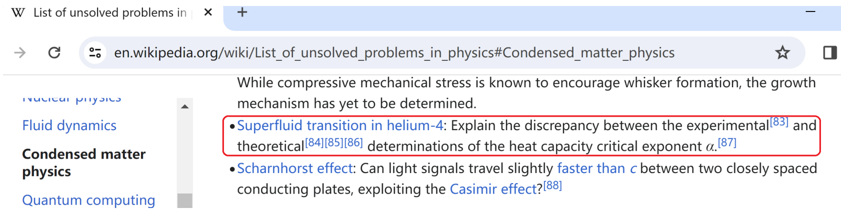
Bootstrap meets Experimental Data

Ning Su
Caltech & MIT

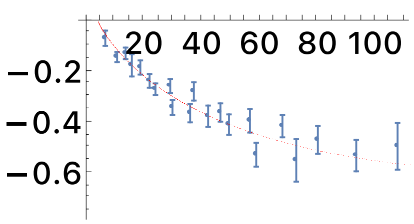
19/02/2024 @ 50+ϵ years of conformal bootstrap

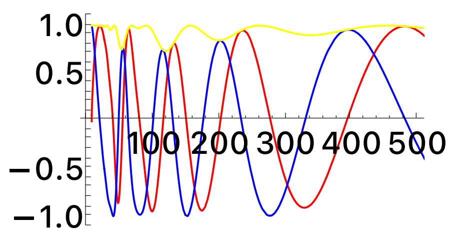
Today’s talk

- Review of numerical techniques for positivity bootstrap
- Application in condensed matter physics : Helium superfluid phase transition



- Application in particle physics : pion-pion scattering



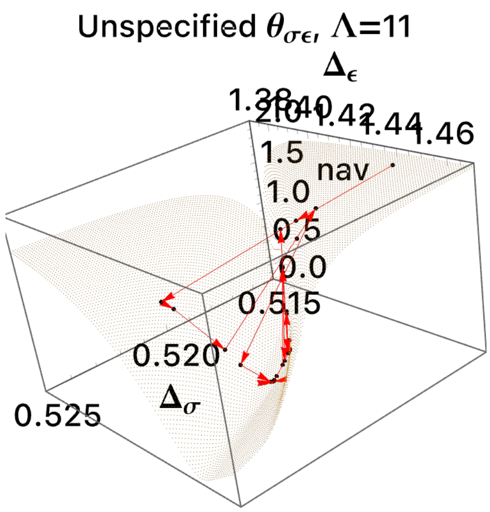


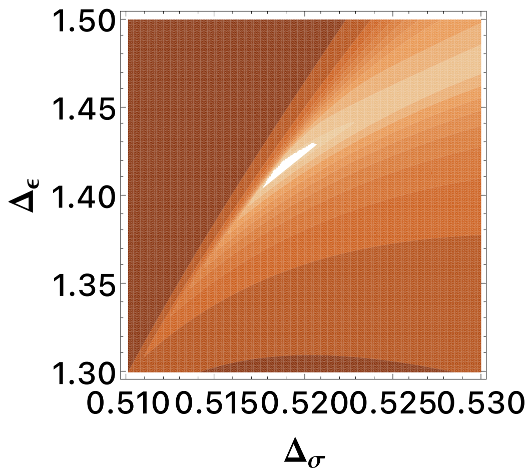
Numerical techniques

1, Cutting surface algorithm [Chester, Landry, Liu, Poland, Simmons-Duffin, SN, Vichi 2019]

Scan over $\left\{\frac{\lambda_{\phi\phi t}}{\lambda_{\phi\phi s}}, \frac{\lambda_{tts}}{\lambda_{\phi\phi s}}, \frac{\lambda_{sss}}{\lambda_{\phi\phi s}}\right\}$ with cost ~ dimension

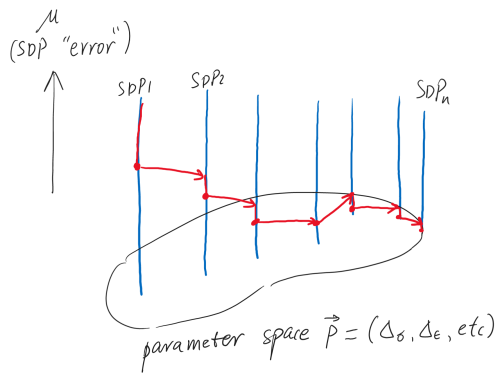
2, Navigator function [Reehorst, Rychkov, Simmons-Duffin, Sirois, SN, van Rees 2021]
SDP compution ⟶ sign of the objective indicating feasible / infeasible





Numerical techniques

3, Skydiving algorithm [Aike Liu, David Simmons-Duffin, NS, Balt van Rees, 2023]
Treat two optimizations (SDP optimization, external parameter search) as a single optimization problem.



