# AdS Three-Body Problem at Large Spin

Jeremy A. Mann, with Petr Kravchuk

$50 + \epsilon$ years of conformal bootstrap, Pisa, 23/02/2024

# 0. Motivation

## 0.1 Multi-twist operators in CFT

- $[\mathcal{O}_1 \ldots \mathcal{O}_n]_{0,J}, \;\; \tau = \Delta - J = \tau_1 + \ldots + \tau_n + \gamma_n(J)$

- $n = 2$: good understanding from analytic bootstrap **[Ferrara,Gatto,Grillo ’71, …, KZ ’12, FKPS-D ’12, …]**

- Intuition: two-body problem in AdS

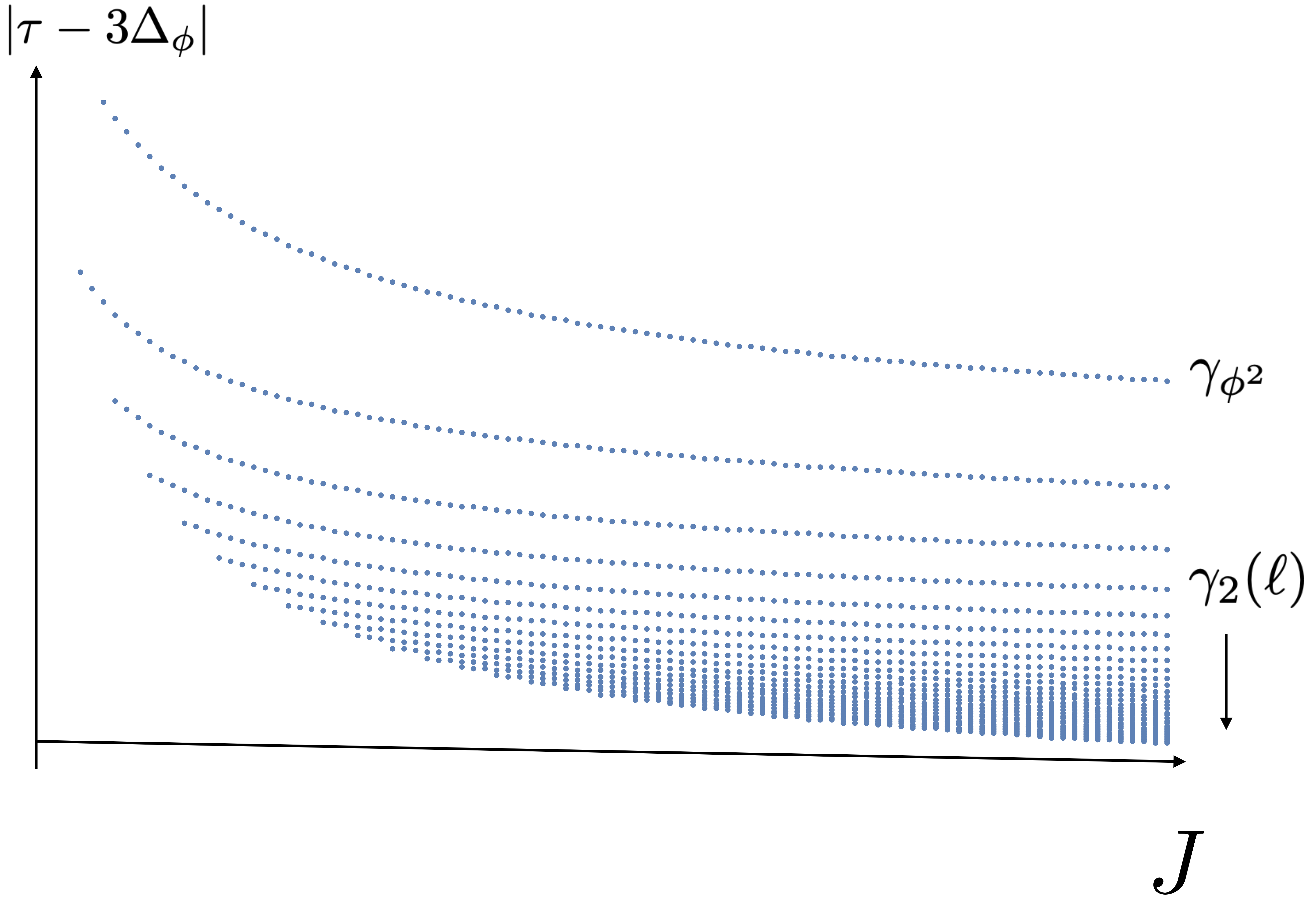
- Consequence: $\gamma_3(J) \to \gamma_2(\ell)$ at large J **c.f. Johan’s talk [HKO ’23]**

- But difficult as $\ell \sim J$ !

- Back to AdS: maybe a tractable three-body problem?

# 0. Motivation

## 0.2 Illustration: triple-twist spectrum at weak coupling



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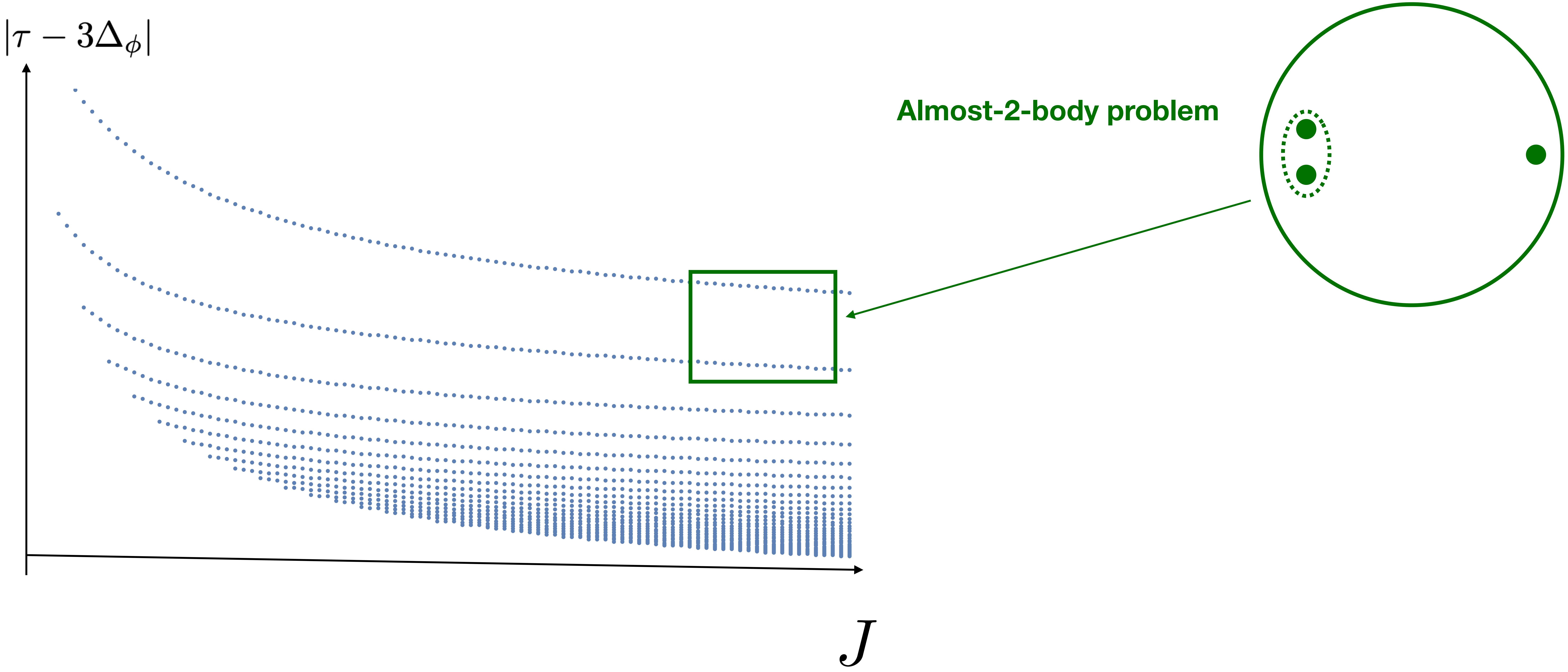

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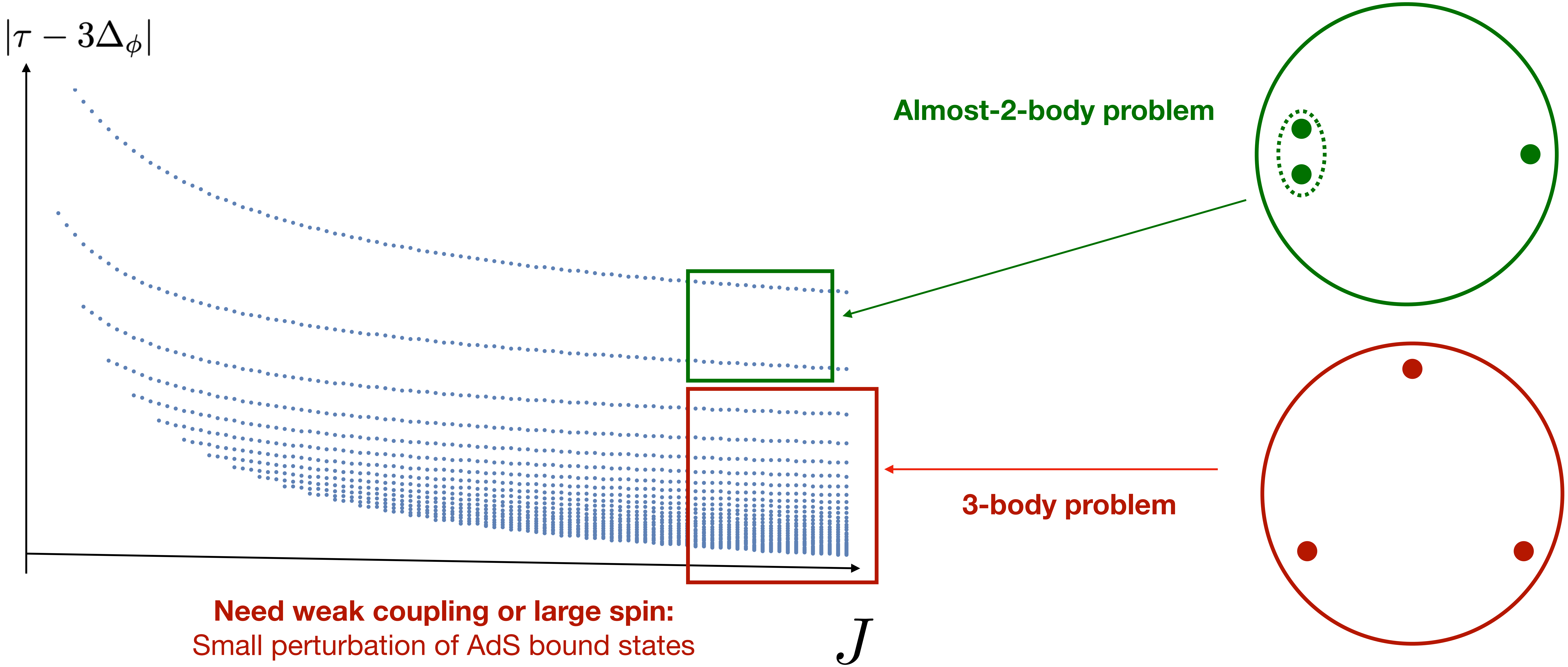
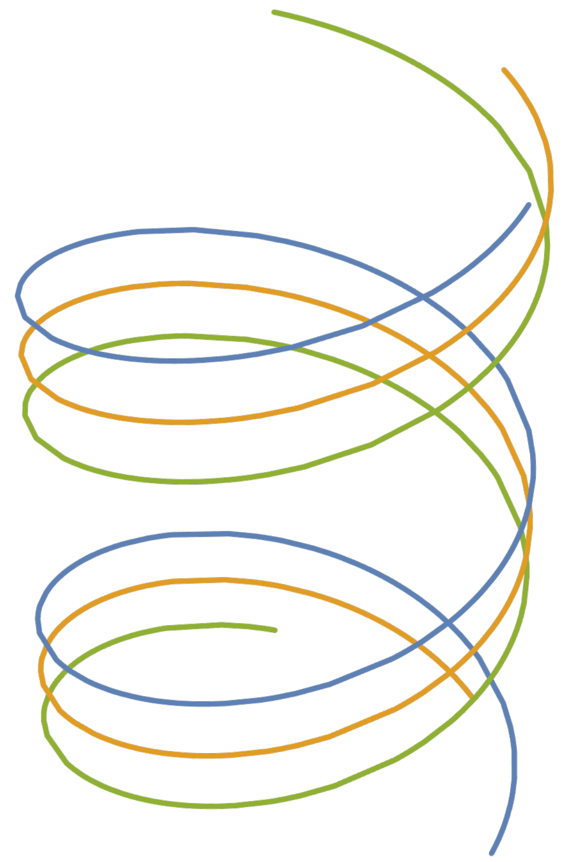


