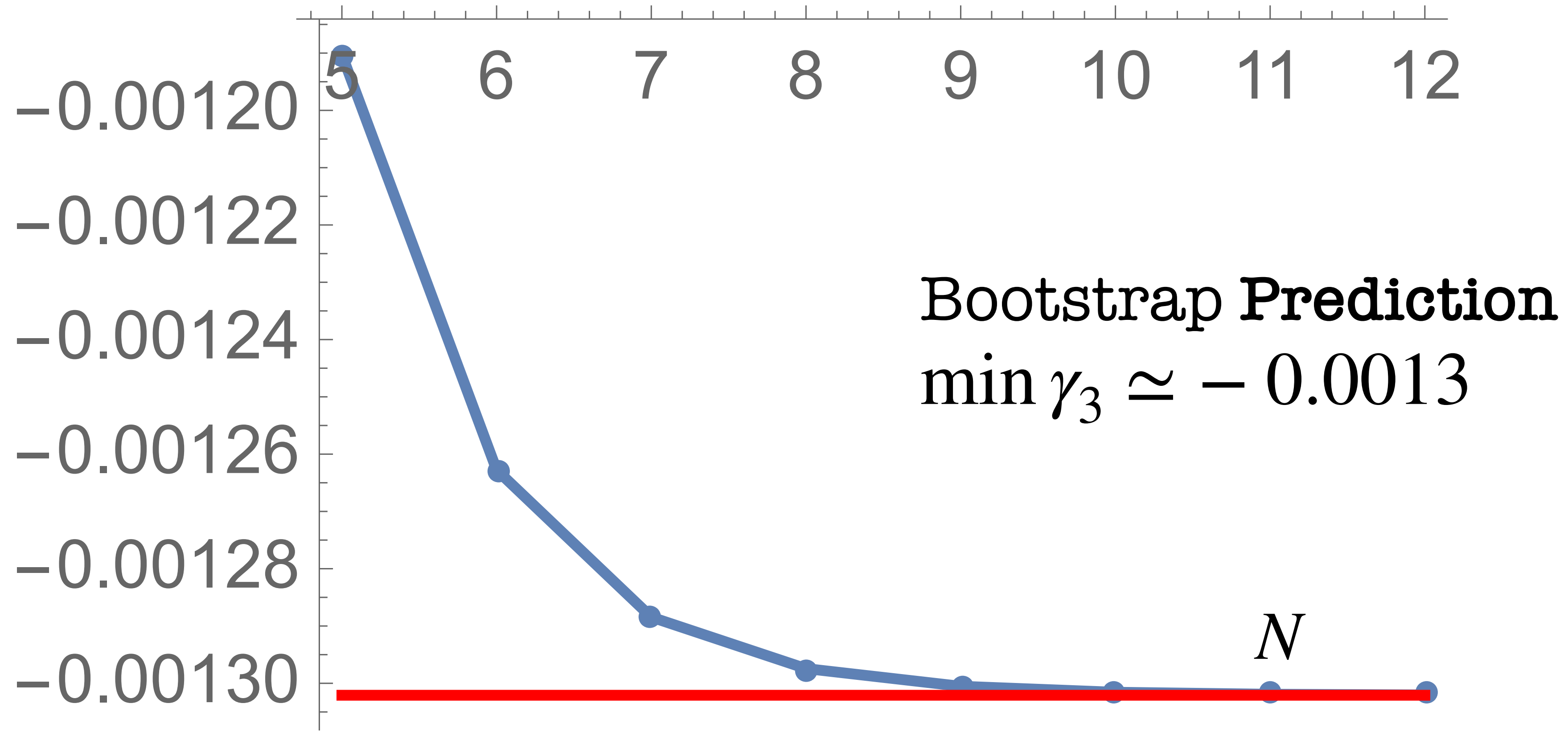




Flux-Tube Bootstrap: What's next?

Q1: The world-sheet QCD axion subject to a triple coincidence, why? $Q_{Lattice} \sim Q_{Bootstrap} \sim Q_{integrable}$

Gaikwad, Gorbenko, ALG (to appear)

Q2: Strings interact with Glueballs, can we inject UV using form factors?

Hebbar, ALG (working in progress)

Q3: Can we go beyond 2->2?

Homrich, ALG, Penedones, Vieira (working in progress)

Naively: Stronger constraints!

$$\sum_n P_{2\to n} = 1 \implies P_{2\to 2} + P_{2\to 4} + \ldots \leq 1$$

The dream: multi-particle Bootstrap

The majority of the bounds so far are consistent with any amount of particle production, even zero.

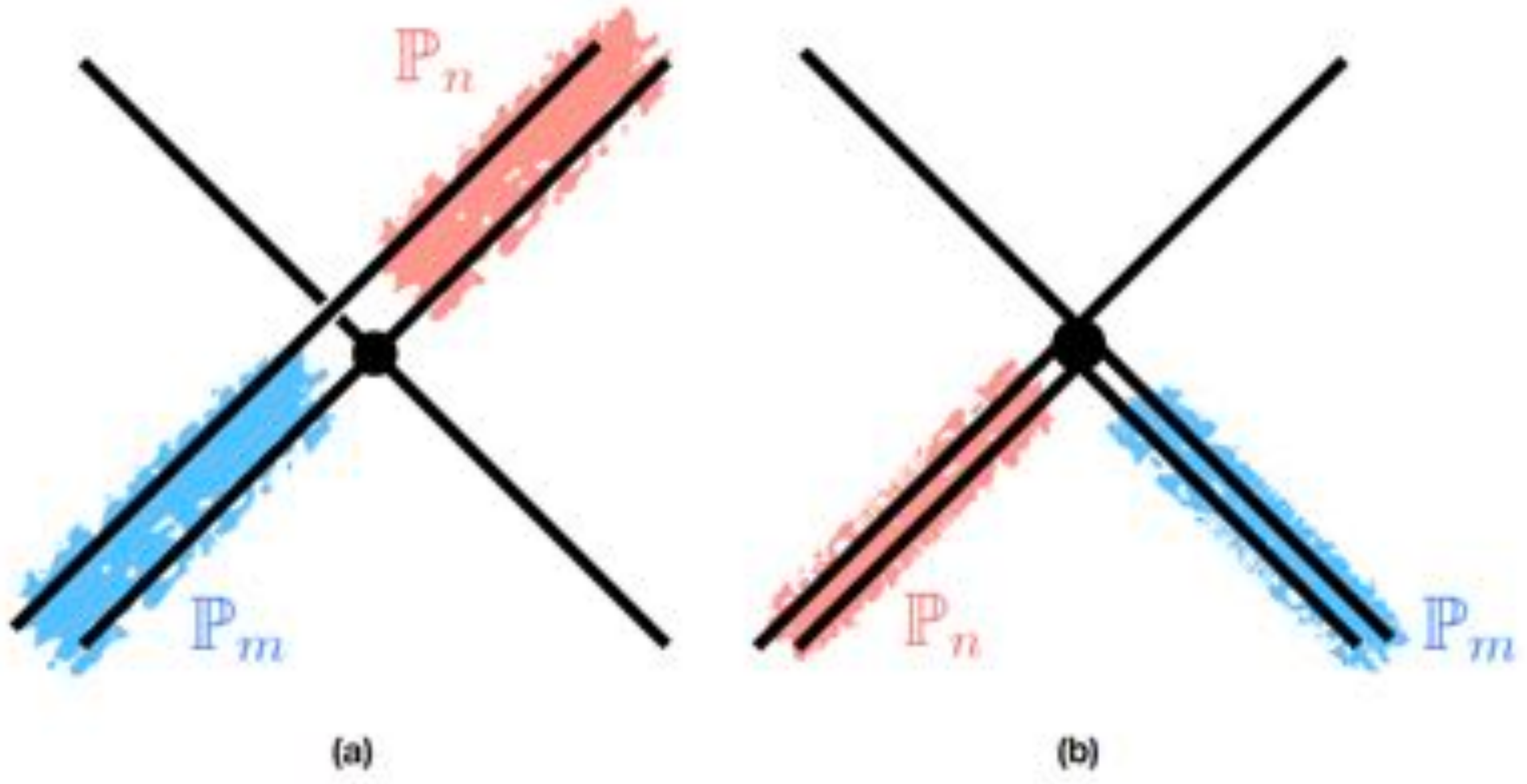
Simplest case: massless Goldstones on Strings in 3D

Idea: project multi-particle states into jet states

No collinear divergences in this theory!

2-particle Jet State

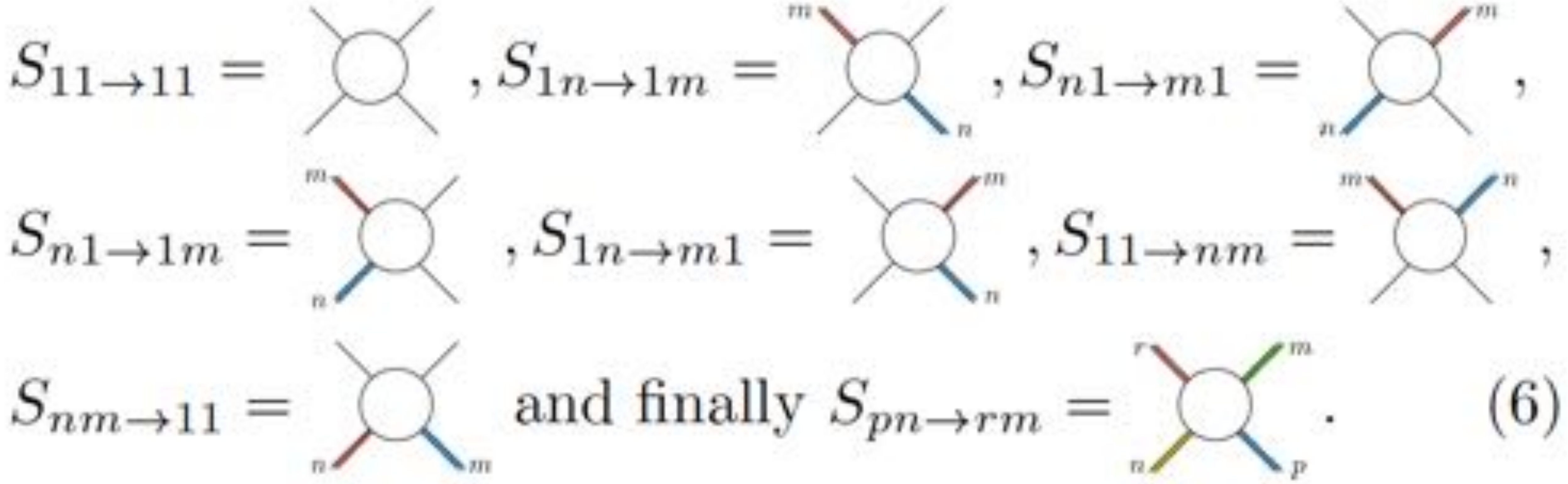
$$|n, P\rangle \equiv \sqrt{2n+1} \int_0^1 d\alpha \frac{P_n(2\alpha-1)}{\sqrt{8\pi\alpha(1-\alpha)}} |\alpha, (1-\alpha), P\rangle_2$$



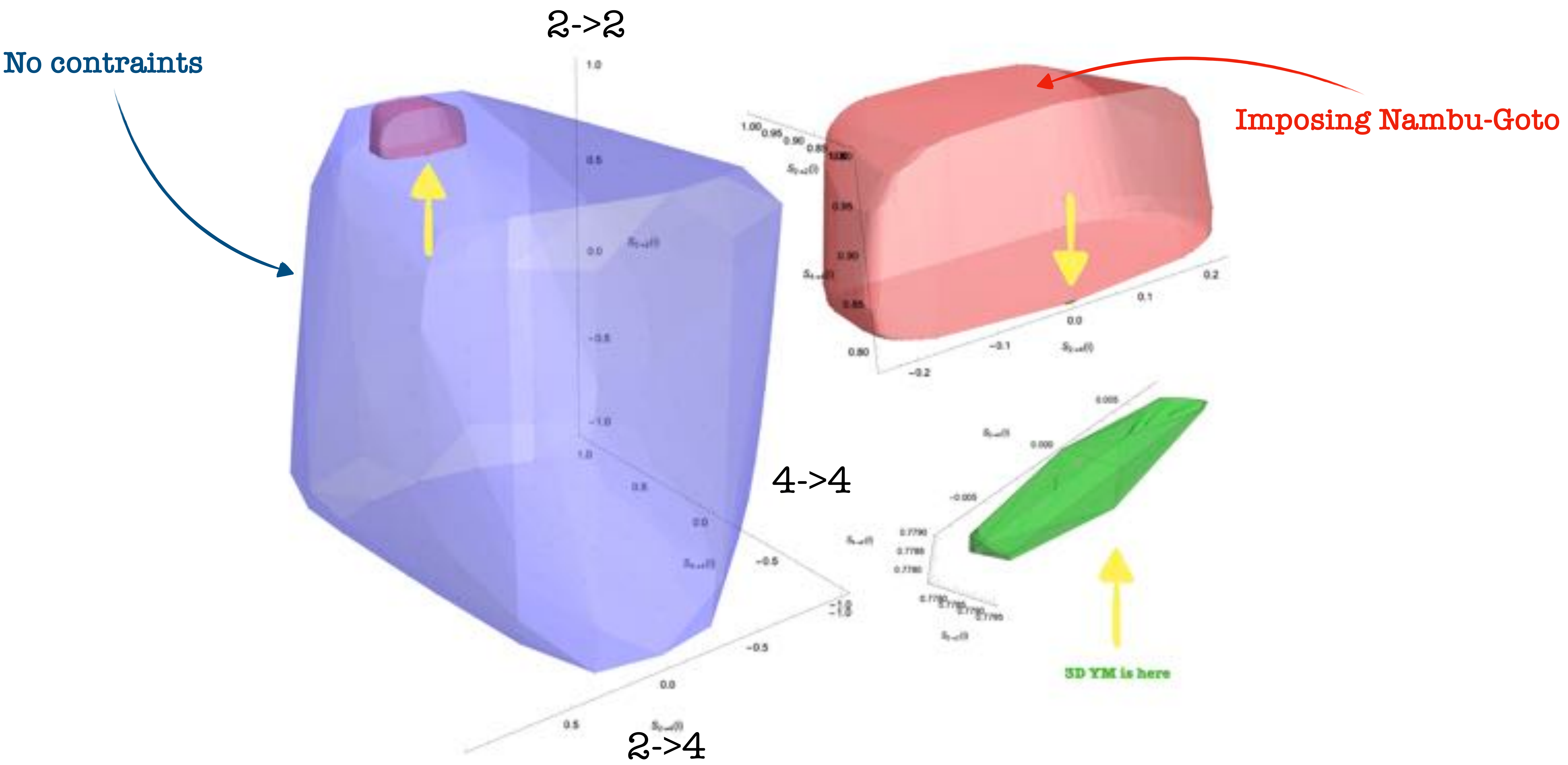
$3 \to 3$ $\quad$ $2 \to 4$

Problem decomposes into a bunch of 2->2 processes

Homrich, ALG, Penedones, Vieira (working in progress)



The Multi-Particle Matrioska coming soon...