Solving the optimization in the parameter $\vec{p}$
and the optimization of SDP ($\mu \to 0$) simultaneously

See [Rychkov, NS 2023] for a review of numerical conformal bootstrap

Today’s talk

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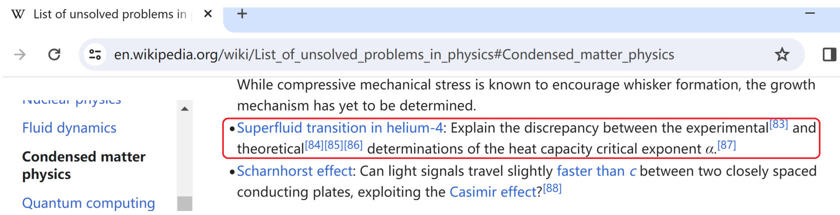
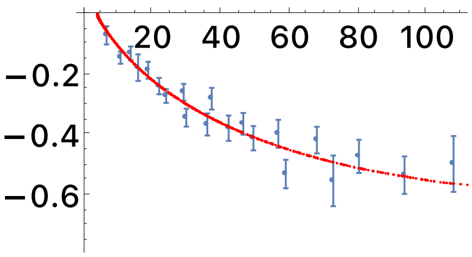

W List of unsolved problems in
en.wikipedia.org/wiki/List_of_unsolved_problems_in_physics#Condensed_matter_physics

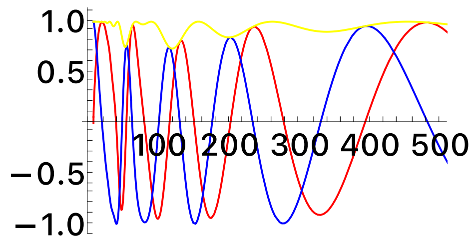
Nuclear physics
Fluid dynamics
Condensed matter physics
Quantum computing

While compressive mechanical stress is known to encourage whisker formation, the growth mechanism has yet to be determined.

- Superfluid transition in helium-4: Explain the discrepancy between the experimental[83] and theoretical[84][85][86] determinations of the heat capacity critical exponent α.[87]
- Scharnhorst effect: Can light signals travel slightly faster than c between two closely spaced conducting plates, exploiting the Casimir effect?[88]

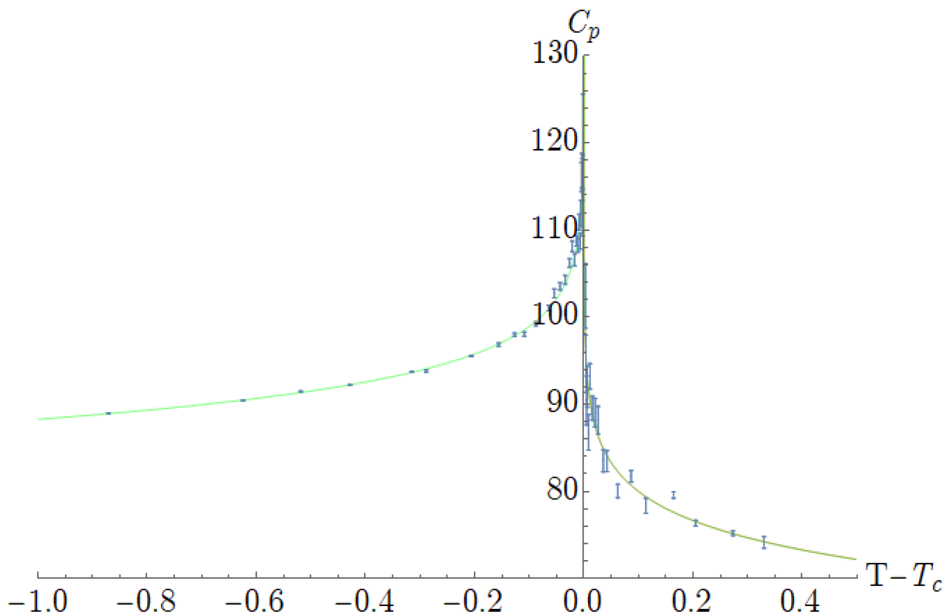
- Application in particle physics : pion-pion scattering





Helium superfluid phase transition

$C_p = A\,(T - T_c)^{-\alpha} + \ldots \qquad \alpha = 2 - \frac{3}{3-\Delta_s}\ , \ \nu = \frac{1}{3-\Delta_s}$



