# REVISITING LATTICE AND MATRIX BOOTSTRAP

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# Outline

- Matrix Quantum Mechanics. (In progress, with Henry Lin)
- SU(2) lattice Yang-Mills theory. (To appear, with Vladimir Kazakov)

Relevant literatures:

[Anderson and Kruczenski, 2017]

[Lin, 2020]

[Han et al., 2020]

[Kazakov and Zheng, 2022]

[Kazakov and Zheng, 2023]

[Cho et al., 2022]

[Lin, 2023]

# One-matrix model

The partition function is chosen to be:

$$Z = \lim_{N\to\infty} Z_N = \lim_{N\to\infty} \int d^{N^2}M\, \mathrm{e}^{-N\mathrm{tr}V(M)}, \quad V(x) = \frac{1}{2}\mu x^2 + \frac{1}{4}gx^4, \quad (1)$$

The integration is over Hermitian matrix.

The basis of operators are:

$$\mathcal{W}_k = \langle \mathrm{Tr}M^k \rangle = \lim_{N\to\infty} \int \frac{d^{N^2}M}{Z_N} \frac{1}{N} \mathrm{tr}M^k \mathrm{e}^{-N\mathrm{tr}V(M)}. \quad (2)$$

And the Schwinger-Dyson equations:

$$\mu \mathcal{W}_{k+1} + g\mathcal{W}_{k+3} = \sum_{l=0}^{k-1} \mathcal{W}_l\ \mathcal{W}_{k-l+1},\ k = 1, 2, 3, ... \quad (3)$$

# Positivity by inner product

Generalization: Any inner products defined on the vector space of operators or its subspace could leads to positivity condition:

$$\langle \mathcal{O}|\mathcal{O}\rangle = \langle \mathcal{O}^\dagger \mathcal{O}\rangle = \alpha^{*\mathrm{T}}\mathcal{M}\alpha \geq 0,\ \forall \alpha \Leftrightarrow \mathcal{M} \succeq 0. \tag{4}$$

Here we do the expansion $\mathcal{O} = \sum \alpha_i \mathcal{O}_i$, $\mathcal{M}_{ij} = \langle \mathcal{O}_i^\dagger \mathcal{O}_j\rangle$.

In the above case of Hermitian matrix integration, we were taking adjoint to be Hermitian conjugation:

$$\mathcal{O}^\dagger = \mathcal{O}^{*\mathrm{T}} = \mathcal{O} \tag{5}$$

# Positivity

Considering the expectations of square of polynomials are always positive semi-definite:

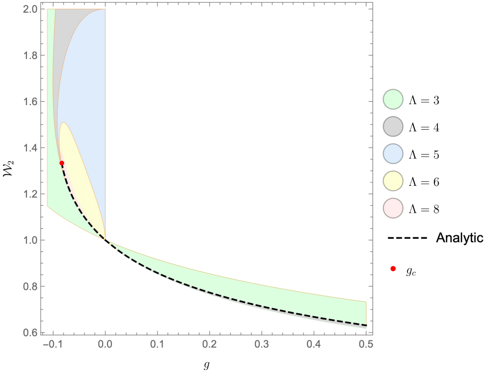
$$\frac{1}{Z}\int_{-\infty}^{\infty} dM \mathrm{Tr}(\sum \alpha_i M^i)^2 \exp(-N\mathrm{tr}V(M)) \geq 0,\ \forall \alpha \tag{6}$$

This is a quadratic form in $\alpha$, its positivity is equivalent to:

$$\mathbb{W} = \begin{pmatrix} \mathcal{W}_0 & \mathcal{W}_1 & \mathcal{W}_2 & \dots \\ \mathcal{W}_1 & \mathcal{W}_2 & \mathcal{W}_3 & \dots \\ \mathcal{W}_2 & \mathcal{W}_3 & \mathcal{W}_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \succeq 0 \tag{7}$$

# Bootstrapping Large N one-matrix model

This is the result of bootstrapping $\mu = 1$ and $\mathbb{Z}_2$ symmetry preserving solution$\mathcal{W}_1 = 0$. From the loop equation and symmetry assumption, all moments are polynomial functions of $\mathcal{W}_2$.



