Bootstrapping Mesons at Large N

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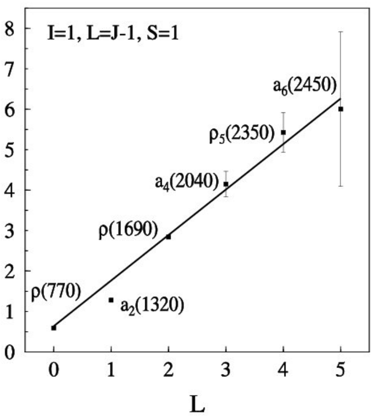

arXiv:2203.11950, 2307.01246 with *J.Albert*
& arXiv:2312.15013 with *J.Albert, J.Henriksson and A.Vichi*

Università di Pisa
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Carving out the space of large N confining gauge theories

A confining gauge theory at $N = \infty$ has an infinite tower of stable hadrons. Meromorphic S-matrix.

Consistency of 2-2 scattering imposes constraints on masses, spins and on-shell 3pt couplings.

Carve out this set of data: $\{m_k, J_k; \lambda_{ijk}\}$

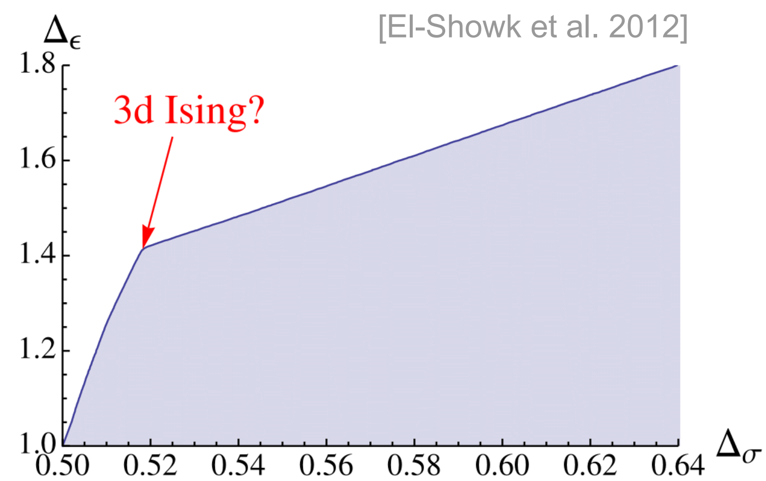
Large N S-matrix Bootstrap:

Carve out large N hadronic data from

- Crossing symmetry
- Unitarity
- Regge boundedness

Does large N QCD sit at a special place?

Conformal Bootstrap:



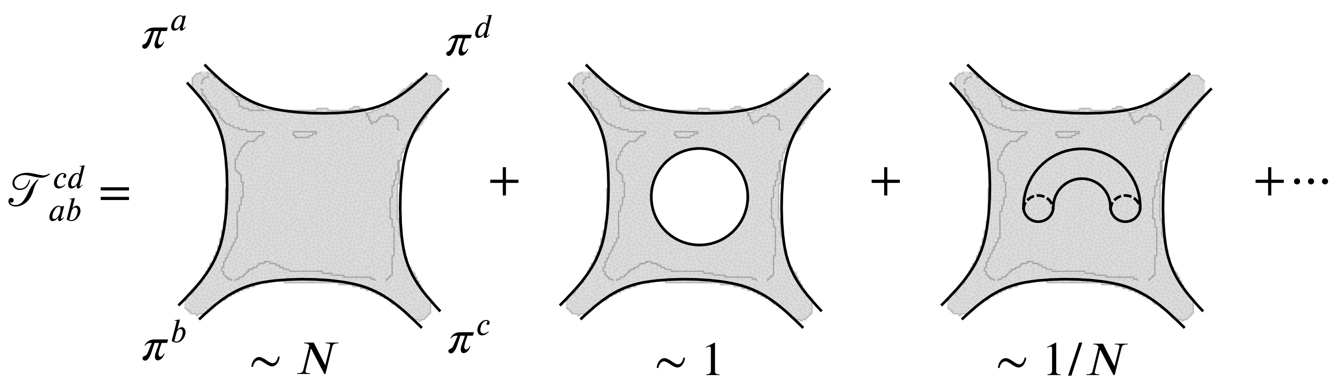
Large N QCD

$D = 4$ $SU(N)$ Yang-Mills with $N_f = 2$ massless quarks in the `t Hooft limit of fixed Λ_{QCD} .

A theory of glueballs, mesons and (heavy) baryons. Infinite tower of stable hadrons.

We focus on the **meson** sector: more constrained, and lots of data.

Pions π^a $=$ Goldstone bosons of $SU(2)_L \times SU(2)_R \to SU(2)_{\text{diag}}$

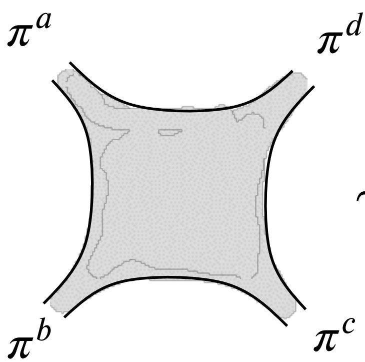


Reminiscent of string theory [`t Hooft], but we won't make any such dynamical assumption.

A new stab at this classic problem.

Modern theory space perspective & new EFT bootstrap methods ideally suited for this problem.

Pion Scattering at large N



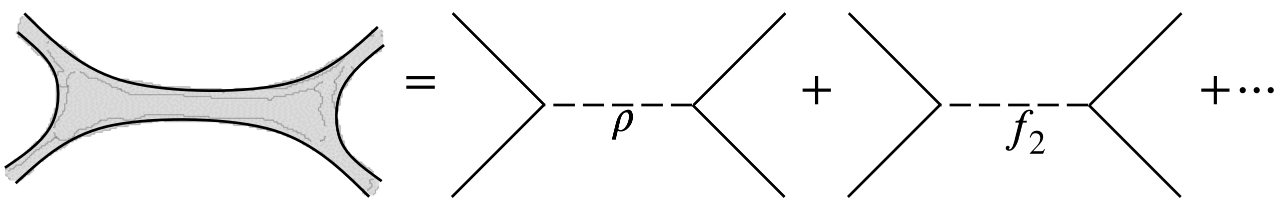
$\sim \mathrm{Tr}\left(\sigma^a\sigma^b\sigma^c\sigma^d\right)$

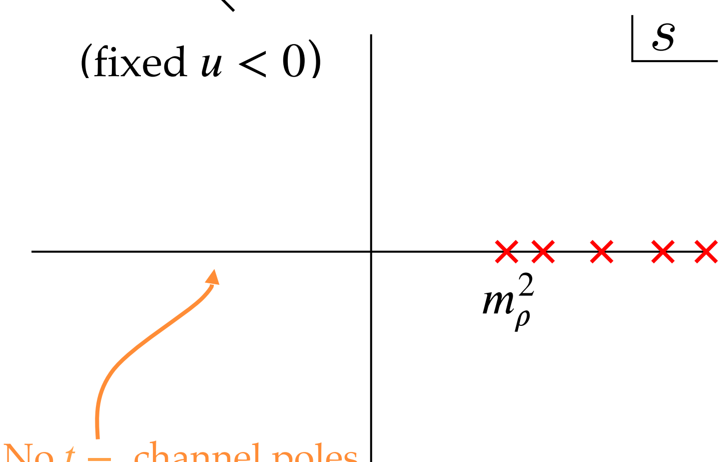
Basic amplitude

$$2\,\mathcal{T}_{ab}^{cd} = \mathrm{Tr}\left(\sigma^a\sigma^b\sigma^c\sigma^d\right)\, M(s,t) + \mathrm{Tr}\left(\sigma^a\sigma^b\sigma^d\sigma^c\right)\, M(s,u) + \mathrm{Tr}\left(\sigma^a\sigma^c\sigma^b\sigma^d\right)\, M(t,u)$$

Crossing symmetry: $M(s,u) = M(u,s)$

Analytic structure: $M(s,u) = \sum$ mesons poles $=$ meromorphic function





Isospin

$\pi\pi: \quad I = 1\times 1 = 0+1+2$

$q\bar{q}: \quad I = 1/2\times 1/2 = 0+1$

Isospin-two channel

$M^{(2)}(s\,|\,t,u) = 2M(t,u)$

OZI rule: No $t-$ channel poles

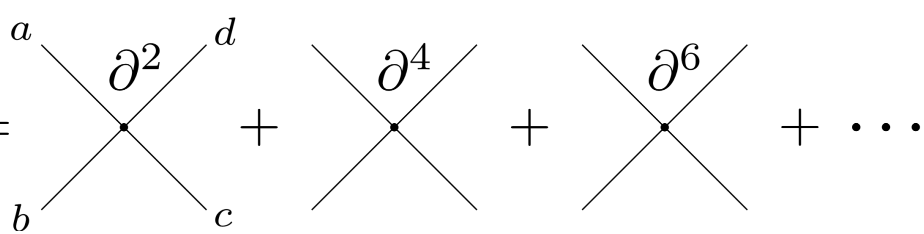
Effective Field Theory

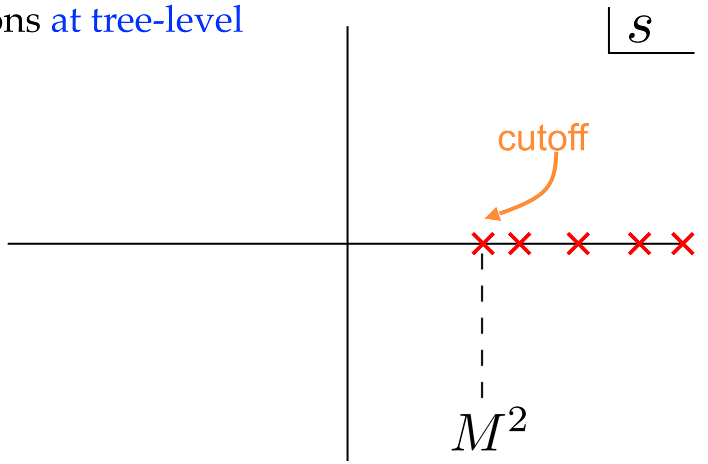
At low energies ($E < M = m_\rho$), we can use EFT, the standard chiral Lagrangian for $U(x) = e^{\frac{i}{f_\pi}\sigma^a \pi^a(x)}$

$$\mathscr{L}_{\rm Ch} = -\frac{f_\pi^2}{4}\left(\partial_\mu U^\dagger \partial_\mu U\right) + \text{higher derivatives}$$

At large N, $\mathscr{L}_{\rm ch}$ arises integrating out the heavy exchanged mesons at tree-level

$\mathcal{T}_{ab}^{cd} =$





$$M_{\rm low}(s,u) = g_{1,0}(s+u) + g_{2,0}\left(s^2+u^2\right) + 2g_{2,1}su + \cdots$$

All $g_{n,\ell} \sim \frac{1}{N}$, EFT is weakly-coupled

$$g_{1,0} \sim \frac{1}{f_\pi^2}$$

Parametrize theory space by $\{g_{n,\ell}\}$

Three Assumptions

Crossing symmetry: $M(s,u) = M(u,s)$

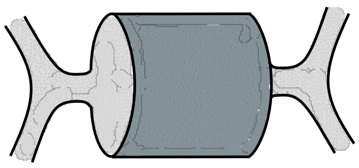
Unitarity → Positivity: $\text{Im}\, M(s,u) = \sum_J \rho_J(s)\, P_J\left(1 + \frac{2u}{s}\right)$

spectral density

Legendre polynomials

$$2 \geq \rho_J(s) \geq 0 \qquad (s > 0)$$

Regge behavior: $M(s,u) \sim s^{\alpha_0(u)}$ for $|s| \to \infty$ and fixed $u < 0$



- **At finite N:** Leading trajectory is the pomeron, with $\alpha_P(0) \sim 1.08$
- **At large N:** Leading trajectory is the rho meson trajectory, with $\alpha_\rho(0) \sim 0.5$

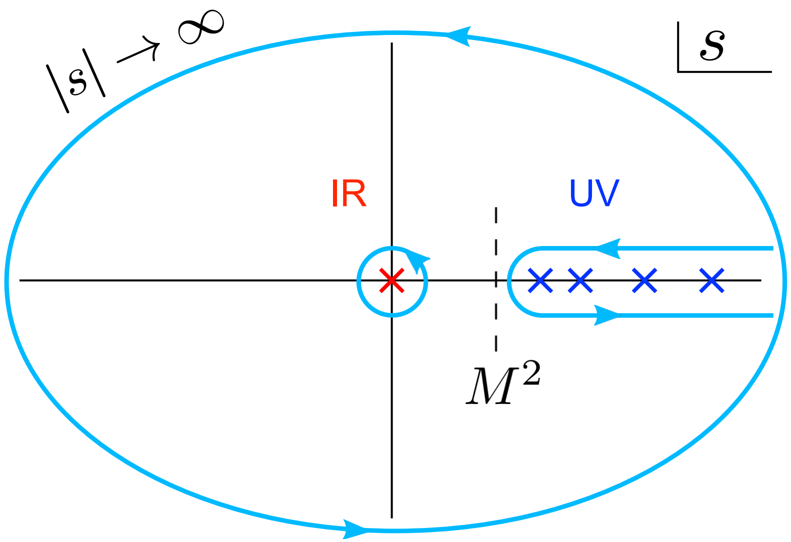
$$\lim_{|s|\to\infty} \frac{M(s,u)}{s} = 0 \qquad \lim_{|s|\to\infty} \frac{M(s,-s-u)}{s} = 0 \qquad (\text{fixed } u < 0)$$

Positivity Bounds

Arkani-Hamed T-C Huang Y-t Huang   Tolley, Wang & Zhou   Bellazzini Mirò Rattazzi Riembau Riva   Caron-Huot & Van Duong

Strategy: use dispersion relations to relate IR to UV.

$$\frac{1}{2\pi i}\oint_{\infty} \frac{ds'}{s'} \frac{M(s',u)}{s'^k} = 0 \qquad k = 1,2,\ldots$$



We get **sum rules** expressing the IR couplings $g_{n,\ell}$ in terms of the (unknown) UV spectral density $\rho_J(s) \geq 0$.