$\alpha = -0.0127\,(3)$

$\nu = 0.6709\,(1)$

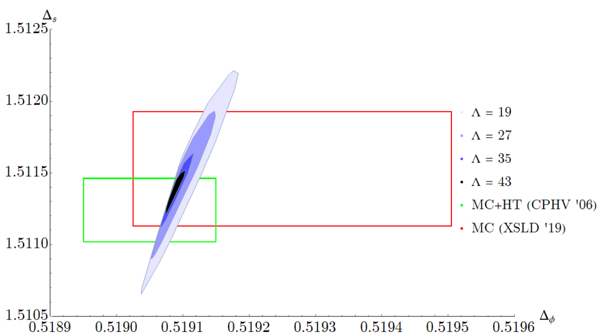
$\Delta_s = 1.50946\,(22)$

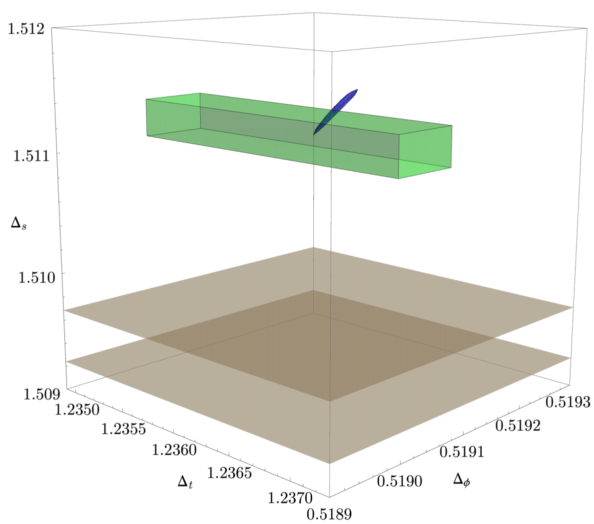
[Lipa, Nissen, Stricker, PHYSICAL REVIEW B 68, 174518 2003]

Bootstrap Helium superfluid phase transition

Bootstrapping $\langle\phi\phi\phi\phi\rangle$, $\langle tttt\rangle$, $\langle ssss\rangle$, $\langle\phi s\phi s\rangle$, $\langle\phi\phi ts\rangle$, $\langle\phi\phi tt\rangle$, $\langle\phi\phi ss\rangle$, $\langle sstt\rangle$

Scan over $\left\{\Delta_\phi, \Delta_s, \Delta_t, \frac{\lambda_{\phi\phi t}}{\lambda_{\phi\phi s}}, \frac{\lambda_{tts}}{\lambda_{\phi\phi s}}, \frac{\lambda_{sss}}{\lambda_{\phi\phi s}}\right\}$

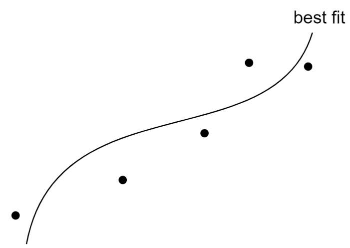


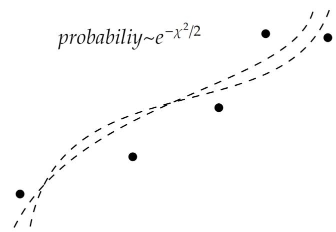


[Chester, Landry, Liu, Poland, Simmons-Duffin, SN, Vichi 2019]

What's wrong in the experimental result?

Local significance v.s. Global significance

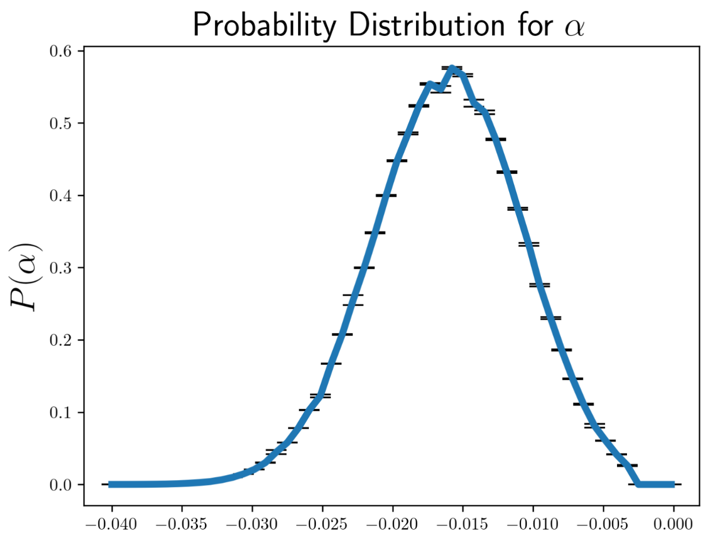




"Look Elsewhere Effect" (LEE)

Experimental error bar in global significance

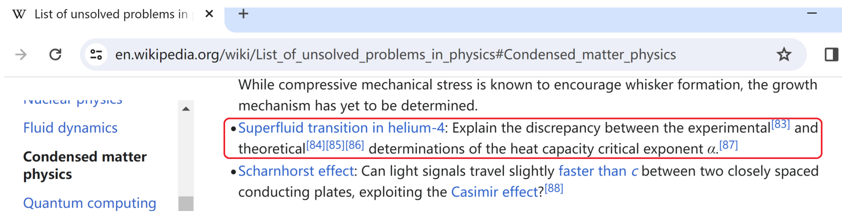
The 95% confidence range : $\alpha \in (-0.031, -0.008)$ Bootstrap: $-0.0152\,(3)$