# Multi-matrix bootstrap: an example

Here we propose to study the following two-matrix model:

$$Z = \lim_{N\to\infty} \int d^{N^2}A\, d^{N^2}B\, \mathrm{e}^{-N\mathrm{tr}\left(-h[A,B]^2/2+A^2/2+gA^4/4+B^2/2+gB^4/4\right)} \tag{8}$$

The integration is over Hermitian matrix. To the best of our knowledge, this model with general g and h value, is not solvable!

$$\mathrm{Tr}A^2,\ \mathrm{Tr}A^4,\ \mathrm{Tr}A^2B^2,\ \mathrm{Tr}ABAB,\ \mathrm{Tr}A^6,\ \mathrm{Tr}A^4B^2,\ \mathrm{Tr}A^3BAB,\ \mathrm{Tr}A^2BA^2B,\ \mathrm{Tr}A^8,$$
$$\mathrm{Tr}A^6B^2,\ \mathrm{Tr}A^5BAB,\ \mathrm{Tr}A^4BA^2B,\ \mathrm{Tr}A^4B^4,\ \mathrm{Tr}A^3BA^3B,\ \mathrm{Tr}A^3BAB^3,\ \mathrm{Tr}A^3B^2AB^2,$$
$$\mathrm{Tr}A^2BABAB^2,\ \mathrm{Tr}A^2BAB^2AB,\ \mathrm{Tr}A^2B^2A^2B^2,\ \mathrm{Tr}ABABABAB\ \ldots \tag{9}$$

$\beta = (\mathrm{Tr}A^2)^2$:

$$
\begin{aligned}
1 &= \mathrm{Tr}A^2 + g\mathrm{Tr}A^4 - h(-2\mathrm{Tr}A^2B^2 + 2\mathrm{Tr}ABAB) \\
0 &= -2\mathrm{Tr}A^2 + \mathrm{Tr}A^4 - h(2\mathrm{Tr}A^3BAB - 2\mathrm{Tr}A^4B^2) + g\mathrm{Tr}A^6 \\
0 &= -\mathrm{Tr}A^2 + \mathrm{Tr}A^2B^2 - h(-\mathrm{Tr}A^2BA^2B + 2\mathrm{Tr}A^3BAB - \mathrm{Tr}A^4B^2) + g\mathrm{Tr}A^4B^2 \\
0 &= -h(2\mathrm{Tr}A^2BA^2B - 2\mathrm{Tr}A^3BAB) + g\mathrm{Tr}A^3BAB + \mathrm{Tr}ABAB \\
\beta &= -2\mathrm{Tr}A^4 + \mathrm{Tr}A^6 - h(2\mathrm{Tr}A^5BAB - 2\mathrm{Tr}A^6B^2) + g\mathrm{Tr}A^8 \\
\beta &= -\mathrm{Tr}A^2B^2 + \mathrm{Tr}A^4B^2 - h(-\mathrm{Tr}A^3B^2AB^2 + 2\mathrm{Tr}A^3BAB^3 - \mathrm{Tr}A^4B^4) + g\mathrm{Tr}A^6B^2 \\
0 &= -2\mathrm{Tr}A^2B^2 - h(-\mathrm{Tr}A^2B^2A^2B^2 + 2\mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3B^2AB^2) + \mathrm{Tr}A^4B^2 + g\mathrm{Tr}A^6B^2 \\
0 &= -\mathrm{Tr}A^4 + \mathrm{Tr}A^4B^2 + g\mathrm{Tr}A^4B^4 - h(-\mathrm{Tr}A^4BA^2B + 2\mathrm{Tr}A^5BAB - \mathrm{Tr}A^6B^2) \\
0 &= \mathrm{Tr}A^3BAB - h(2\mathrm{Tr}A^2BAB^2AB - \mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3BAB^3) + g\mathrm{Tr}A^5BAB - \mathrm{Tr}ABAB \\
0 &= \mathrm{Tr}A^3BAB + g\mathrm{Tr}A^5BAB - 2\mathrm{Tr}ABAB - h(-2\mathrm{Tr}A^2BABAB^2 + 2\mathrm{Tr}ABABABAB) \\
0 &= \mathrm{Tr}A^3BAB + g\mathrm{Tr}A^3BAB^3 - h(-\mathrm{Tr}A^3BA^3B + 2\mathrm{Tr}A^4BA^2B - \mathrm{Tr}A^5BAB) \\
0 &= g\mathrm{Tr}A^3BA^3B + \mathrm{Tr}A^3BAB - h(2\mathrm{Tr}A^3B^2AB^2 - 2\mathrm{Tr}A^3BAB^3) \\
0 &= -\mathrm{Tr}A^2B^2 + \mathrm{Tr}A^2BA^2B - h(-\mathrm{Tr}A^2BAB^2AB + 2\mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3B^2AB^2) + g\mathrm{Tr}A^4BA^2B \\
\beta &= \mathrm{Tr}A^2BA^2B + g\mathrm{Tr}A^3B^2AB^2 - h(2\mathrm{Tr}A^3BA^3B - 2\mathrm{Tr}A^4BA^2B).
\end{aligned}
\tag{10}
$$

