Gradient Properties of RG Flows

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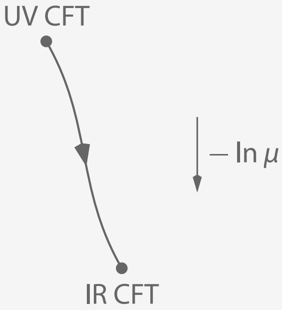
(Based on work with William Pannell)

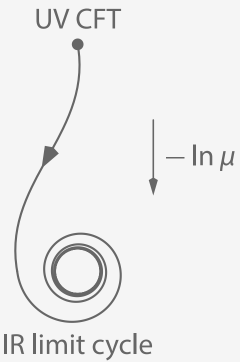
Renormalisation group flows

In relativistic QFT we typically think of RG flows as flows in theory space, with a particularly interesting case that of a CFT in the UV flowing to a CFT in the IR after a relevant deformation:



An essential constraint is unitarity.

Limit cycles



Limit cycles have been suggested as possible endpoints of RG flows in the early ’70s by Wilson, but they have never been found in relativistic unitary QFT.

Limit cycles would arise for unitary theories that are scale-invariant but not conformal. (Fortin, Grinstein & AS, 2012)

a-theorem

Strongest version: Is there a positive-definite metric G_{IJ} in the space of couplings such that $\partial_I A = G_{IJ}\beta^J$ for some scalar A?

Strong version: Is there a quantity that decreases monotonically in the flow from the UV to the IR?

Weak version: In the flow between a UV and an IR fixed point, is there a quantity a such that $a_{\text{UV}} > a_{\text{IR}}$?

A monotonically-decreasing quantity was found in $d = 2$ by Zamolodchikov in 1986.

At the RG flow endpoints it becomes the central charge of the corresponding CFT.

The RG flow in $d = 2$ is gradient in conformal perturbation theory.

a-theorem in $d = 4$

4d CFT in curved space: $T^{\mu}{}_{\mu} = a E_4 + c W^{\mu\nu\rho\sigma} W_{\mu\nu\rho\sigma}$

It was suggested by Cardy in 1988 that the coefficient of the Euler term in the trace anomaly, called a, may be the quantity that satisfies a (weak) a-theorem in $d = 4$.

There have been lots of successful checks of Cardy's suggestion over the years.

A nice chain of arguments by Komargodski and Schwimmer proved the weak version of the a-theorem in 2011.

The relevant quantity is indeed a: $a_{UV} > a_{IR}$.

In perturbation theory, the strong version of the a-theorem was established by Jack and Osborn in 1990.

The quantity they considered also becomes a at fixed points.

Multiscalar models

We will be interested in the strongest version of the *a*-theorem.

We will first consider multiscalar theories in $d = 4$, defined by

$$\mathscr{L} = \tfrac{1}{2}\partial^{\mu}\varphi_i\partial_{\mu}\varphi_i + \tfrac{1}{2}\kappa_{ij}\varphi_i\varphi_j + \tfrac{1}{4!}\lambda_{ijkl}\varphi_i\varphi_j\varphi_k\varphi_l \,.$$

The beta function of the quartic coupling in these theories has been recently computed to six loops. (Bednyakov & Pikelner, 2021; using results of Kompaniets & Panzer, 2017)

We want to see if

$$\beta^I = G^{IJ}\partial_J A\,, \qquad \partial_I = \frac{\partial}{\partial\lambda^I}\,, \qquad I = (ijkl)\,.$$

We will also discuss this problem in $d = 4 - \varepsilon$.

The metric

We will demand that the metric G_{IJ} is Riemannian, i.e.

- symmetric, $G_{IJ} = G_{JI}$, and
- positive-definite, $G \succ 0$.

The symmetry property is necessary for one to be able to choose local coordinates such that

$$G_{IJ} = \delta_{IJ} \quad \Rightarrow \quad \beta_I = \partial_I A \,,$$

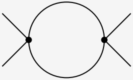
and this implies that the eigenvalues of the matrix

$$\partial_I \beta_J = \partial_I \partial_J A \,,$$

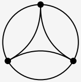
are real at real fixed points and unitarity is maintained.

These eigenvalues determine scaling dimensions of quartic operators in φ at fixed points.

