Twist-4 trajectories and missing local operators

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IPhT Saclay

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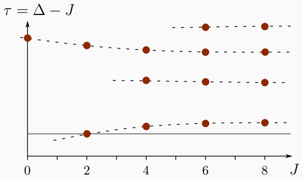
Work with Petr Kravchuk & Brett Oertel [2312.09283]

A mystery with twist families

- Local CFT operators come in twist families
- Light-ray operators are the natural language for describing the spectrum at continuous spin
- Lorentzian inversion formula gives data for local/light-ray operators that are analytic in J, convergent for $\mathrm{Re}\, J > 1$



$$C(\Delta, J) \;\Rightarrow\; \quad \gamma[J], \quad \lambda_{\mathcal{O}_1\mathcal{O}_2\mathbb{O}}\lambda_{\mathcal{O}_3\mathcal{O}_4\mathbb{O}}[J]$$

- **Problem:** *Many twist families have their first local operator at some* $J_0 \gg 1$. *But light-ray operators should not cease to exist at isolated points.*
- Naïve resolution: functions $\lambda_{\mathcal{O}_1\mathcal{O}_2\mathbb{O}}\lambda_{\mathcal{O}_3\mathcal{O}_4\mathbb{O}}[J]$ have zeros precisely at even integer $J < J_0$.
  Requires infinitely many vanishing conditions for every trajectory and J.

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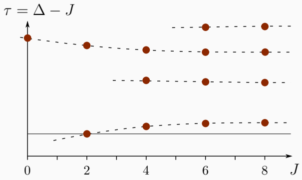

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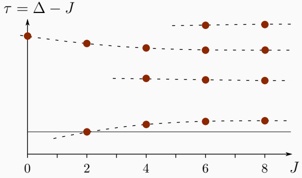

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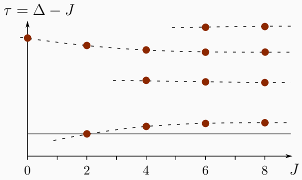

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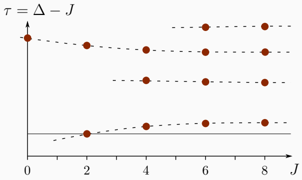

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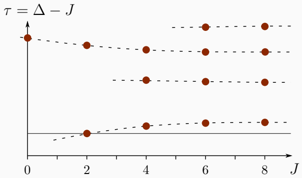

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- Why do functions $\lambda_{\mathcal{O}_1\mathcal{O}_2\mathbb{O}}\lambda_{\mathcal{O}_3\mathcal{O}_4\mathbb{O}}[J]$ have zeros precisely at even integers $J < J_0$?
- Example from interpolation in $\mathcal{N} = 4$ SYM
  [Homrich, Simmons-Duffin, Vieira 2022]
- Resolution of mystery

$$\lambda_{\mathcal{O}_1\mathcal{O}_2\mathbb{O}}\lambda_{\mathcal{O}_3\mathcal{O}_4\mathbb{O}}[J] = \sin\left(\frac{\pi J}{2}\right)\frac{\langle\mathcal{O}_1|\mathbb{O}|\mathcal{O}_2\rangle\langle\mathcal{O}_3|\mathbb{O}|\mathcal{O}_4\rangle}{\langle\mathbb{O}\mathbb{O}\rangle}$$

$\langle\mathbb{O}\mathbb{O}\rangle$ has simple zeros when $\mathbb{O}$ is related to a local operator at $J = J_0$

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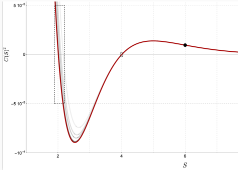

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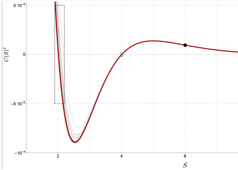