# Plan

1. Multi-particle states in free theory



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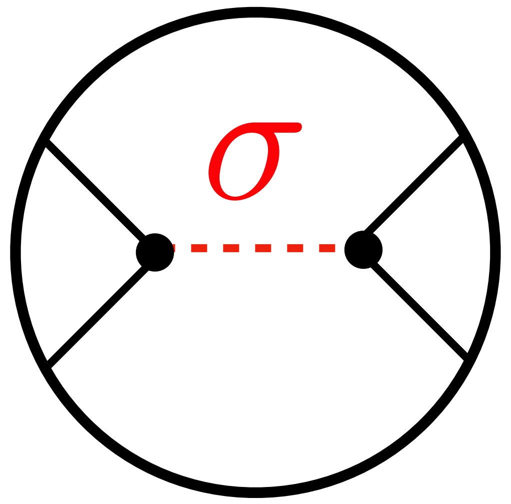
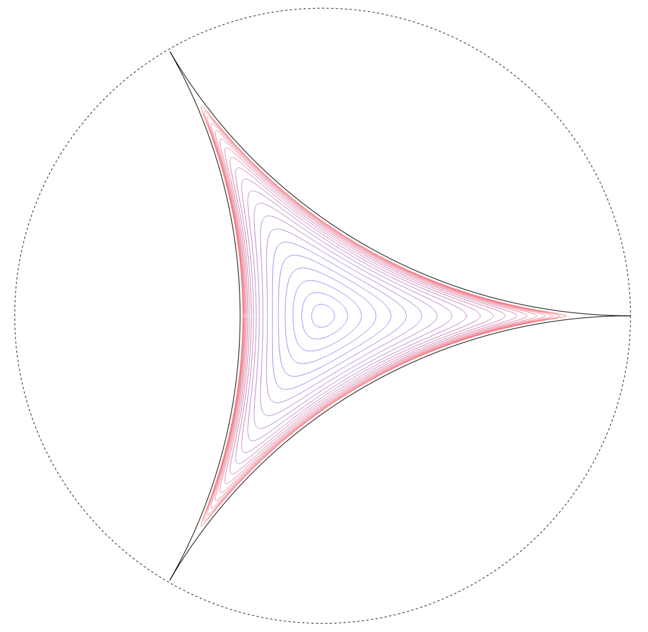



3. Semi-classics at large spin



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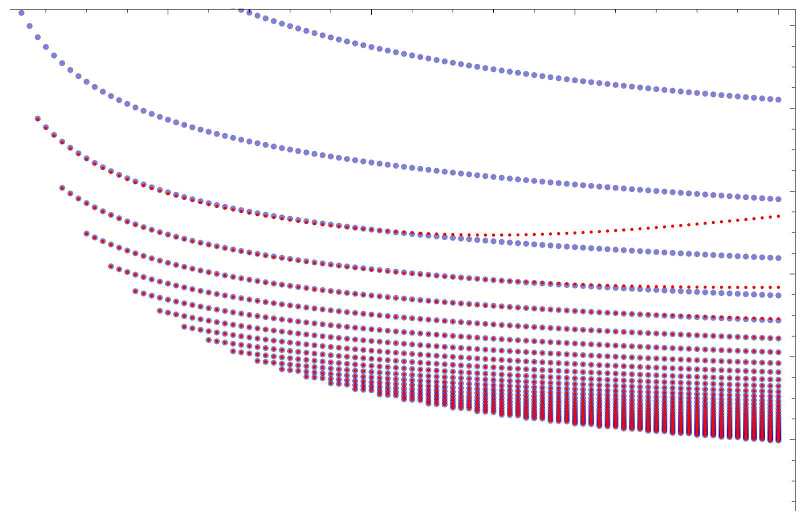


5. Conclusion and outlook

# 1. States in Free Theory

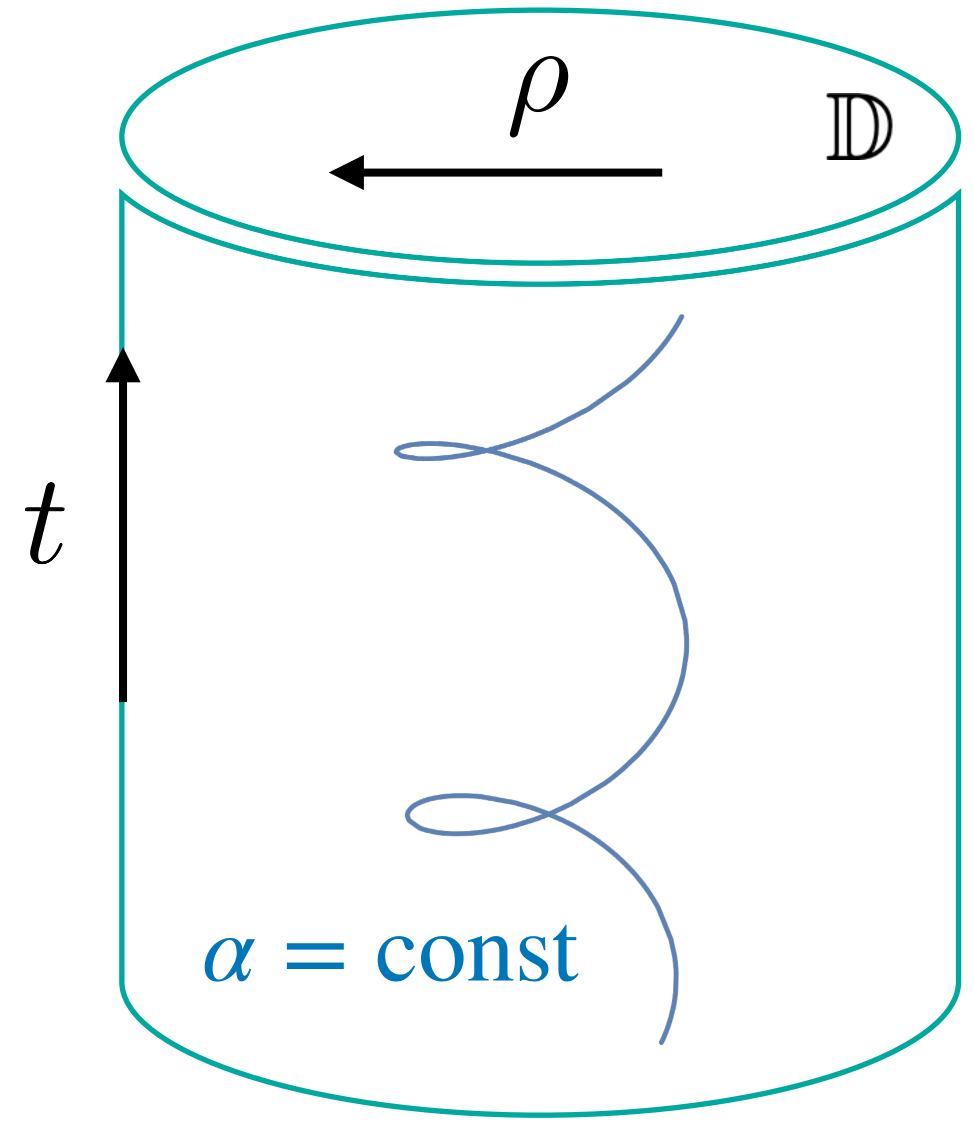
## 1.1 One-particle states

- Goal: organize states by twist and spin

$$ds^2_{\mathrm{AdS}_3} = -\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, d\varphi^2$$

- Rotating frame: $\phi := \varphi - t, \quad \alpha := e^{i\phi}\tanh\rho \in \mathbb{D}$

$$ds^2_{AdS_3} = -dt^2 + d\rho^2 + \sinh^2\rho\, d\phi^2 - 2\sinh^2\rho d t d\phi$$



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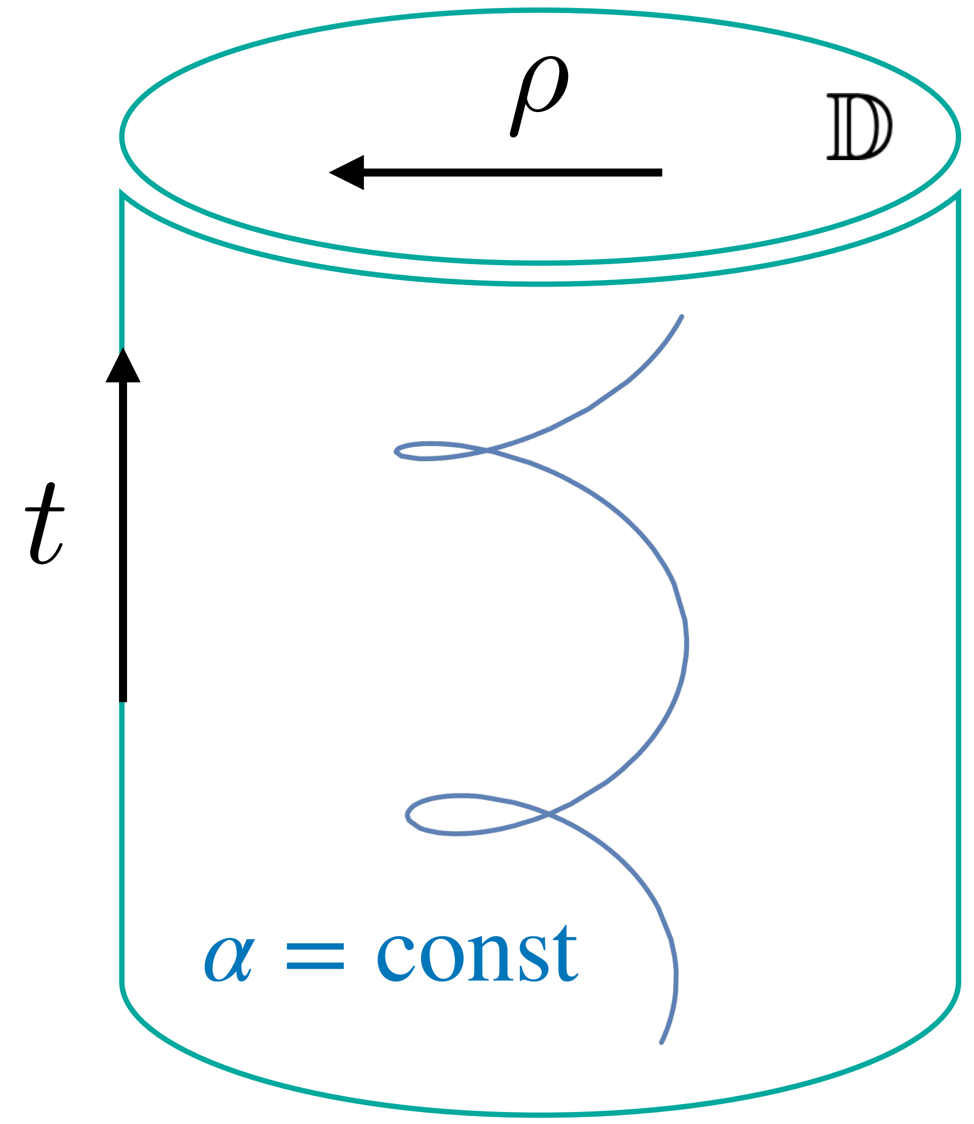
Magnetic field ⇒ Landau levels