Toy model: max spin-2 coupling

The maximum residue at the spin-2 pole is a hard problem ($\mathbb{Z}_2$ symmetry, no $s = m^2$ pole)

$$M \supset \frac{-g^2}{s - m_b^2} P_2\left(1 + \frac{2t}{m_b^2 - 4m^2}\right) + \ldots \sim t^2$$

AG, Hebbar, van Rees 2312.00127

Without Regge it would violate unitarity!

They must restore $M(t \to \infty, s \leq 0) < t \log^2 t$

Toy model: max spin-2 coupling

The maximum residue at the spin-2 pole is a hard problem ($\mathbb{Z}_2$ symmetry, no $s = m^2$ pole)

$$M \supset \frac{-g^2}{s - m_b^2} P_2\left(1 + \frac{2t}{m_b^2 - 4m^2}\right) + \ldots \sim t^2$$

AG, Hebbar, van Rees 2312.00127

Without Regge it would violate unitarity!

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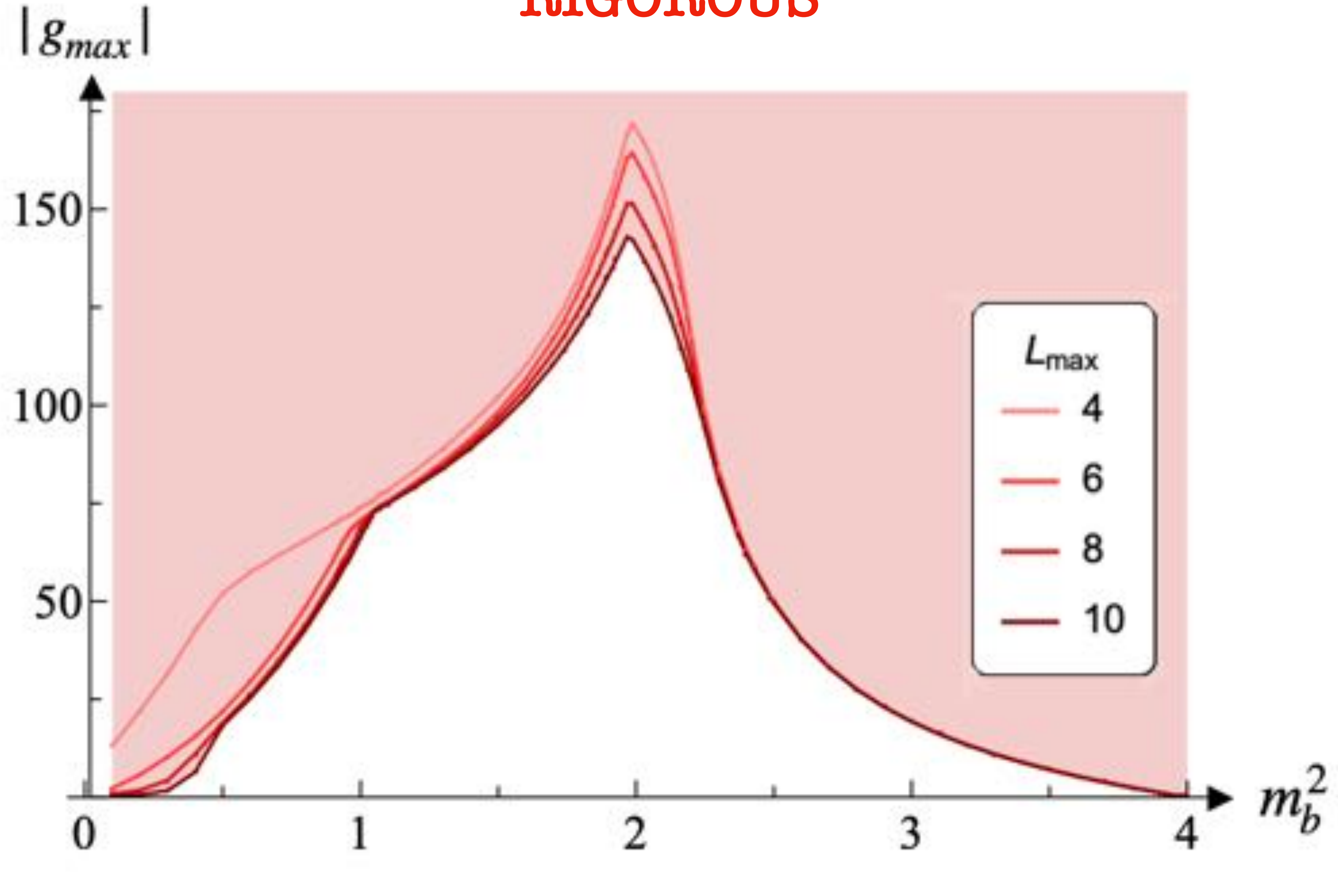
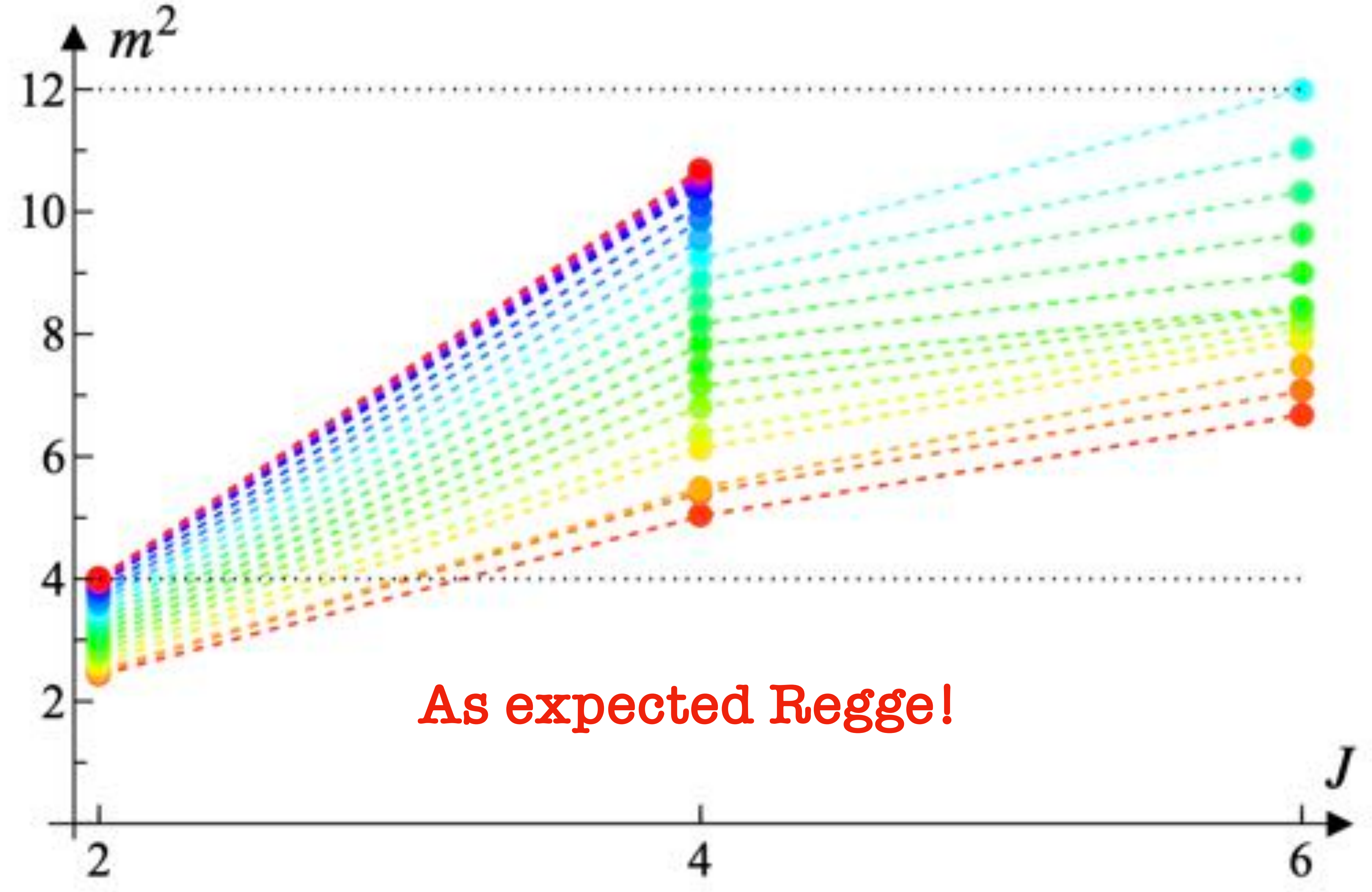
RIGOROUS



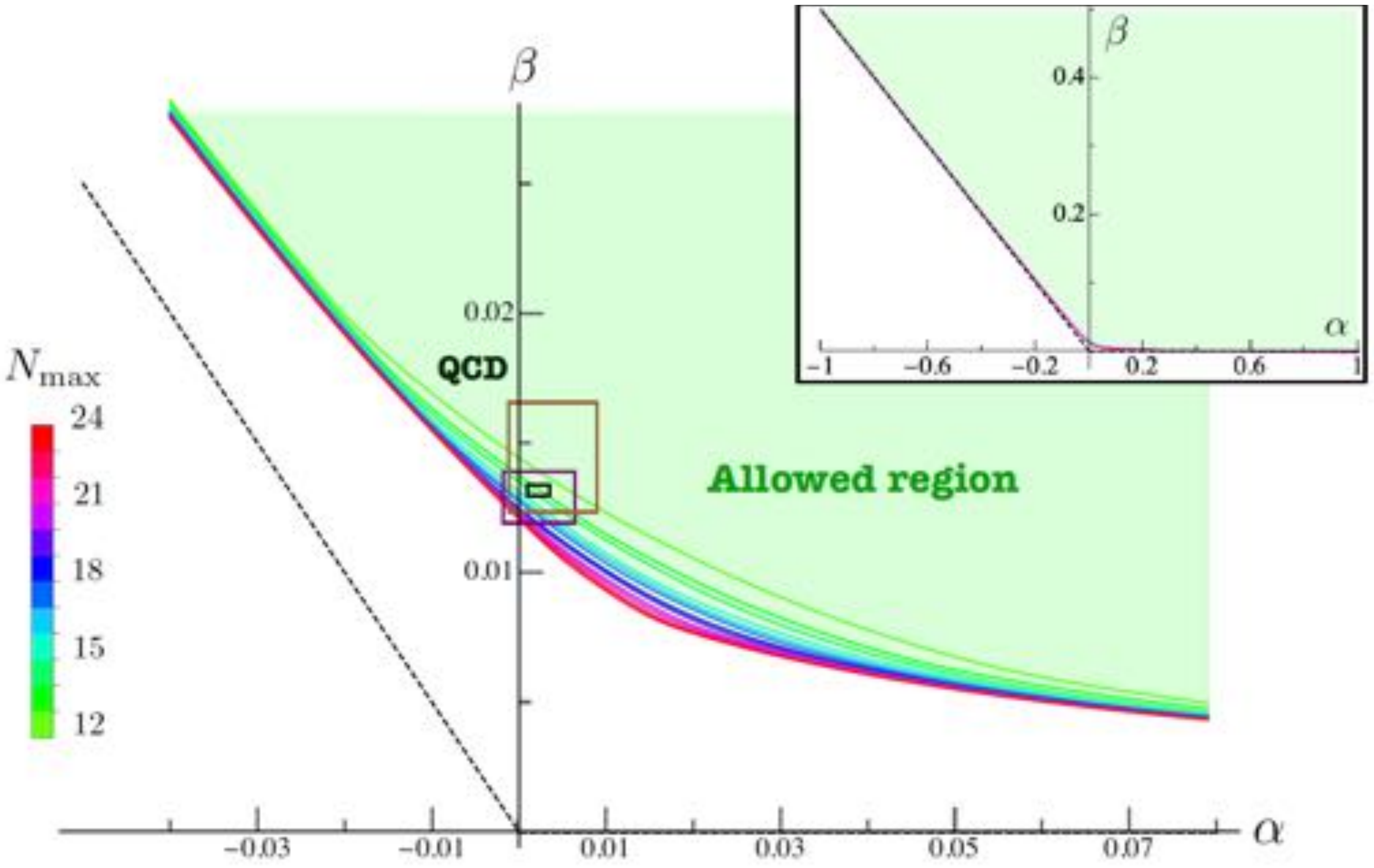


As expected Regge!

Low energy QCD

In QCD dynamical mass generation, non-perturbative RG flow

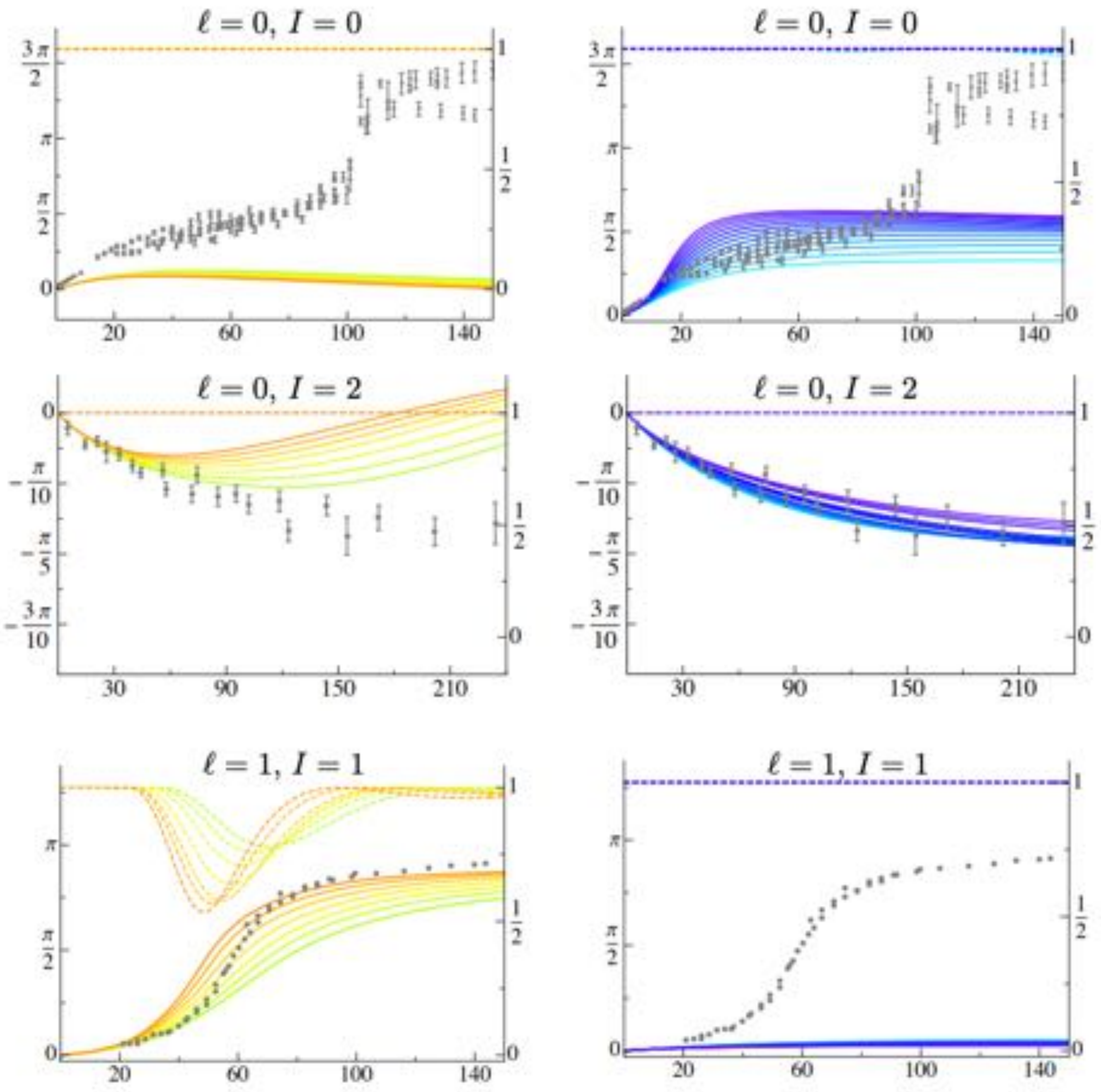
Amplitude = $\frac{s}{f^2} + \alpha \frac{s^2}{f^4} + \beta \frac{t^2 + u^2}{f^4} + \text{logs}$ + UV completion



ALG, Penedones, Vieira 1810.12849 (gapped), 2011.02802 (gapless)

α, β can be only computed using lattice QCD today or extracted from data!!!