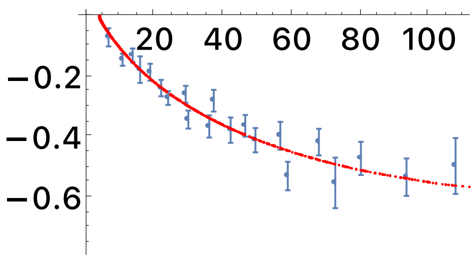
[Landry, Liu, Poland, Simmons-Duffin, SN, to appear]

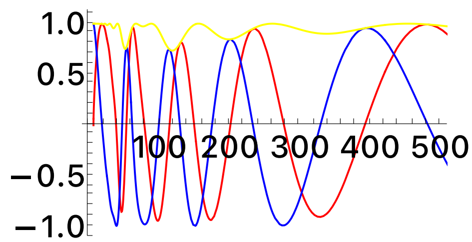
Today’s talk

❧ Application in condensed matter physics : Helium superfluid phase transition



❧ Application in particle physics : pion-pion scattering





pion-pion scattering : the S matrix

Unitary :

$$1 = \langle p_1\, p_2 \,|\, S\, S^\dagger \,|\, p_3\, p_4 \rangle = \sum_X \langle p_1\, p_2 \,|\, S \,|\, X \rangle \langle X \,|\, S^\dagger \,|\, p_3\, p_4 \rangle$$

$$|S_{2\to 2}|^2 = 1 - \sum_{X \in \text{rest}} |S_{2\to X}|^2 \implies |S_{2\to 2}|^2 \le 1$$

$$\mathbb{1} + \begin{pmatrix} -\text{Im}[S_{2\to 2}] & \text{Re}[S_{2\to 2}] \\ \text{Re}[S_{2\to 2}] & \text{Im}[S_{2\to 2}] \end{pmatrix} \ge 0$$

Consistency condition for S matrix

$\pi + \pi \to \pi + \pi$ S matrix : $S \equiv S_{2\to 2}(s, t, u) = 1 + i\, T$, $s + t + u = 4$ (unit : $m_\pi = 1$)

Partial waves : $T(s, t, u) = \sum_\ell T_\ell(s)\, P_\ell(x)$ with $x = \cos(\theta) = 1 + \frac{2\,t}{s-4}$

Unitarity : $2\,\mathrm{Im}[T_\ell] \geq |T_\ell|^2$ or $|S_\ell(s)|^2 \leq 1$ for all ℓ ($S_\ell = 1 + i\, T_\ell$)

Analyticity : $S_\ell(s)$ is analytic, except real $s \geq 4$ and its image under crossing

Exact O(3) isospin symmetry : vector $\otimes$ vector $\to$ singlet $\oplus$ antisym $\oplus$ sym ($0 \oplus 1 \oplus 2$ isospin)

$\langle \pi_i\, \pi_j \,|\, \pi_k\, \pi_l \rangle = \sum_{r=0,1,2} P^{(r)}_{ijkl}\, S^{(r)}(s, t, u) = \sum_{r=0,1,2} P^{(r)}_{ijkl} \sum_\ell S^{(r)}_\ell(s)$ (Example : $P^{(r=0)}_{ijkl} = \delta_{ij}\, \delta_{kl}$)

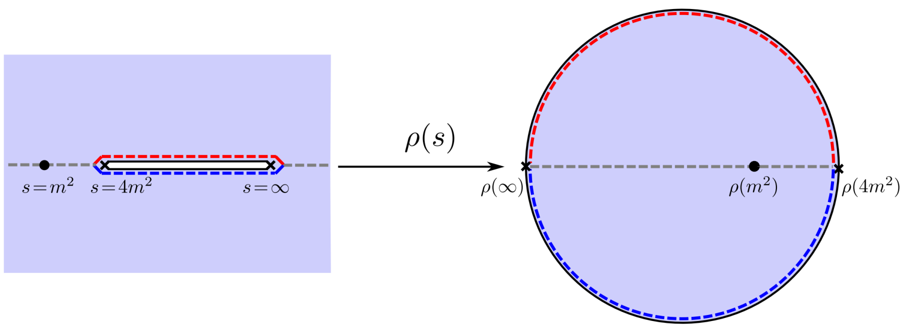
Crossing : $S(s, t, u)$ has crossing symmetry

Translate to SDP : analyticity

[Paulos, Penedones, Toledo, van Rees, Vieira 1708.06765; Guerrieri, Penedones, Vieira 1810.12849]

Analyticity :

$S(s)$ is analytic in s complex plane except $[4, \infty)$ $\Longrightarrow$ $S(s) = \sum a_n \rho_s^n$, $\rho_s = \frac{\sqrt{m^2-4} - \sqrt{4-s}}{\sqrt{m^2-4} + \sqrt{4-s}}$