# Relaxation

Our general strategy: we treat the quadratic terms in the loop equations as independent variable, and replace the algebraic equality by the convex inequality:

$$Q = xx^{\mathrm{T}} \tag{11}$$

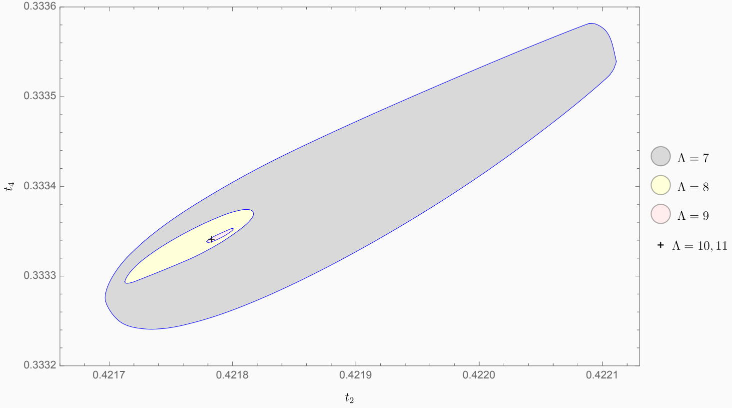
to:

$$\mathcal{R} = \begin{pmatrix} 1 & x^{\mathrm{T}} \\ x & Q \end{pmatrix} \succeq 0. \tag{12}$$

For the previous situation, we have a simple matrix:

$$\mathcal{R} = \begin{pmatrix} 1 & \mathrm{Tr} A^2 \\ \mathrm{Tr} A^2 & \beta \end{pmatrix} \succeq 0. \tag{13}$$

# Results




$$\Lambda = 11,\ g = h = 1: \begin{cases} 0.421783612 \leq \langle \mathrm{Tr} A^2 \rangle \leq 0.421784687 \\ 0.333341358 \leq \langle \mathrm{Tr} A^4 \rangle \leq 0.333342131 \end{cases} \tag{14}$$

# Compare with MC

Compared to the MC study of the same model 2111.02410 (Jha), we are convinced that for this model bootstrap is at least two order of magnitude more efficient than MC.

- MC: 80-85 hours for N=800 simulation to get 4.5 digits.
- Bootstrap: less than 1 hour to get 6 digits.

The Hamiltonian is chosen to be:

$$H = \mathrm{tr}(P^2 + X^2 + gX^4) \tag{15}$$

Here $X$ is a large N Hermitian matrix:

$$[X_{ij}, P_{kl}] = i\delta_{il}\delta_{jk} \tag{16}$$

The ground state is known to be solvable.

# Loop equations

The corresponding loop equations are:

$$\langle [H, \mathcal{O}] \rangle = 0, \; \forall \mathcal{O} \tag{17}$$

$$\langle \mathrm{tr}(G\mathcal{O}) \rangle = 0, \; \forall \mathcal{O} \tag{18}$$

together with the cyclicity of $\mathrm{tr}\mathcal{O}$. $G = i[X, P] + I$ is the generator of the $SU(N)$ gauge symmetry.

Result: general words in $P$ and $X$ can be reduced to polynomials of $\mathrm{tr}X^m$.

$$\begin{aligned} \mathrm{tr}P^2X^2P^2X^4 = {} & \frac{12}{77}g^2\mathrm{tr}X^{14} - \frac{2}{3}g\mathrm{tr}X^2\mathrm{tr}X^6 - \frac{1}{5}g\mathrm{tr}X^8 + \frac{40}{231}g\mathrm{tr}X^{12} \\ & + \frac{\mathrm{tr}X^2}{24} - \frac{1}{3}\mathrm{tr}X^2\mathrm{tr}X^4 - \frac{\mathrm{tr}X^6}{10} + \frac{\mathrm{tr}X^{10}}{21} \end{aligned} \tag{19}$$

# Positivity

For the ground state, or more generally, any stationary state, the corresponding loop equations are:

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0, \; \forall \mathcal{O} \tag{20}$$

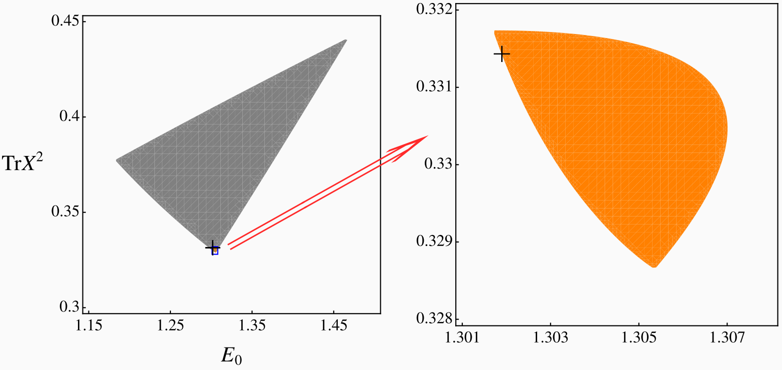
$$\langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle \geq 0, \; \forall \mathcal{O} \tag{21}$$