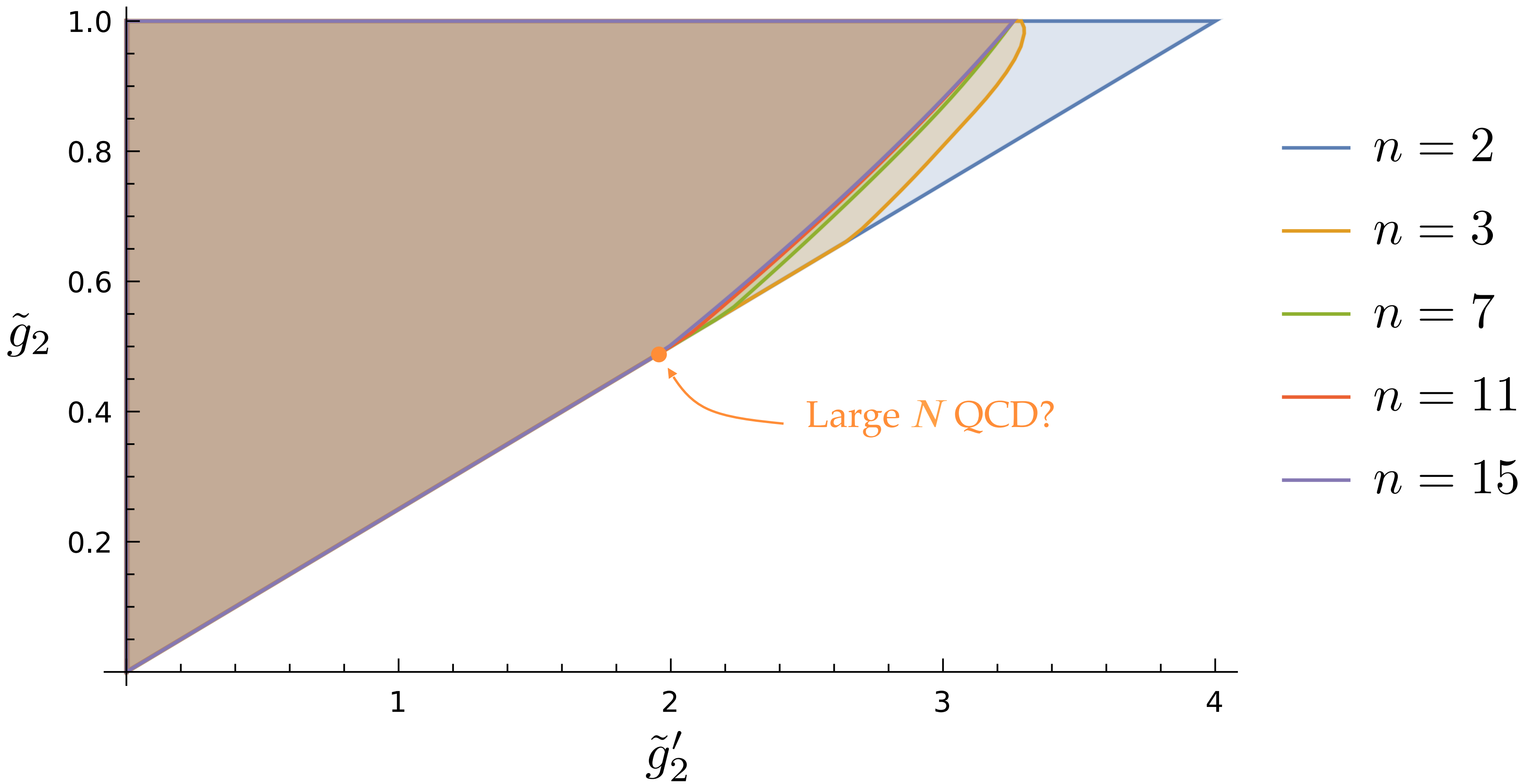
There are also **"null constraints"** for UV density that encode crossing symmetry.

Semidefinite programming can be used to derive **two-sided bounds** for homogeneous ratios of the couplings, in units of the cutoff $M = m_\rho$, such as

$$\tilde{g}_2 = \frac{g_{2,0}M^2}{g_{1,0}} \qquad \tilde{g}_2' = \frac{2g_{2,1}M^2}{g_{1,0}}$$

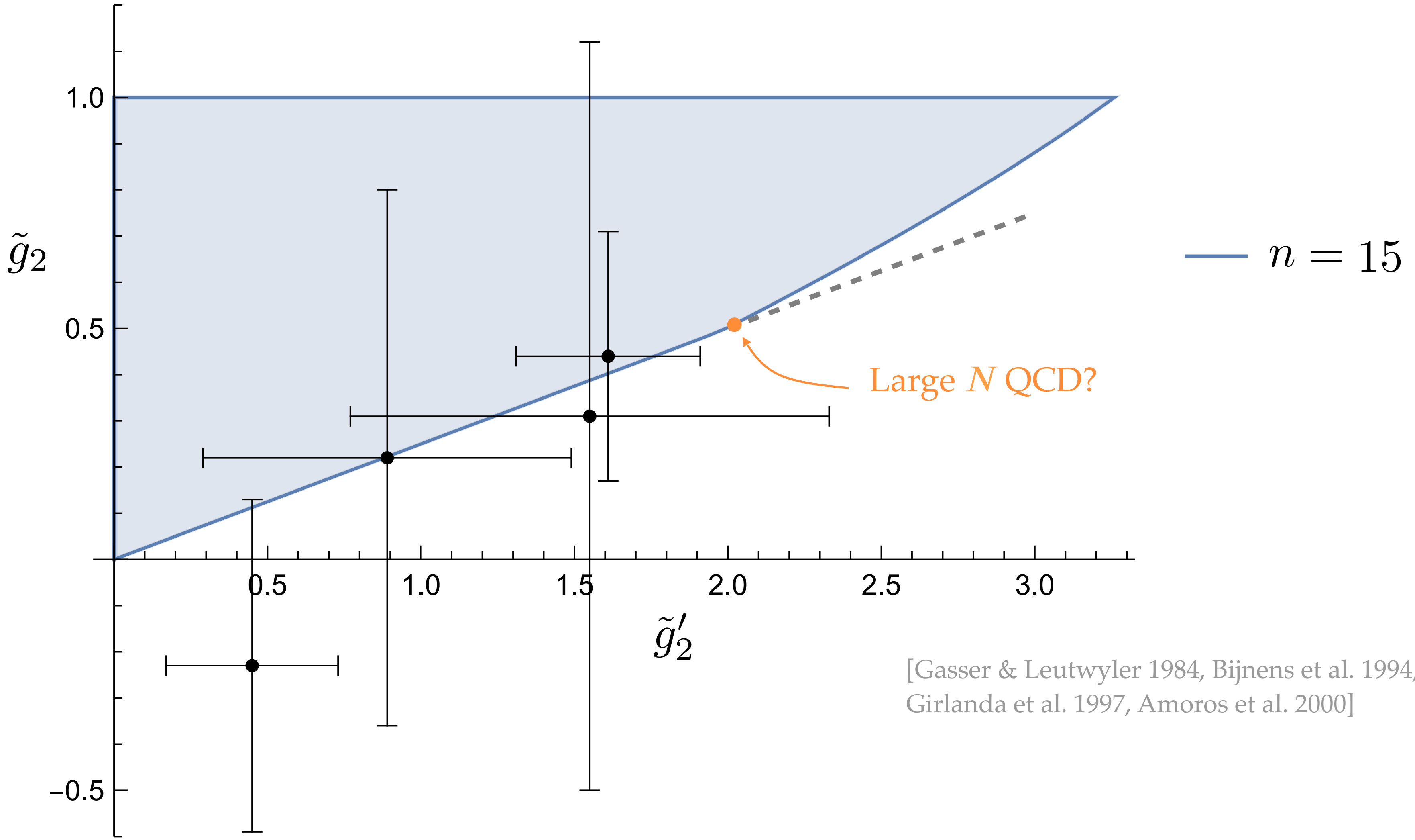
Exclusion plot

[Albert, LR, arXiv:2203.11950]



Allowed region in the space of two-derivative couplings.

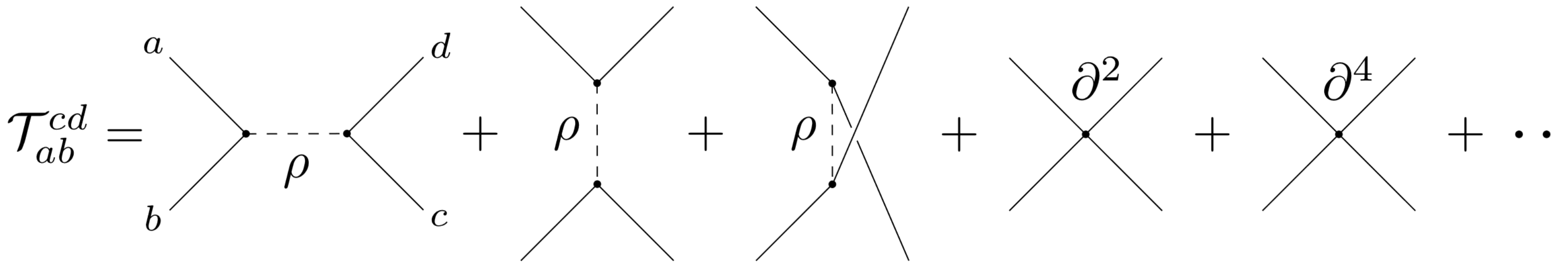
Healthy theories must lie in the colored region.



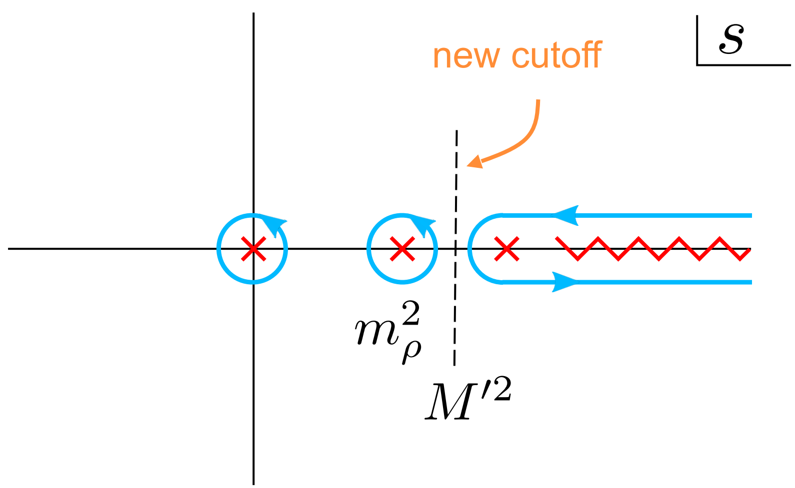
Comparison of the region allowed by unitarity to experiment.