One loop

$$^1\beta_{ijkl} = \lambda_{ijmn}\lambda_{mnkl} + \text{permutations} =$$ 

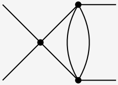
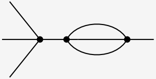
Obviously

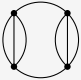
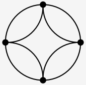
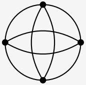
$$^1A = \lambda_{ijkl}\lambda_{klmn}\lambda_{mnij} =$$ 

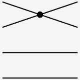
$$^1G_{ijkl;mnpq} = \delta_{ijkl;mnpq} =$$ 

The solution is not unique as we can rescale A by any constant and the metric by the inverse of that constant.

Two loops

$^2\beta = {}^2b$  $+ {}^2c$ 

$^2A = {}^2a_1$  $+ {}^2a_2$  $+ {}^2a_3$ 

$^2G = {}^2g$ 

Demanding $\beta = G\partial A$ now gives

$$
\begin{aligned}
{}^2a_1 - {}^2c &= 0\,,\\
{}^2a_2 + \tfrac{1}{4}\,{}^1b\,{}^2g &= 0\,,\\
2({}^2a_3) - 3({}^2b) + {}^1b\,{}^2g &= 0\,.
\end{aligned}
$$

These can be solved with $^2g = 0$.

Higher loops

The number of independent terms that contribute to the various quantities discussed here grows rapidly with the loop order.

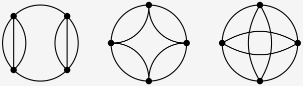
	1	2	3	4	5	6
β	1	2	7	23	110	571
A	1	3	5	17	42	177
G	1	1	7	18	97	453

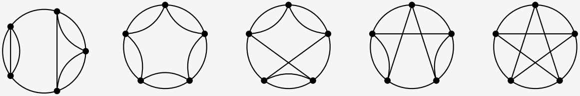
As we saw, up to two loops a globally flat metric suffices.

At three loops it turns out that we need a correction to the metric to satisfy $\beta = G\partial A$.(Wallace & Zia, 1974)

Need for metric at three loops

Contrast diagrams needed at two vs. three loops:





Diagrams with inequivalent vertices give rise to distinct contributions to the beta function.

If those beta function contributions have incompatible coefficients, then only a metric can restore the gradient property.

Four loops and beyond

	1-loop	2-loop	3-loop	4-loop	5-loop	6-loop
A	1	3	5	17	42	177
G_{IJ}	1	1	7	18	97	453
Equations	1	3	10	36	164	819

Starting at four loops, it appears that we don't have enough freedom to satisfy all equations. At this order naive counting shows that we do, in fact, as some freedom remains from undetermined lower order coefficients.

Naive counting fails, however, as some of our degrees of freedom appear only in specific combinations.

At four loops $\beta = G\partial A$ cannot be satisfied unless the beta function coefficients satisfy four constraints.(Jack & Poole, 2018)

They do, so the flow is gradient at four loops.

Scheme dependence

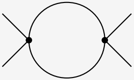
Coefficients in the beta function beyond leading order are scheme-dependent.

When a Feynman diagram has subdivergences, these need to be subtracted.

Of course one may decide to subtract subdivergences in different ways, i.e. one may choose to remove finite parts along with the divergences.

Such choices cannot possibly affect physical quantities or statements.

In the present context, scheme changes are described by coupling redefinitions:

$\lambda \to \lambda + r$  $+ O(\lambda^3)$.

Relations like the ones satisfied by the four-loop beta function coefficients are scheme-independent.

Results at five loops

At five loops there are 110 beta function terms, 42 bubble diagrams and 97 contributions to the symmetric metric.

We found 37 contstraints among beta function coefficients that need to be satisfied for the flow to be gradient.

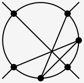
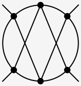
Some are simple, but most appear rather complicated, e.g.