Non perturbative S-matrices from Bootstrap



Left side of the boundary

Right side of the boundary

What can we add to nail down QCD?

Work in progress with H. Murali

An analytic bound on scattering

Goal: we bound $c_4 \Longleftrightarrow$ we bound Δ_3

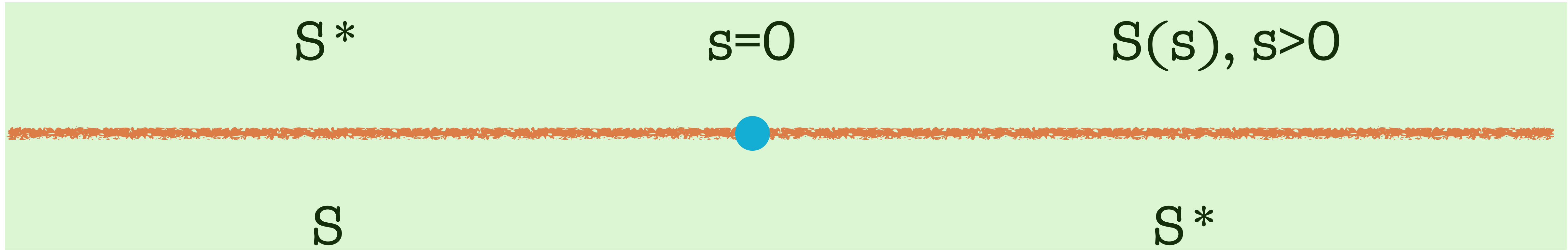
What are the non-perturbative properties of the branons scattering amplitude?

Unitarity: define $S(s) = 1 + \frac{i}{2s} T_{2\to 2}(s)$ then $|S(s)|^2 \leq 1$ for s>0

Crossing: $S(s) = S(u)$ where t=0, u=-s

Analyticity

Analytic away from the real axis



Low Energy Constraints: $S(s) = 1 + i\frac{s}{4} - \frac{s^2}{32} + i(\gamma_3 - \frac{1}{384})s^3 + \dots$

Solution (Schwarz-Pick Theorem)

$$\gamma_3 \geq -\frac{1}{768}$$

$$S(s) = \frac{8i - s}{8i + s}$$

gauge group	$\mathbb{Z}_2$	$SU(2)$	$SU(6)$	$SU(\infty)$
$\gamma_3 \times 768$	-0.4 [4]	-0.3 [5]	0.2 [1, 6]	0.3

[4] Baffigo, Caselle '23

[5] Caristo, Caselle, Magnoli, Nada, Panero '21

[1,6] Dubovsky, Gorbenko, et al

A New Kink

Undergoing search in a 4 parameter family of amplitudes

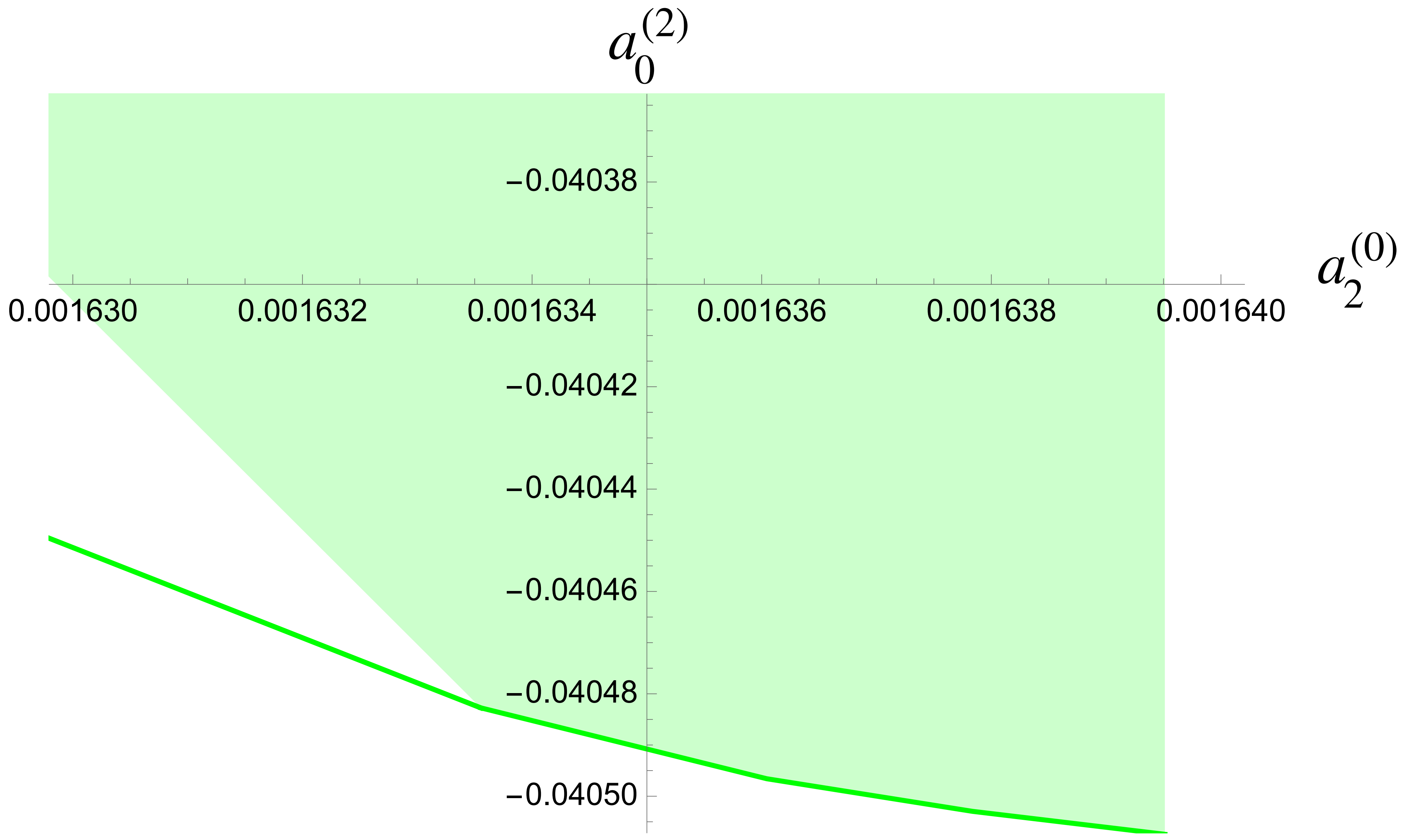
AG, Haring, Su (work in progress)