The later positivity is specialized for the ground state. For more general thermal state with inverse temperature $\beta$,

$$\langle \mathcal{O}^\dagger \mathcal{O} \rangle \log \frac{\langle \mathcal{O}^\dagger \mathcal{O} \rangle}{\langle \mathcal{O} \mathcal{O}^\dagger \rangle} \leq \beta \langle \mathcal{O}^\dagger [H, \mathcal{O}] \rangle, \; \forall \mathcal{O} \tag{22}$$

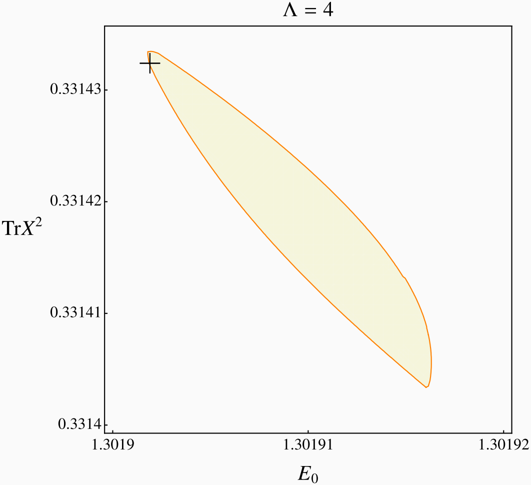
Mathematically, these positivities together with the loop equations is necessary and sufficient.

The illustration of convergence, the left one is $\Lambda = 2$, whereas the right one corresponds to $\Lambda = 3$. The size of the SDP matrix is $2, 2, 2$ and $3, 3, 2, 3$, respectively,



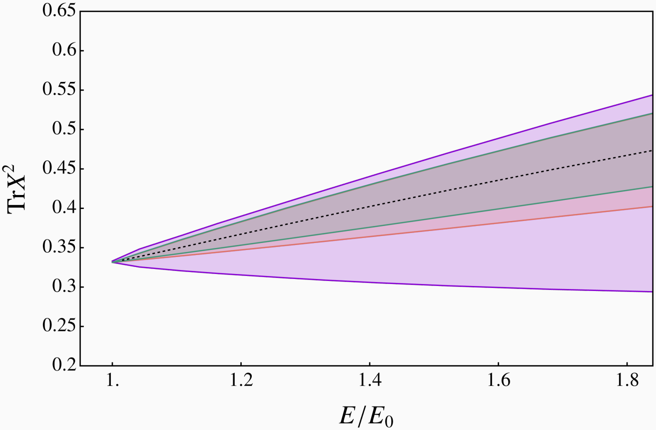
# Convergence

The size of the SDP matrices are $5, 4, 4, 4$.



# Above ground state

The dashed line is the thermal state with the corresponding energy expectation. Different colors correspond to $\Lambda = 8, 18, 26$.



# BFSS model

The Hamiltonian is chosen to be:

$$H = \frac{1}{2}\,\mathrm{Tr}\left(g^2 P_I^2 - \frac{1}{2g^2}\left[X_I, X_J\right]^2 - \psi_\alpha \gamma^I_{\alpha\beta}\left[X_I, \psi_\beta\right]\right) \tag{23}$$

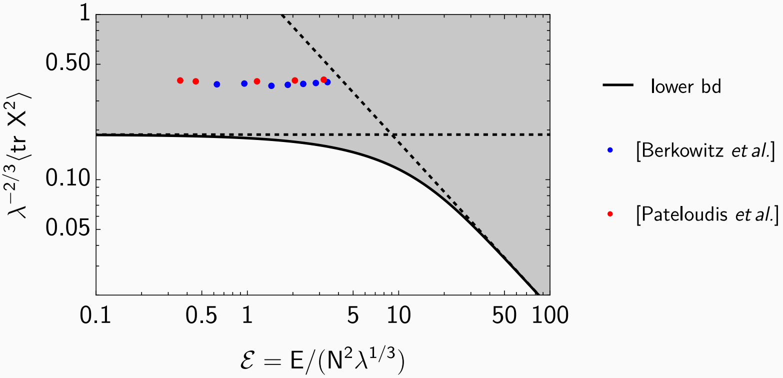
Here:

$$[X_{ij}, P_{kl}] = i\delta_{il}\delta_{jk},\ \{\psi_{\alpha,ij}, \psi_{\beta,kl}\} = \delta_{\alpha\beta}\delta_{il}\delta_{kj} \tag{24}$$

Dual to the dynamics of the D0-brane.

The matrices are in multiples of the $SO(9)$ symmetry.