$$\beta_{116} + 2\,\beta_{136} + 2\,\beta_{137} + \beta_{13}\,\beta_3 + 2\,\beta_{22}\,\beta_3 + 2\,\beta_{37} + \frac{\beta_4{}^2}{4} + \beta_{40}$$
$$+ \beta_{10}\left(2\,\beta_3{}^2 + \beta_4 + \frac{\beta_5}{3}\right) + \frac{\beta_3{}^2\,\beta_5}{3} + \beta_{52} + \beta_{63} + \frac{\beta_{64}}{2} + \frac{\beta_{65}}{2} + 4\,\beta_8{}^2 ==$$
$$2\,\beta_{10}{}^2 + 2\,\beta_{113} + 4\,\beta_{138} + 2\,\beta_{140} + 2\,\beta_{25}\,\beta_3 + 4\,\beta_3\,\beta_{30} + 2\,\beta_3\,\beta_{32}$$
$$+ \frac{\beta_3{}^2\,\beta_4}{2} + 2\,\beta_{42} + \beta_{44} + \frac{\beta_{47}}{2} + \frac{\beta_4\,\beta_5}{6} + \frac{\beta_{51}}{2} + \beta_{54} + \beta_{56} + \beta_{68} + \frac{5\,\beta_4\,\beta_8}{2}$$

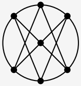
They are all scheme-independent and satisfied!

A simple constraint at five loops

Consider

$$ {}^{5}\beta \supset {}^{5}b_{67} $$  $$ + \; {}^{5}b_{69} $$  .

It turns out that these diagrams both come from

$$ {}^{5}A \supset {}^{5}a_{42} $$ 

and there is no other way to generate them in $G\partial A$.

This means that

$$ {}^{5}b_{67} = 4\,{}^{5}a_{42} \quad \text{and} \quad {}^{5}b_{69} = 3\,{}^{5}a_{42}\,, $$

or

$$ 3\,{}^{5}b_{67} = 4\,{}^{5}b_{69}\,. $$

Results at six loops

At six loops there are 571 beta function terms, 177 bubble diagrams and 453 contributions to the symmetric metric.

We found 234 contstraints among beta function coefficients that need to be satisfied for the flow to be gradient.

Again, few are simple, and most appear rather complicated.

They are all scheme-independent.

All but 5 are satisfied.

This constitutes a direct proof that the beta function is not a gradient vector field.

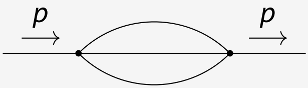
But is the RG flow gradient or not?

Renormalisation subtleties

In dim-reg, beta functions arise from the simple poles of counterterms introduced via

$$\varphi_B = \mu^{-\varepsilon/2} Z^{1/2} \varphi_R \,, \qquad \lambda_B = \mu^{\varepsilon}(\lambda_R + L) \,.$$

But there is a subtlety: we never actually compute $Z^{1/2}$. Rather, we compute $(Z^{1/2})^T Z^{1/2}$, e.g. via



But this is ambiguous, because one can introduce an arbitrary orthogonal matrix between $(Z^{1/2})^T$ and $Z^{1/2}$. Then,

$$\gamma \to \gamma - \omega \,, \qquad \beta_I \to \beta_I + (\omega\lambda)_I \,,$$

with ω in the Lie algebra of the orthogonal group.

RG flow

Usually this subtlety is implicitly resolved by requiring that the anomalous dimension matrix be symmetric.

Nevertheless, physical statements should not depend on such choices.

It immediately follows that the beta function is not the vector field whose gradient properties we should be checking.

What is the correct vector field?

Jack & Osborn gave the answer in 1990.

The problematic equation turns out to be

$$T^{\mu}{}_{\mu} = \beta^I \mathcal{O}_I \,.$$

Trace of stress-energy tensor

This is true only for zero-momentum insertions of $T^\mu_{\ \mu}$. More generally,

$$T^\mu_{\ \mu} = \beta^I \mathcal{O}_I + \partial_\mu J^\mu .$$

If we have scalar fields, for example, $J^\mu = S_{ij}\varphi_i \partial^\mu \varphi_j$, with S_{ij} anti-symmetric.

Using the equations of motion we see then that

$$T^\mu_{\ \mu} = B^I \mathcal{O}_I, \qquad B^I = \beta^I - (S\lambda)^I .$$

S can be computed in perturbation theory using dim-reg.

B^I is invariant since $\beta^I \to \beta^I + (\omega\lambda)^I$ and $S \to S + \omega$.

The lesson is that the RG flow is governed by the B function, not the beta function.

How to compute S

This S is perhaps unfamiliar.

It appears due to the need to renormalise correlation functions involving both T^μ_μ and $\mathcal{O}_I$.

The renormalisation of such correlation functions can be systematised by considering local couplings, $\lambda \to \lambda(x)$, in addition to considering the theory in curved space.

New counterterms are then necessary, to cancel new divergences that arise in this background, e.g.