Translate to SDP : isospin symmetry

$\langle \pi_i \pi_j | \pi_k \pi_l \rangle = S(s\,|\,t, u)\,\delta_{ij}\,\delta_{kl} + S(t\,|\,s, u)\,\delta_{ik}\,\delta_{jl} + S(u\,|\,s, t)\,\delta_{il}\,\delta_{jk}$

Under $i \leftrightarrow j$, $k \leftrightarrow l \implies t \leftrightarrow u \implies S(s\,|\,t, u)$ is symmetric under $t \leftrightarrow u$

Example : $S(s\,|\,t, u) = \sum a_{n\,m}\,(\rho_t^n\,\rho_u^m + \rho_u^n\,\rho_t^m) + \sum b_{n\,m}(\rho_t^n + \rho_u^n)\,\rho_s^m$ (constraint : $s + t + u = 4$)

$S^{(r)}$ is linear combination of $S(\square\,|\,\square, \square)$

Example : $S^{(r=1)}(s, t, u) = S(t\,|\,s, u) - S(u\,|\,s, t)$

Translate to SDP

$S_\ell^{(r)}(s) = \sum a_{n\,m}\,(\ldots) + \sum b_{n\,m}\,(\ldots)$ (...) : functions in s

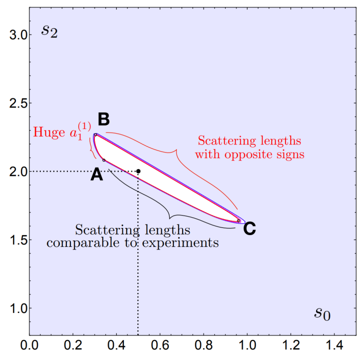
$$\mathbb{1} + \begin{pmatrix} -\mathrm{Im}\left[S_\ell^{(r)}(s)\right] & \mathrm{Re}\left[S_\ell^{(r)}(s)\right] \\ \mathrm{Re}\left[S_\ell^{(r)}(s)\right] & \mathrm{Im}\left[S_\ell^{(r)}(s)\right] \end{pmatrix} \geq 0 \quad \text{for } r = 0, 1, 2;\ \ell = 0, 1, \ldots;\ s \in [4, \infty)$$

SDP : search for α_i such that $\sum_i \alpha_i M_i^{(j)} > 0$ for $j = 1, \ldots J$

Found α_i : we have a candidate $S_\ell(s)$ that satisfies (1) unitarity; (2) analyticity; (3) crossing; (4) isospin symmetry

Bootstrap pion scattering : old result

[Guerrieri, Penedones, Vieira 1810.12849]



“Adler zeros” s_0, s_2 : $T_0(s_0) = 0$, $T_2(s_2) = 0$ ($S_\ell(s) = 1 + i\, T_\ell(s)$)

Shaded : Found α_i . White : Can’t found α_i

Bootstrap pion scattering

Bootstrap is the right way to analyze data in particle scattering experiment

How people found meson (resonance) mass in the past?

A model + experimental constraints + some theory constraints (unitarity/crossing)

$\Downarrow$

An analytic continuation of $S_\ell(s)$ to s complex plane

$\Downarrow$

Find pole on the complex plane : $M = m - i\,\frac{\Gamma}{2}$

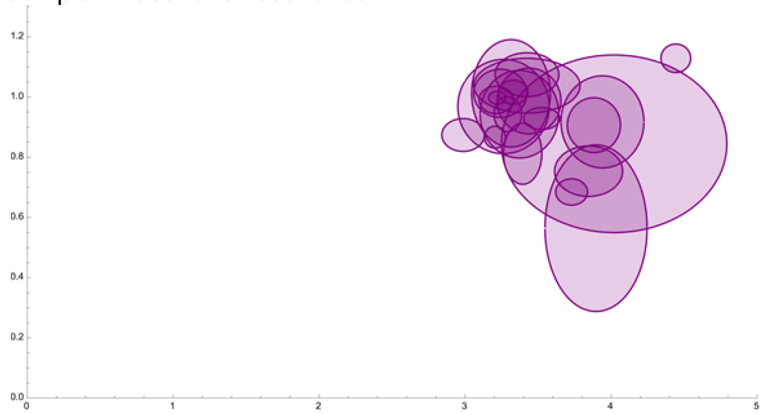
Problem : different continuation $\longrightarrow$ different mass $\longrightarrow$ biggest source of error in PDG

Pion scattering : old approaches

A model + experimental constraints + some theory constraints (unitarity/crossing)

Problem : different continuation ⟶ different mass ⟶ biggest source of error in PDG

Complex mass for σ resonance :



PDG : Re: 400 to 800 MeV   (2.9 to 4.0) ,   Im: 100 to 175 MeV   (0.7 to 1.3)