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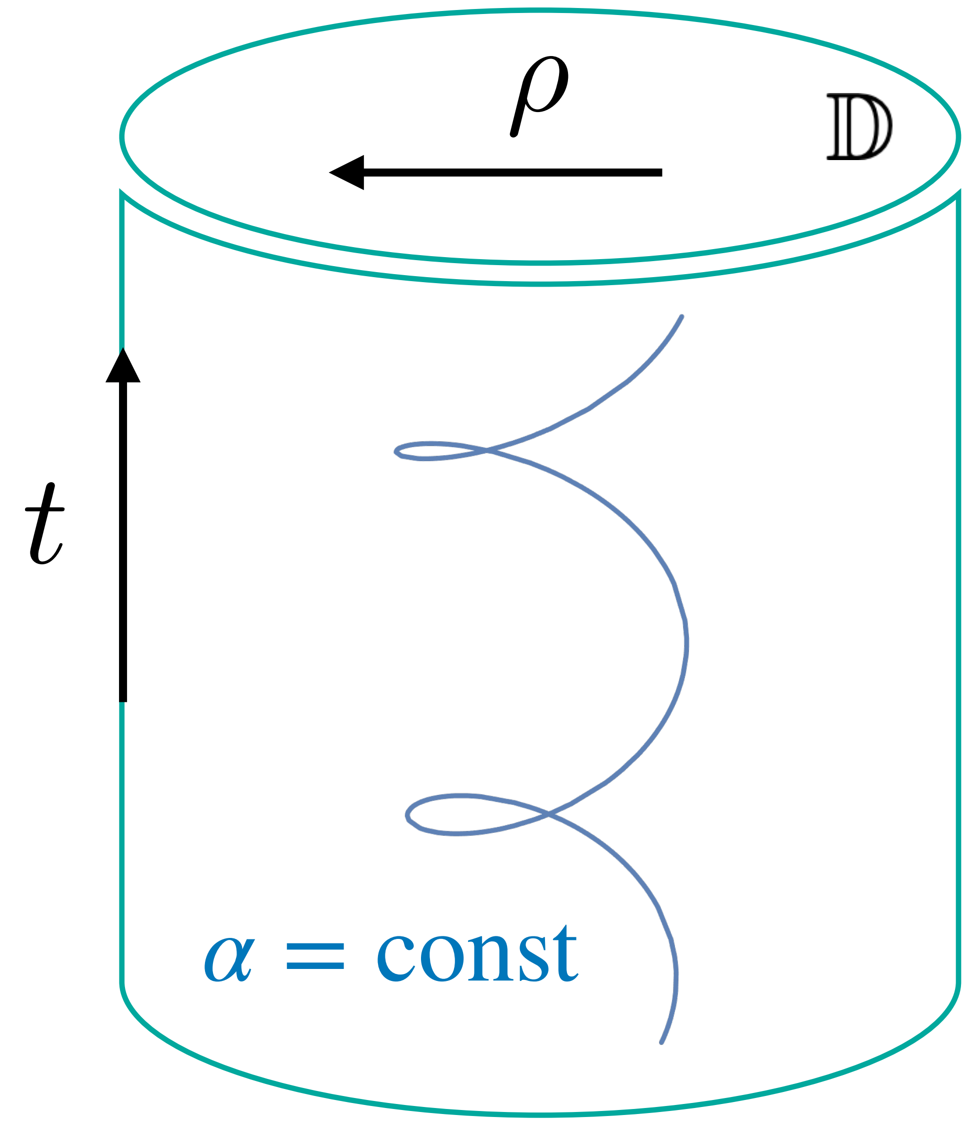


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**Magnetic field ⇒ Landau levels**

- LLL Hilbert space: $\langle 0|\Phi(x)|\Psi_J\rangle = e^{-i\Delta_\phi t}(1-|\alpha|^2)^{\frac{\Delta_\phi}{2}}\alpha^J$ **[e.g. Comtet '87]**

- "fuzzy disk": $\alpha, \bar{\alpha}$ parameterize phase space **[c.f. Yin-Chen's talk]**

- **Nota bene**: Oscillations on disk/transverse directions require O(1) twist corrections or transverse spin

# 1. States in Free Theory

## 1.2 Multi-particle states

- Wavefunctions: $\langle 0|\Phi(x_1)\dots\Phi(x_n)|\Psi\rangle = e^{-in\Delta_\phi t}\prod_{i=1}^{n}(1-|\alpha_i|^2)^{\frac{\Delta_\phi}{2}}\psi(\alpha_1,\dots,\alpha_n)$
- CFT primary states: $\psi(\lambda\alpha_i+\beta)=\lambda^J\psi(\alpha_i)$ **su(1,1) lowest weight**
- Scalar product: $C^n_{\Delta_\phi}\langle\psi_1,\psi_2\rangle = \left(C_{\Delta_\phi}\frac{\Delta_\phi-1}{\pi}\right)^n\prod_i\int_{\mathbb{D}} d^2\alpha_i(1-\alpha_i\bar{\alpha}_i)^{\Delta_\phi-2}\bar{\psi}_1\psi_2$

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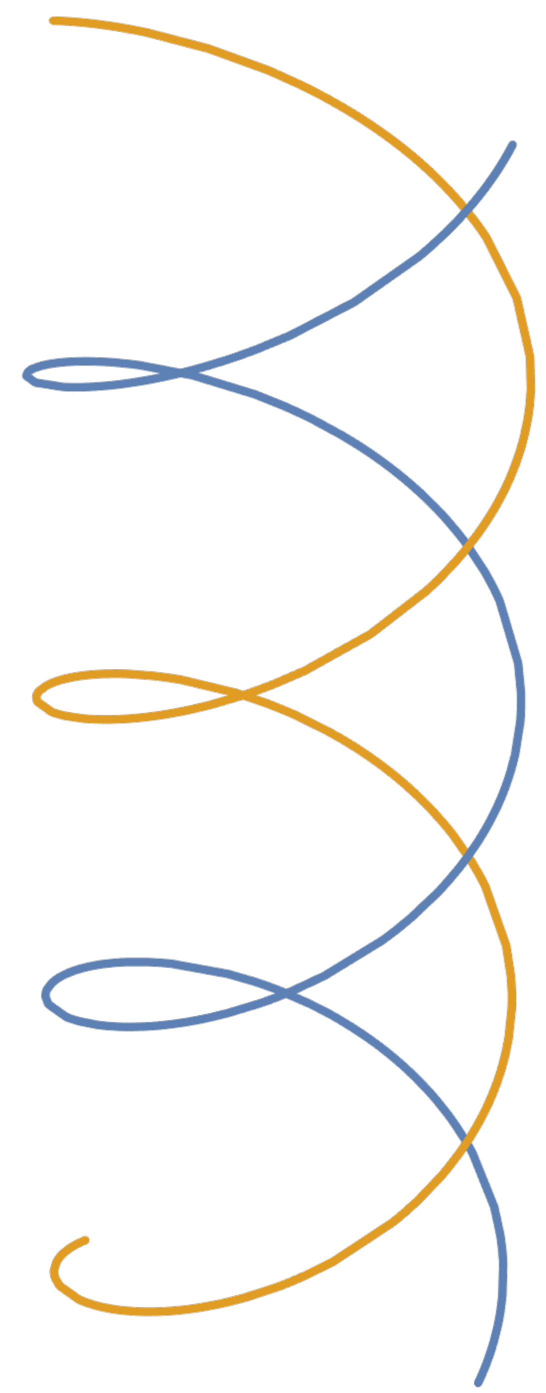
- Ex: n=2: $\psi(\alpha_1,\alpha_2)=(\alpha_1-\alpha_2)^J$ $\quad |\alpha_{\max}| = 1-\frac{\Delta_\phi}{2J}+O(J^{-2})$

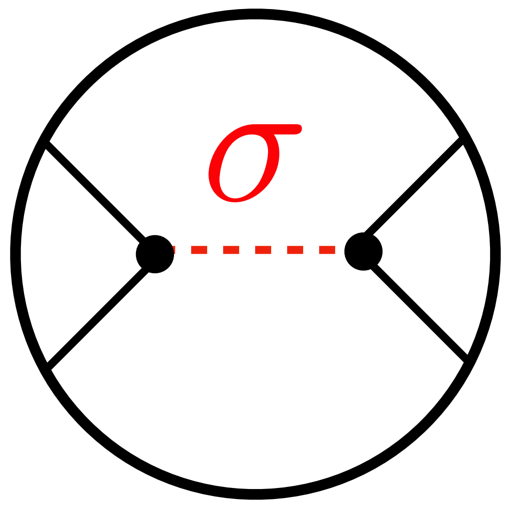


c.f. [Alday-Maldacena '07]

- **Nota bene**: Equivalent to canonical quantization of GFF in AdS
  + restriction to leading twist states

# 2. Toy Model & Effective Hamiltonian

- Two scalars, cubic coupling




$$\mathcal{L}_{int} = g\,\Phi^2\sigma$$

Integrate out sigma

$$V = g^2 \int d^dx\sqrt{-g}d^dy\sqrt{-g}\,K_\sigma(x,y)\ :\Phi(x)^2\Phi(y)^2:$$

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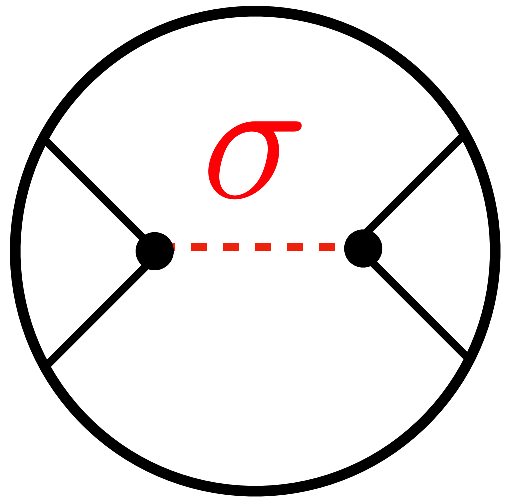