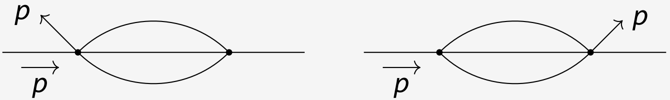
$$\mathscr{L}_{\text{c.t.}} \supset (N_I)_{ij} \partial^\mu \lambda^I \varphi_i \partial_\mu \varphi_j \,, \qquad N_I = \sum_n \frac{N_I^{(n)}}{\varepsilon^n} \,.$$

A careful treatment shows that

$$S_{ij} = (N_I^{(1)})_{ij} \lambda_I \,.$$

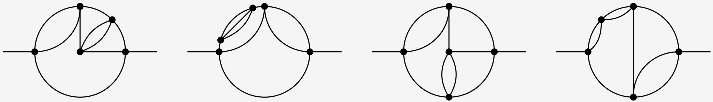
How to compute S

Diagrammatically, to determine N we have to look for diagrams that contribute to the anomalous dimension of the field, but we need to compute them allowing momentum to flow out of the coupling, e.g.



This specific diagram gives contributions to N, but not S. This is because it is symmetric under exchange of the external legs.

In multiscalar models, the first potential contribution to S is at five loops. There are four relevant asymmetric diagrams:



From β to B

At five loops we had found 37 constraints among beta function coefficients that need to be satisfied for the flow to be gradient.

It turns out that the four contributions to S at five-loop order do not affect these constraint equations.

At six loops there are 19 contributions to S.

We had found that the beta function was not a gradient vector field.

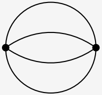
After including the five- and six-loop S, we find that with four conditions on S the B vector field is indeed gradient:

$$
\begin{aligned}
{}^5s_2 &= -\frac{259}{4608}\,, \\
192\,{}^5s_3 - 384\,{}^5s_4 &= 36\zeta_3 - 31\,, \\
{}^5s_1 - 12\,{}^5s_2 - {}^5s_4 &= -12\,{}^6s_8 + 6\,{}^6s_9 + 6\,{}^6s_{16}\,, \\
4\,{}^5s_1 - 2\,{}^5s_3 &= 2\,{}^6s_3 - {}^6s_6 - 4\,{}^6s_{10} + 2\,{}^6s_{15} - {}^6s_{17}\,.
\end{aligned}
$$

$4 - \varepsilon$ dimensions

In $d = 4 - \varepsilon$ the beta function has one more term, linear in λ:

$$\beta_{ijkl} = -\varepsilon\, \lambda_{ijkl} + \mathrm{O}(\lambda^2)\,.$$

This term is easy to accommodate via the bubble  .

However, such a contribution to A will then get multiplied with terms from the metric and will start appearing all over the place at higher orders.

These contributions can be absorbed in order-ε terms in A and G_{IJ}.

Both are needed, and then ε^2 terms need to cancel. This sets further constraints among beta function coefficients at five and six loops.

All of them (4 at five and 32 at six loops) are scheme-independent, independent of the constraints we found in $d = 4$ and satisfied!

Line defects

In the ε expansion $\Delta_\varphi = 1 - \frac{1}{2}\varepsilon < 1$, and so one can consider

$$S_{\mathrm{CFT}} \to S_{\mathrm{CFT}} + h_i \int_{-\infty}^{\infty} d\tau\, \varphi_i(\tau, \mathbf{0}) \,.$$

S_{CFT} could involve only scalars, or scalars and fermions.

The question is if there exists an IR defect CFT, where the couplings h_i flow to a fixed point.

Recently the so-called g-theorem was established non-perturbatively for line defects in any d.(Komargodski, Cuomo & Mezei; 2021)

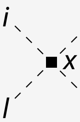
The g-theorem says that there exists a monotonically-decreasing function for RG flows on a defect.