Statistical error bar may have uncontrollable systematic errors

Bootstrap + experimental constraints

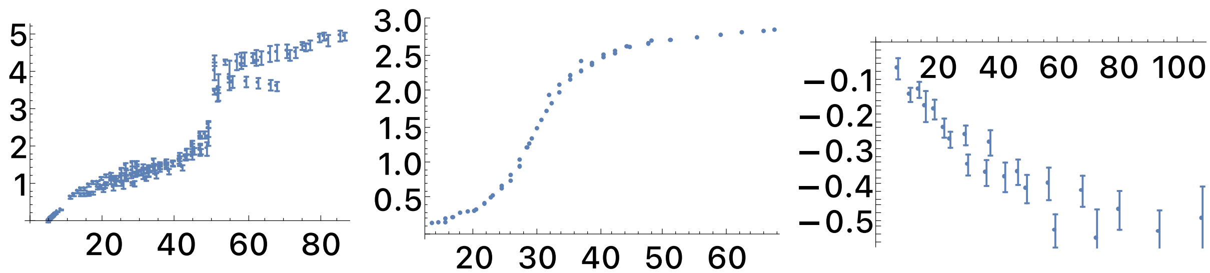
[Kelian Haring, Andrea Guerrieri, NS, ongoing work]

Bootstrap + exp input : systematic analysis of experimental constraints + full theory constraints

$$S = a_{n\,m}\,(\rho_t^n\,\rho_u^m + \rho_u^n\,\rho_t^m) + \ldots$$

Find $a_{n\,m}$ such that $|S| \leq 1$, crossing symmetric, AND matching experimental data

Experimental data : $\mathrm{Arg}\left[S_{\ell=0}^{(r=0)}\right]$, $\mathrm{Arg}\left[S_{\ell=1}^{(r=1)}\right]$, $\mathrm{Arg}\left[S_{\ell=0}^{(r=2)}\right]$



Bootstrap + experimental constraints

New ideas:

1, Navigator function

Scan over parameters $(s_0, s_2, ...)$ such that the result matches experimental data

$$\text{SDP}(s_0, s_2, \ldots) \rightarrow \text{fitness score on } (s_0, s_2, \ldots) \xrightarrow{\text{minimize score}} \text{best fit } w.r.t.\ (s_0, s_2, \ldots)$$

Bootstrap + experimental constraints

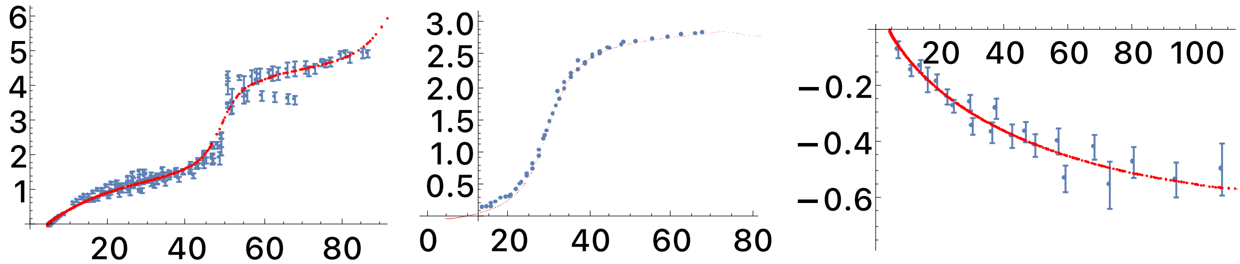
New ideas:

2, An objective that encoded experimental data

$$\text{objective} = \sum_s (\text{Im}[S^{\text{ans}}(s)],\ \text{Re}[S^{\text{ans}}(s)]) \,.\, (\text{Sin}[\phi(s)],\ \text{Cos}[\phi(s)])$$

SDP tries to matches experimental data as much as possible, subject to theory constraints

Bootstrap pion : Preliminary results



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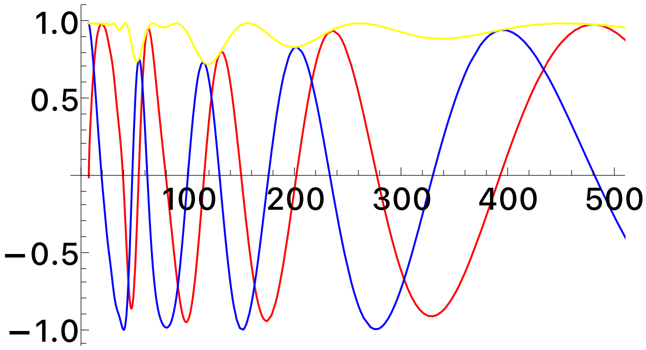
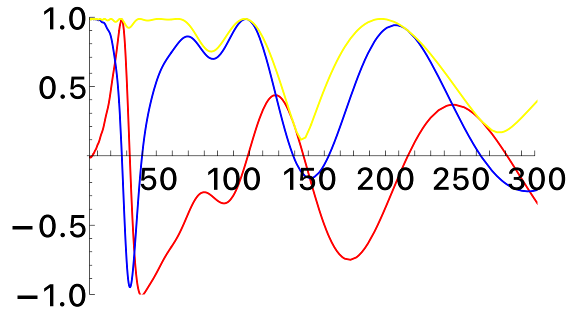
Bootstrap pion : Preliminary results