$$a_0^{(0)} = 0.22$$

$$a_1^{(1)} = 0.038$$

$$z_0 = 0.36$$

$$z_2 = 2.04$$



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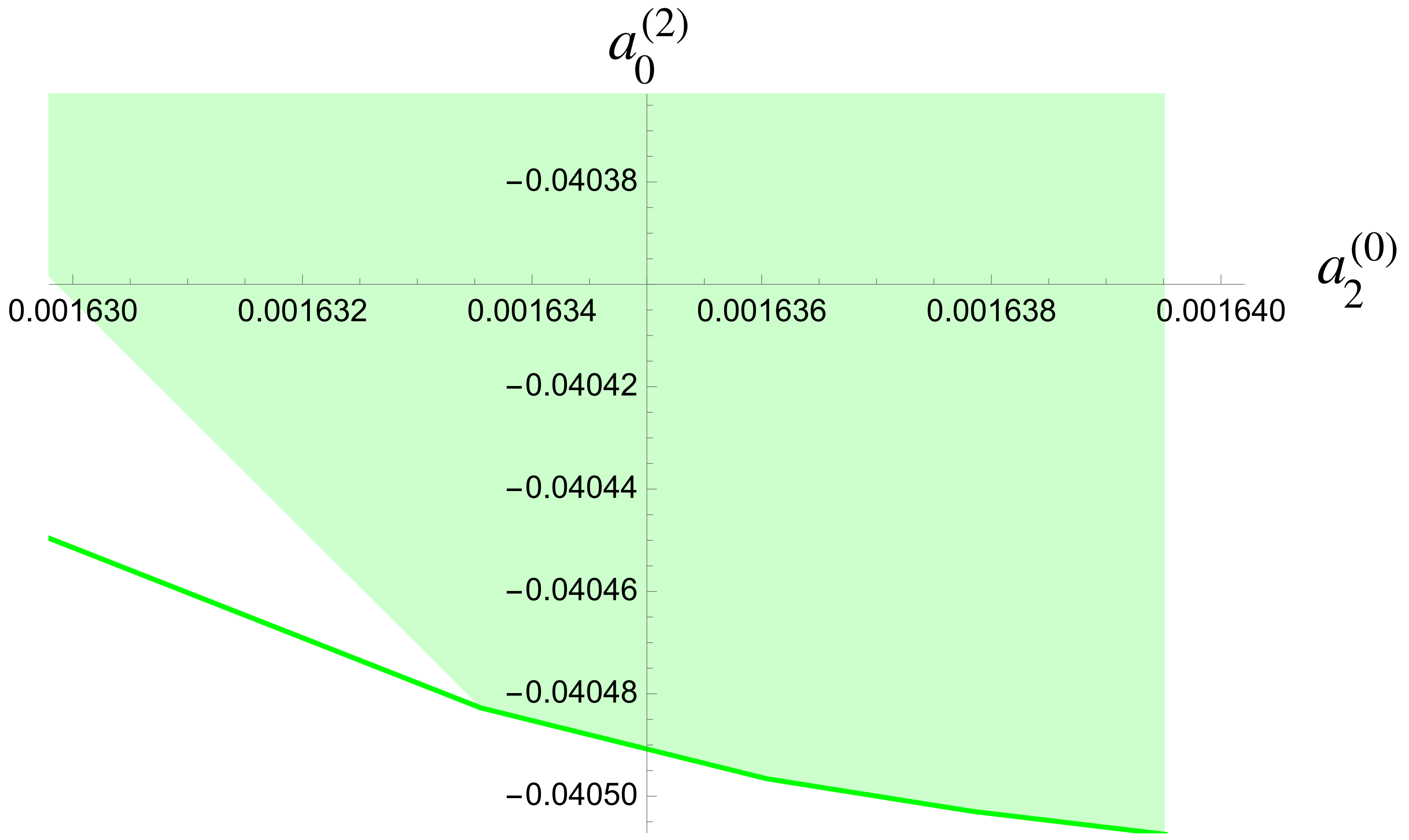
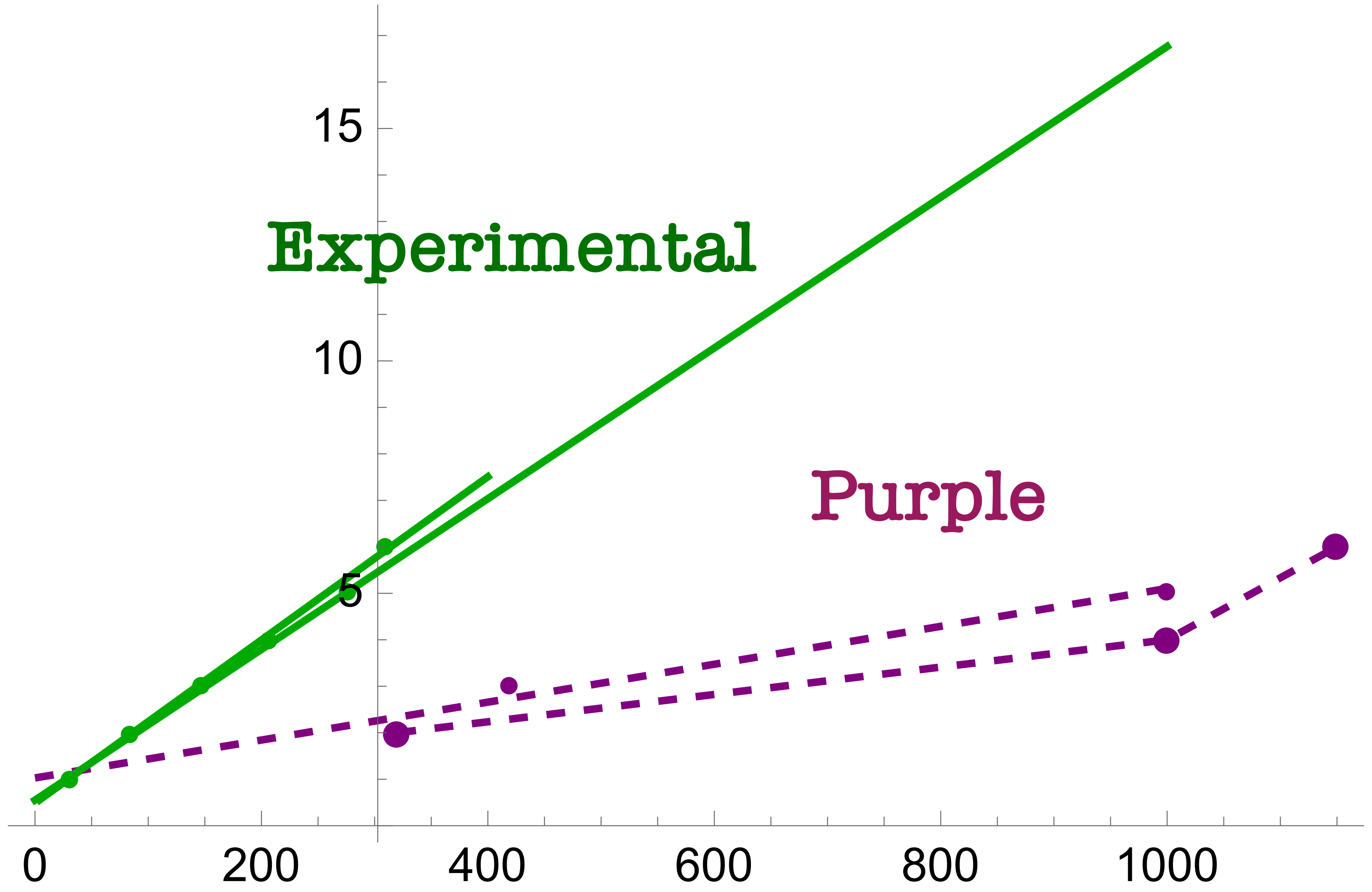
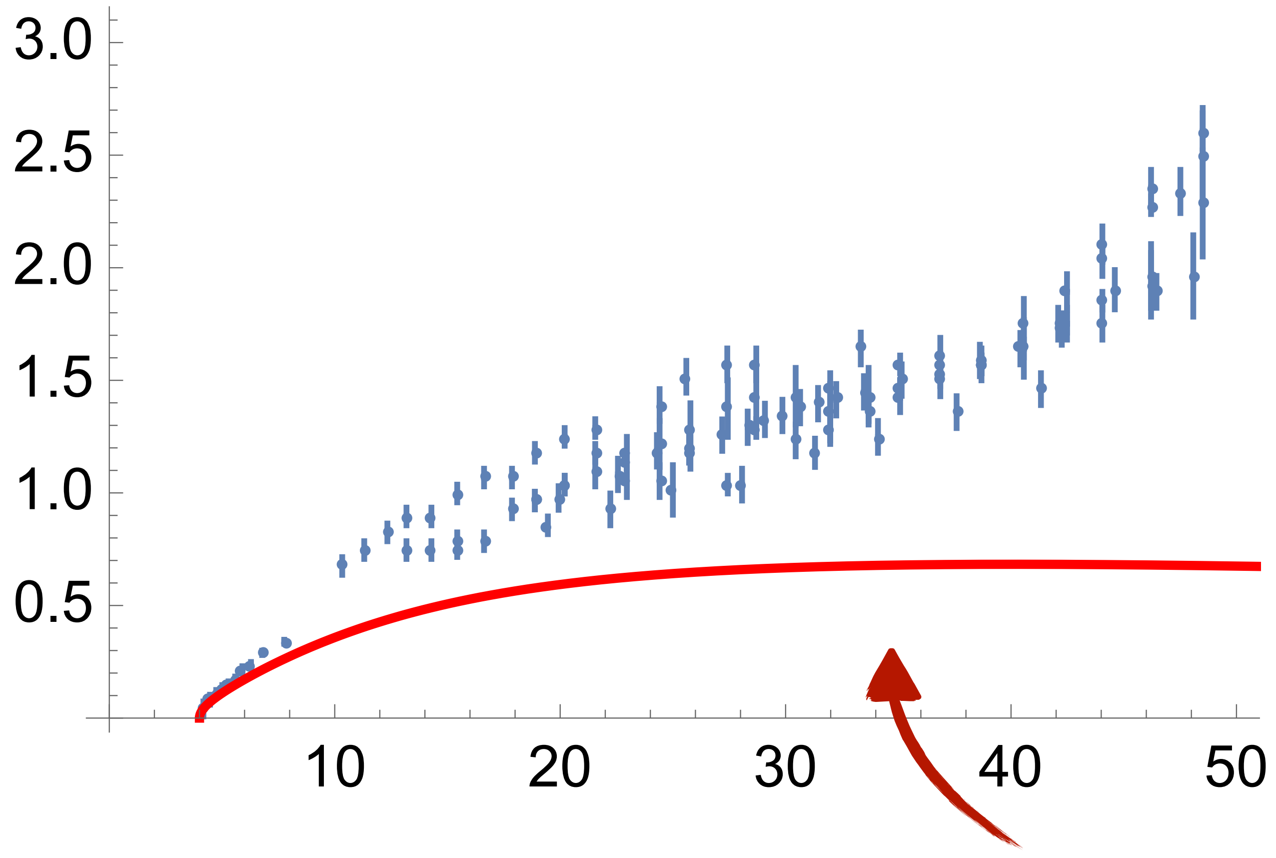
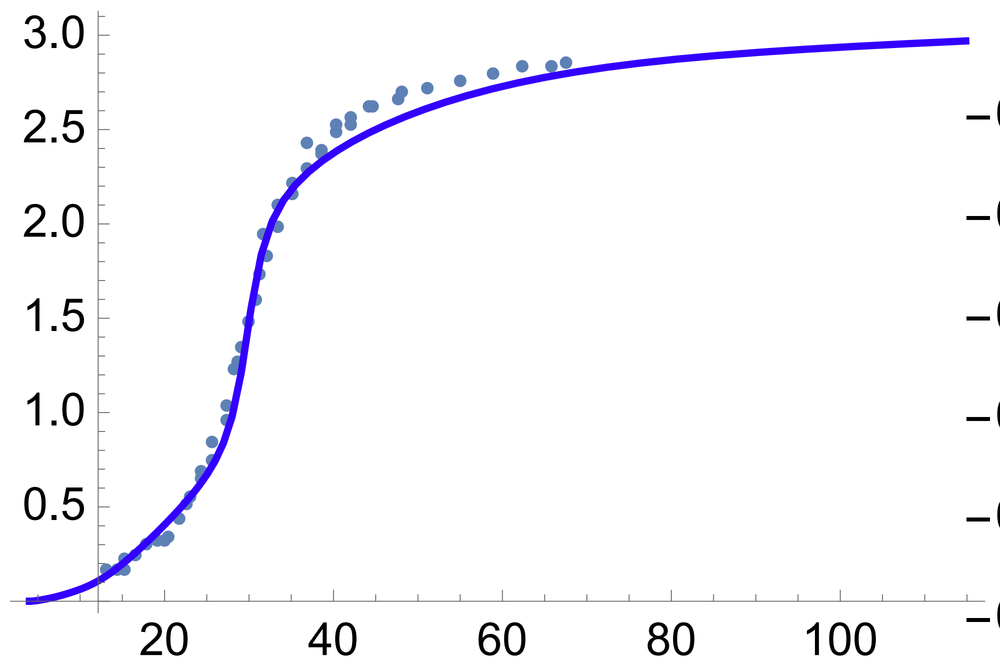
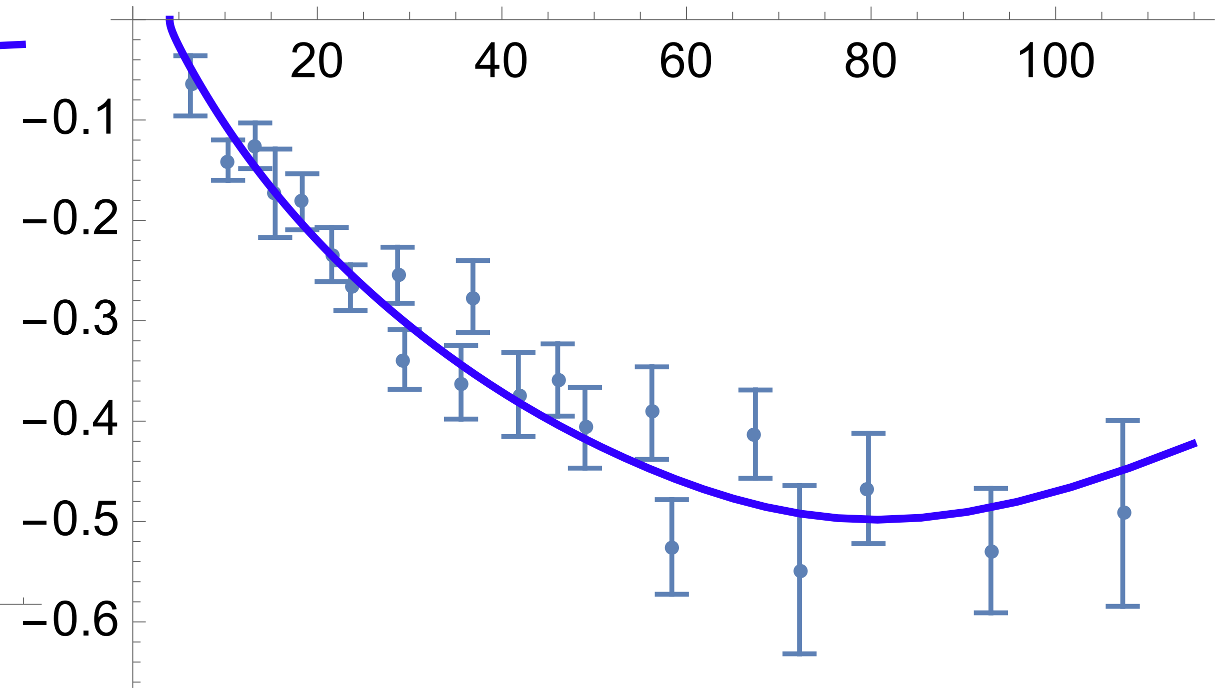




Much better χ^2



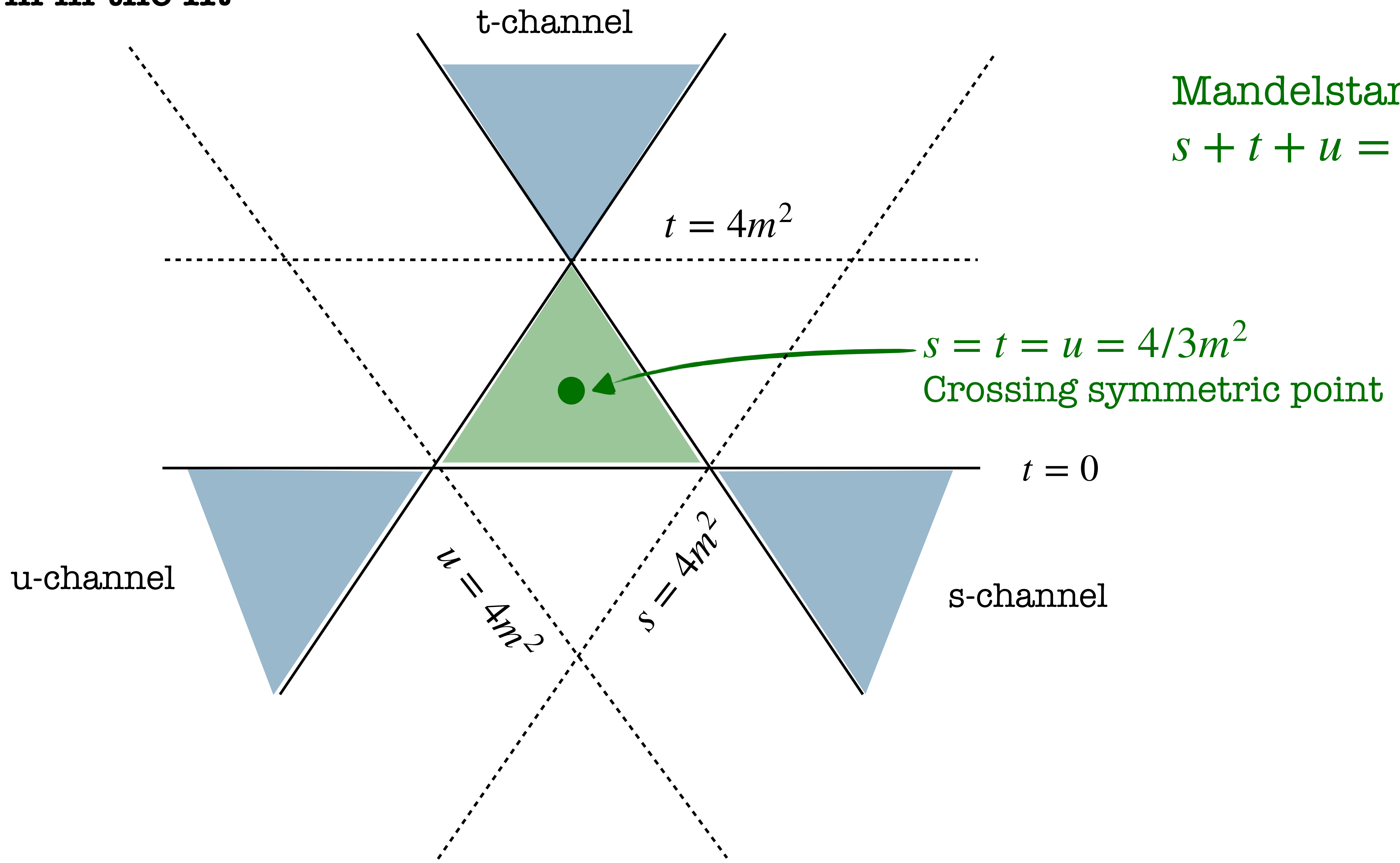




Still not perfect!