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**Integrate out sigma**

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- Hamiltonian formalism

  $\longrightarrow$ interaction energy in 1st order perturbation theory **[Fitzpatrick, Shih ’11]**

$$\gamma_{ij} = \langle\Psi_i|V[\phi]|\Psi_j\rangle, \quad \langle\Psi_i|\Psi_j\rangle = \delta_{ij}$$

- Effective potential: $\langle\Psi_1|\,V\,|\Psi_2\rangle = \langle\psi_1(\alpha_i), H(\alpha_i,\bar{\alpha}_i)\psi_2(\alpha_i)\rangle$

- Symmetries of the disk: $H \equiv H(s_{ij}),\ s_{ij} := \dfrac{|\alpha_i-\alpha_j|^2}{(1-|\alpha_i|^2)(1-|\alpha_j|^2)}$

# 2. Toy Model & Effective Hamiltonian

- For 2 particles: $$H(s_{12}) = U_2(s_{12}) = \sum_{n=0}^{\infty} b_n(\Delta_\phi, \Delta_\sigma, d)\, s_{12}^{-\frac{\Delta_\sigma}{2}}\, {}_2F_1\left(\frac{\Delta_\sigma}{2}, \frac{\Delta_\sigma}{2}; \Delta_\sigma; -s_{12}^{-1}\right)$$

- Expected large-spin asymptotics for double-twist operators: $$\frac{\langle \alpha_{12}^J, s^{-h} \alpha_{12}^J \rangle}{\langle \alpha_{12}^J, \alpha_{12}^J \rangle} = \frac{\Gamma(\Delta_\phi + h - 1)^2}{\Gamma(\Delta_\phi - 1)^2} J^{-2h} + O(J^{-2h-1})$$

- Agreement with L.I.F. at d=3 and finite integer J **[Albayrak, Meltzer, Poland ’19]**

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- For n>2: $$H(s_{ij}) = \sum_{i<j} U_2(s_{ij})$$

- Finite matrix at finite J: $\mathrm{dim}(J) = J/6 + O(1)$

⟶ **exact diagonalization**

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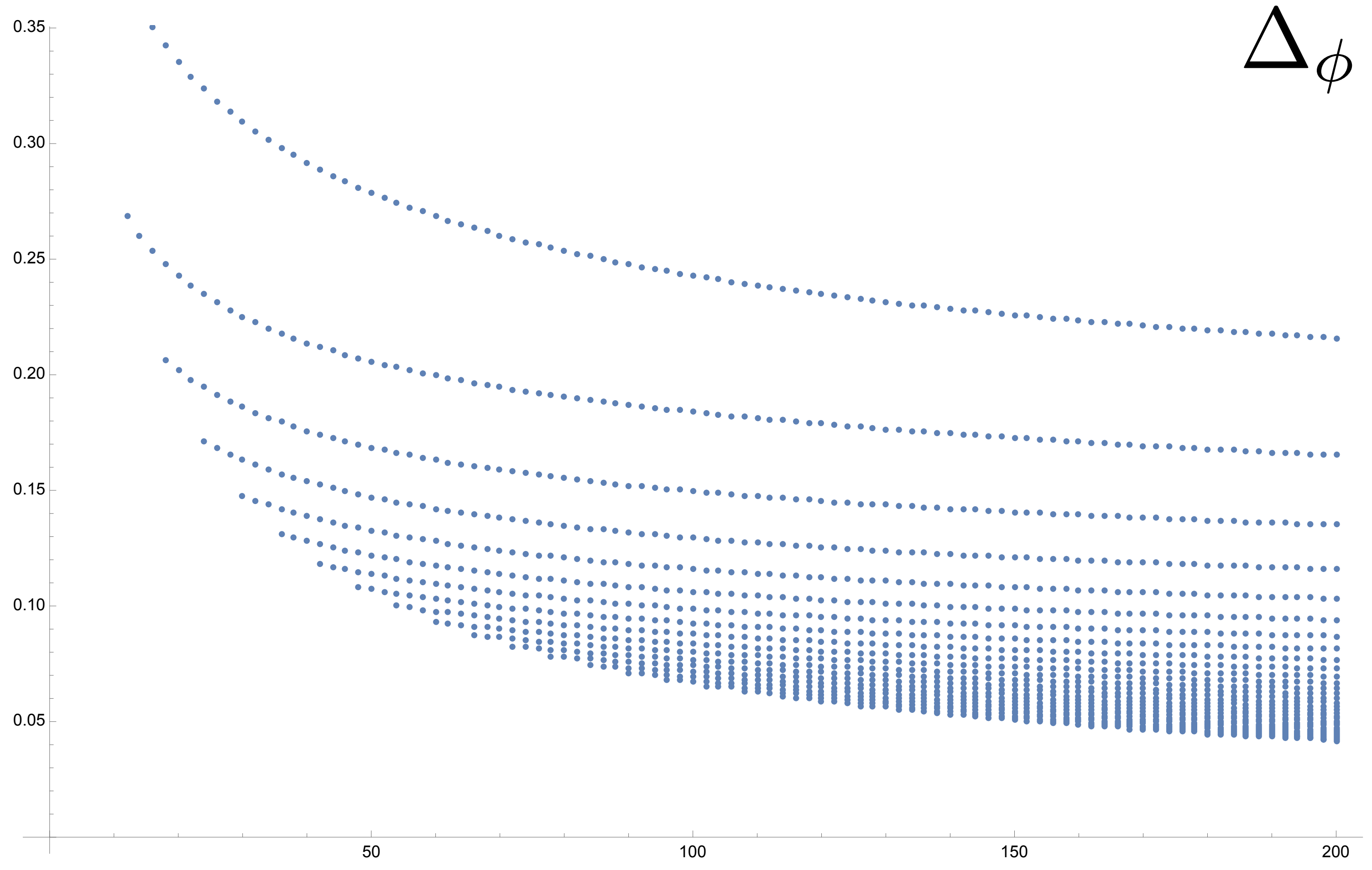


$$\Delta_\phi = 1.234, \quad \Delta_\sigma = 0.6734$$

$$J \leq 200$$

$$J = \text{even}$$

# 3. 3-particle semiclassics

## 2.1 Reduction to one d.o.f. and classical limit

- Goal: explicitly reduce to one d.o.f. using primary property: $\psi(\lambda\alpha_i + \beta) = \lambda^J \psi(\alpha_i)$, $(\lambda, \beta) \in \mathbb{C}^* \ltimes \mathbb{C}$

  $\longrightarrow$ expect $\mathcal{M}_{\text{class}} \cong \mathbb{C}P^1 / S_3$

- **But only SU(1,1) is manifest** in scalar product & Hamiltonian:

$$\langle \psi_1, \psi_2 \rangle = \mathcal{N} \prod_i \int_{\mathbb{D}} d^2\alpha_i (1 - \alpha_i \bar{\alpha}_i)^{\Delta_\phi - 2} \bar{\psi}_1 \psi_2, \quad H = \sum_{i<j} U_2 \left( \frac{|\alpha_i - \alpha_j|^2}{(1 - \alpha_i \bar{\alpha}_i)(1 - \alpha_j \bar{\alpha}_j)} \right)$$

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- Solution: integrate out the orbits $\quad d^2\alpha_1d^2\alpha_2d^2\alpha_3\,(\dots)\longrightarrow d^2\lambda d^2\beta d^2z\,(\dots)$

$$\langle-,-\rangle\longrightarrow\int_{\mathbb{C}P^1}d^2z\,N_J(z,\bar{z})\overline{(-)}(-)\qquad H\longrightarrow H_J(z,\bar{z})$$

# 3. 3-particle semiclassics

## 2.1 Reduction to one d.o.f. and classical limit

- Together: ingredients for Berezin-Toeplitz quantization: **[Berezin '75,..., Charles '06,..., Le Floch '12, ....]**

1. Holomorphic wave functions $L^J \to \mathbb{C}P^1$
2. Inner product
3. Hamiltonian = "Toeplitz operator": $\Pi \, H_J(z, \bar{z}) \, \Pi$

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**Classical limit:** $\omega = dz \wedge d\bar{z}\, \partial\bar{\partial} \log \lambda_0^2, \quad H_0(z,\bar{z}) = E$

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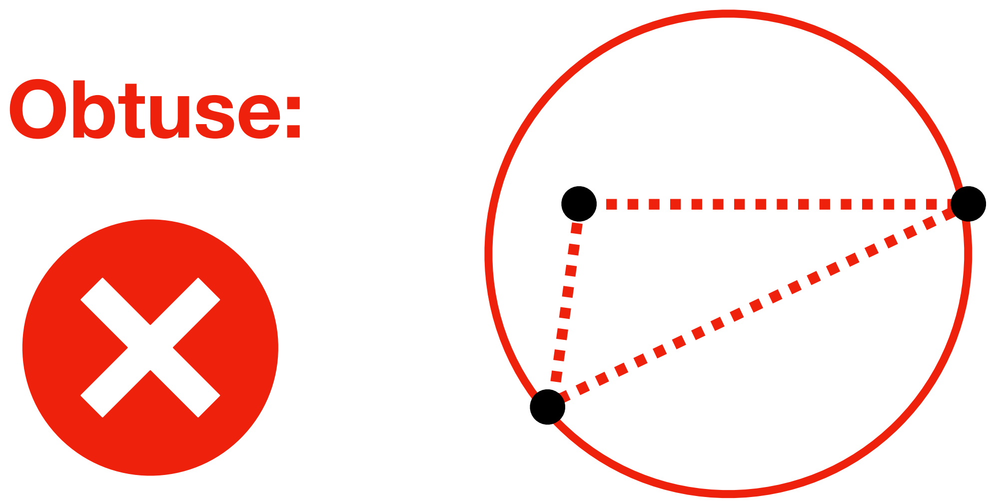
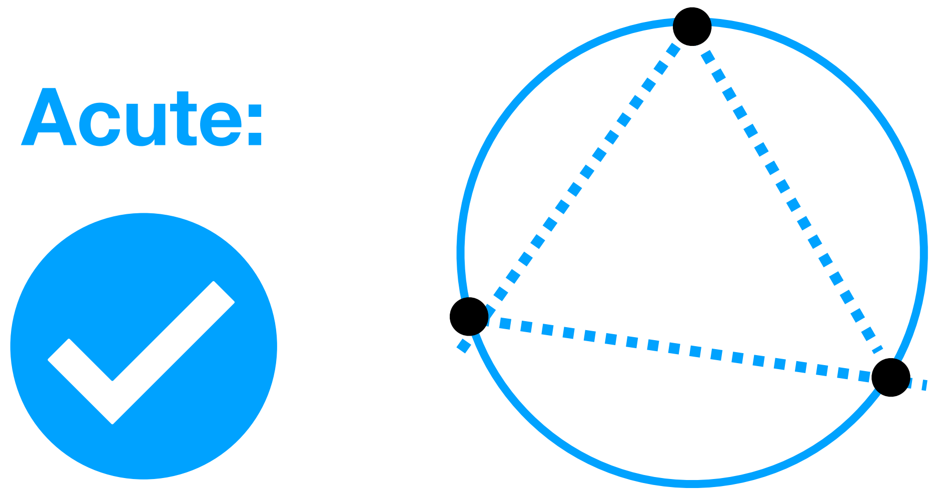
- Here: $\lambda_0 =$ smallest possible $|\lambda|$ such that $|\lambda^{-1}(\alpha_i - \beta)| \leq 1$