- Infinite resonance production





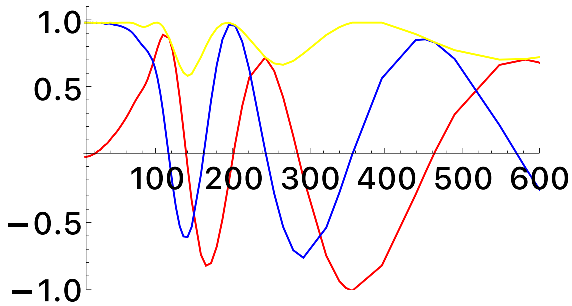
- precise data for σ, ρ, $f(980)$, $f(1350)$

σ : $3.09 + 1.12\,i$ PDG : (2.9 to 4.0) + (0.7 to 1.3) i

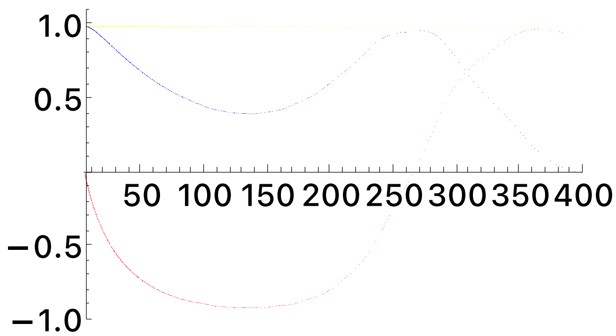
$f(1350)$: 9.60 PDG : (8.7, 10.9) (1200 MeV to 1500 MeV)

Bootstrap pion : Preliminary results

❧ Observation of higher spin Regge trajectory



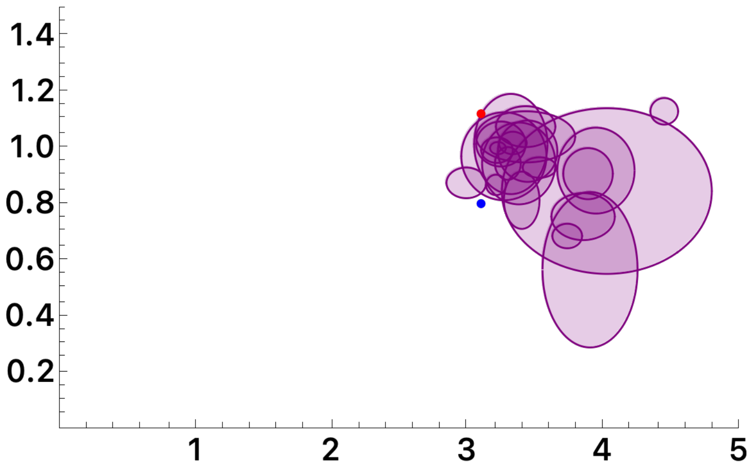
❧ Observation of isospin 2 spin 0 meson (must be at least tetra-quark or more quarks)



Error bar for sigma

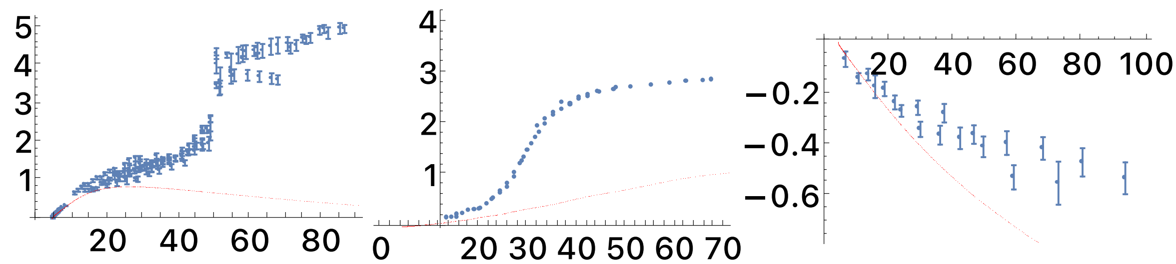
Statistical error bar may have uncontrollable systematic errors

PDG : Re: 400 to 800 MeV (2.9 to 4.0) , Im: 100 to 175 MeV (0.7 to 1.3)



Bootstrap value : $m_\sigma = 3.09 + 1.12\,i$ (Red) . What's if σ is away from $3.09 + 1.12\,i$?