Including the rho meson

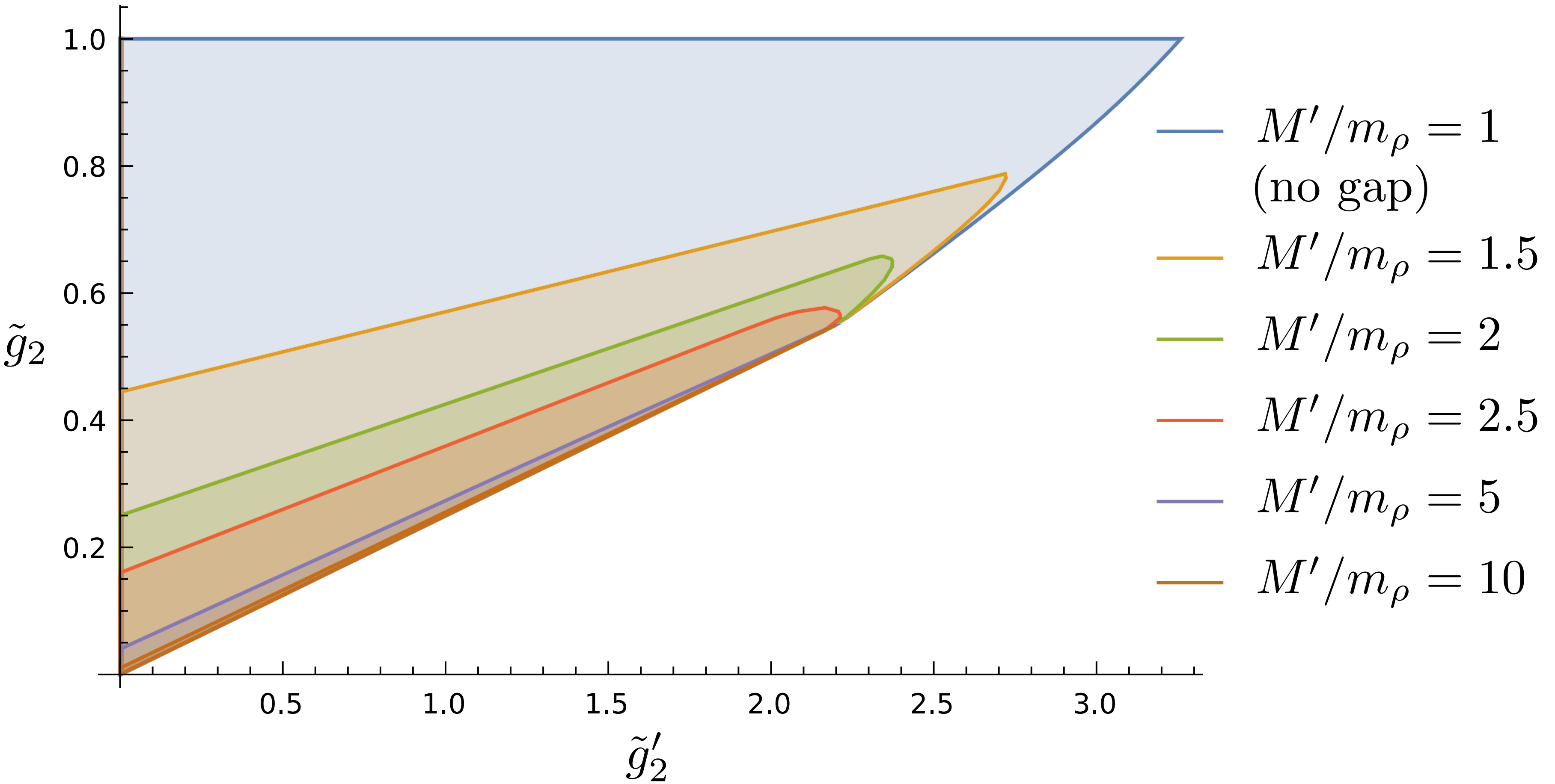
New EFT: We account for the ρ_μ^a, an isospin triplet of spin $J = 1$ and mass m_ρ.



$$M_{\text{low}}^{(\rho)}(s,u) = \frac{1}{2} g_{\pi\pi\rho}^2 \left(\frac{m_\rho^2 + 2u}{m_\rho^2 - s} + \frac{m_\rho^2 + 2s}{m_\rho^2 - u} \right) + \sum_{n=0}^{\infty} \sum_{\ell=0}^{[n/2]} \hat{g}_{n,\ell} \left(s^{n-\ell} u^{\ell} + u^{n-\ell} s^{\ell} \right)$$



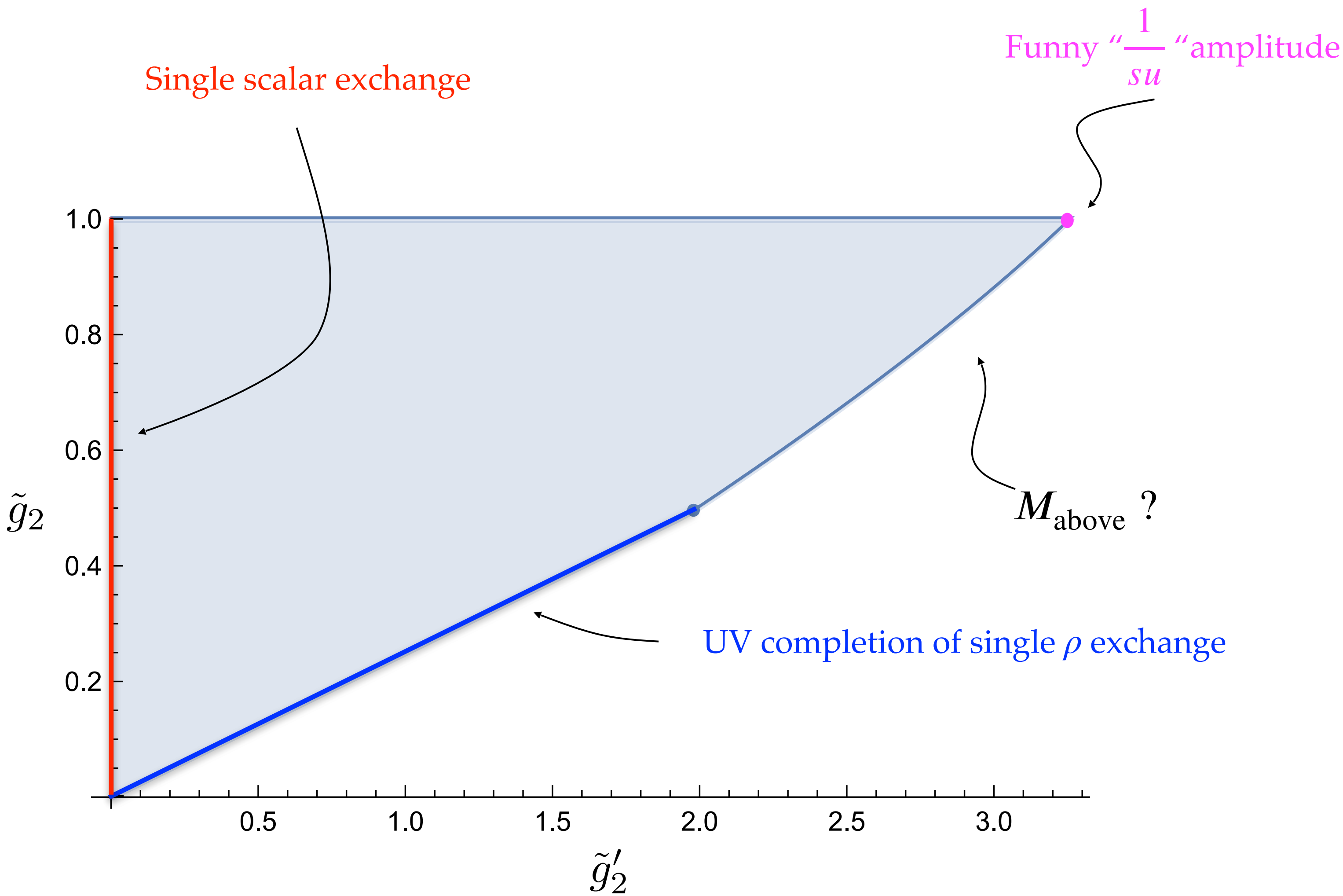
New exclusion plot



Allowed region in the space of two-derivative couplings, as a function of the gap above the rho meson. For reference, $m_{f_2}/m_\rho \cong 1.64$

Analytically ruling in

[Caron-Huot & Van Duong] [Albert, LR] [Fernandez et al]

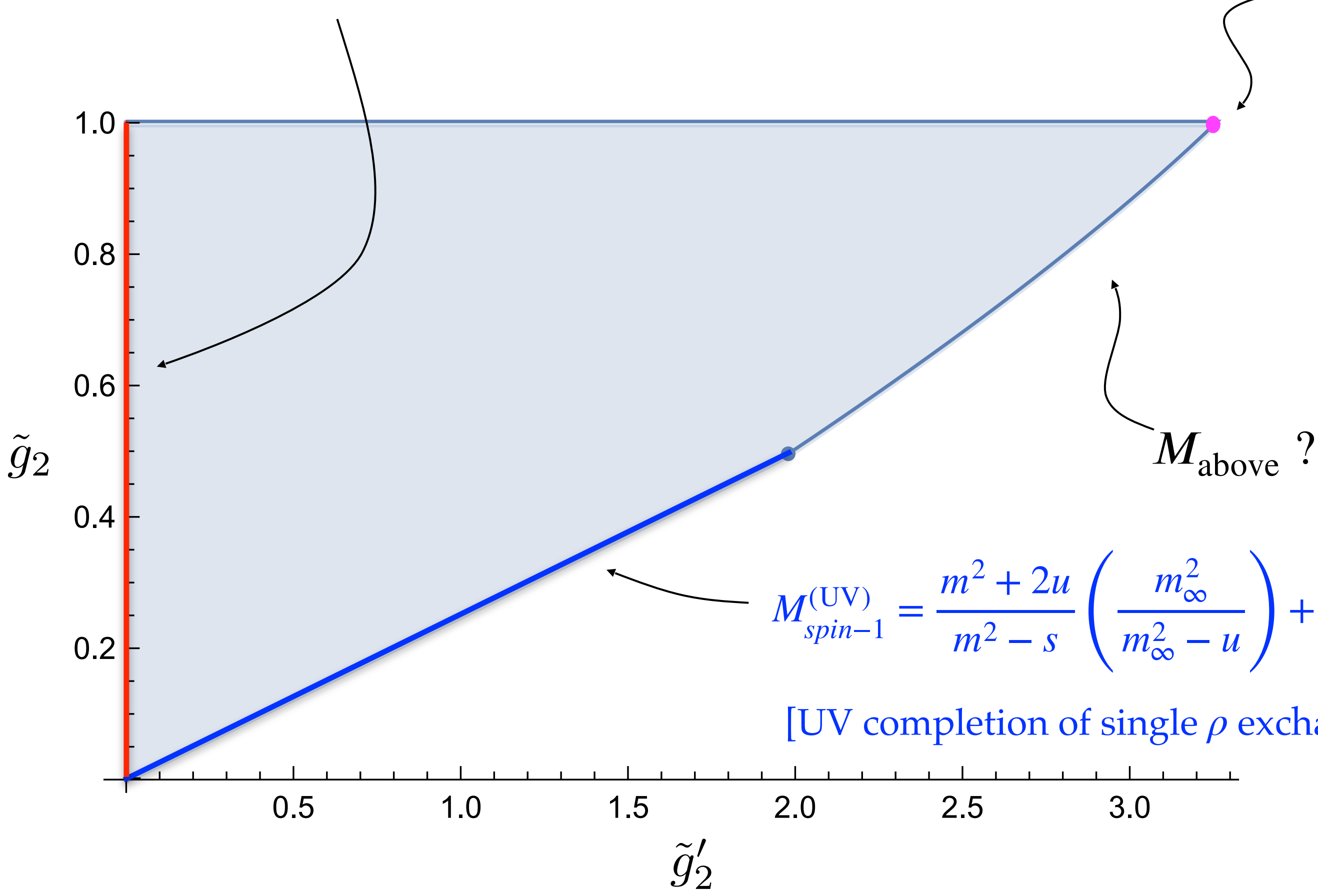


Simple solutions to crossing turn out to saturate (some of) the bounds.

Analytically ruling in

$$M_{spin-0} = \frac{m^2}{m^2 - s} + \frac{m^2}{m^2 - u}$$

$$M_{su-pole} = \frac{M^4}{(M^2 - s)(M^2 - u)} - \alpha_0 M_{spin-0}$$



$$M^{(\text{UV})}_{spin-1} = \frac{m^2 + 2u}{m^2 - s}\left(\frac{m_\infty^2}{m_\infty^2 - u}\right) + \frac{m^2 + 2s}{m^2 - u}\left(\frac{m_\infty^2}{m_\infty^2 - s}\right)$$

[UV completion of single ρ exchange]

The kink is perhaps explained by a change of dominance between two unphysical Ms.

Regge behavior and UV completion

Recall our Regge-limit assumption:

$$\lim_{|s|\to\infty} \frac{M(s,u)}{s} = 0 \qquad \text{(fixed } u < 0\text{)}$$

A spin-J exchange contributes $\sim s^J$ in the Regge limit.

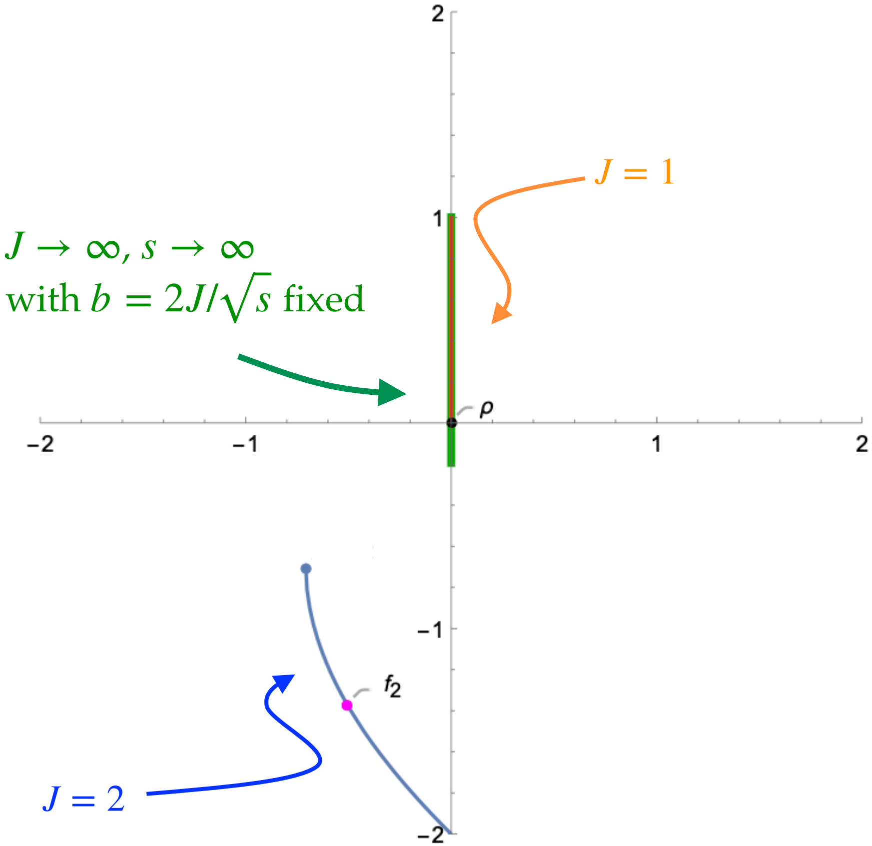
Intuitively, the possibility of a cheap UV completion of the single ρ exchange by states at very large mass is due to $J = 1$ being "marginally allowed".