# Numerical result



# Numerical result $\Lambda = 2$

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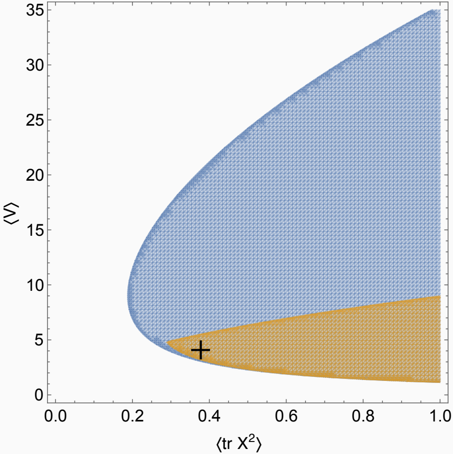



Please take the MC result with a pinch of salt.

# Numerical result $\Lambda = 3$

# Numerical result $\Lambda = 3$

In progress.

# Finite N

General $2 \times 2$ matrix as an example:

$$\epsilon_{\alpha_1\alpha_2\alpha_3}\epsilon_{\beta_1\beta_2\beta_3}M_1^{\alpha_1\beta_1}M_2^{\alpha_2\beta_2}M_3^{\alpha_3\beta_3} = 0 \tag{25}$$

$$\epsilon_{\alpha_1\alpha_2\alpha_3}\epsilon_{\beta_1\beta_2\beta_3} = \delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3} + (-1)^P\sum_P(\dots) \tag{26}$$

$$\begin{aligned}&\mathrm{tr}M_1\mathrm{tr}M_2\mathrm{tr}M_3 = \mathrm{tr}M_1M_2\mathrm{tr}M_3 + \mathrm{tr}M_1\mathrm{tr}M_2M_3 + \mathrm{tr}M_1M_3\mathrm{tr}M_2\\&-\mathrm{tr}M_1M_2M_3 - \mathrm{tr}M_1M_3M_2\end{aligned} \tag{27}$$

So the loop equation of $N \times N$ matrix model must truncate at N-trace loop variables.

# Unit modularity condition

General $SU(2)$ matrix as an example:

$$\mathrm{tr}U^m \mathrm{tr}U^n = \mathrm{tr}U^{m+n} + \mathrm{tr}U^{m-n} \tag{28}$$

This is obvious since we can diagonolize $U$ as $\mathrm{diag}\{\lambda, \lambda^-\}$. Less non-trivially, this is true for any $U$ and $V$:

$$\mathrm{tr}U \mathrm{tr}V = \mathrm{tr}UV + \mathrm{tr}U^\dagger V \tag{29}$$

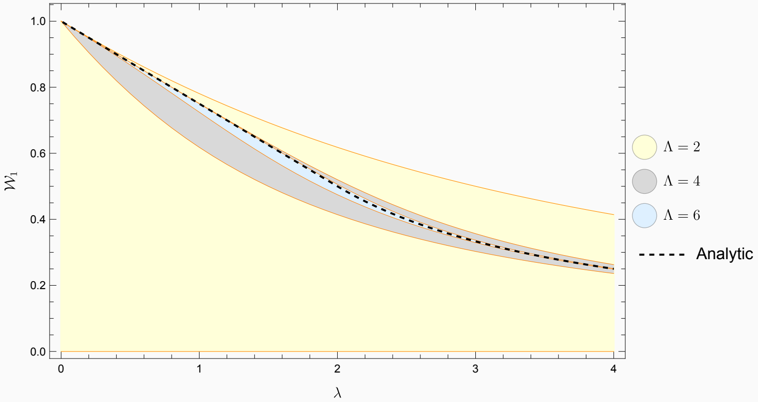
General result: The loop equation of $SU(N)$ matrix model must truncate at (N-1)-trace loop variables.

# Plaquette model

To illustrate the method, we consider the following model, in the ensemble of $SU(2), U(2)$ and $U(\infty)$:

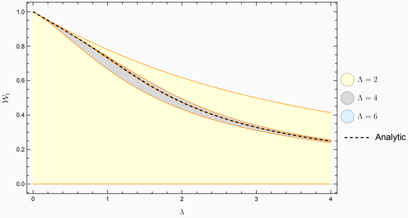
$$Z = \int \mathrm{d}U \exp(-S), \quad S = -\beta N \left(\mathrm{Tr}U + \mathrm{Tr}U^\dagger\right) \tag{30}$$

$U(\infty)$:

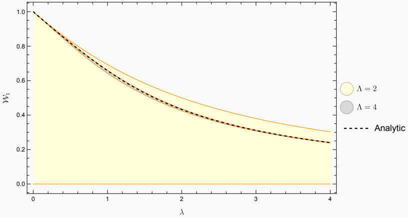



Here $\lambda = \frac{1}{\beta}$, $\Lambda$ is the highest moment in the Toeplitz matrix.