The beta function of h_i was recently computed to next-to-leading order including fermions in the bulk.(Pannell & AS, 2023)

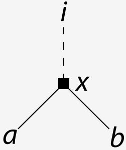
Feynman rules

$$x_1 \;\text{- - - - - -}\; x_2 \;=\; \frac{\Gamma(\frac{1}{2}d-1)}{4\pi^{d/2}} \frac{1}{(x_{12}{}^2)^{\frac{1}{2}d-1}} \,, \qquad x_{12} = x_1 - x_2 \,,$$

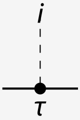
$$x_1, a \;\text{———}\; x_2, \dot{\alpha} \;=\; \frac{(d-2)\Gamma(\frac{1}{2}d-1)}{4\pi^{d/2}} \frac{x_{12}{}^{\mu}\sigma_{\mu a\dot{\alpha}}}{(x_{12}{}^2)^{d/2}} \,,$$



$$= -\mu^{\varepsilon}\lambda_{ijkl} \int d^d x \,,$$

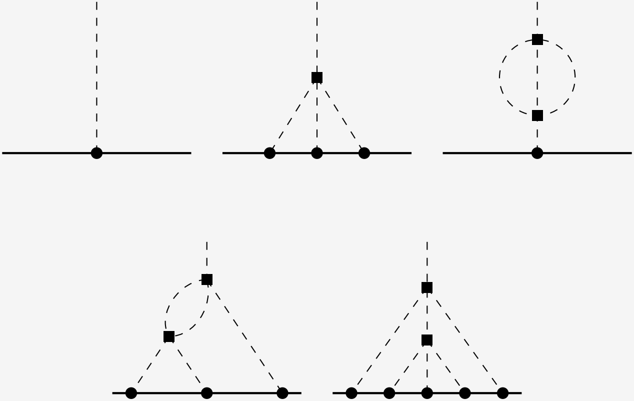


$$= -\mu^{\varepsilon/2} y_{iab} \int d^d x \,,$$

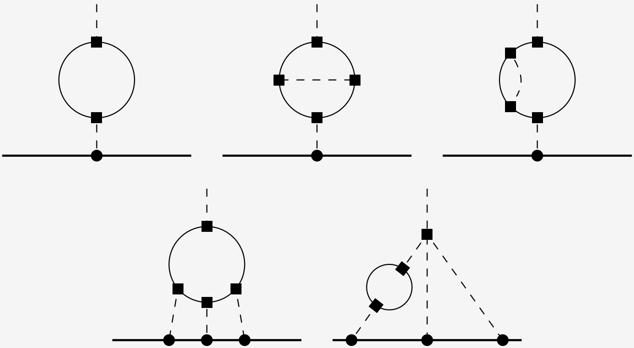


$$= -\mu^{\varepsilon/2} h_i \int_{-\infty}^{\infty} d\tau \,.$$

Diagrams with scalars only



Diagrams with scalars and fermions



Defect beta function

The defect beta function is

$$\beta_i = -\tfrac{1}{2}\varepsilon h_i + \tfrac{1}{6}\lambda_{ijkl}h_j h_k h_l + \tfrac{1}{12}\lambda_{ijkl}\lambda_{jklm}h_m$$
$$- \tfrac{1}{4}\lambda_{ijkl}\lambda_{klmn}h_j h_m h_n - \tfrac{1}{12}\lambda_{ijkl}\lambda_{jmnp}h_k h_l h_m h_n h_p$$
$$+ \tfrac{1}{2}Y_{ij}h_j - \tfrac{1}{4}\widetilde{Y}_{ijkj}h_k - \tfrac{3}{8}\widetilde{Y}_{ijjk}h_k$$
$$+ (1 - \tfrac{1}{6}\pi^2)\widetilde{Y}_{ijkl}h_j h_k h_l - \tfrac{1}{4}\lambda_{ijkl}Y_{lm}h_j h_k h_m \,,$$

with

$$Y_{ij} = \mathrm{Tr}(y_i y_j^* + y_i^* y_j)\,,$$
$$\widetilde{Y}_{ijkl} = \mathrm{Tr}(y_i y_j^* y_k y_l^* + y_i^* y_j y_k^* y_l)\,.$$

This flow is also gradient,

$$\beta_i = G_{ij}\partial^j H\,, \qquad \partial^i = \frac{\partial}{\partial h_i}\,,$$

with $G_{ij} = \delta_{ij} + \delta G_{ij}$, where the correction is symmetric and needed only for the last term in the beta function.

Conclusion

Perturbative computations in a variety of contexts support a gradient property for the RG flow.

This is often highly non-trivial and follows from relations among beta function coefficients.

In the highest-loop case, the gradient property of the RG flow rests on relations among five- and six-loop contributions to S.

It is crucial to identify the correct vector field that describes the RG flow.

As a future direction we may consider the gradient question in theories with scalars, fermions and gauge fields in $d = 4$ and $d = 4 - \varepsilon$. Some work already exists in the literature.