**Obtuse:**



**OR**

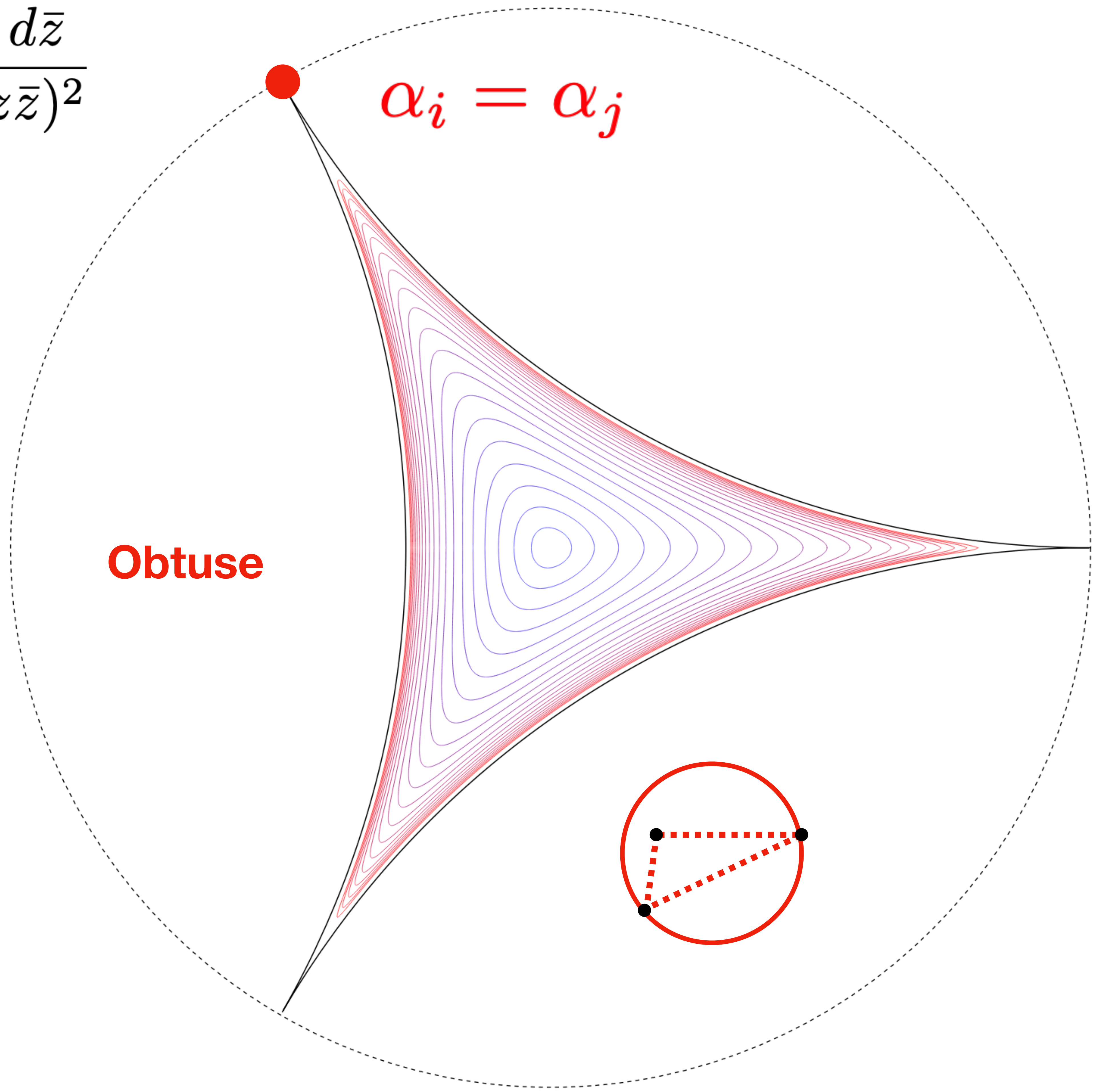
**Acute:**



# 3. 3-particle semiclassics

## 2.2 Phase space and classical Hamiltonian

$\mathcal{M}_{\text{class}} \cong \mathbb{C}P^1/S_3,\ \omega = 2i\frac{dz \wedge d\bar{z}}{(1-z\bar{z})^2}$



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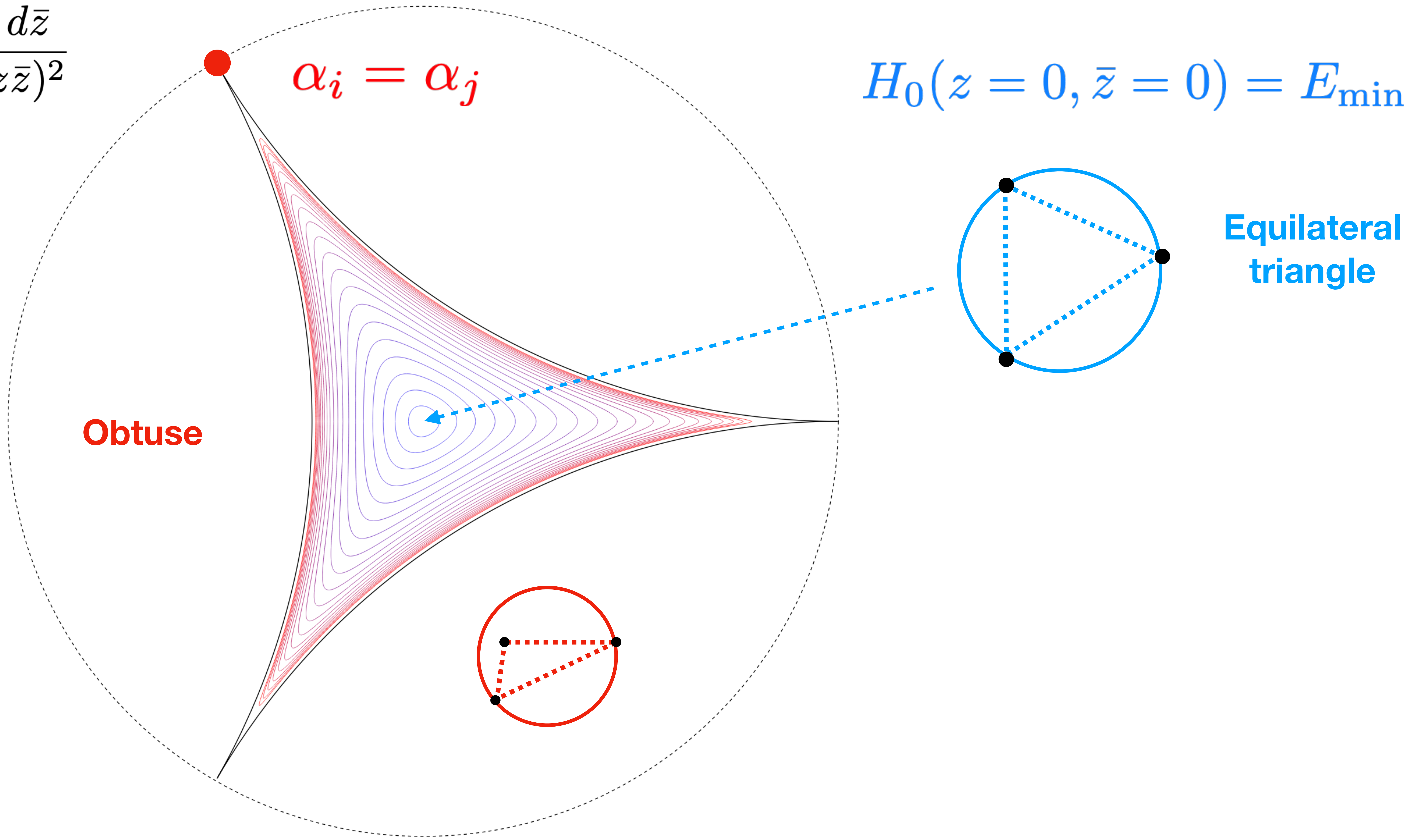

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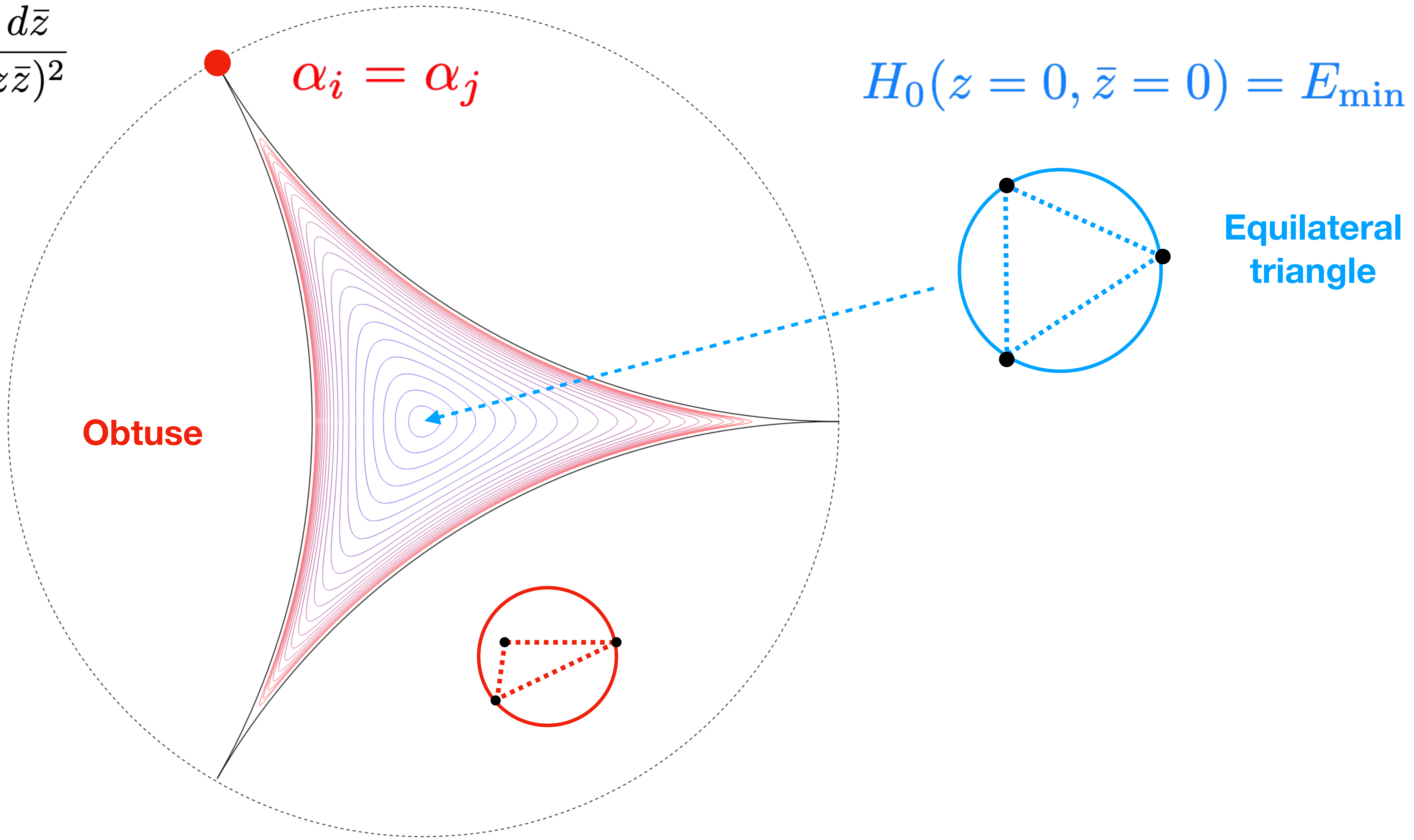
$$\mathcal{M}_{\text{class}} \cong \mathbb{C}P^1/S_3,\ \omega = 2i\frac{dz \wedge d\bar{z}}{(1-z\bar{z})^2}$$