# U(2) & SU(2)




(a) U(2)




(b) SU(2)

**Figure:** Left is U(2), right one is SU(2). Λ is the sum of abs of the power of double trace operator.

# Lattice Gauge theory

We are going to bootstrap the $SU(2)$ lattice gauge theory:

$$Z = \int \prod_{x,\,\mu} \mathrm{d}U_\mu(x) \exp(-S) \tag{31}$$

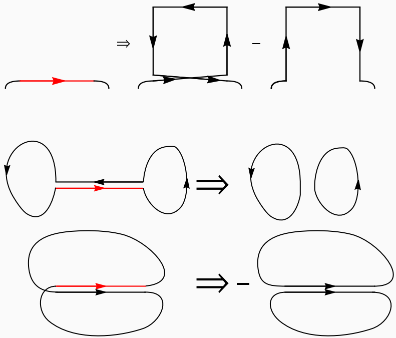
$$S = -\frac{N_c}{2\lambda} \sum_P \mathrm{Re}\,\mathrm{tr} U_P \tag{32}$$

where $U_P$ is the product of four unitary link variables around the plaquette $P$ and we sum up over all plaquettes $P$, including both orientations. In our last work we bootstrap the one plaquette average:

$$u_P = \frac{1}{N_c} \langle \mathrm{tr} U_P \rangle \tag{33}$$

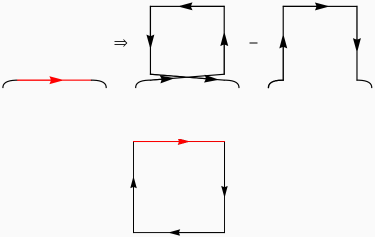
Doing the following infinitesimal transformation $U_\mu(x) \to U_\mu(x)(1 + i\epsilon)$ to the Wilson loop $\mathcal{W}[C]$, we can get the following loop equations schematically:

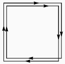
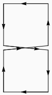
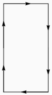
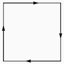
$$(\text{linear}) + 2\lambda\mathcal{W}[C] = 2\lambda(\text{nonlinear}) \tag{34}$$



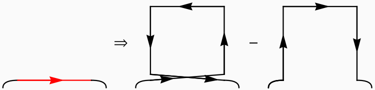
# Makeenko-Migdal loop equations

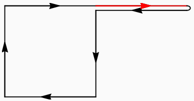
$$(\text{linear}) + 2\lambda \mathcal{W}[C] = 2\lambda(\text{nonlinear}) \qquad (35)$$

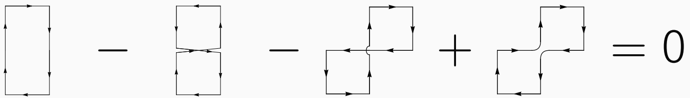


 $-1+$  $-$  $+2\lambda$  $=0$

$$(\text{linear}) + 2\lambda\mathcal{W}[C] = 2\lambda(\text{nonlinear}) \tag{36}$$







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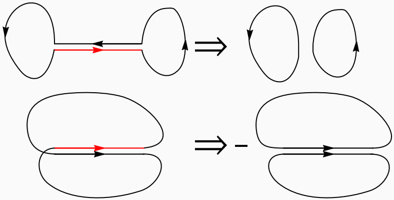
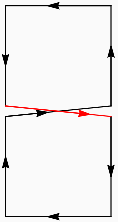


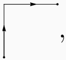
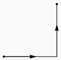


# Positivity by Hermitian Conjugation

In parallel to the bootstrap for Hermitian matrix model, we have:

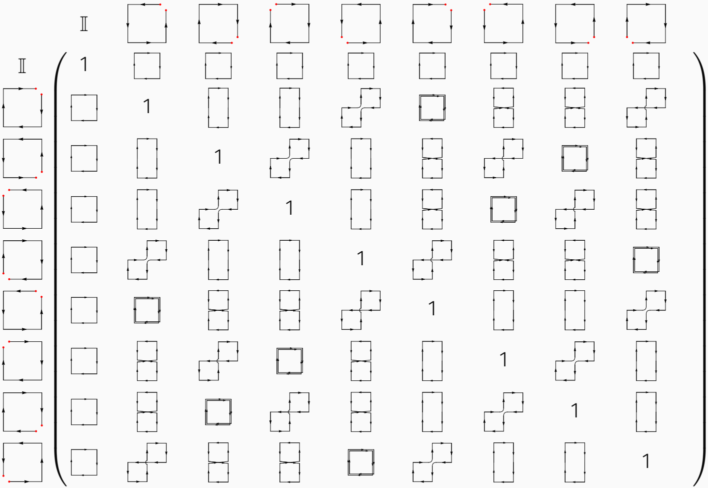
$$\mathrm{Path}^{*\mathrm{T}} = \mathrm{Reverse} \circ \mathrm{Path} \tag{38}$$

For a simplest example:

$\mathrm{Path}_1 =$  , $\quad \mathrm{Path}_2 =$  (39)