Analytic properties

Example: 1 scalar field of mass m in the IR



Mandelstam plane
$s + t + u = 4m^2$

Analytic properties

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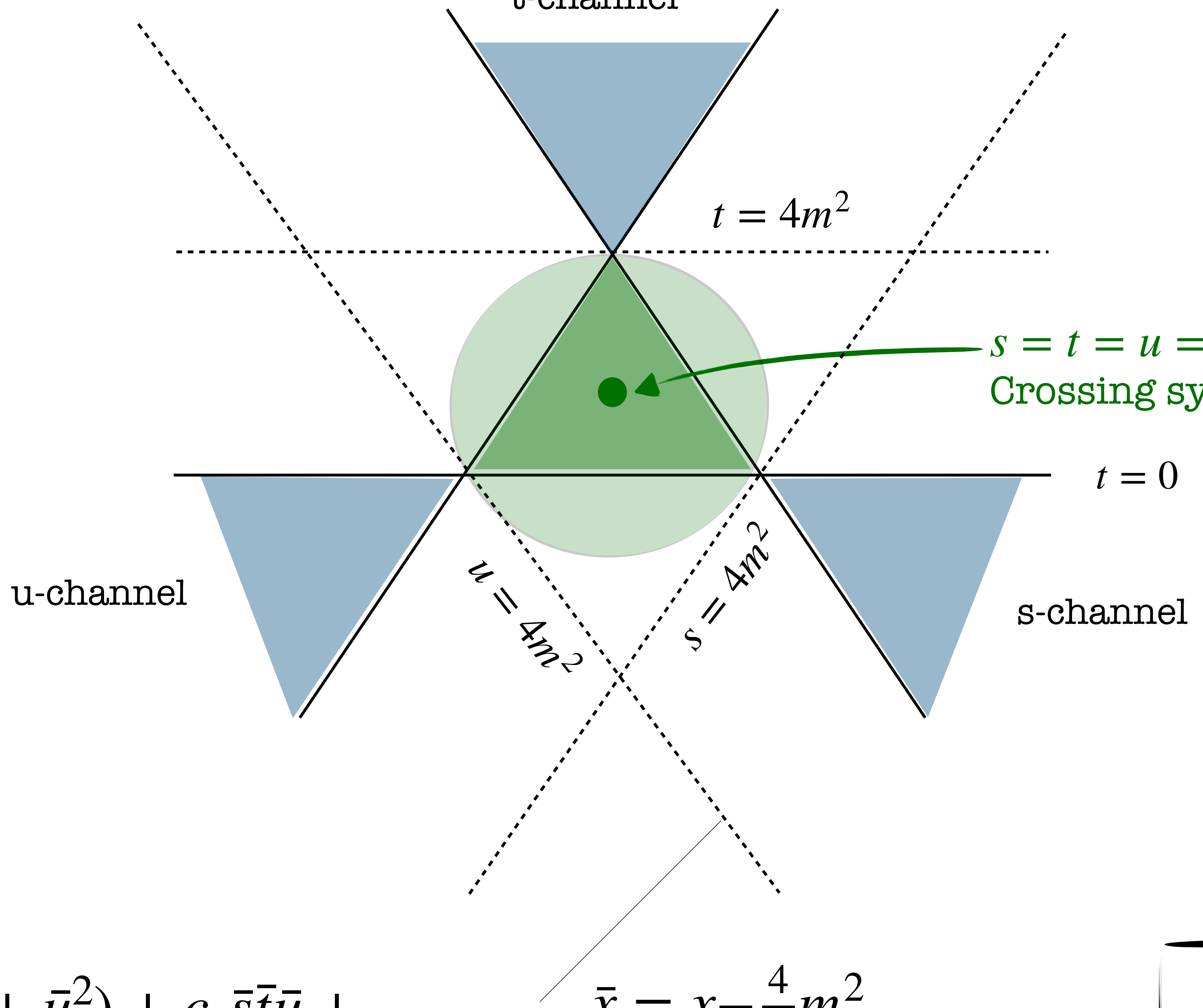


Mandelstam plane
$s + t + u = 4m^2$

$$T(s,t,u) = -c_0 + c_2(\bar{s}^2 + \bar{t}^2 + \bar{u}^2) + c_3 \bar{s}\bar{t}\bar{u} + \ldots \qquad \bar{x} = x - \frac{4}{3}m^2$$

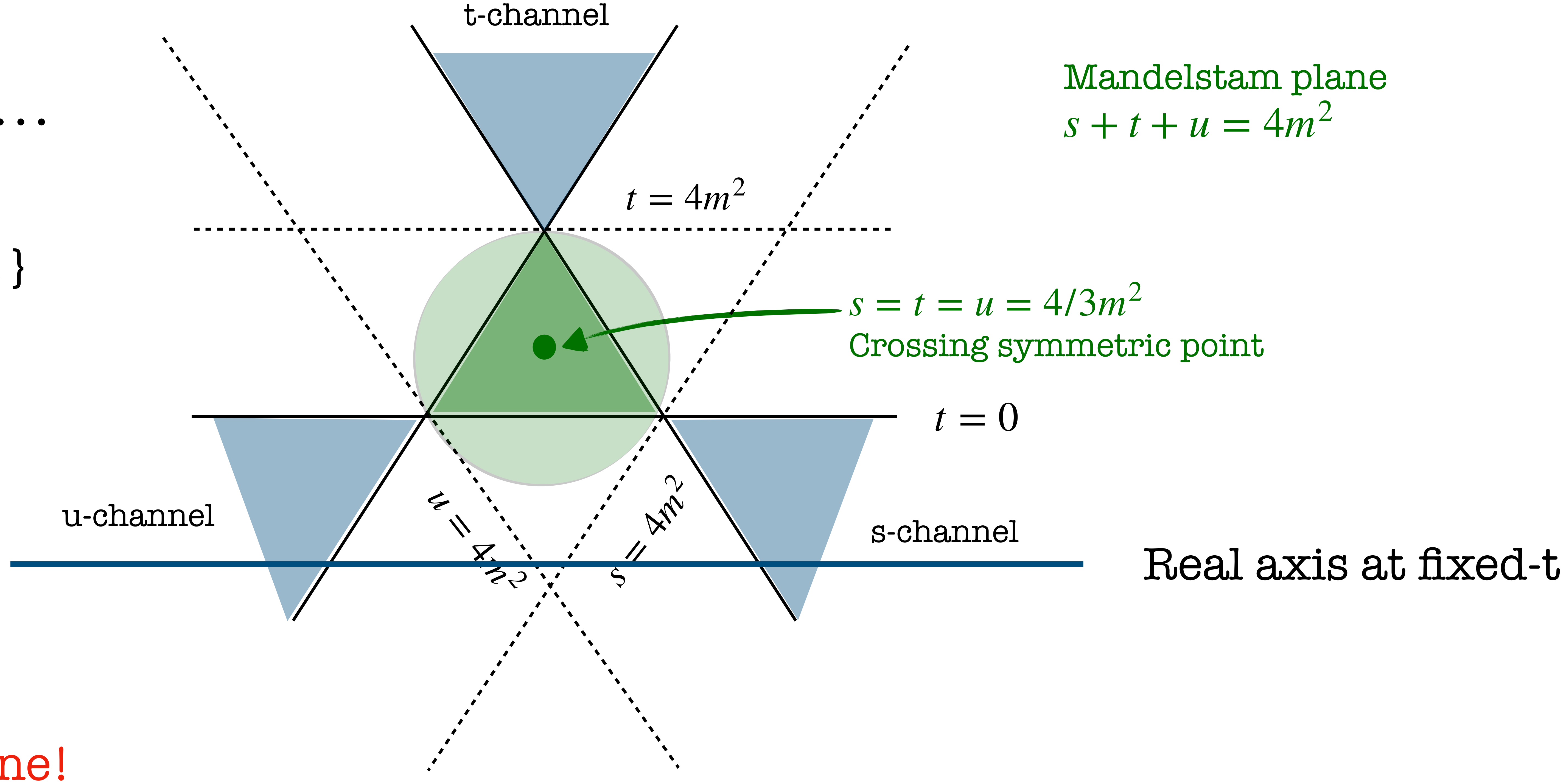
The set $\{c_0, c_2, c_3, \ldots\}$ parametrizes the space of amplitudes

S-matrix Data

Analytic properties

$$T(s,t,u) = -c_0 + c_2(\bar{s}^2 + \bar{t}^2 + \bar{u}^2) + c_3\bar{s}\bar{t}\bar{u} + \ldots$$

Space of amplitudes parametrized by $\{c_0, c_2, c_3, \ldots\}$

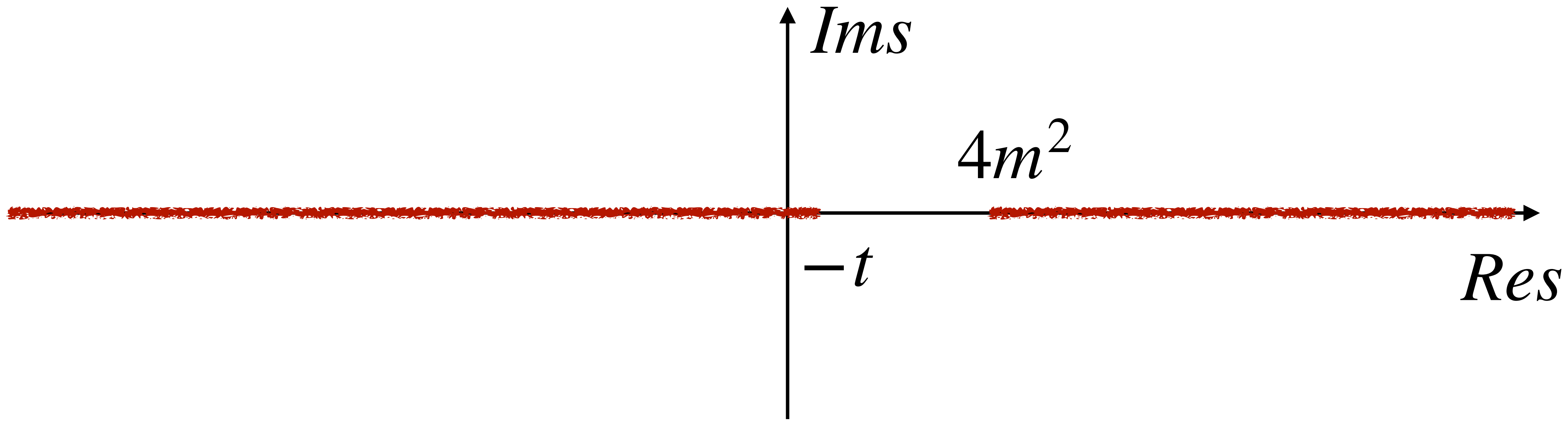


Mandelstam plane
$s + t + u = 4m^2$

Analyticity tells how to go into the complex plane!

Analytic in the s-plane away from the cuts for all $-28m^2 < t < 4m^2$

Martin, Jin, Froissart, Mandelstam, Lehmann, and many many others



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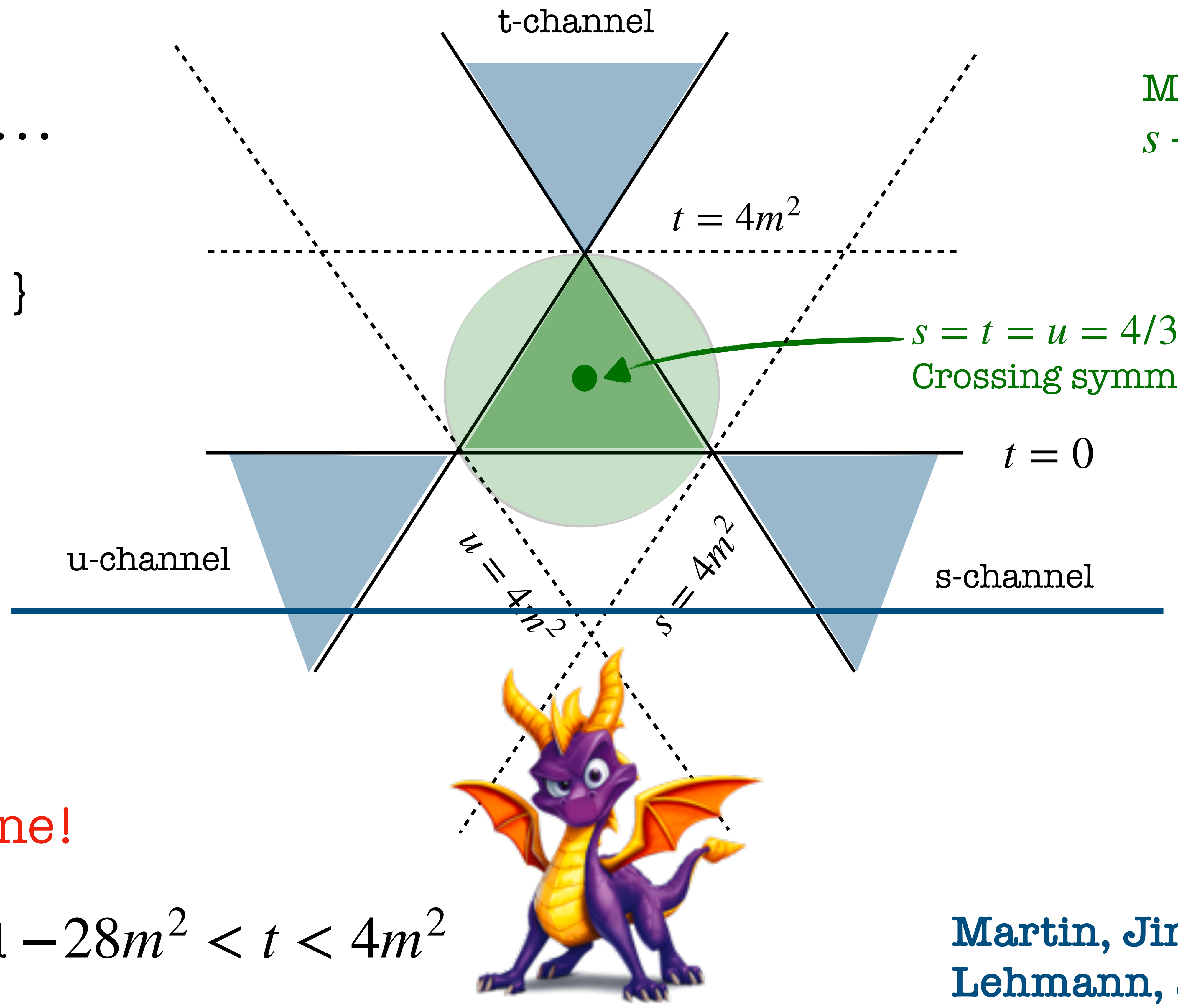

Mandelstam plane
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Real axis at fixed-t

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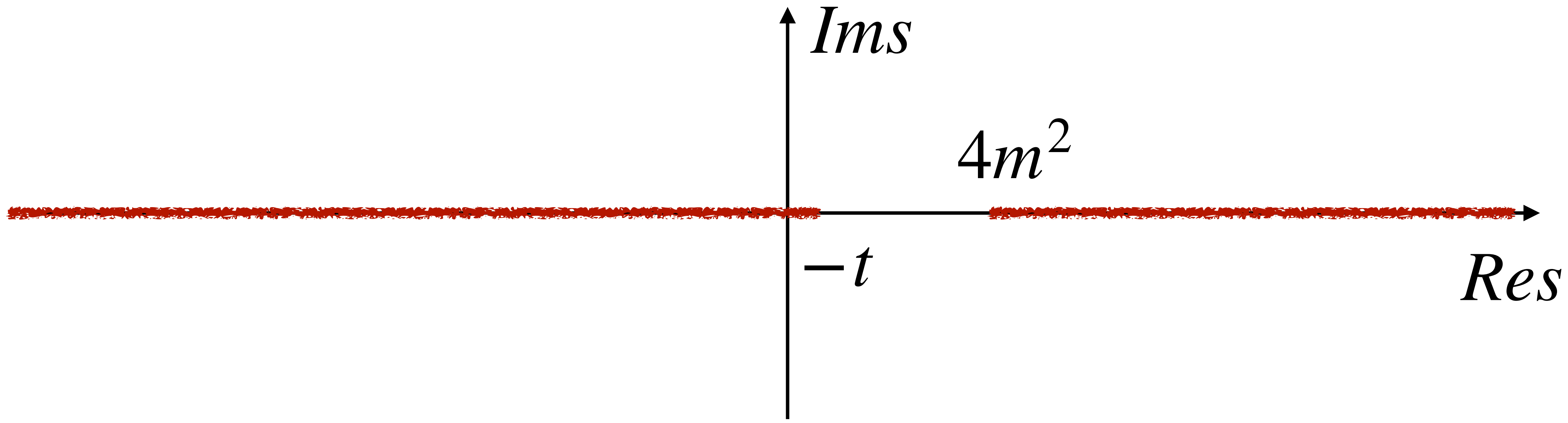

$t < -28m^2$: we hit the double discontinuity!