We expect *no s*uch simple UV completion for states with $J > 1$. In fact we expect an infinite tower of higher-spins to be needed. [See the causality thought experiments of CEMZ]

We can make this precise with a graphical bootstrap.

Spin two (f_2 meson) *cannot* UV completed at $m_\infty \to \infty$

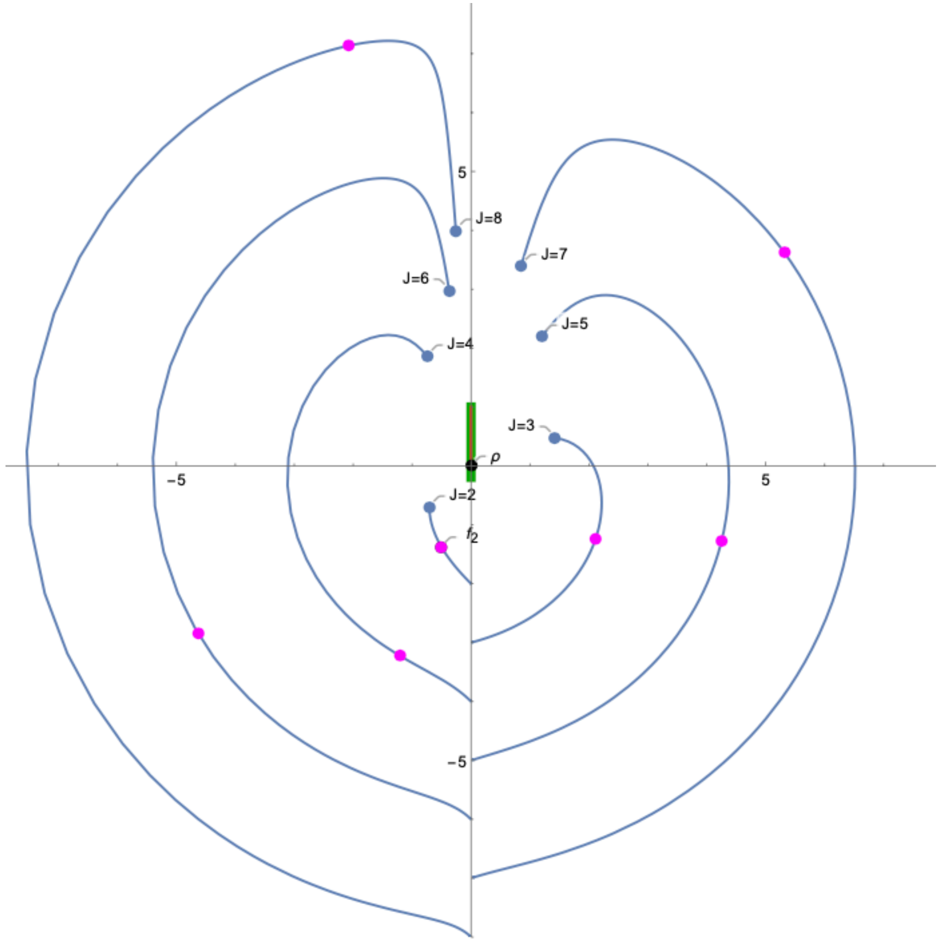
A slice in the space of null constraints:



No solution with just states at m_∞

[Albert Henriksson LR Vichi]

Spin two (f_2 meson) *cannot* UV completed at $m_\infty \to \infty$



Need states with odd spin at finite mass

[Albert Henriksson LR Vichi]

New strategy

[Albert Henriksson LR Vichi, arXiv:2312.15013]

Spectral assumptions:

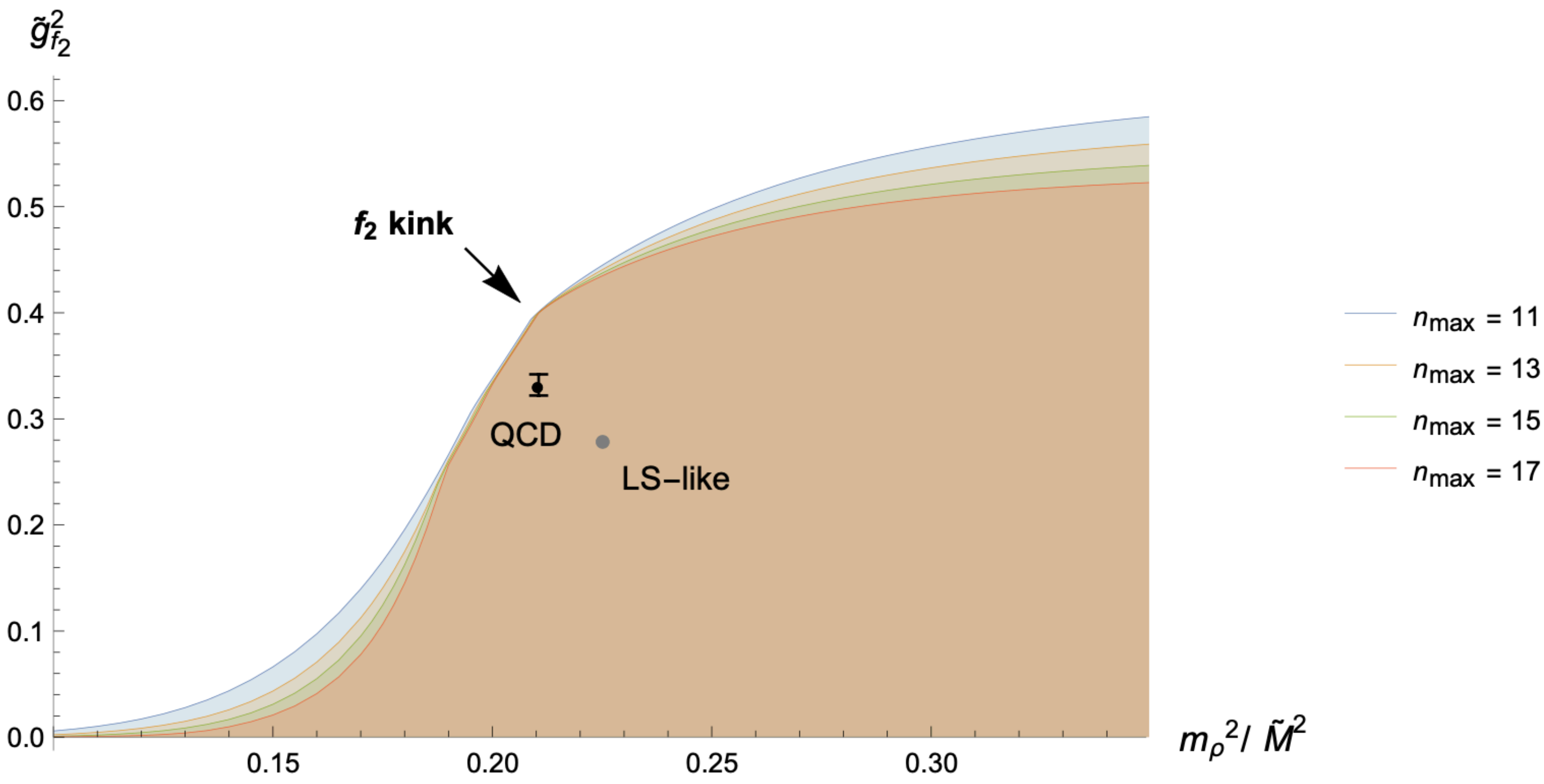
- 1st exchanged state: $J = 1$, mass m_ρ. Agnostic about $g_{\pi\pi\rho}$.
- 2nd exchanged state: $J = 2$, mass $m_{f_2} > m_\rho$. Fixed coupling $g_{\pi\pi f_2}$.
- New cut-off $\widetilde{M} \geq m_{f_2}$.

For definiteness we pick the physical value $\dfrac{m_{f_2}^2}{m_\rho^2} = (1.65)^2$.

We know from the graphical bootstrap that for any $g_{\pi\pi f_2} \neq 0$ we cannot push $\widetilde{M} \to \infty$.

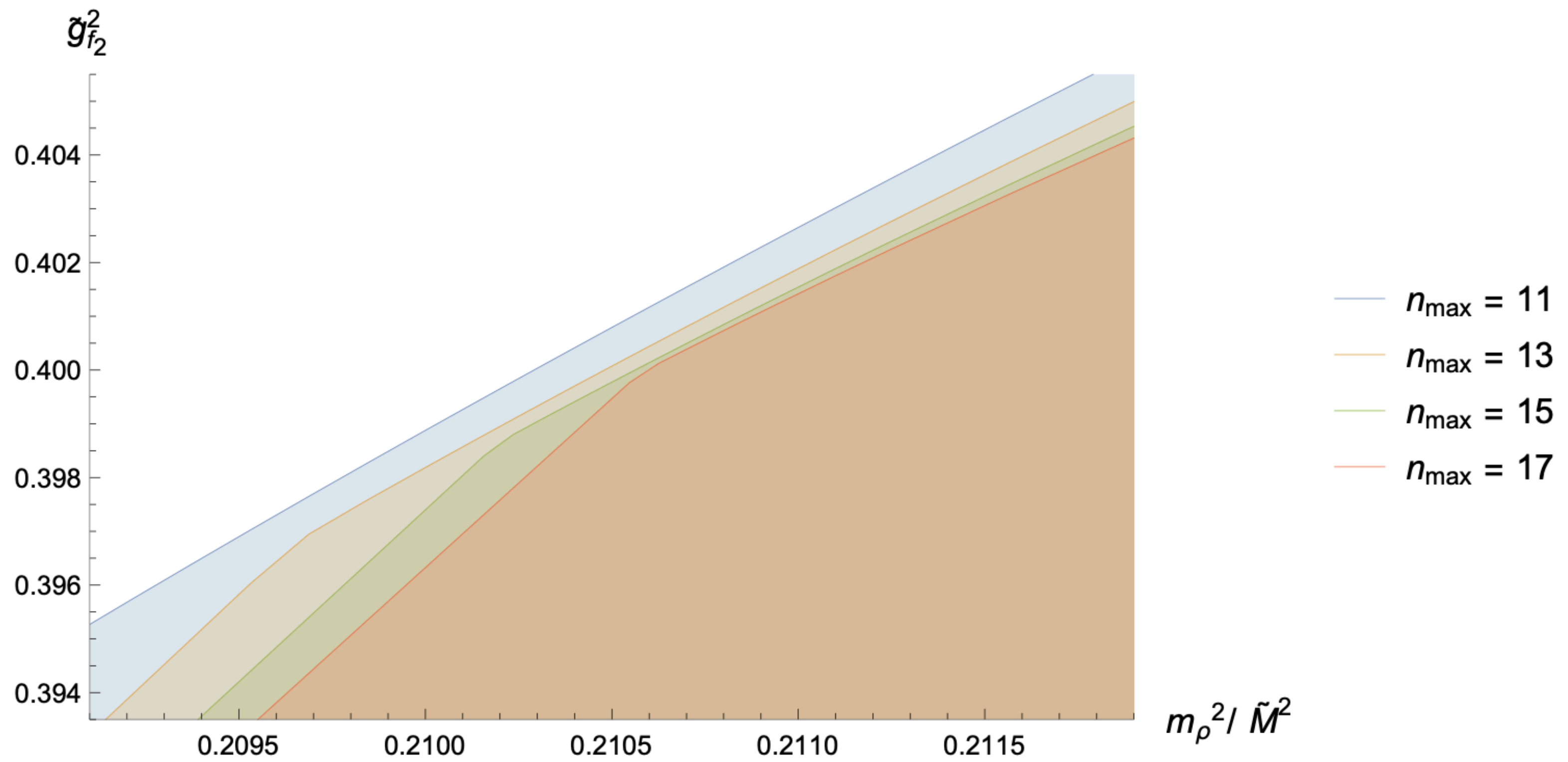
A new intriguing kink

[Albert Henriksson LR Vichi]



Numerically stable kink at a special value of $\widetilde{M}$. Novel extremal solution.