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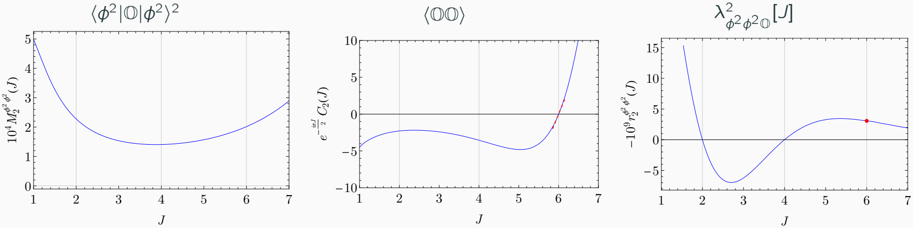
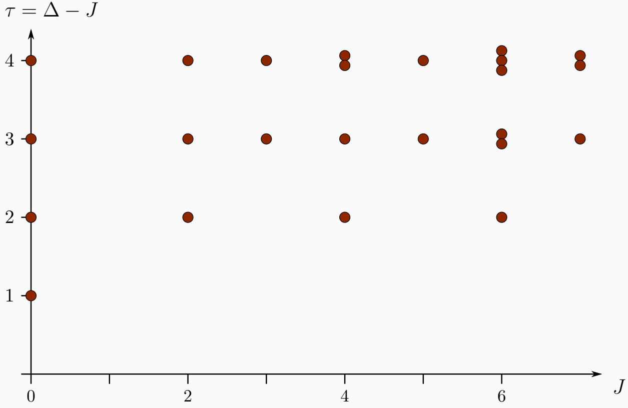
- Preview with $J_0 = 6$



Wilson–Fisher theory

[Wilson & Fisher 1972; Wilson & Kogut 1974] $d = 4 - \varepsilon$



Degeneracies in free-theory spectrum [Henning, Lu, Melia, Murayama 2017]

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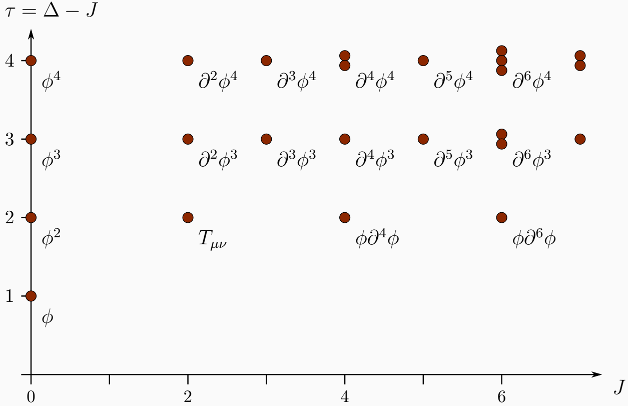

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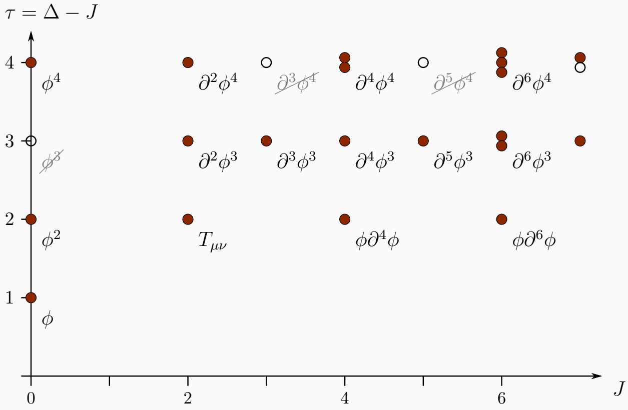

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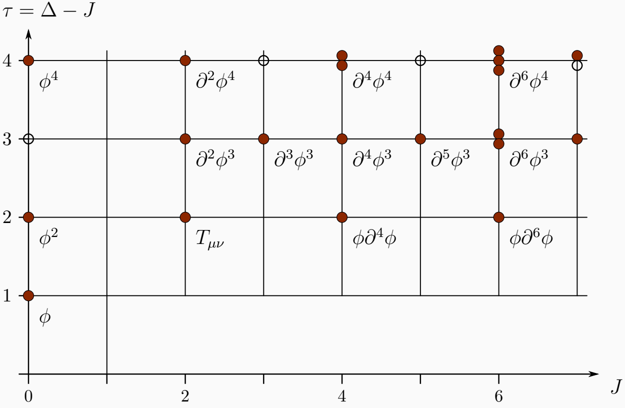

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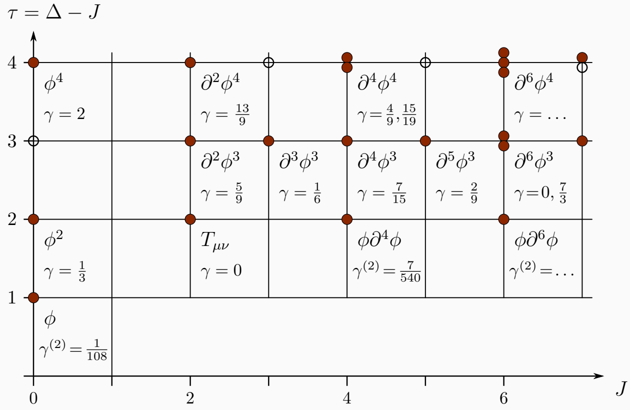