Correia, Sever, Zhiboedov 2111.12100
Tourkine, Zhiboedov 2303.08839

Unitarity

Froissart bound $$\lim_{s\to\infty} \frac{T(s, t < t_0)}{|s|^2} = 0$$

$$M(s,t,u) = -c_0 + c_2(\bar{s}^2 + \bar{t}^2 + \bar{u}^2) + c_3\bar{s}\bar{t}\bar{u} + \dots$$

Dispersive parameters ≡ operators of dimension ≥ 8

Non Dispersive

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Dispersive parameters ≡ operators of dimension ≥ 8

$$c_2 = \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{T_v(v, t_0)}{\bar{v}^3} \geq 0$$

$t_0 = s_0 = 4/3m^2$

Subtraction point

$$T_v(v, t_0) \equiv 16\pi \sum_{\ell=0}^{\infty} (2\ell + 1) Im f_\ell(s) P_\ell(1 + 2t_0/(s-4)) \geq 0$$

Positivity

Legendre positivity

$P_\ell(x) > 0, \quad x \geq 1$

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NOT POSITIVE!

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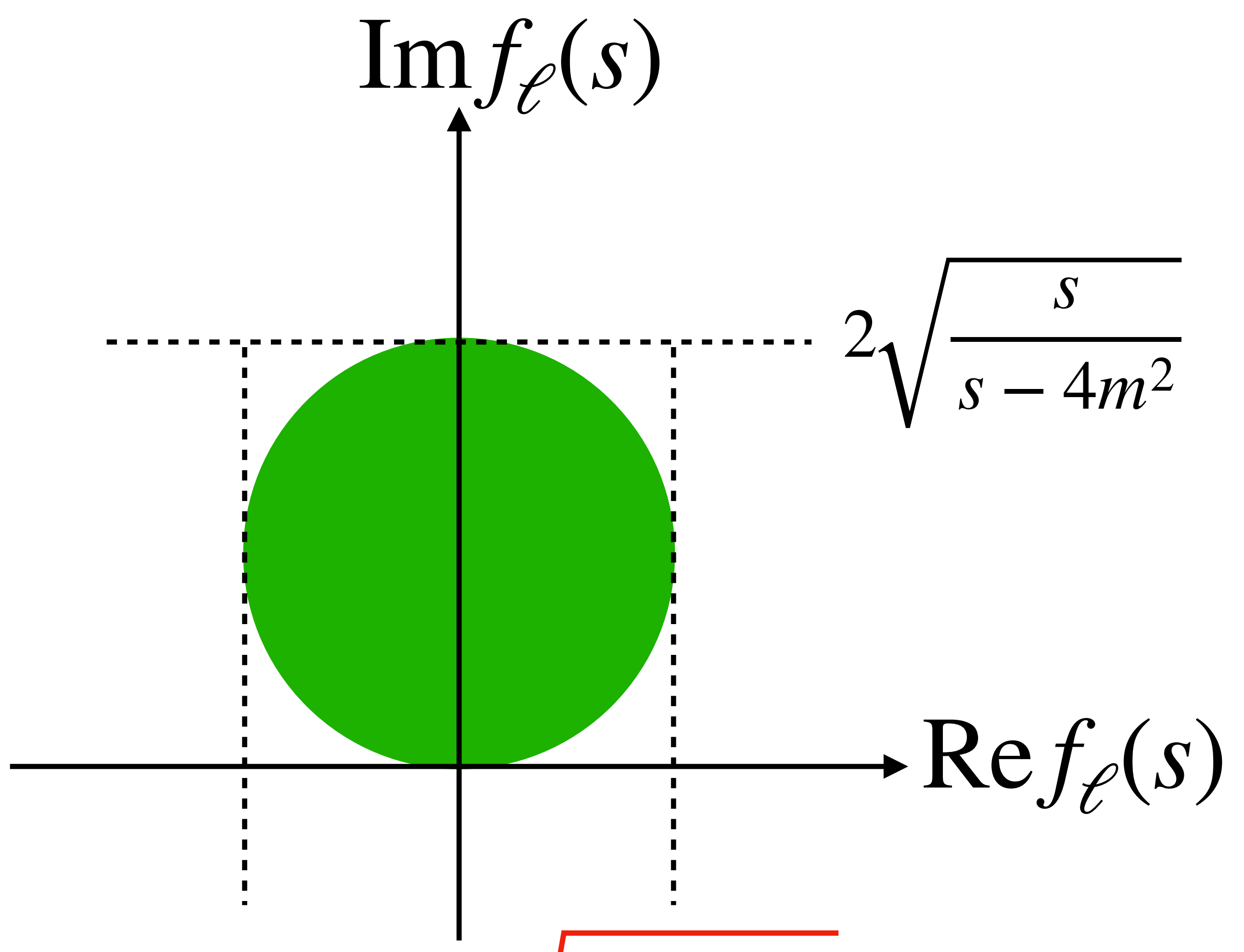
Unitarity saves the day!



Unitarity: $2Imf_\ell \geq \sqrt{\frac{s-4m^2}{s}} |f_\ell|^2$

The Island of 4d scalar amplitudes

We bound c_0, c_2 using dispersion relations and unitarity!
It is an exercise in constrained optimization theory.

Bonnier, Lopez, Mennessier, '70s

AG, Sever 2106.10257

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AG, Sever 2106.10257

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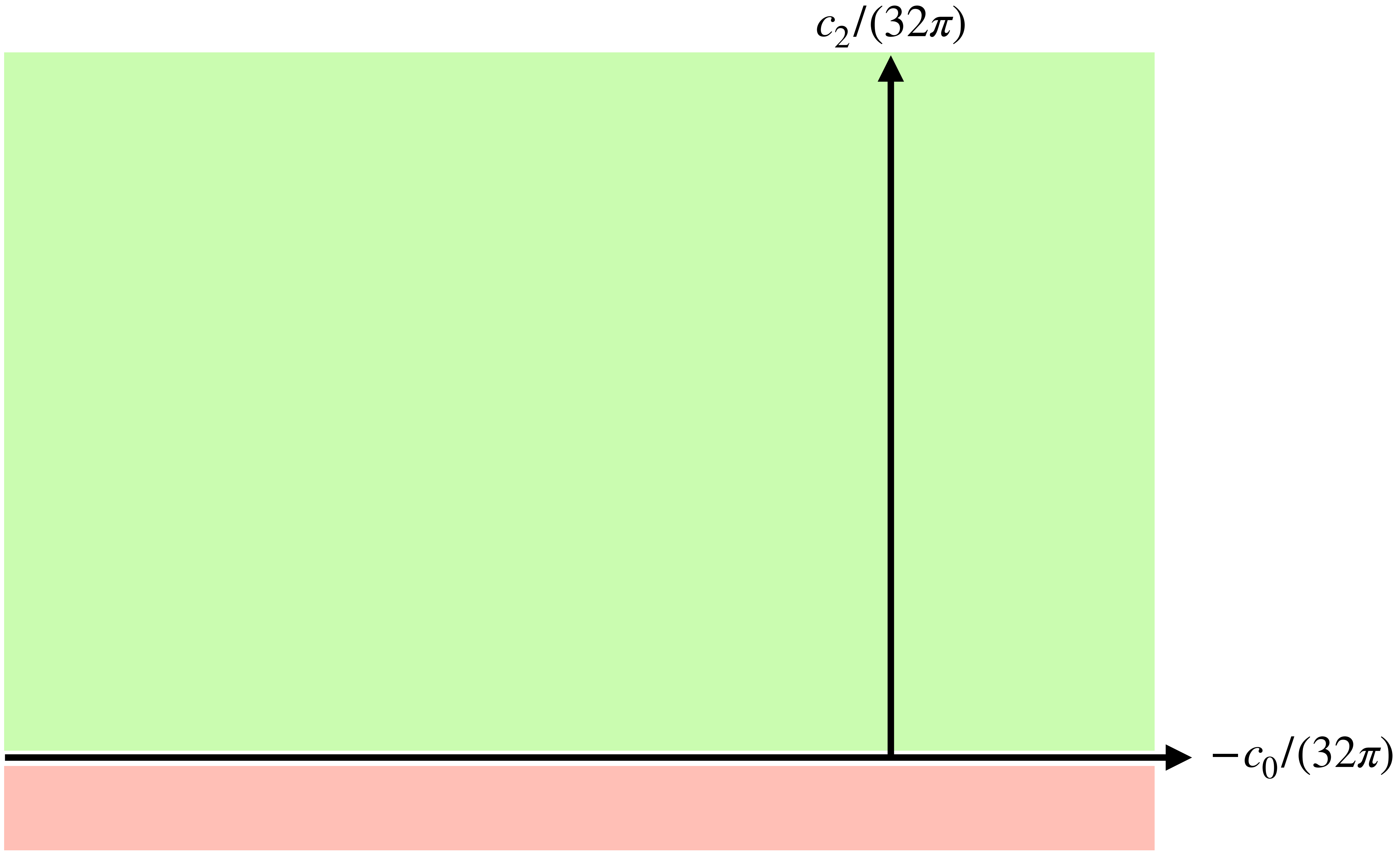

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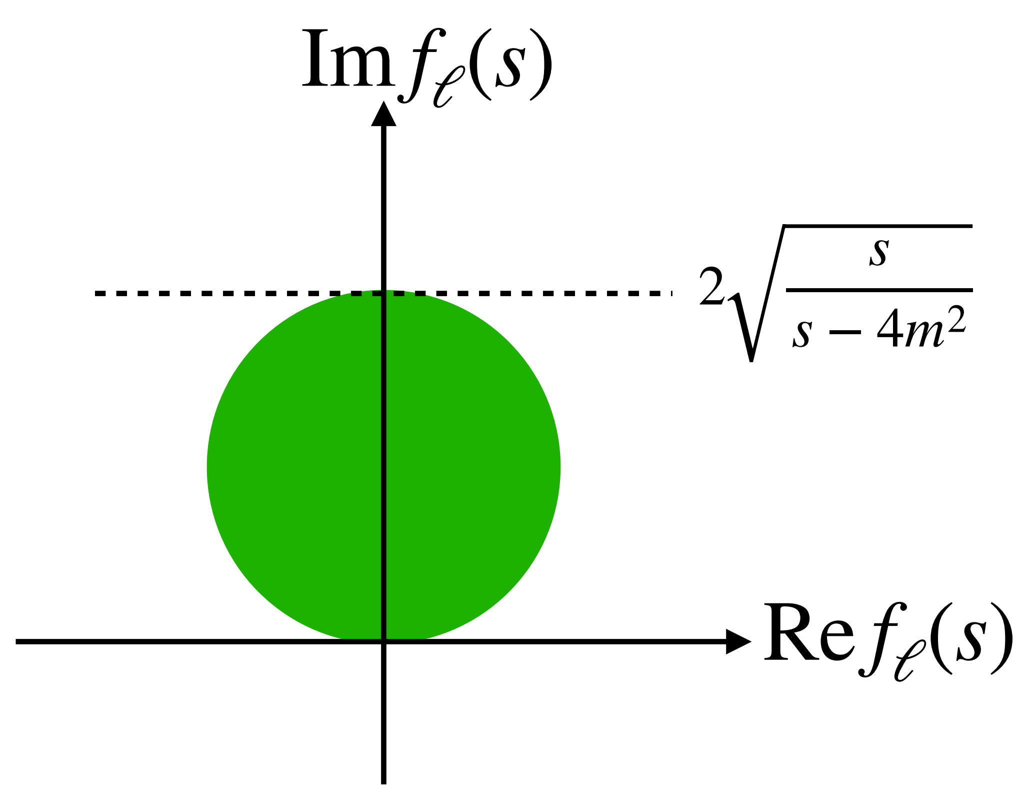
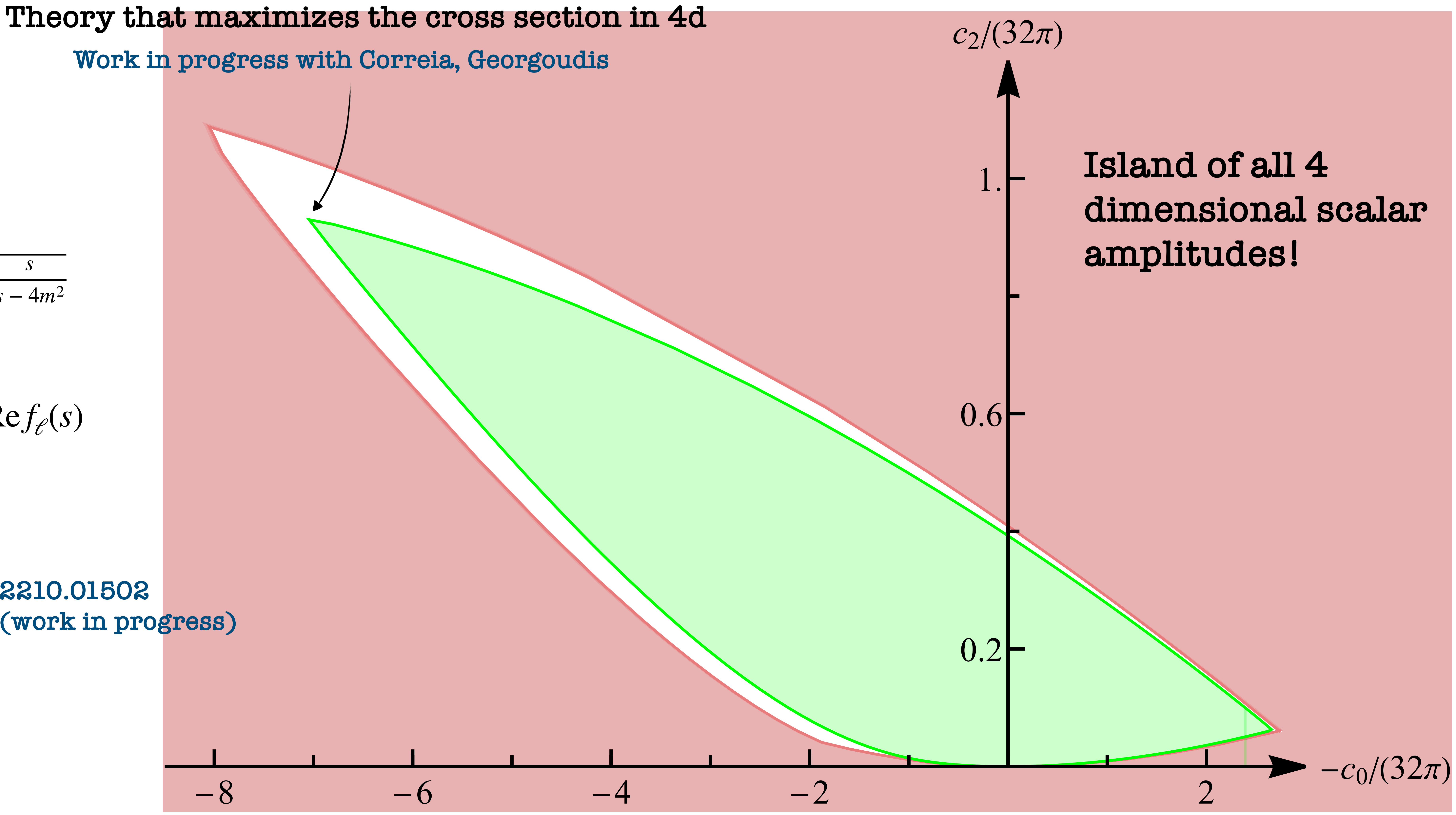