A new intriguing kink

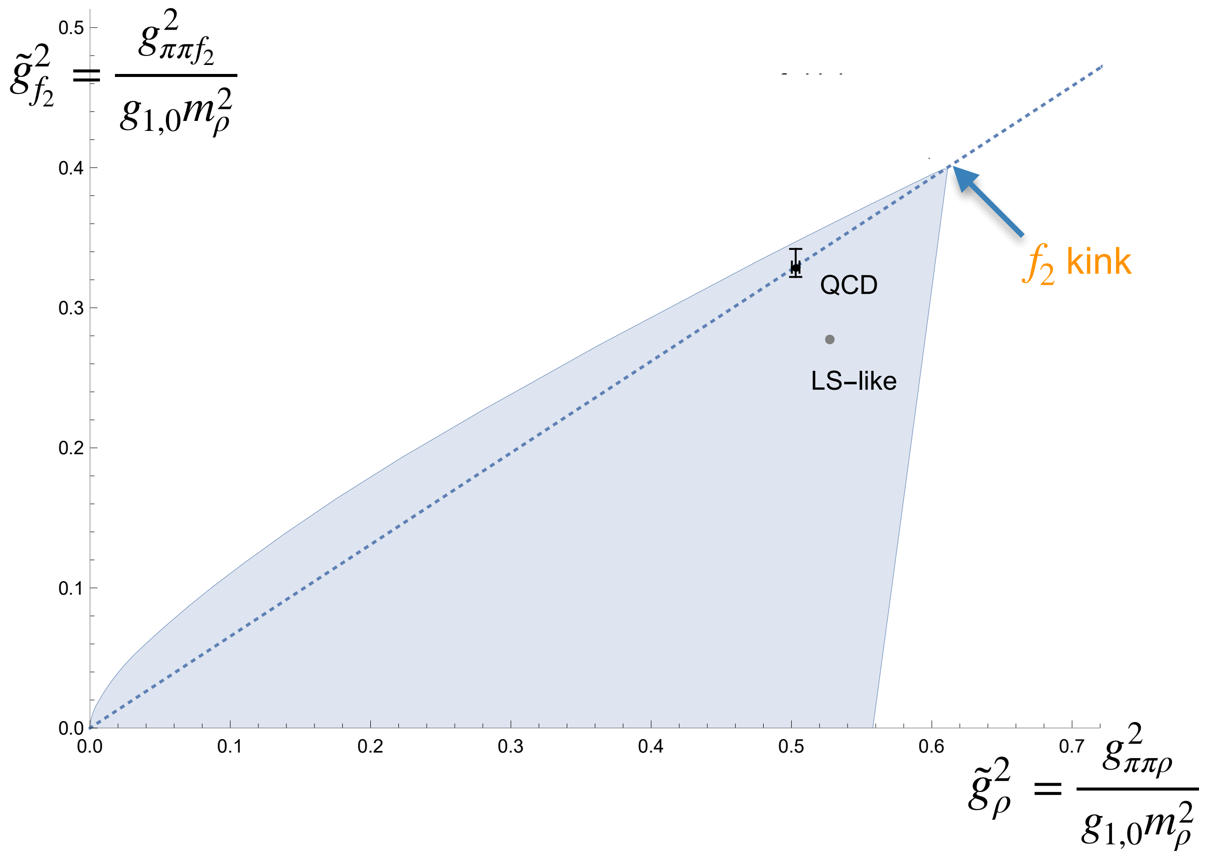


At the kink, $\dfrac{m_\rho^2}{\widetilde{M}^2} \approx 0.2106$

In QCD, $\dfrac{m_\rho^2}{m_{\rho_3}^2} \approx 0.2107$ (the first state above the f_2 is the ρ_3 meson with $J = 3$)

Not bad!!!

Exclusion plot for the normalized ρ and f_2 couplings



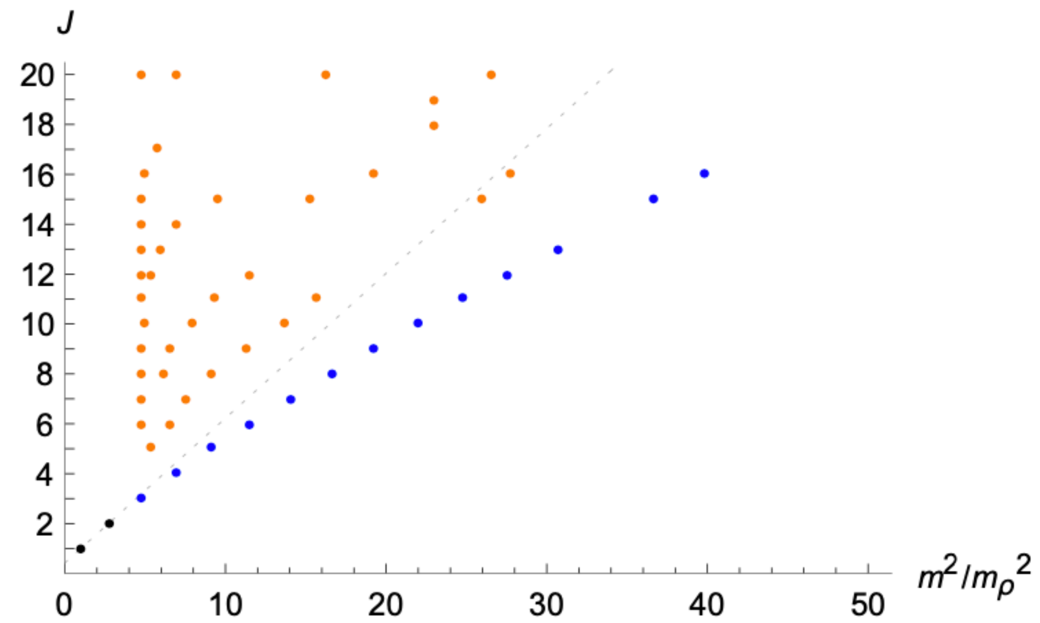
QCD is a bit away from the f_2 kink, but sits precisely on the dotted line.

Because of subtractions, our extremal solutions have no exchanged $J = 0$ states.

Removing $J = 0$ states from the QCD amplitude would push the normalized couplings towards the f_2 kink, but the ratio $\tilde{g}_{f_2}^2 / \tilde{g}_\rho^2$ would not change.

Extremal spectrum

The naive extremal spectrum from SDPB is messy, polluted by spurious numerical artifacts



Extremal spectrum

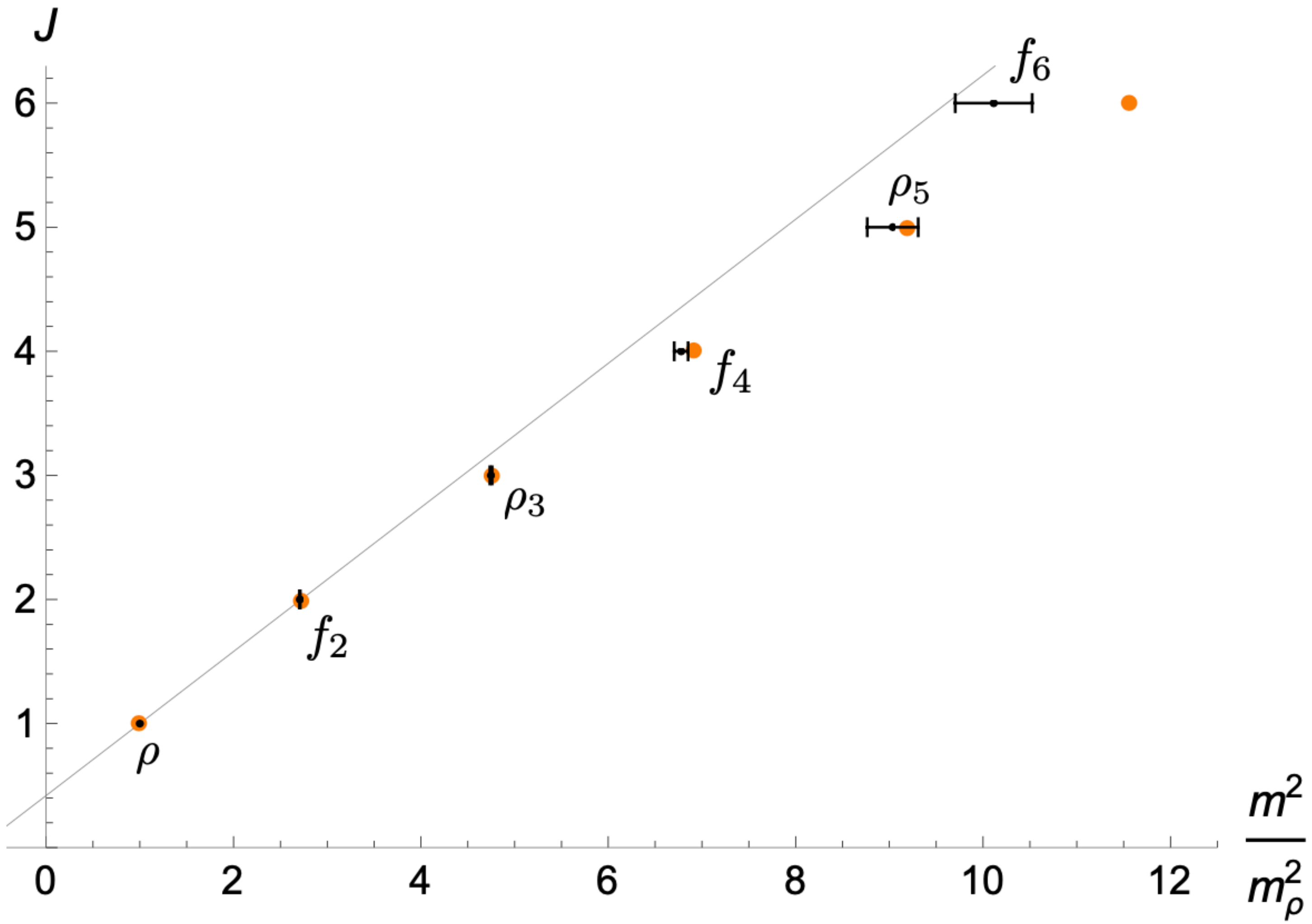
The naive extremal spectrum from SDPB is messy, polluted by spurious numerical artifacts

	Dominant state		Other states					
J	m^2	$\tilde{g}_X^2$	m^2	$\tilde{g}_X^2$	m^2	$\tilde{g}_X^2$	m^2	$\tilde{g}_X^2$
3	4.747774	0.33527						
4	6.902792	0.28933						
5	9.181336	0.25334	5.278811	1.515×10^{-4}				
6	11.54579	0.22174	4.747774	3.297×10^{-6}	6.582414	1.773×10^{-4}		
7	14.01378	0.19857	4.835758	7.862×10^{-7}	7.581251	1.237×10^{-4}		
8	16.67318	0.18599	4.747774	8.771×10^{-8}	6.235041	1.265×10^{-6}	9.207674	1.352×10^{-4}
9	19.28674	0.16358	4.808180	1.895×10^{-8}	6.571938	4.367×10^{-7}	11.31167	2.537×10^{-4}
10	21.93016	0.14912	5.019793	7.308×10^{-9}	7.879411	3.754×10^{-7}	13.61458	4.242×10^{-4}
11	24.82063	0.11649	4.825621	6.643×10^{-10}	9.289181	1.875×10^{-6}	15.69828	1.554×10^{-4}
12	27.53345	0.10811	4.747774	8.380×10^{-11}	5.390215	7.235×10^{-11}	11.48907	7.067×10^{-6}

Various assumptions remove the spurious states while preserving the extremal solution

When the dust settles: one beautiful, curved Regge trajectory, and probably no other states

Low-lying states



[Albert Henriksson LR Vichi]

Discussion

Have we cornered large N QCD? Tantalizing close:

- A Regge trajectory, at last!
- Astonishing numerical agreement for the first few states.
- *But*, the spectrum is too sparse. No evidence of daughter trajectories.

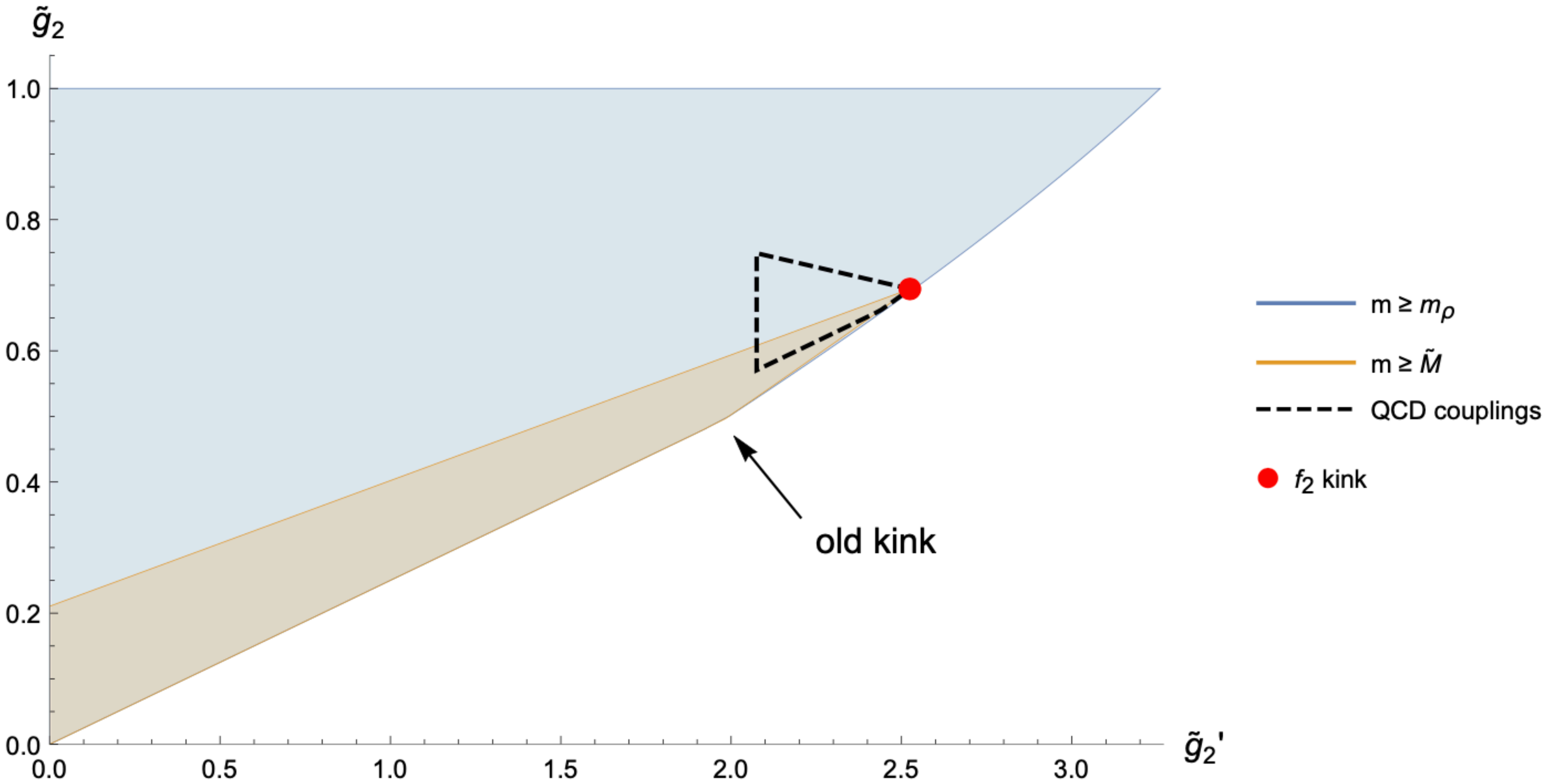
Perhaps daughters would appear if we could dramatically increase the number of constraints.