- Acute region: ideal triangle in the unit disk with area $\pi$
- Infinite potential wall at acute/obtuse boundary
- Total number of states:

$$\frac{\pi/3}{2\pi J^{-1}} = \frac{J}{6}$$



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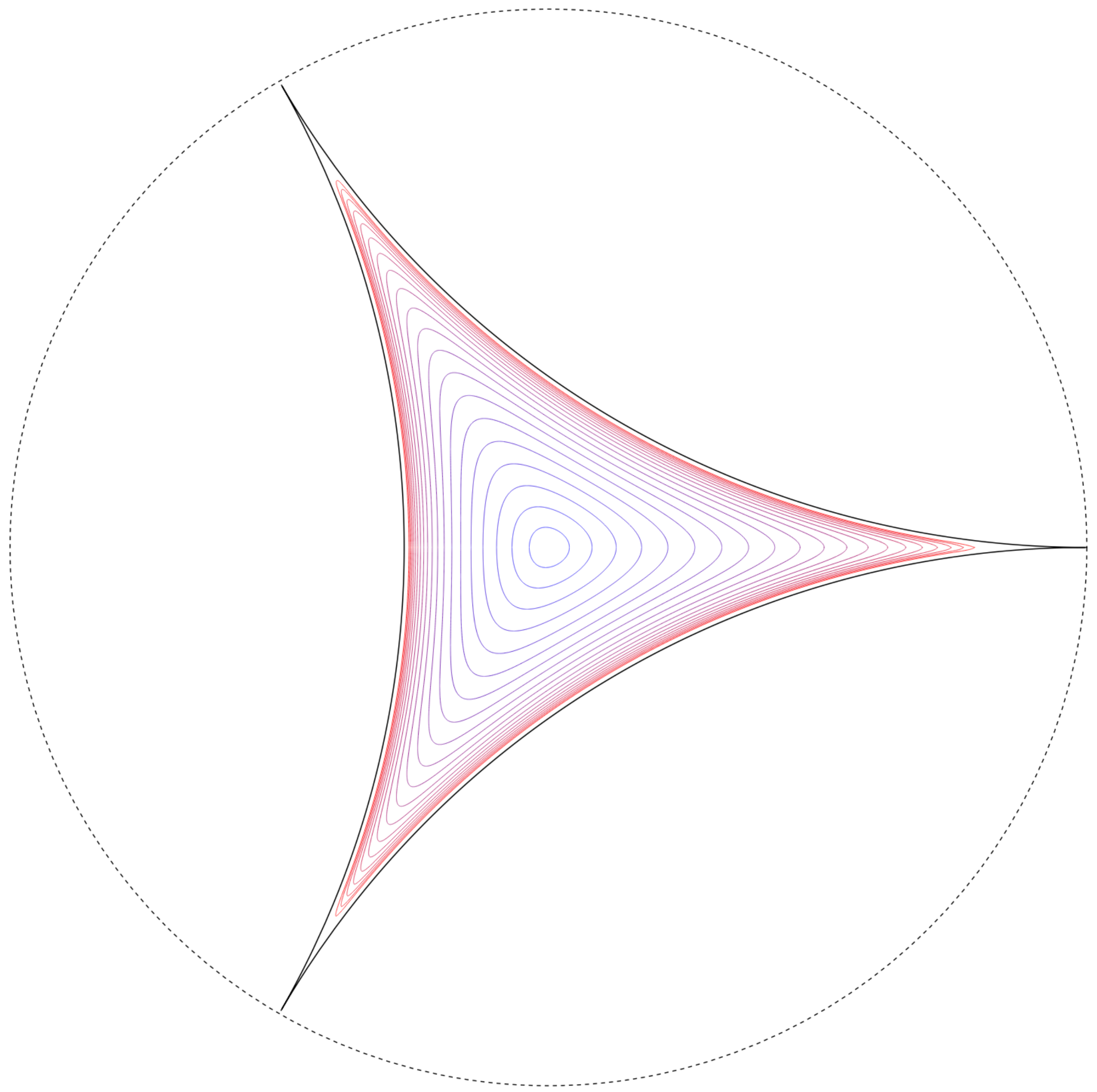
## 2.3 Bohr-Sommerfeld quantization conditions



- WKB approximation: $\psi(w) = \exp(iJS_0(w) + iS_1(w) + \ldots)$
- Wavefunction must be **single-valued**:

$$\oint_{\Gamma_E} d\log\psi = 2\pi i n, \quad \Gamma_E := \{H_0(z,\bar{z}) = E\}$$

- Up to subleading order: $E(\epsilon) := E_{\min}(1+\epsilon)$

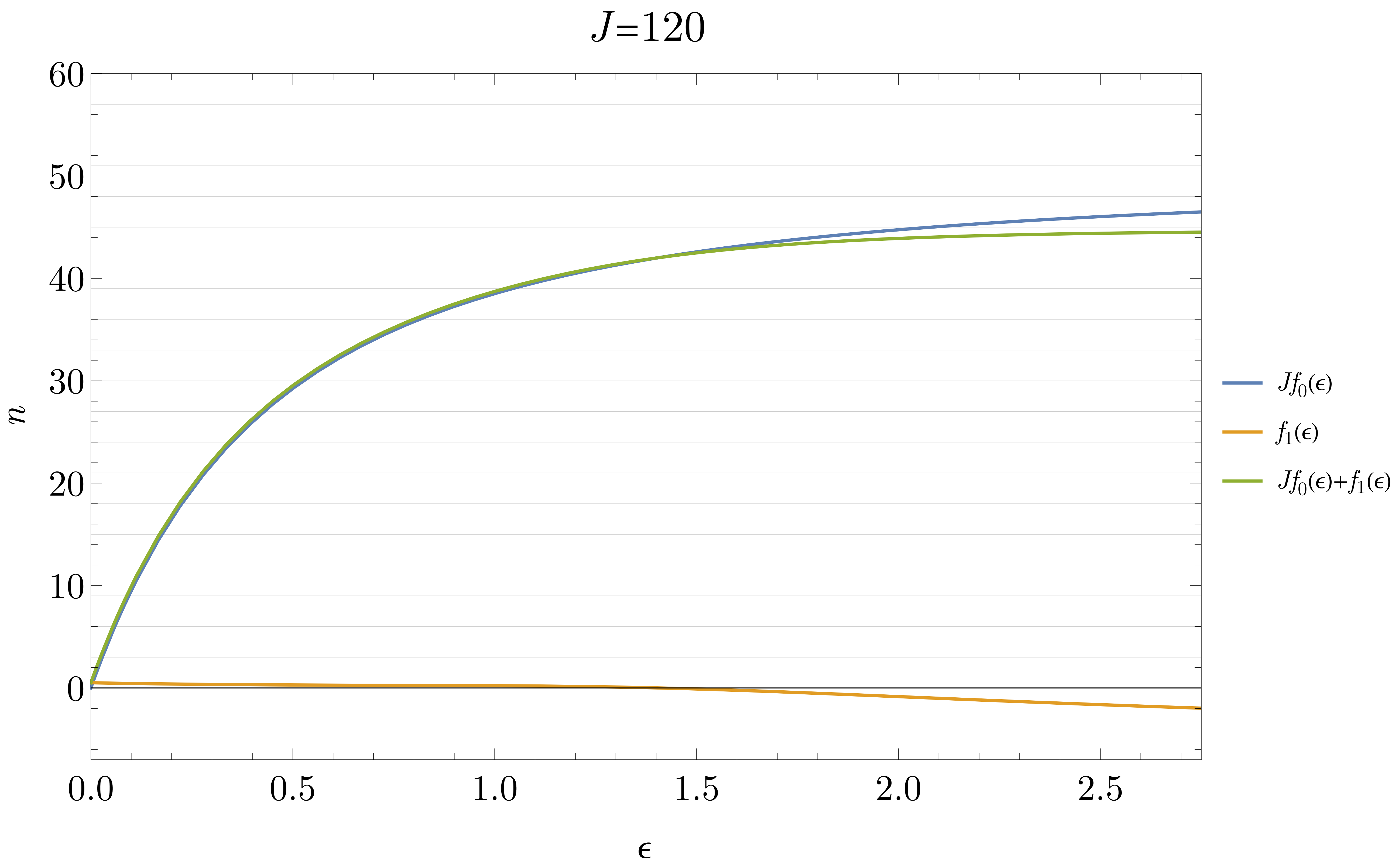
$$J f_0(\epsilon) + f_1(\epsilon) = n \qquad f_0(\epsilon) := \frac{1}{2\pi}\mathrm{Area}\left(\mathrm{Int}\,\Gamma_{E(\epsilon)}\right)$$

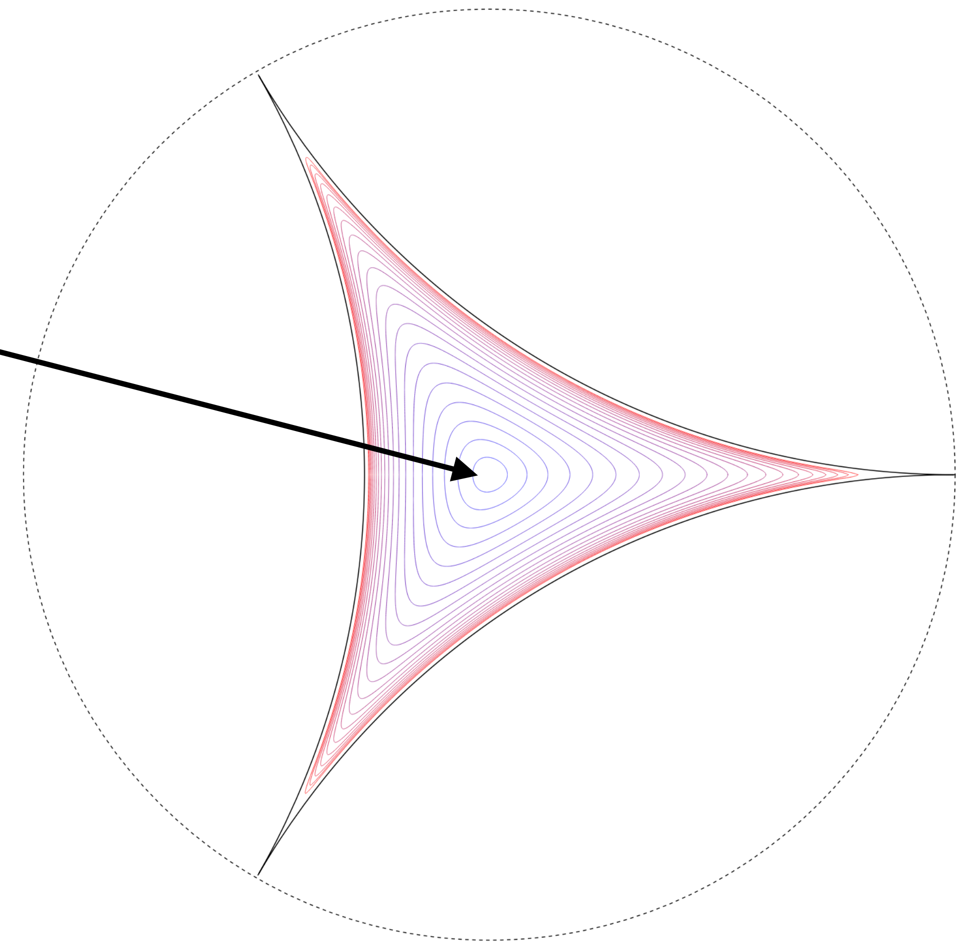
- Cyclic permutation symmetry: $J + n = 0 \bmod 3$