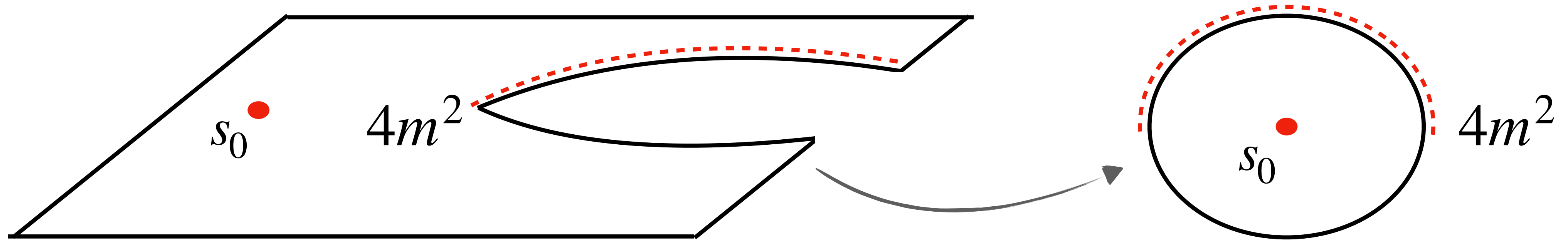
Elias-Miro, AG, Gumus 2210.01502
Elias-Miro, AG, Gumus (work in progress)



Methodology: example max c_0

PRIMAL

$$T(s,t,u) = \sum_{a,b,c}^{N_{max}} \alpha_{(abc)} \rho_s^a \rho_t^b \rho_u^c$$



$$\rho_s = \frac{\sqrt{4m^2 - s_0} - \sqrt{4m^2 - s}}{\sqrt{4m^2 - s_0} + \sqrt{4m^2 - s}}$$

MANIFEST CROSSING + MAXIMAL ANALYTICTY

$$S_\ell = 1 + i\sqrt{\frac{s-4}{s}} f_\ell(s)$$

$$|S_\ell|^2 \leq 1 \qquad s_{grid} > 4m^2, \quad \ell = 0,\ldots,L_{max}$$

Truncated set of semidefinite-positive constraints

$$N_{max} \to \infty, L_{max} \to \infty, s_{grid} \to s$$

Paulos, Penedones, Toledo, van Rees, Vieira '17

FIXED-t DUAL

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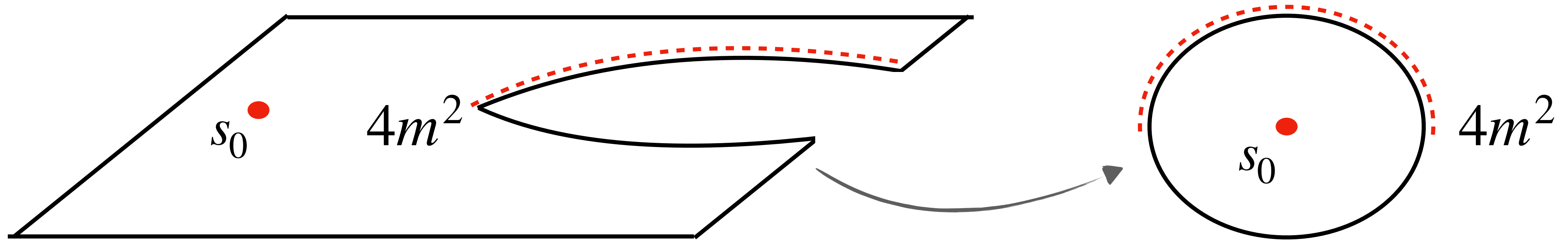

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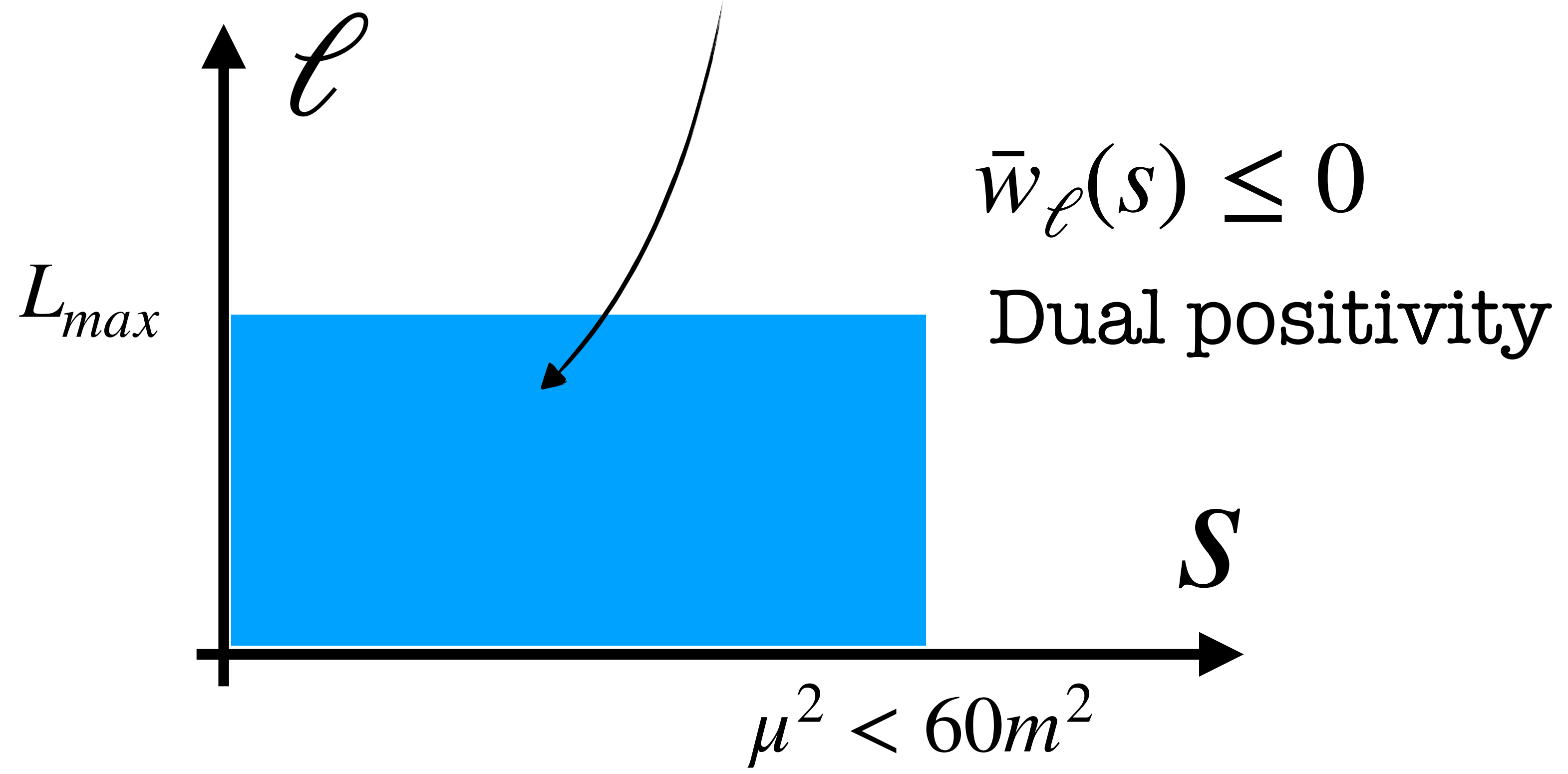
FIXED-t DUAL

$$W(s,t,u) = \sum_{\ell}^{L_{max}} P_\ell(1 + 2t/(s-4m^2)) w_\ell(s)$$

$$c_0 \le \sum_{\ell}^{L_{max}} \int_{4m^2}^{\mu^2 < 60m^2} \bar{w}_\ell(s) + \sqrt{\bar{w}_\ell(s)^2 + w_\ell(s)^2}$$



ALGORITHMICALLY AND THEORETICAL RIGOROUS

(Bounds follow from Wightman axioms)

BSM Applicaiton: Dimension 6 operators

The most enigmatic piece of the Standard Model!

$$m_H \approx 125.35\, GeV$$
$$\Gamma_H \approx 4\ MeV$$

ATLAS, CMS 4/7/2012

$$\frac{m_H}{\Gamma_H} \approx 3 \times 10^{-5}$$

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ATLAS, CMS 4/7/2012

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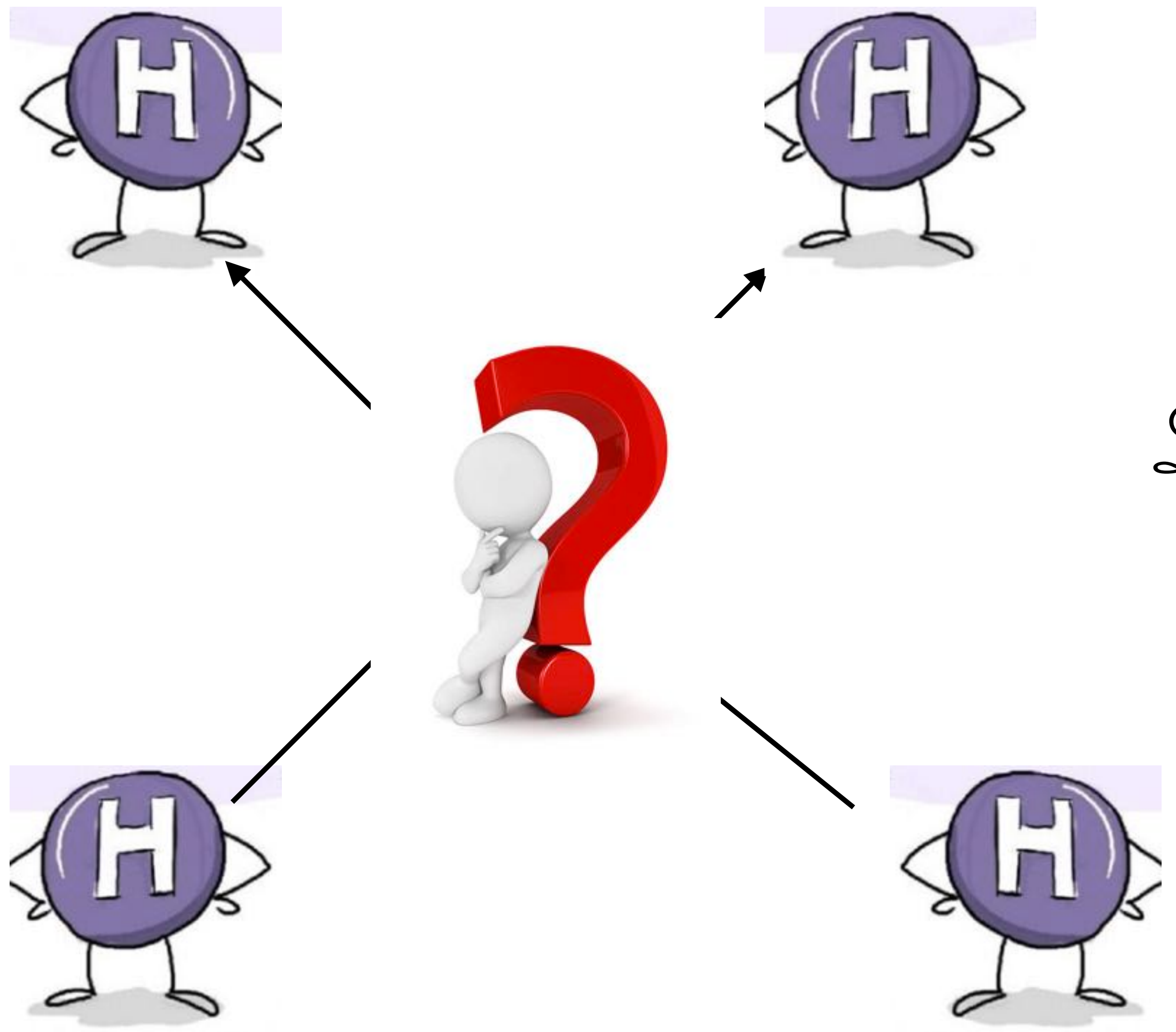
Assumptions same as SILH: $g_{SM} \ll g_{BSM}$, custodial symmetry $SO(4) \simeq SU(2)_L \times SU(2)_R$

Giudice, Grojean, Pomarol, Rattazzi hep-ph/0703164

$$H_i H_j \to H_k H_\ell$$



$$\mathscr{L}_H \supset \frac{g_H}{2f^2} \partial_\mu |H|^2 \partial^\mu |H|^2 + \dots$$

Island of O(4) amplitudes

$$\frac{M}{(4\pi)^2} = c_\lambda + c_H \bar{s} + c_2 \bar{s}^2 + c_2'(\bar{t}^2 + \bar{u}^2) + \ldots$$

Dispersive parameters

Let's be rigorous, "Wightman axiom style"

$$c_H \frac{\pi}{3}(s-4) = \mathrm{Re} f_1^{(3)}(s) - \int_4^\infty dv\, k_{1,\ell}^{(3,J)}(s,v) \mathrm{Im} f_\ell^{(J)}(v)$$

NOT POSITIVE!