Or perhaps we have stumbled upon a curious solution to crossing.
Maximizing the normalized f_2 might lead to a solution with as sparse a spectrum as possible.

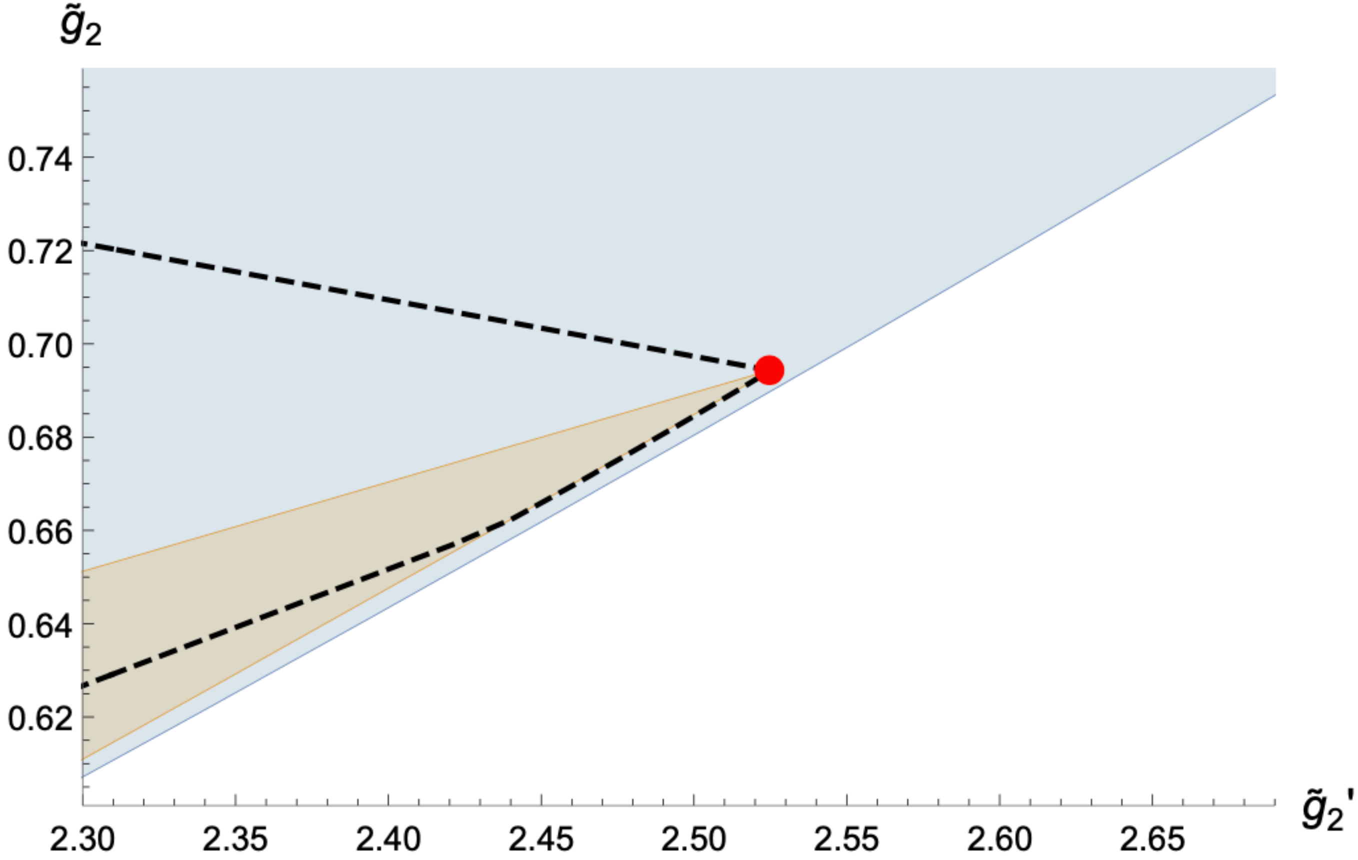
The main lesson seems to be the power of Regge assumptions.

Speculation: with higher-spin states on the *external* legs, a much more powerful bootstrap.

EFT couplings



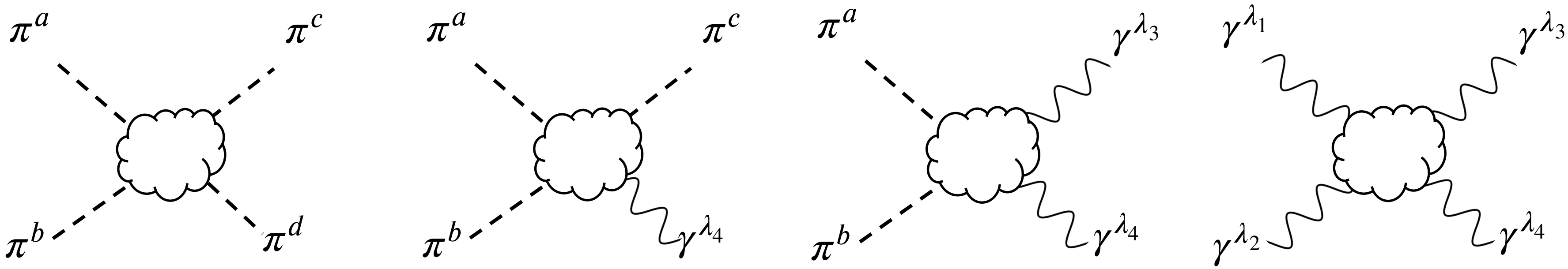
EFT couplings



Mixed Pion / Photon System

[Albert, LR, arXiv:2307.01246]

Photons only external, i.e. we're computing form factors of the U(1) electromagnetic current

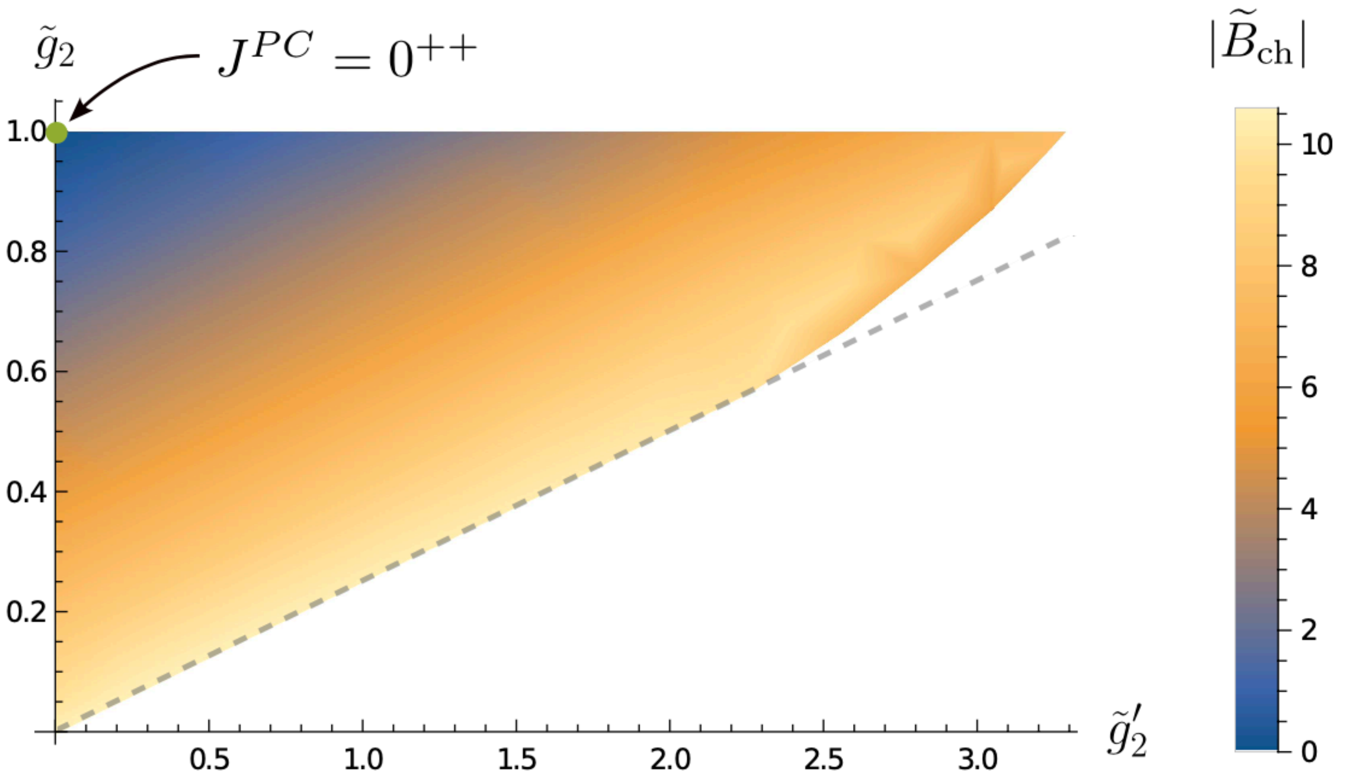


Much more involved system! Payoffs:

- Much more sensitive to large N selection rules for mesons that generalize OJI
- Sum rules that encode Goldstone boson nature of the pions
- Knows about coefficient of WZW, which can be treated as any other EFT coupling

  or matched with the **chiral anomaly** of large N QCD

[Albert, LR, arXiv:2307.01246]



Upper bound on **chiral anomaly**, normalized by f_π and by an (unknown) $\pi\pi \to \gamma\gamma$ EFT coupling

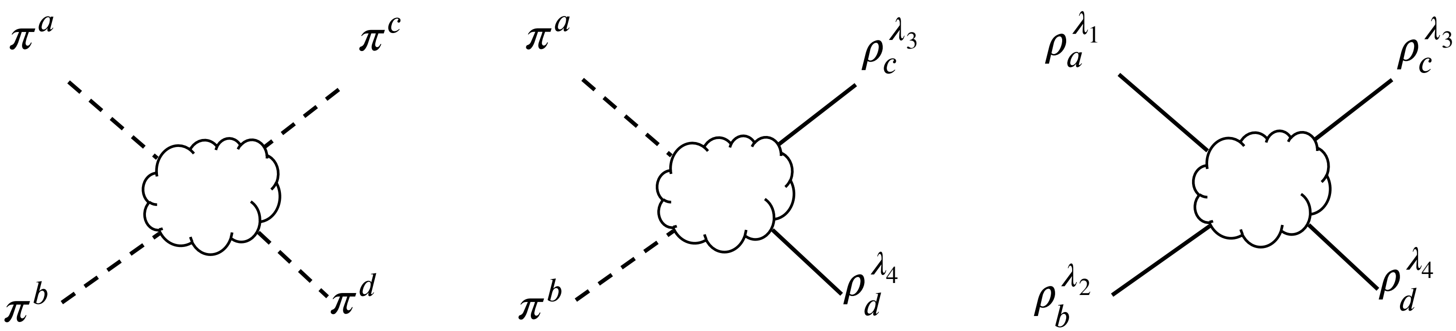
Inhomogeneous bounds involving N, e.g. $\sqrt{-c_2^{(1)}/e^2} \gtrsim \frac{1}{10.583} \frac{2N}{3\pi^2 f_\pi^2}$.

Goal: a lower bound on $\frac{f_\pi^2}{m_\rho^2 N}$.

Mixed Pion/Rho System

[in progress: Albert, Henriksson, LR, Vichi]

External rho mesons: There are three independent processes involving pions and rho mesons.



Unitarity: Positivity of the spectral density is now given in terms of a matrix.

$$\begin{pmatrix} \rho_{\pi\pi\to\pi\pi} & 0 & \rho_{\pi\pi\to\rho\rho} \\ 0 & \rho_{\pi\rho\to\pi\rho} & 0 \\ \rho_{\rho\rho\to\pi\pi} & 0 & \rho_{\rho\rho\to\rho\rho} \end{pmatrix} \succcurlyeq 0$$

Matrices for $\begin{cases} a = 1,2,\ldots,N_f \\ \lambda = +,-,0 \end{cases}$

[Caron-Huot, Li, Parra-Martinez, Simmons-Duffin 2022]

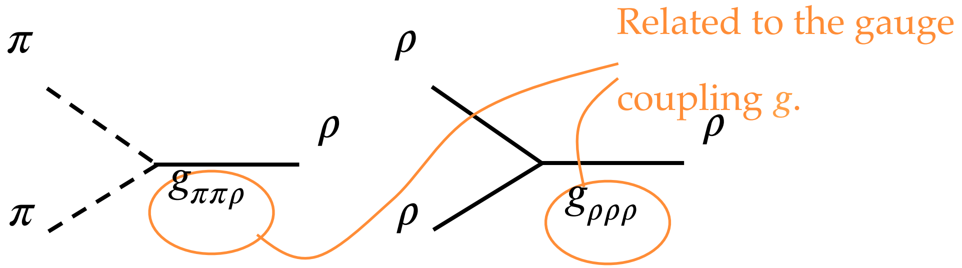
We will be able to access the full set of on-shell 3pt couplings of the pion/rho system.