One-loop spectrum [Kehrein & Wegner 1994, Hogervorst et al 2015, JH 2022]

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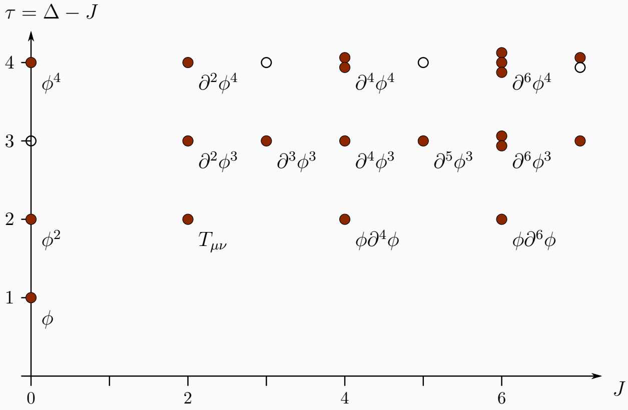

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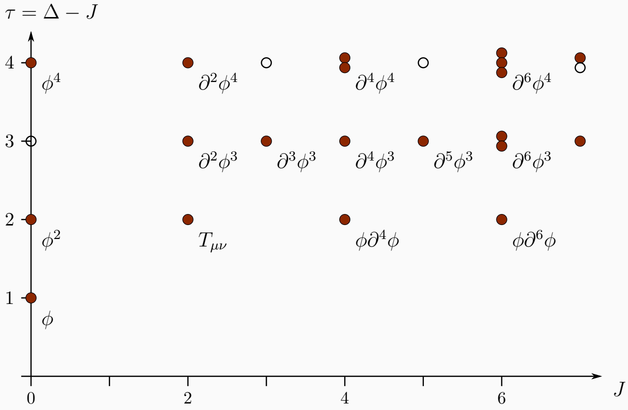
$\tau = \Delta - J$
J
0
2
4
6
1
2
3
4
ϕ
ϕ^2
ϕ^4
$T_{\mu\nu}$
$\phi\partial^4\phi$
$\phi\partial^6\phi$
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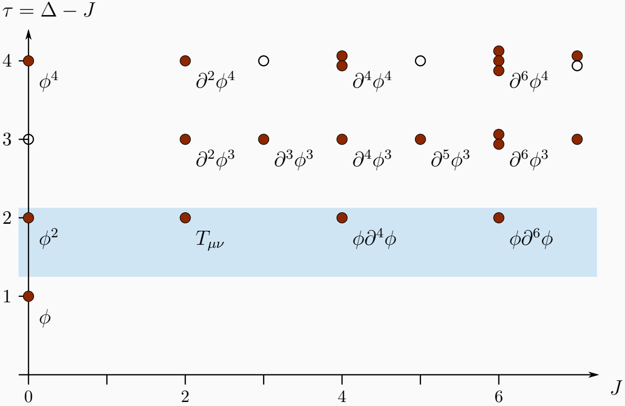
Twist-2 operators



Local twist-2 operators

Index-free definition

$$\mathcal{O}_J(x,z) = \phi(x)(z.\partial)^J \phi(x), \qquad J \in 2\mathbb{N},$$

Anomalous dimensions $\tau = \Delta - J = 2 - \varepsilon + \gamma_J$ [Derkachov et al 1997]

$$\gamma_J = \frac{\varepsilon^2}{54}\left(1 - \frac{6}{J(J+1)}\right) + \frac{\varepsilon^3}{5832}\left(\frac{109J^4+218J^3+373J^2-384J-324}{J^2(J+1)^2} - \frac{432S_1(J)}{J(J+1)}\right) + \dots$$

- γ_J moments of splitting functions $\gamma_J = -\int_0^1 dx\, x^{J-1} P(x)$
- Positive+bounded DIS cross-section $\Rightarrow$ γ_J convex function [Nachtmann 1973]
- Reciprocity: γ_J expands in powers of $[\frac{\Delta+J}{2}\frac{\Delta+J-2}{2}]^{-1}$ [. . . ; Alday, Bissi, Łukowski 2015]
- Lorentzian inversion formula [Caron-Huot 2017] gives γ_J, λ_J^2 [Alday, JH, van Loon 2017]

$$\lambda_J^2 = a_J^{GFF}\left(1 + \frac{\varepsilon^2}{9J(J+1)}\left(\frac{1}{J+1} + S_1(2J) - S_1(J)\right) + \dots\right)$$

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Twist-2 light-ray operators

Light-ray operators defined for $J \in \mathbb{C}$ [Balitsky & Braun 1989, ...]