# 4. Comparison with exact diagonalization

## 4.1 Ground state and first excitated states

$$H_J(z,\bar{z}) \sim U_0\, 3^{1-\Delta_\sigma} J^{-\Delta_\sigma} \left(1 + \omega z\bar{z} + \ldots\right)$$



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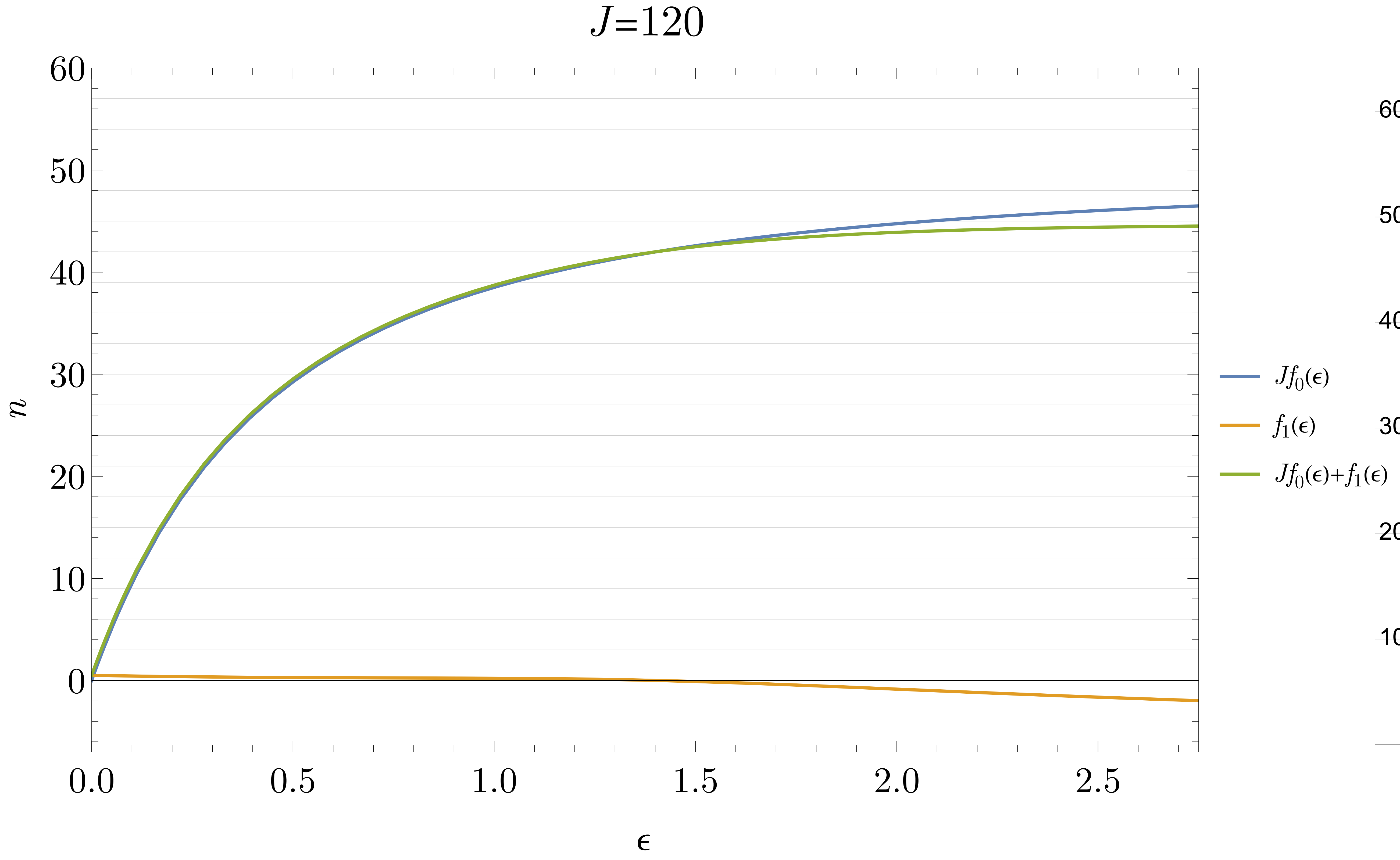
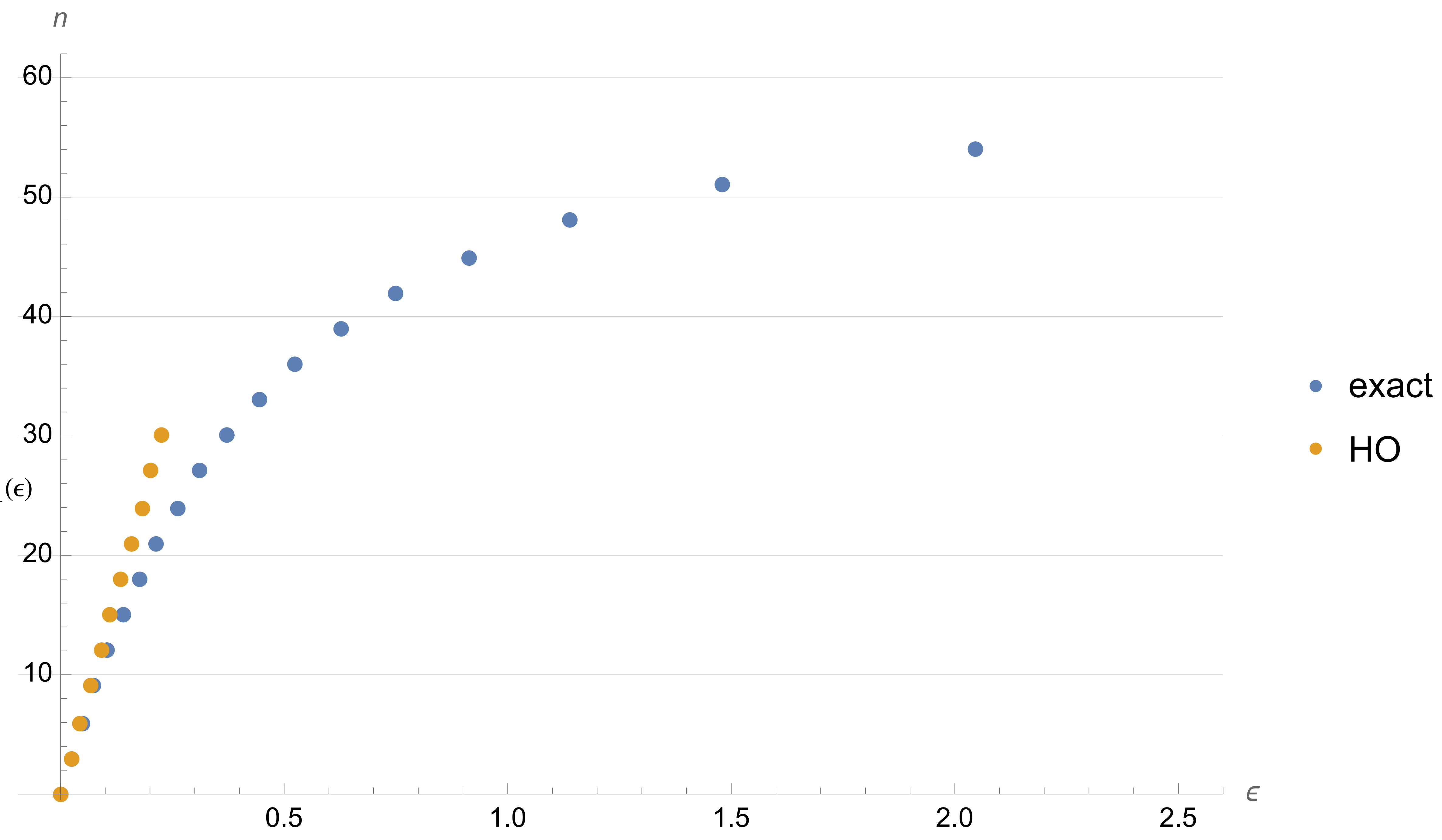
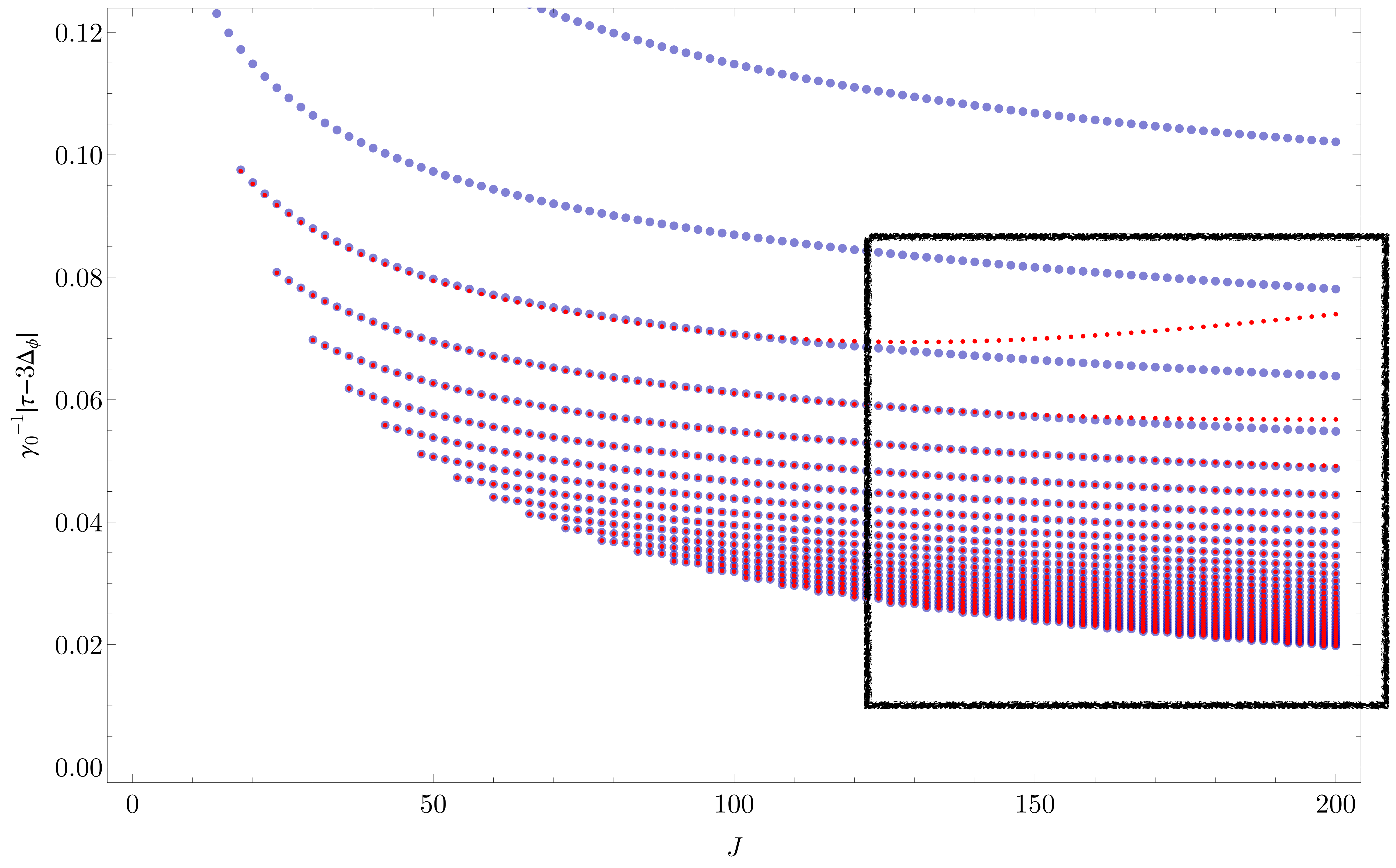