Compare with various phenomenological models, e.g. with

"Hidden local symmetry": Rho meson is introduced as the gauge boson of a "hidden" symmetry.

$$U(x) = \xi_L(x)\xi_R(x)^\dagger$$

Local symmetry
$$\xi_L(x) \to \xi_L(x)h(x)$$
$$\xi_R(x) \to \xi_R(x)h(x)$$



[Bando, Kugo, Uehara, Yamawaki & Yanagida 1985]

Outlook

Summary:

- A new stab at a very old problem
- Exclusion plots must be interpreted with care. Analytic ruling-in.
- Forcing higher-spin exchanged mesons leads to Regge trajectories!
- Almost-too-good agreement with the real world.
- Perhaps not yet large N QCD, but tantalizing close.

In progress:

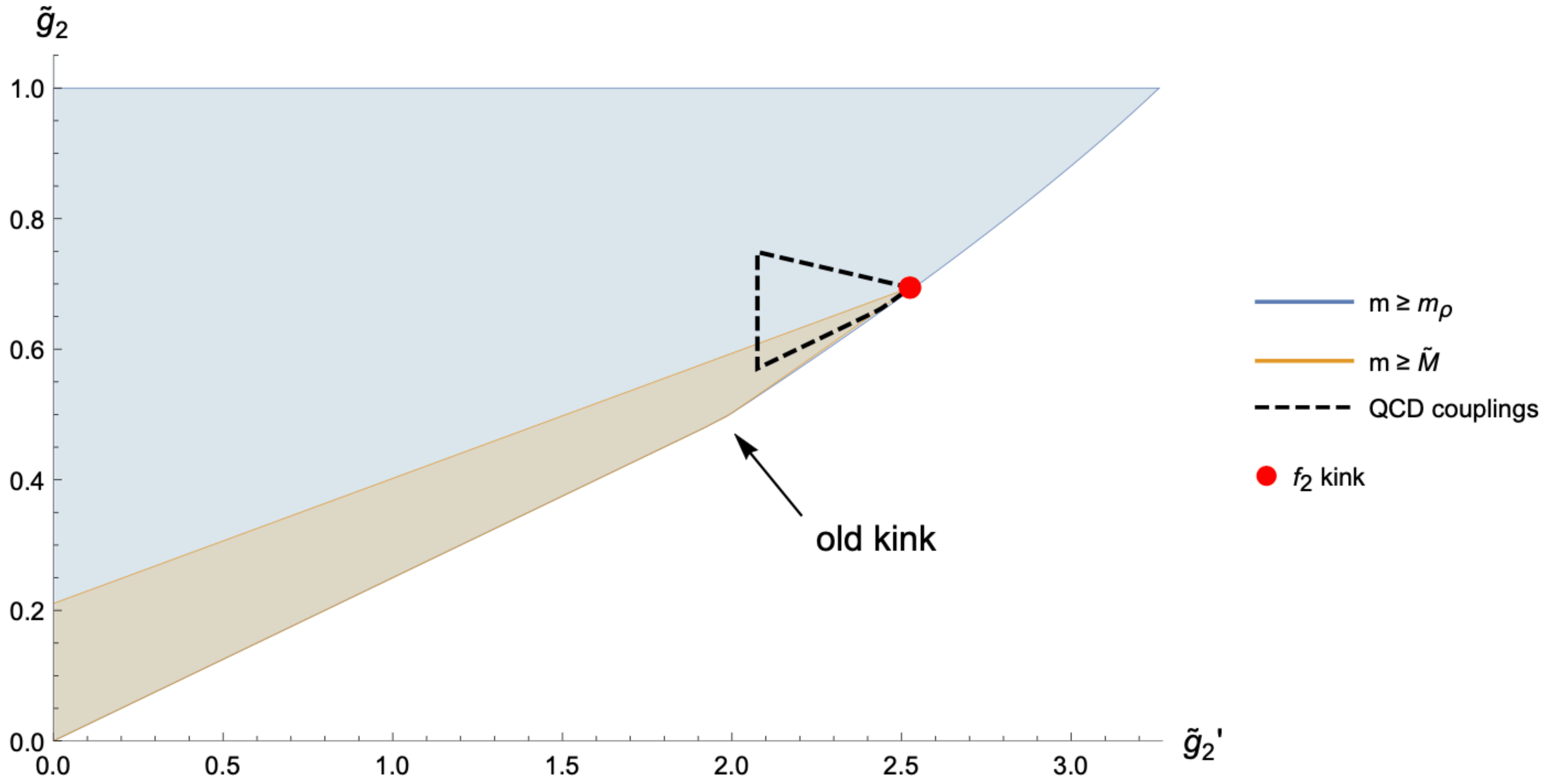
- Mixed amplitudes with external rhos and pions
- Off-shell external photons

Future:

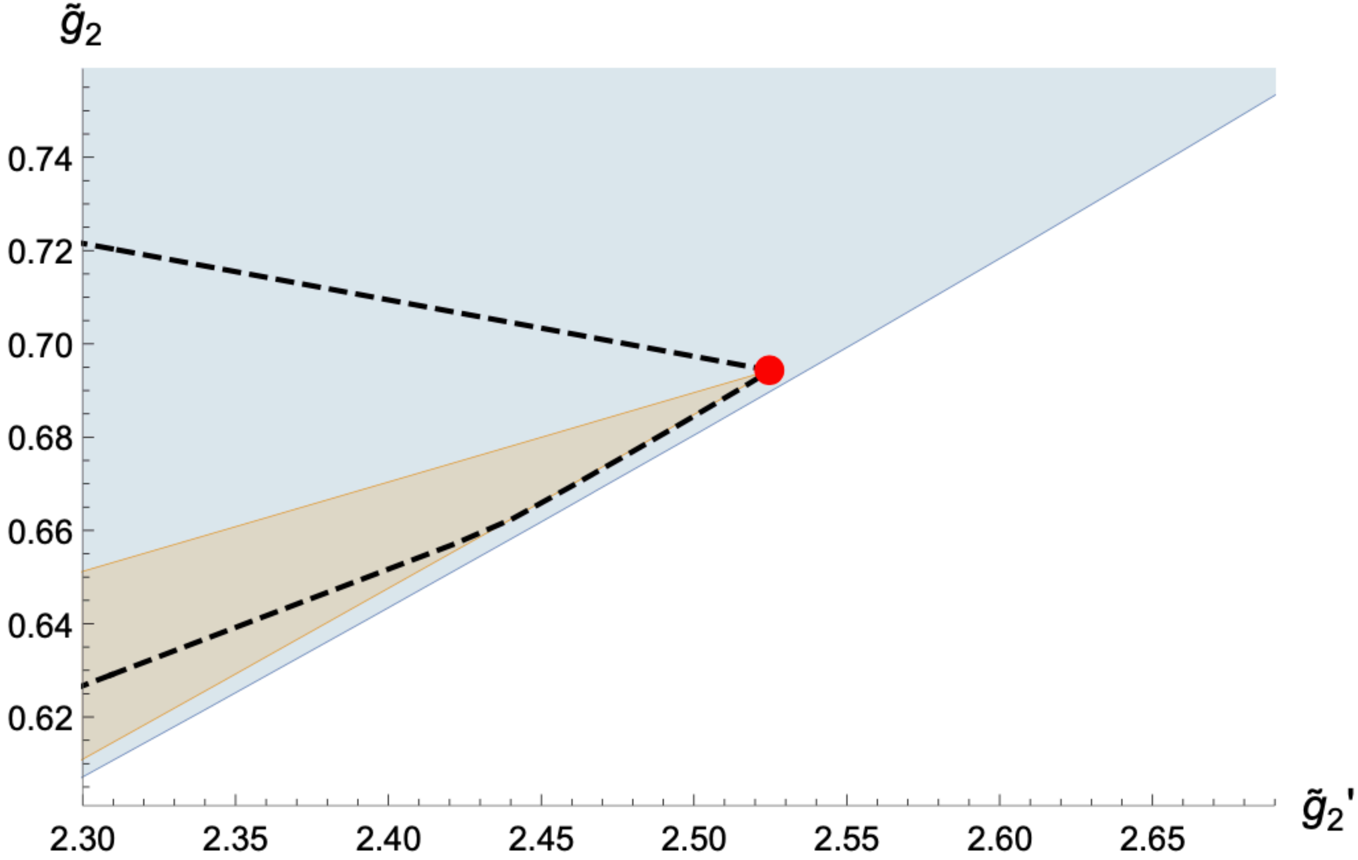
- Which assumptions are needed to corner large N QCD or other theories of interest?
- Relation with worldsheet bootstrap?
- $D = 3$, susy, …

Bonus Slides

EFT couplings



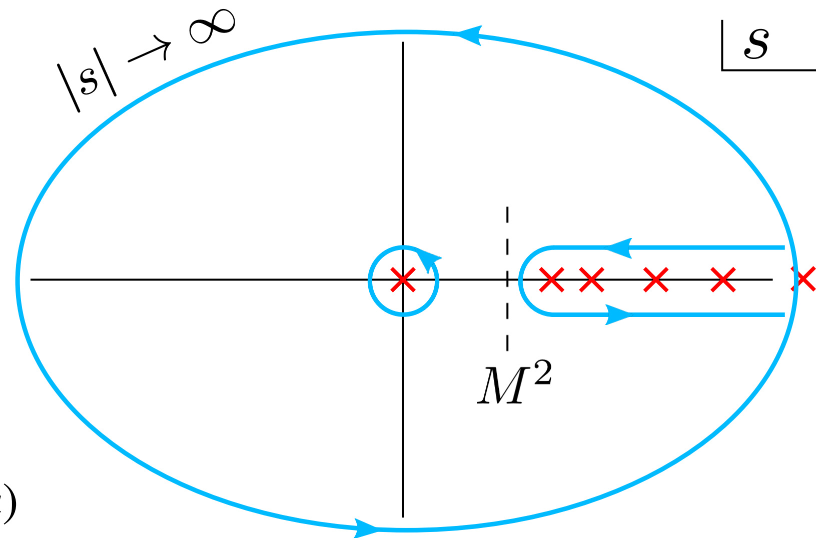
EFT couplings



Dispersion Relations: UV = IR

By the Regge behavior,

$$\frac{1}{2\pi i}\oint_{\infty} ds' \frac{M(s',u)}{s'^{k+1}} = 0 \qquad k = 1,2,\ldots$$



Deforming the contour,

$$\frac{1}{2\pi i}\oint_{0} ds' \frac{M_{\text{low}}(s',u)}{s'^{k+1}} = \frac{1}{\pi}\int_{M^2}^{\infty} ds' \frac{\text{Im } M_{\text{high}}(s',u)}{s'^{k+1}}$$

$$= \frac{1}{\pi}\sum_J n_J^{(D)} \int_{M^2}^{\infty} \frac{dm^2}{m^2}\, m^{4-D} \underbrace{\rho_J(m^2)}_{\geq 0} \frac{P_J\left(1+\frac{2u}{m^2}\right)}{m^{2k}} = \left\langle \frac{P_J\left(1+\frac{2u}{m^2}\right)}{m^{2k}} \right\rangle$$

Expanding around $u \sim 0$,

$$k = 1: \qquad g_{1,0} + 2g_{2,1}u + g_{3,1}u^2 + \ldots = \left\langle \frac{P_J(1)}{m^2} + 2\frac{P_J{}'(1)}{m^4}u + 2\frac{P_J{}''(1)}{m^6}u^2 + \ldots \right\rangle$$

$$k = 2: \qquad g_{2,0} + g_{3,1}u + \ldots = \left\langle \frac{P_J(1)}{m^4} + 2\frac{P_J{}'(1)}{m^6}u + \ldots \right\rangle$$

Sum Rules and Null Constraints

Sum rules:

$$g_{1,0} = \left\langle \frac{1}{m^2} \right\rangle, \quad g_{2,0} = \left\langle \frac{1}{m^4} \right\rangle, \quad 2g_{2,1} = \left\langle \frac{J(J+1)}{m^4} \right\rangle, \ \cdots$$

$$g_{3,1} = \left\langle 2\frac{P_J{}'(1)}{m^6} \right\rangle = \left\langle 2\frac{P_J{}''(1)}{m^6} \right\rangle$$

Null constraints:

$$\mathcal{X}_{3,1}\left(m^2, J\right) = \frac{P_J{}'(1)}{m^6} - \frac{P_J{}''(1)}{m^6} \qquad \left\langle \mathcal{X}_{3,1}\left(m^2, J\right) \right\rangle = 0$$

(Due to crossing symmetry)

[Caron-Huot & Van Duong 2021,
Tolley, Wang & Zhou 2021]

Additional set of dispersion relations

$$\frac{1}{2\pi i} \oint_{\infty} ds' \frac{M(s', -s'-u)}{s'^{k+1}} = 0 \quad \longrightarrow \quad \left\langle \mathcal{Y}_{n,k}\left(m^2, J\right) \right\rangle = 0$$