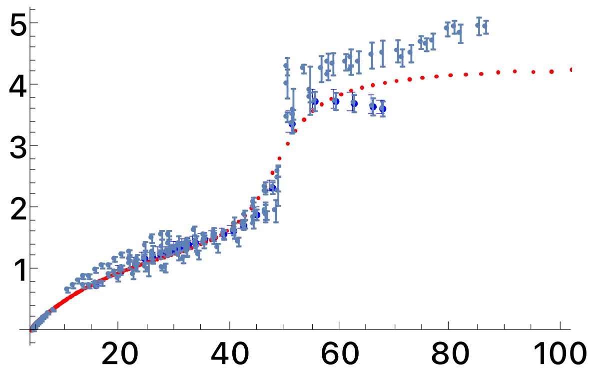
Error bar for sigma

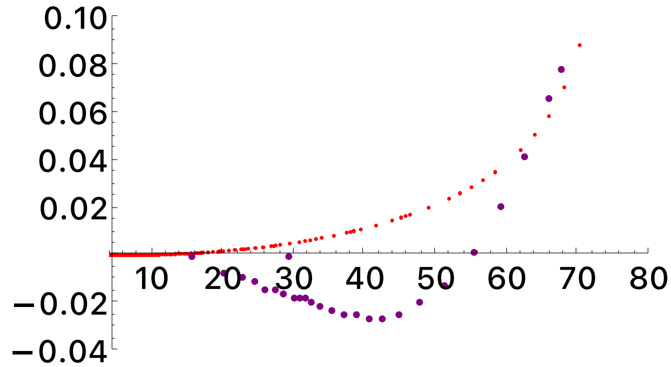


“robust error bar” : points outside the error bar can’t match experimental data

Verifying experimental data

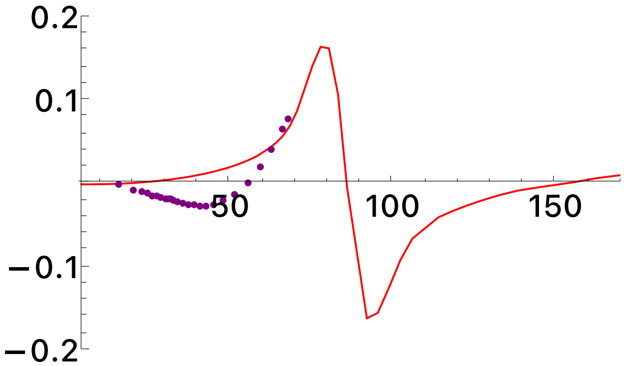
- Some experimental data are likely wrong!

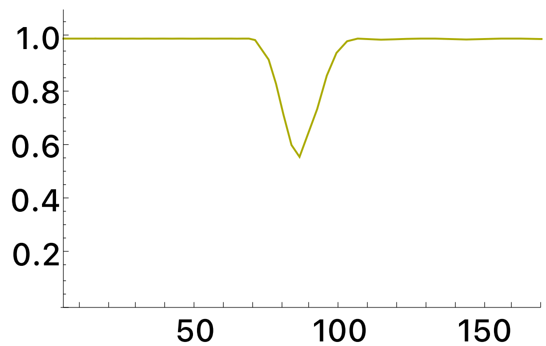




"Experimental" data : $N + N \overset{\pi}{\rightarrow} \pi + \pi + \pi \xrightarrow{\text{extrapolate}}$ on shell $\pi + \pi \rightarrow \pi + \pi$

Verifying experimental data





Matches with José R. Peláez 's analysis

The original dream of bootstrap

en.wikipedia.org/wiki/Bootstrap_model

spin greater than 1 and without the then undiscovered phenomenon of confinement, it is naively inconsistent with the rved Regge behavior of hadrons.

Chew and followers believed that it would be possible to use crossing symmetry and Regge behavior to formulate a consistent S-matrix for infinitely many particle types. The Regge hypothesis would determine the spectrum, crossing and analyticity would determine the scattering amplitude (the forces), while unitarity would determine the self-consistent quantum corrections in a way analogous to including loops. The only fully successful implementation of the program required another assumption to organize the mathematics of unitarity (the narrow resonance approximation). This meant that all the hadrons were stable particles in the first approximation, so that scattering and decays could be thought of as a perturbation. This allowed a bootstrap model with infinitely many particle types to be constructed like a field theory — the lowest order scattering amplitude should show Regge behavior and unitarity would determine the loop corrections order by order. This is how Gabriele Veneziano and many others constructed string theory, which remains the only theory constructed from general consistency conditions and mild assumptions on the spectrum.

Many in the bootstrap community believed that field theory, which was plagued by problems of definition, was fundamentally inconsistent at high energies. Some believed that there is only one consistent theory which requires infinitely many particle species and whose form can be found by consistency alone. This is nowadays known not to be true, since there are many theories which are nonperturbatively consistent, each with their own S-matrix. Without the narrow-resonance approximation, the bootstrap program did not have a clear expansion parameter, and the consistency equations were often complicated and unwieldy, so that the method had limited success. It fell out of favor with the rise of quantum chromodynamics, which described mesons and baryons in terms of elementary particles called quarks and gluons.