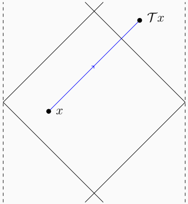
$$\mathbb{O}_J(x,z) = \int_{-\infty}^{\infty} d\alpha_1 (-\alpha_1)^{-\Delta_\phi} d\alpha_2 (-\alpha_2)^{-\Delta_\phi} \psi(\alpha_1,\alpha_2) \, :\phi(x - \tfrac{z}{\alpha_1})\phi(x - \tfrac{z}{\alpha_2}):$$

Wave function $\psi(\alpha_1,\alpha_2) = \dfrac{|\alpha_1 - \alpha_2|^{-1-J}}{\Gamma(-\frac{J+2}{2})}$

For $J = 0, 2, 4, \ldots$, $\psi \propto \delta^{(J)}(\alpha_1 - \alpha_2)$

$$\mathbb{O}_J(x,z) = \int_{-\infty}^{\infty} d\alpha (-\alpha)^{-\Delta-J} (\phi\partial^J\phi)(x - \tfrac{z}{\alpha}, z) = \mathbf{L}[\phi\partial^J\phi]$$

Light transform of local operator [Kravchuk, Simmons-Duffin 2018]



Lorentzian inversion formula $\Rightarrow$ data of light-ray operators [Kravchuk & Simmons-Duffin 2018]

$$C(\Delta, J) \sim \frac{\lambda^2_{\phi\phi\mathbb{O}}[J]}{\Delta - \Delta_i[J]}$$

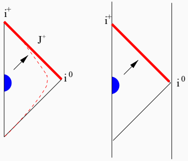
Aside: dimension: $1 - J$, spin: $1 - \Delta$

Twist-2 light-ray operators as detectors

"Conformal colliders:" light-ray operators as detectors (calorimeters) [Hofman & Maldacena 2008]

Energy (ANEC) operator $\mathcal{E} = \mathbb{O}_2 = \mathbf{L}[T_{\mu\nu}]$

$$\mathcal{E}(\vec{n}) = \lim_{r\to\infty} r^2 \int_{-\infty}^{\infty} dt\, n^i T^0{}_i(t, r\vec{n})$$



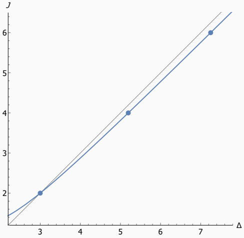
Energy correlators inside state

$$\langle \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\rangle = \frac{\langle 0|\mathcal{O}^\dagger \mathcal{E}(\theta_1)\cdots\mathcal{E}(\theta_n)\mathcal{O}|0\rangle}{\langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle}$$

- Free theory, $\mathbb{O}_J(\vec{n})$ counts particles in direction $\vec{n}$, weighted by E^{J-1}
- Interacting theory: soft (IR) divergences $\Rightarrow$ renormalise light-ray operators

Wilson–Fisher ϕ^4 theory [Caron-Huot et al 2022]

$$\gamma[J] = \frac{\varepsilon^2}{54}\left(1 - \frac{6}{J(J+1)}\right) + \ldots$$



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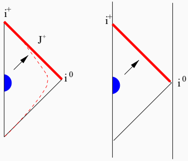

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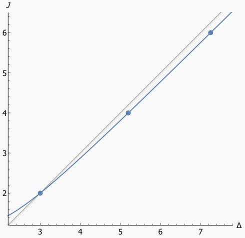
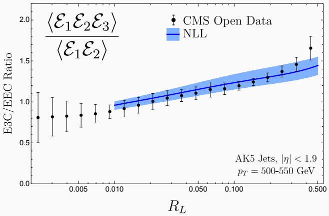


Light-ray data is observable!