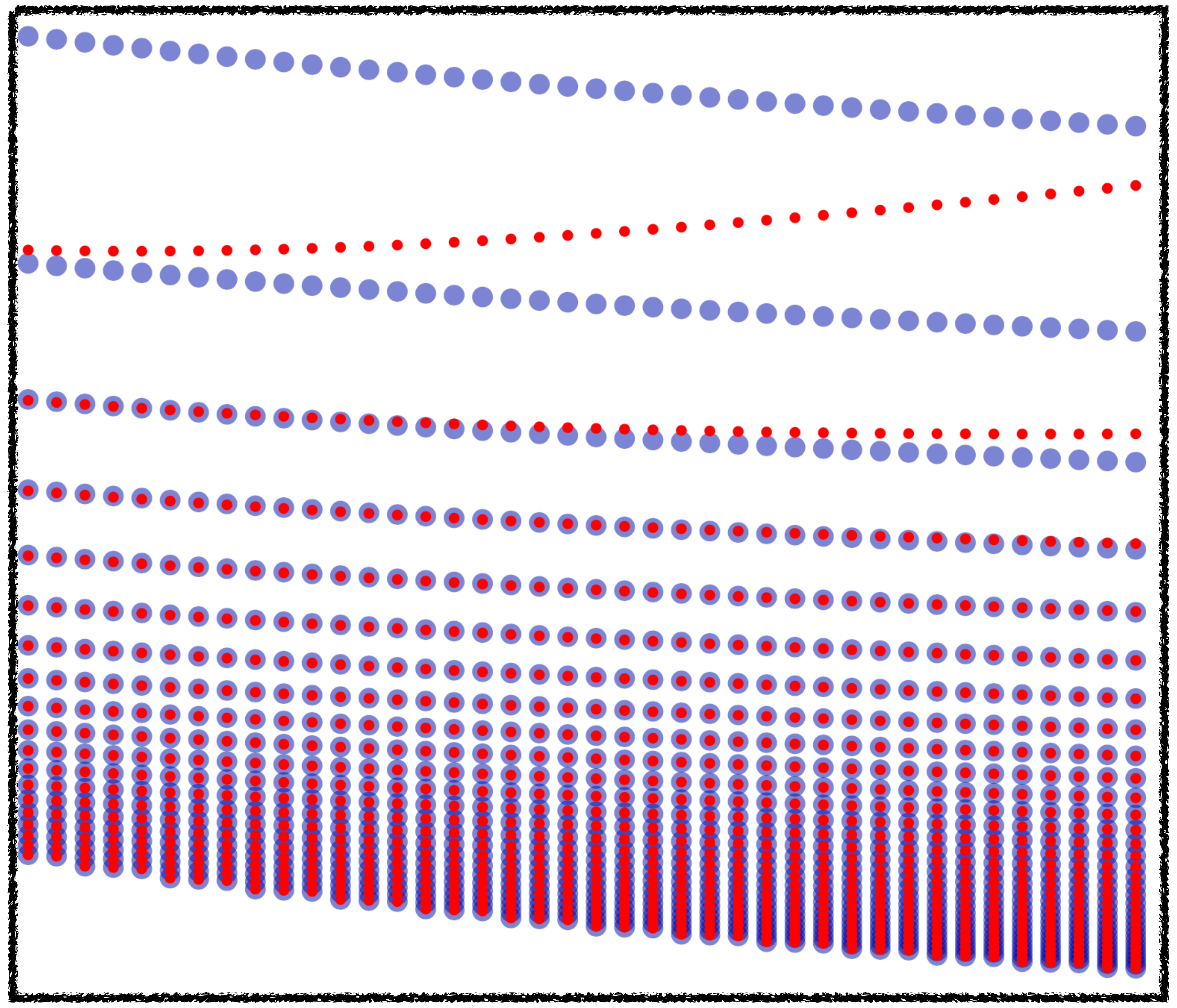



# 4. Comparison with exact diagonalization

## 4.2 Bohr-Sommerfeld quantization



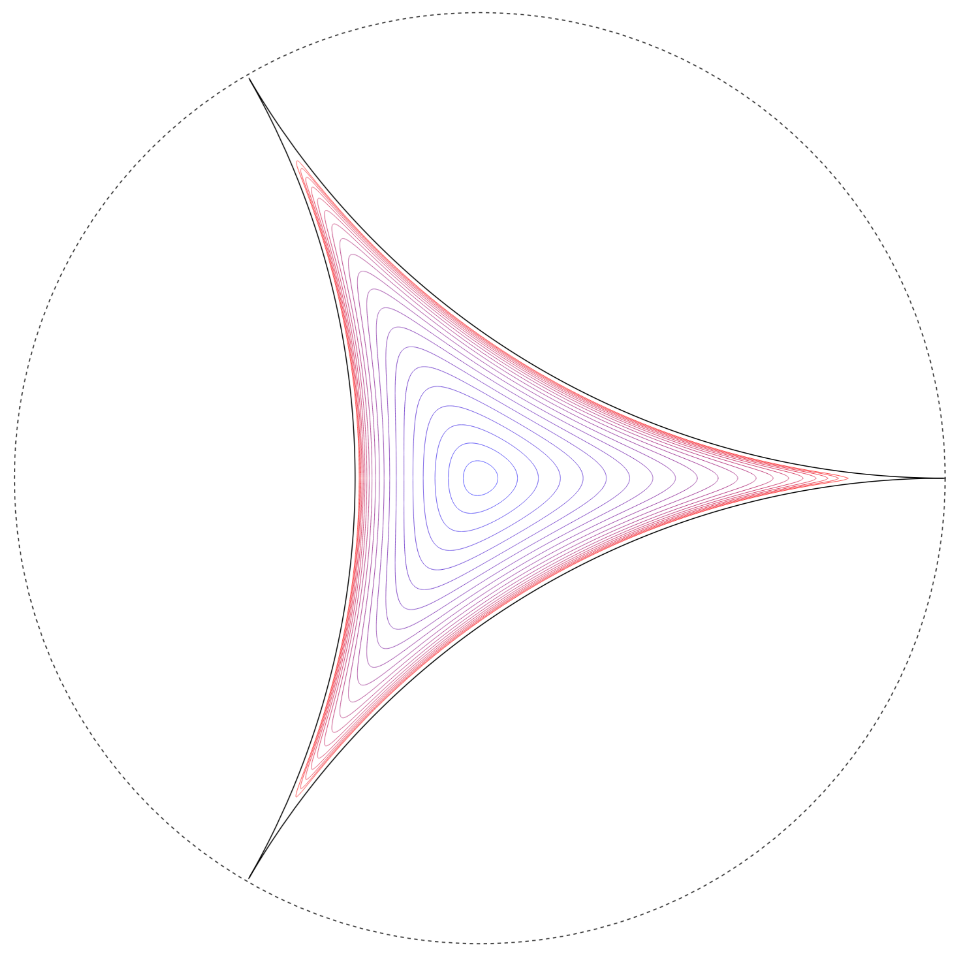

Semiclassical approximation breaks down away from triple-twist accumulation point



**Almost all states at large spin are covered**

# 5. Conclusion and Outlook

- Semi-classical approximation works well for most states at large J
- Expect breakdown at $\gamma = O(J^{-\frac{\Delta_\sigma}{2}}) \leftrightarrow \ell = O(\sqrt{J})$ **[c.f. Volker's talk]**
- Bootstrap derivation of the semiclassical regime
- “Many-body” problem n>3:
  – Berezin-Toeplitz quantization with more d.o.f.s
  – Large charge?



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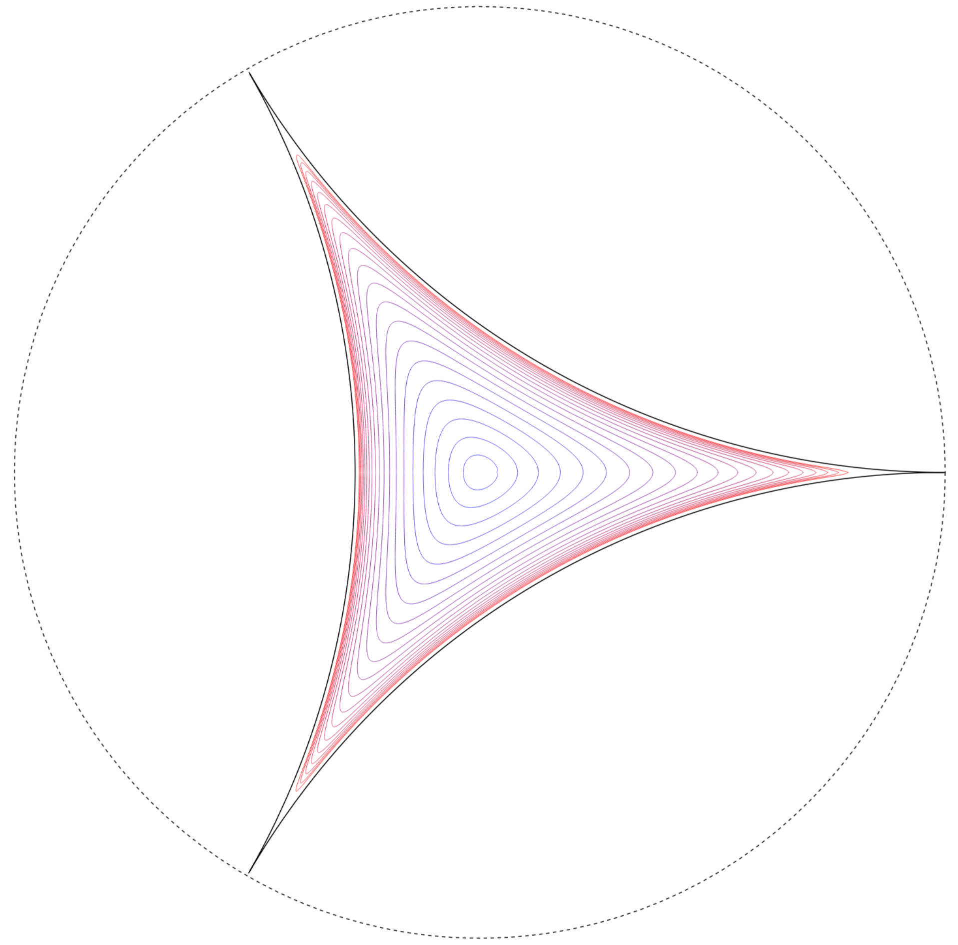

**Thank you for your time!**

# 4. Bonus slide

## 4.2 Bohr-Sommerfeld quantization