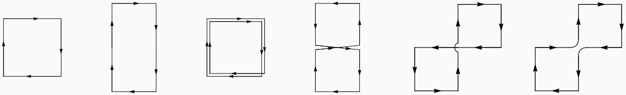
$$\begin{array}{c} \\ \mathrm{Path}_1^\dagger \\ \mathrm{Path}_2^\dagger \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{Path}_1 & \mathrm{Path}_2 \end{array} \\ \begin{pmatrix} 1 & u_P \\ u_P & 1 \end{pmatrix} \end{array} \succeq 0. \tag{40}$$

# Positivity by Hermitian Conjugation

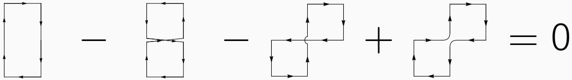


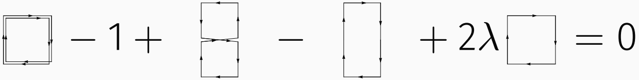

$\succeq 0.$

There are actually 6 Wilson loops in the matrix:



(41)




After the optimization, we get ($\lambda = 1$):

# Reflection Positivity

We can also define the inner product by reflection positivity:

$$\mathcal{O}^\dagger = \Theta\mathcal{O} \tag{42}$$

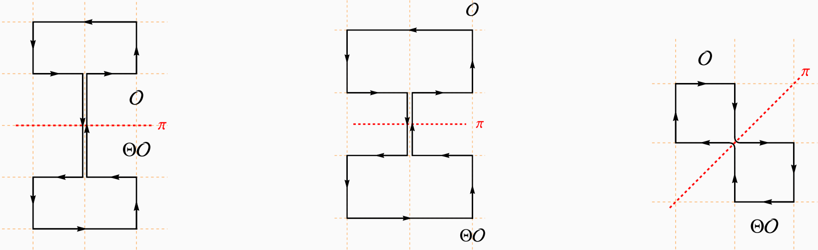



**Figure:** Three reflection symmetries on the lattice allowing new positivity conditions on Wilson loops combining the original and reflected Wilson lines.

# Final scheme

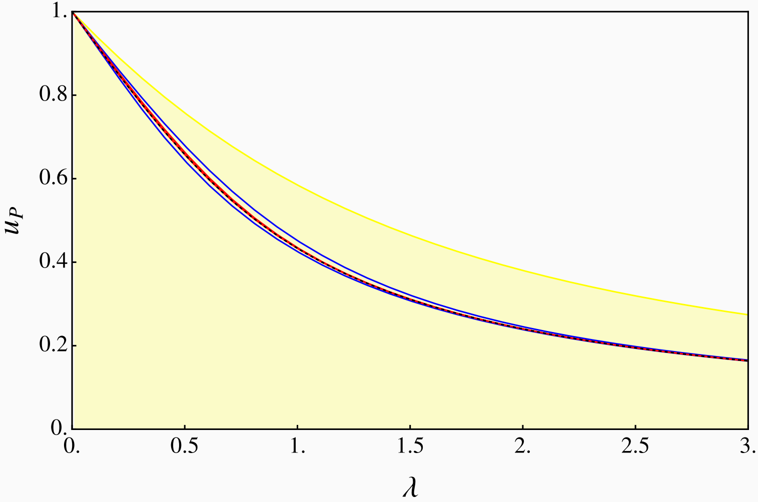
min / max ,

subject to MM loop equations

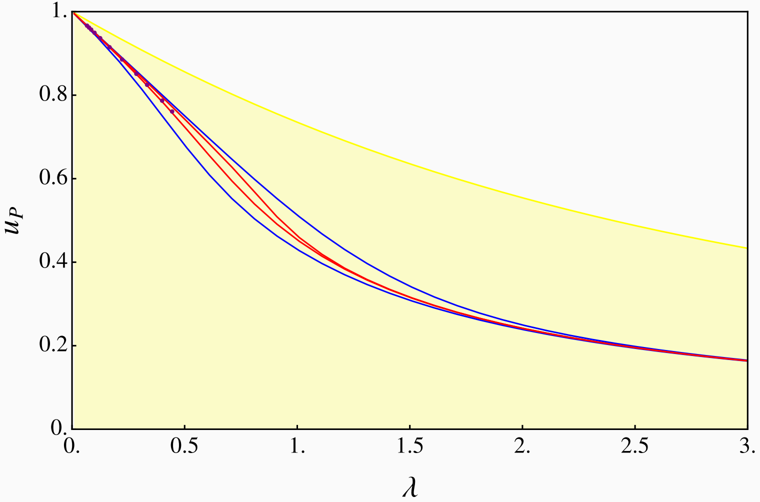
$\text{HerM}^{\text{irrep}} \succeq 0,$

$\text{RefM}^{\text{irrep}} \succeq 0$ (43)