New set of null constraints

Sum Rules and Null Constraints

Sum rules: $g_{1,0} = \left\langle \frac{1}{m^2} \right\rangle, \quad g_{2,0} = \left\langle \frac{1}{m^4} \right\rangle, \quad 2g_{2,1} = \left\langle \frac{J(J+1)}{m^4} \right\rangle, \ \ldots$

(Due to low-energy crossing symmetry)

$$g_{3,1} = \left\langle 2\frac{P_J{}'(1)}{m^6} \right\rangle = \left\langle 2\frac{P_J{}''(1)}{m^6} \right\rangle$$

[Caron-Huot & Van Duong 2021, Tolley, Wang & Zhou 2021]

Null constraints: $\mathcal{X}_{3,1}(m^2, J) = \frac{P_J{}'(1)}{m^6} - \frac{P_J{}''(1)}{m^6}$ $\quad \left\langle \mathcal{X}_{3,1}(m^2, J) \right\rangle = 0$

Additional set of dispersion relations

$$\frac{1}{2\pi i} \oint_{\infty} ds' \frac{M(s', -s'-u)}{s'^{k+1}} = 0$$

→ New set of null constraints $\left\langle \mathcal{Y}_{n,k}(m^2, J) \right\rangle = 0$

Two-sided bounds: By unitarity, $\langle \ldots \geq 0 \rangle \Rightarrow \langle \ldots \rangle \geq 0.$ $\quad g_{i,j} \geq 0$ [Pham & Truong 1985]

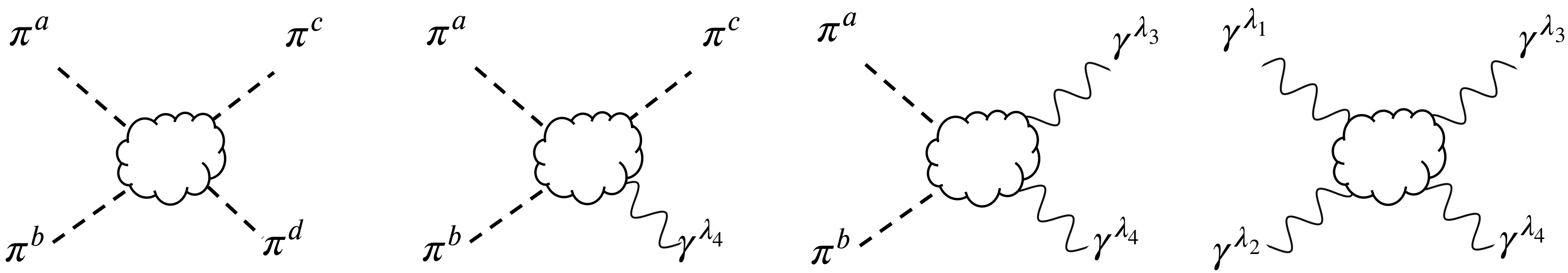
$$0 \leq \tilde{g}_2 \equiv \frac{g_{2,0} M^2}{g_{1,0}} = \frac{\left\langle \frac{M^4}{m^4} \right\rangle}{\left\langle \frac{M^2}{m^2} \right\rangle} \leq 1 \qquad 0 \leq \tilde{g}_2' \equiv \frac{2 g_{2,1} M^2}{g_{1,0}} \leq \ ?$$

$m \geq M$

Mixed Pion / Photon System

[Albert, LR, arXiv:2307.01246]

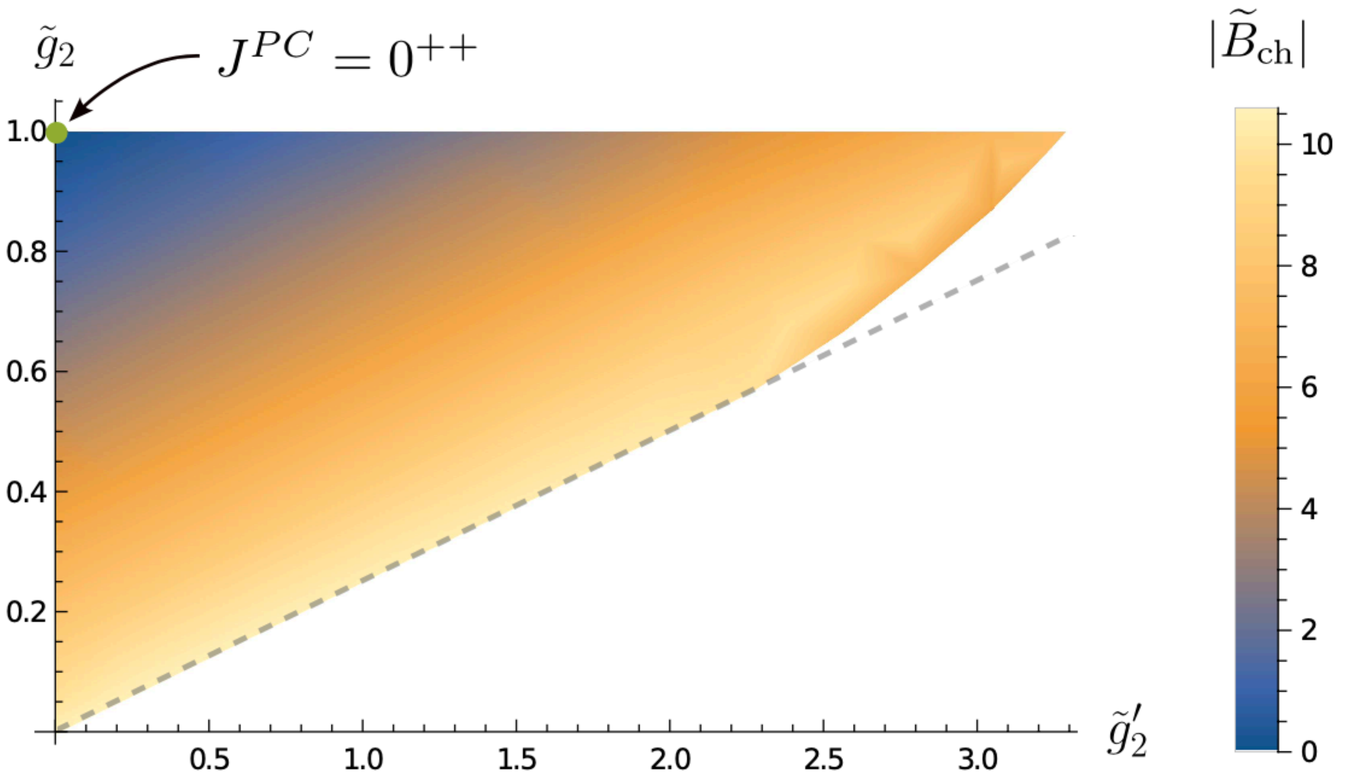
Photons only external, i.e. we're computing form factors of the U(1) electromagnetic current



Much more involved system! Payoffs:

- Much more sensitive to large N selection rules for mesons that generalize OJI
- Sum rules that encode Goldstone boson nature of the pions
- Knows about coefficient of WZW, which can be treated as any other EFT coupling **or** matched with the **chiral anomaly** of large N QCD

[Albert, LR, arXiv:2307.01246]



Upper bound on **chiral anomaly**, normalized by f_π and by an (unknown) $\pi\pi \to \gamma\gamma$ EFT coupling

Inhomogeneous bounds involving N, e.g. $\sqrt{-c_2^{(1)}/e^2} \gtrsim \frac{1}{10.583} \frac{2N}{3\pi^2 f_\pi^2}$.

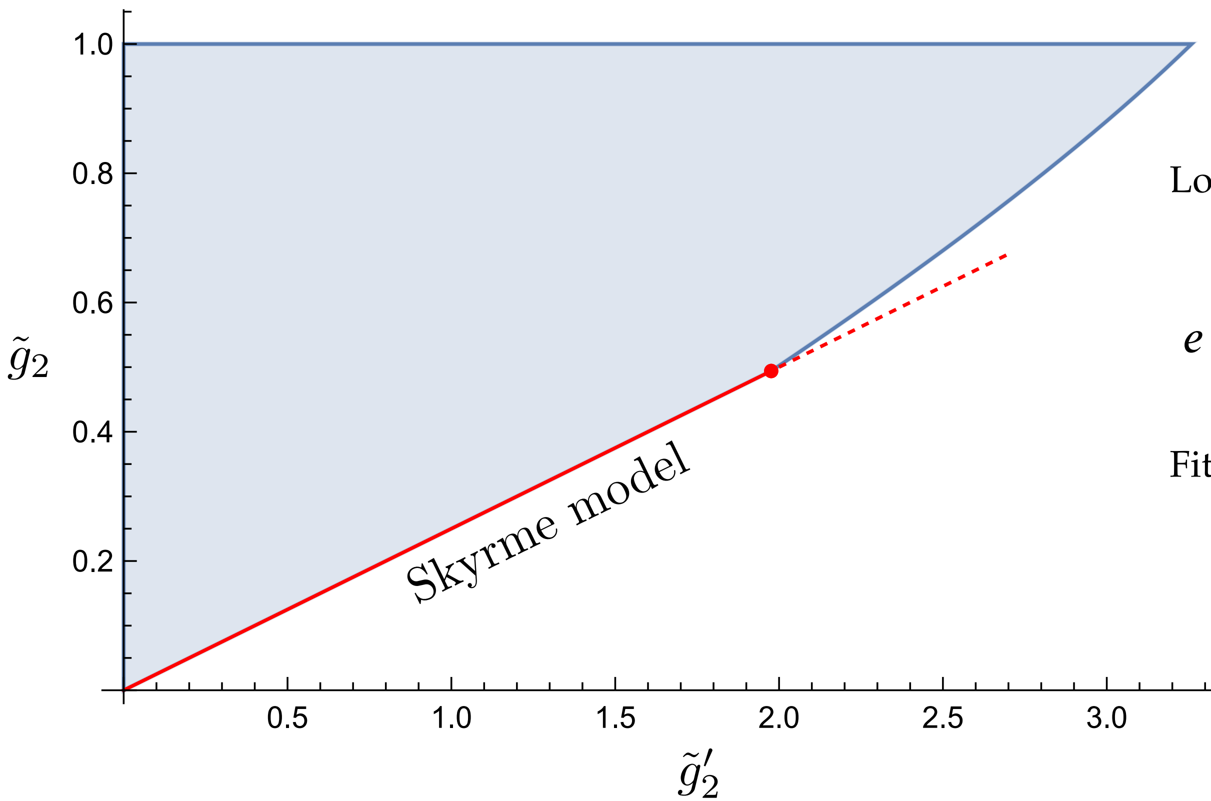
Goal: a lower bound on $\frac{f_\pi^2}{m_\rho^2 N}$.

Skyrme model

Used to describe baryons as solitons of the chiral Lagrangian.

$$\mathscr{L}_{\text{Skyr}} = -\frac{f_\pi^2}{4}\mathrm{Tr}\Big(\partial_\mu U^\dagger \partial^\mu U\Big) + \frac{1}{32e^2}\mathrm{Tr}\Big(\big[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U\big]\big[U^\dagger\partial^\mu U, U^\dagger\partial^\nu U\big]\Big)$$

Particular choice: $-\ell_1 = \ell_2 = \frac{1}{4e^2}$, $\quad \tilde{g}_2' = 4\,\tilde{g}_2 = \frac{1}{e^2}\frac{M^2}{f_\pi^2}$



Lower bound on the coupling:

$$e\ f_\pi \geq \sqrt{\frac{M^2}{4\ \tilde{g}_2^{(\text{kink})}}} \simeq 551\ \text{MeV}$$

Fitting the nucleon and Δ mass:

$$e\ f_\pi \simeq 352\ \text{MeV}$$

[Adkins, Nappi & Witten 1983]

Hidden local symmetry and rho dominance

Hidden local symmetry: $U(x) = \xi_L(x)\xi_R(x)^\dagger \qquad \xi_L, \xi_R \in SU(2)$

Global symmetry $\quad \xi_L(x) \to g_L \xi_L(x) \qquad \xi_R(x) \to g_R \xi_R(x)$

Local (hidden) symmetry $\quad \xi_L(x) \to \xi_L(x)h(x) \qquad \xi_R(x) \to \xi_R(x)h(x)$

Rho meson = gauge field of this local symmetry. [Bando, Kugo, Uehara, Yamawaki & Yanagida 1985]

$$\mathscr{L}_{\mathrm{HLS}} = -\frac{1}{2g^2}\mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \frac{f_\pi^2}{4}\mathrm{Tr}\left(\left(\xi_L^\dagger\partial_\mu\xi_L - \xi_R^\dagger\partial_\mu\xi_R\right)^2\right) - a\frac{f_\pi^2}{4}\mathrm{Tr}\left(\left(2\rho_\mu^a T_a - i\left(\xi_L^\dagger\partial_\mu\xi_L + \xi_R^\dagger\partial_\mu\xi_R\right)\right)^2\right) + $$

$$g_{\pi\pi\rho} = \frac{1}{2}ga \qquad m_\rho^2 = ag^2 f_\pi^2$$

Rho dominance: $a = 2$

KSRF: $\quad m_\rho^2 = 2g_{\pi\pi\rho}^2 f_\pi^2$

Integrating out the rho,

$$\mathscr{L}_{\mathrm{HLS}} \longrightarrow \mathscr{L}_{\mathrm{Skyr}}$$

$$\tilde{g}_2' = 4\tilde{g}_2 = 2$$