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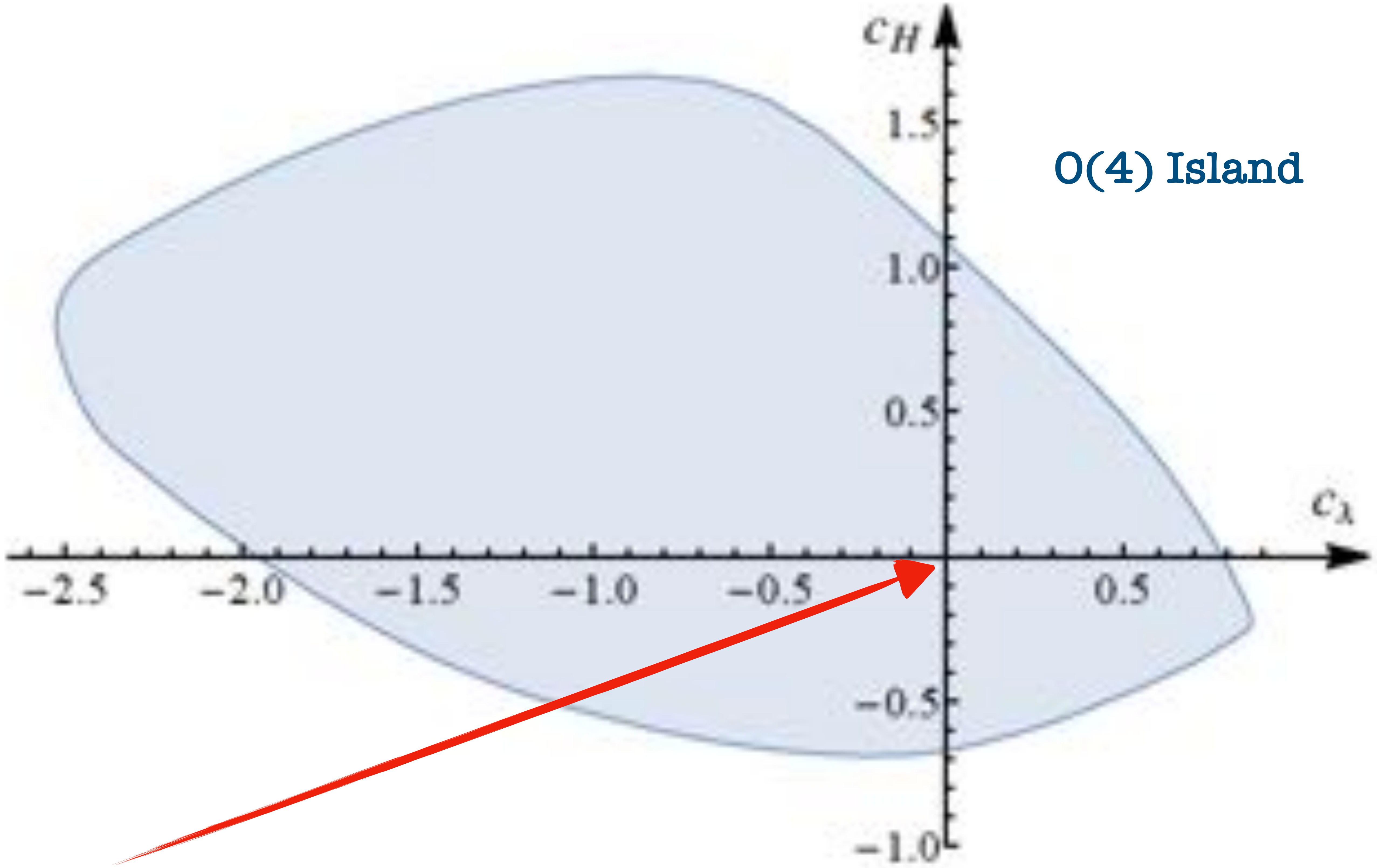


O(4) Island

Higgs coupling expected to be deep inside!

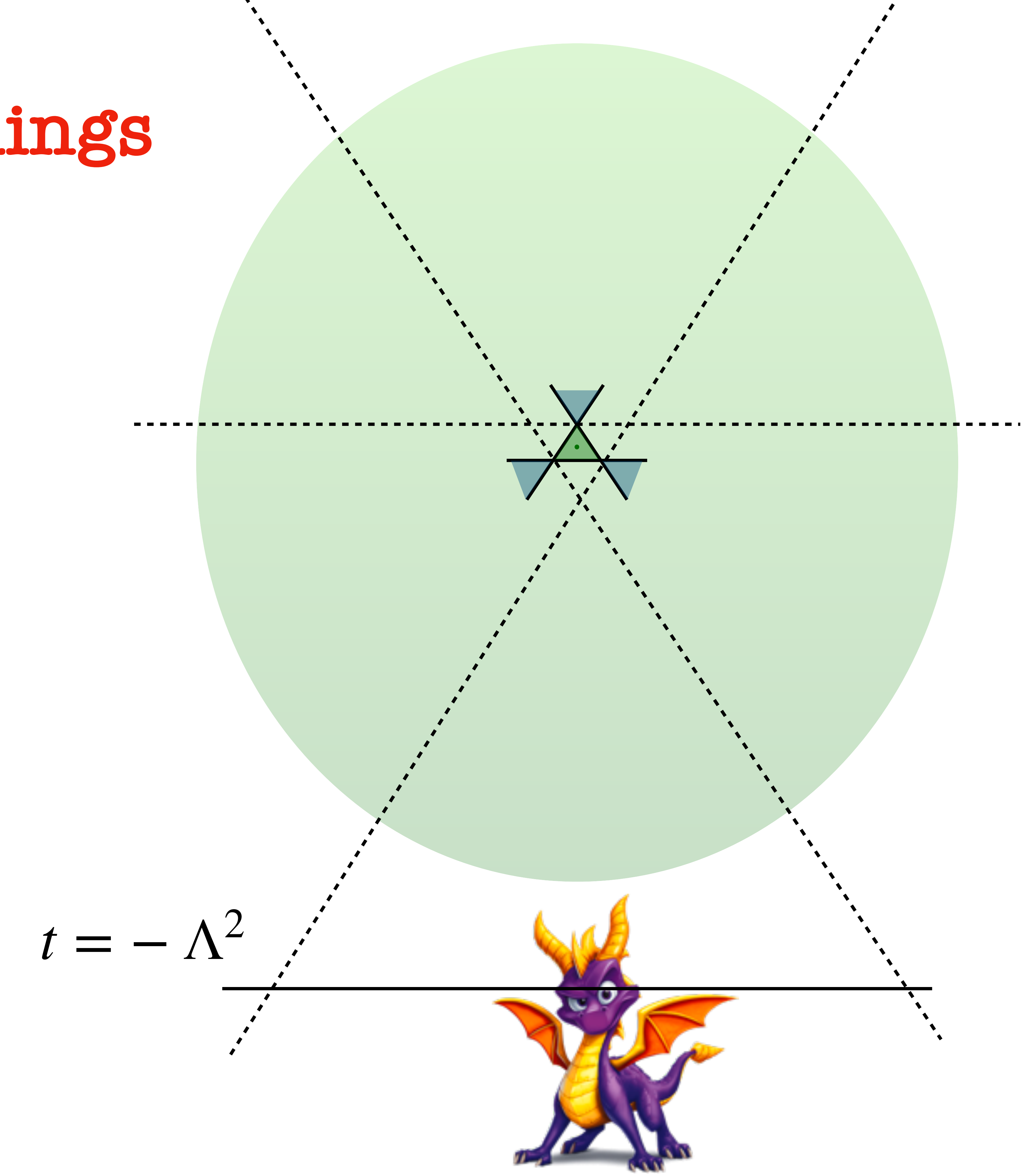
Elias-Miro, AG, Gumus 2311.09283

Bounds on Extremal EFT couplings

SILH Assumptions into the Bootstrap: $m_H \to 0, \quad c_\lambda \to 0$
Λ to sets the units

Technical assumption:
We consider only UV dragons (more perturbative analyticity)

$$\frac{\pi}{3} c_H s = \mathrm{Re} f_1^{(3)}(s) - \int_{\Lambda^2}^{\infty} dv\, k_{1,\ell}^{(3,J)}(s,v) \mathrm{Im} f_\ell^{(J)}(v)$$



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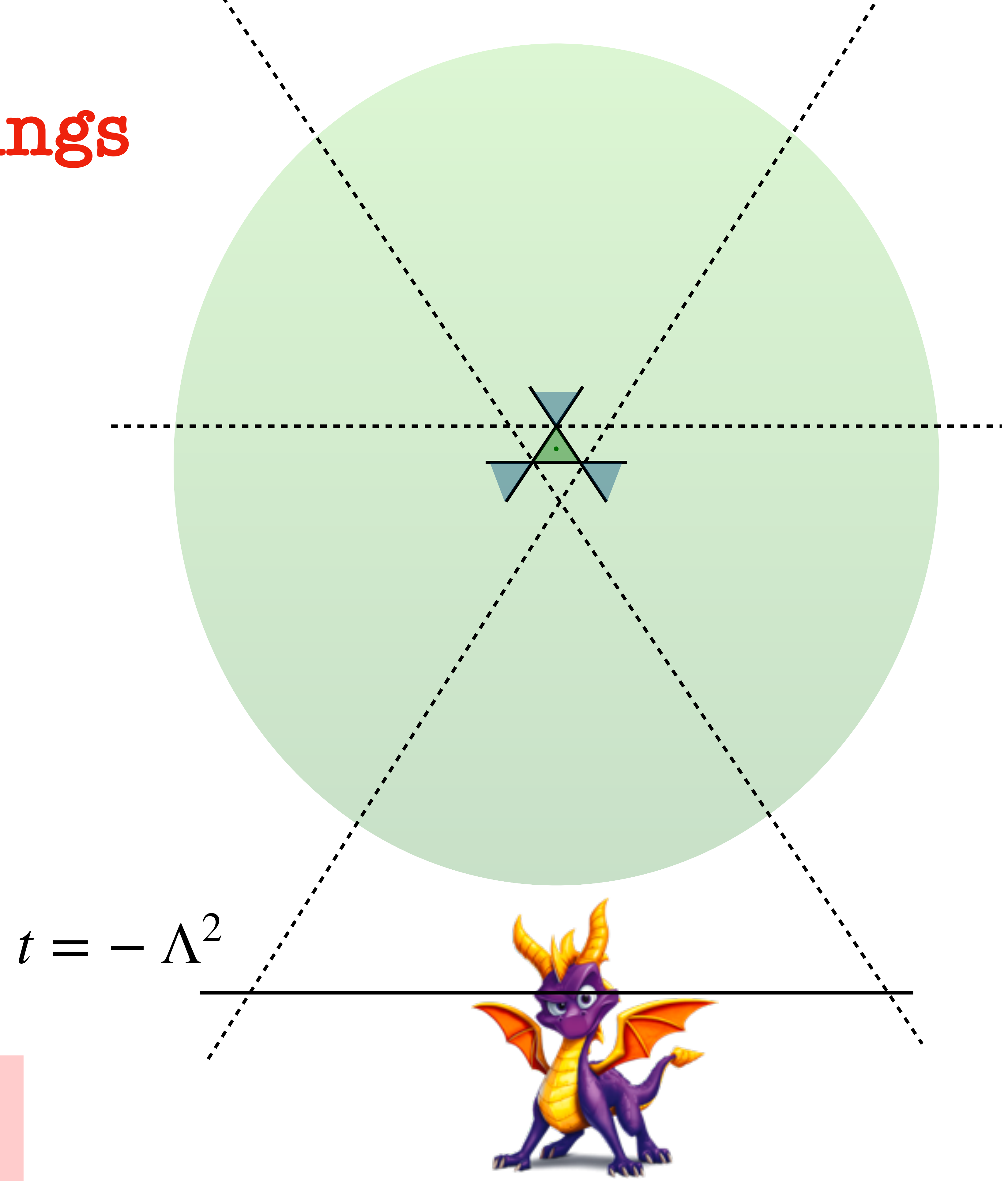

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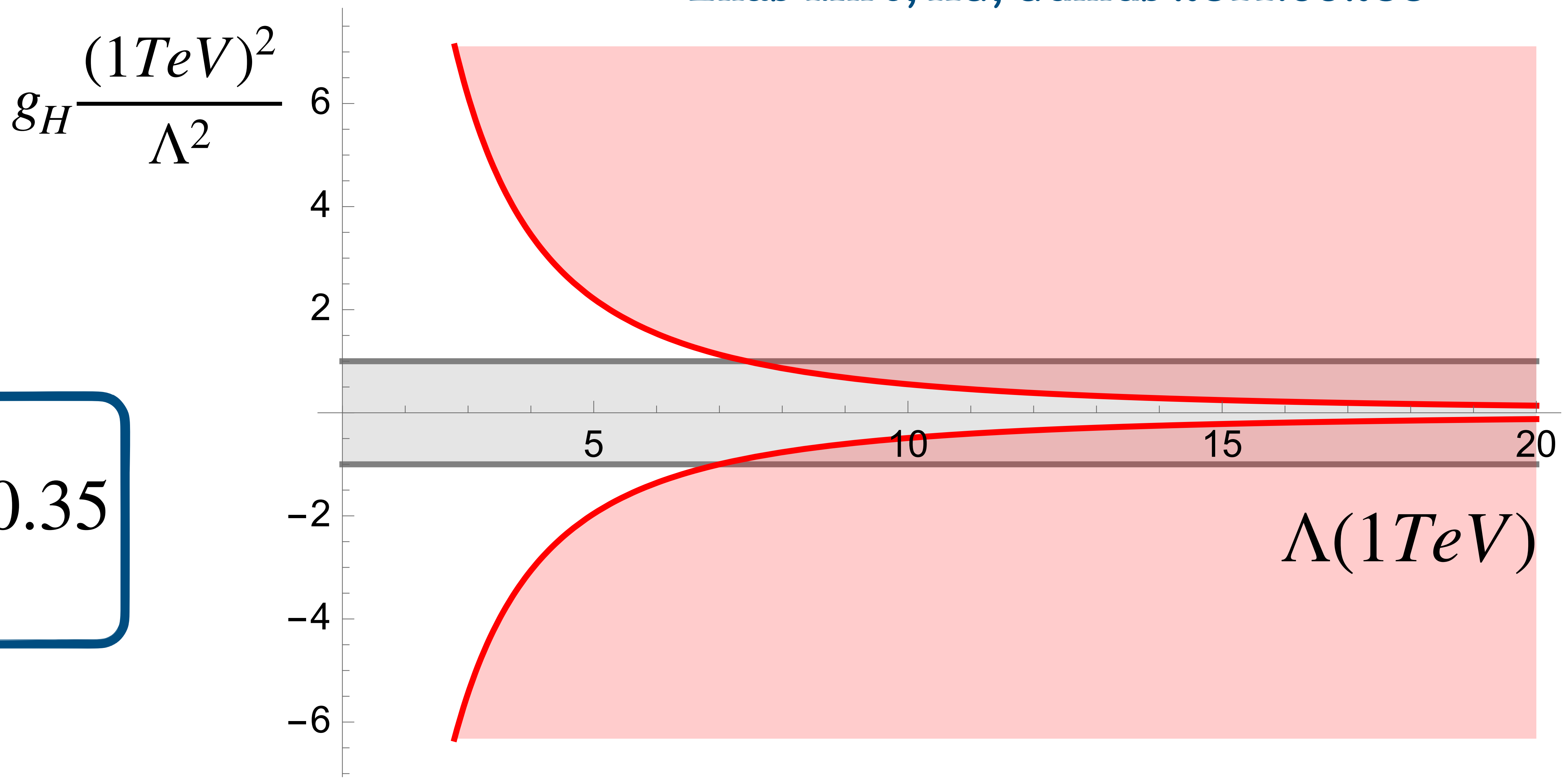
Elias-Miro, AG, Gumus 2311.09283



$$-0.31 < \frac{g_H}{(4\pi)^2} < 0.35$$

SMEFIT Collaboration 2105.00006

$$g_H \leq \frac{\Lambda^2}{(1TeV)^2}$$