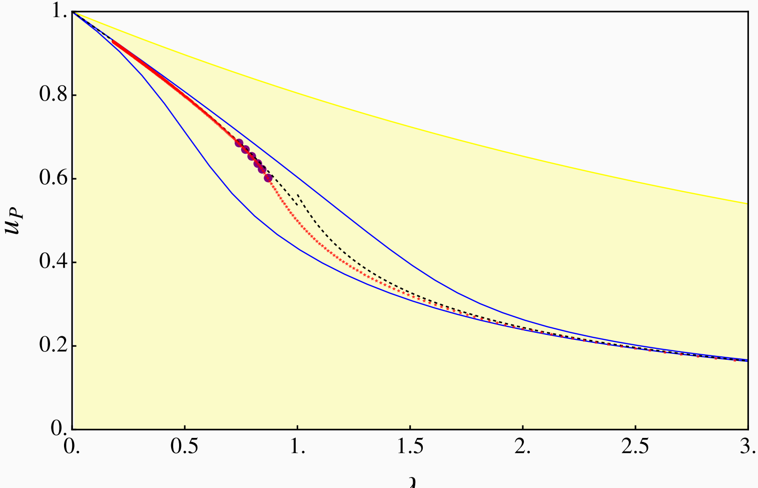
# 2D

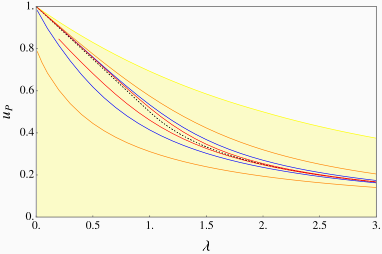


# 3D

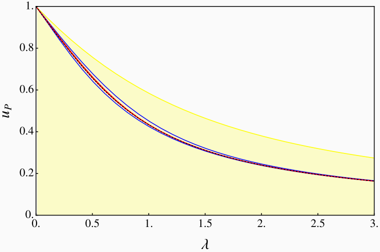


Going to $L_{\max} = 24$ is numerically challenging.



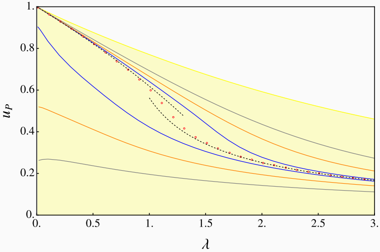



(a) $U(\infty)$

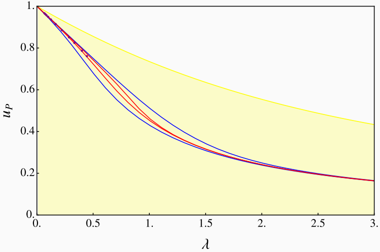



(b) $SU(2)$

# Large N comparison: 3d

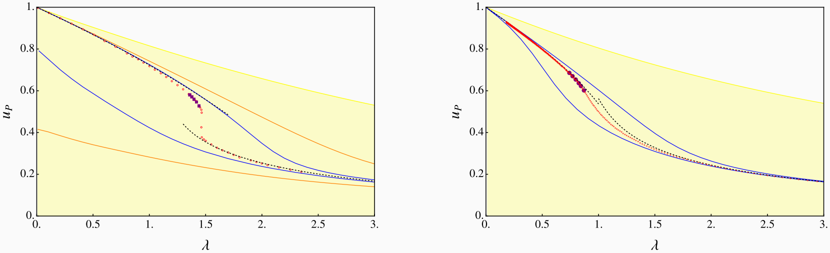



(a) $U(\infty)$




(b) $SU(2)$

# Large N comparison: 4d

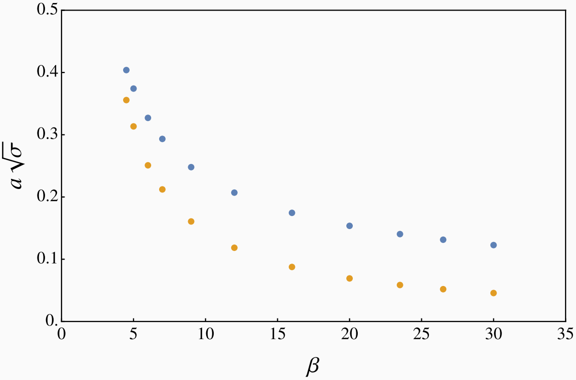



(a) $U(\infty)$ (b) $SU(2)$

# String tension

Here we use the following formula for the string tension:

$$a\sqrt{\sigma} = \sqrt{-\log \frac{\mathcal{W}_{23}\mathcal{W}_{12}}{\mathcal{W}_{22}\mathcal{W}_{13}}} \tag{44}$$



# Prospect

- Corse graining.
- Sign problem.
- Bootstrap Yang-Mills theory in the continuum directly.

# Questions?

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