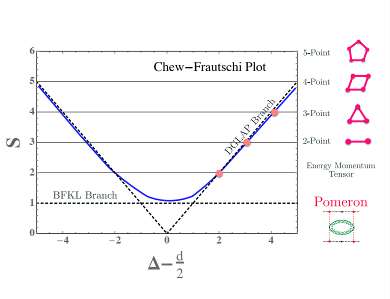
Energy–energy correlators **[Basham et al 1978; Hofman & Maldacena 2008; . . .]**

$$\mathcal{E}(\theta)\mathcal{E}(0) \sim \theta^{-2+\gamma[3]\frac{\alpha_S}{\pi}+\cdots}\mathbb{O}_3(0) + \dots$$

In general $\mathbb{O}_{J_1} \times \mathbb{O}_{J_2} \sim \mathbb{O}_{J_1+J_2-1}$ **[Hofman & Maldacena 2008; Koloğlu et al 2019]**



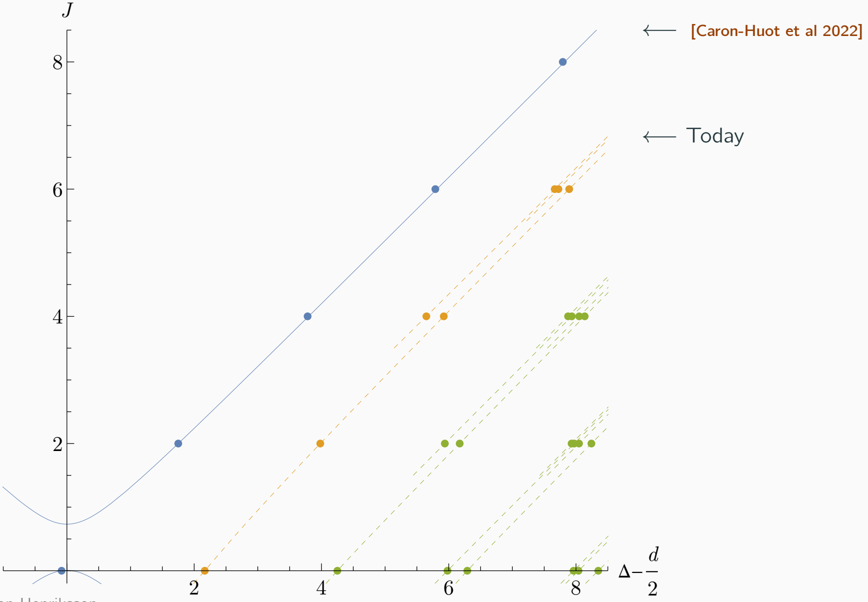
Jet substructure in LHC data **[Chen et al 2020; Lee, Meçaj, Moult 2022]**



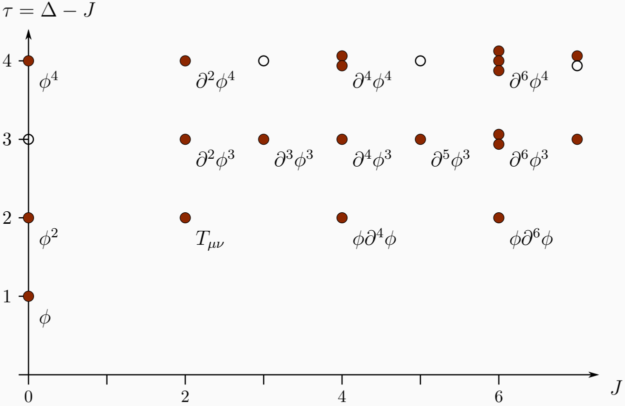
$$\mathcal{O}_q \sim \bar{q}(z.\gamma)(z.\partial)^{J-1}q,$$
$$\mathcal{O}_g \sim z^\mu F_{\alpha\mu}(z.\partial)^{J-2}z_\nu F^{\alpha\nu}$$

$$\gamma_{qq}[J] = \tfrac{1}{2}C_2(R)\left(4S_1(J) - 3 - \tfrac{2}{J(J+1)}\right)$$
$$\gamma_{qg}[J] = -2T(R)\tfrac{J^2+J+2}{J(J+1)(J+2)}$$
$$\gamma_{gq}[J] = -C_2(R)\tfrac{J^2+J+2}{(J-1)J(J+1)}$$
$$\gamma_{gg}[J] = \tfrac{2}{3}N_f T(R) + 2C_2(A)\left(S_1(J) - \tfrac{1}{(J+1)(J+2)} - \tfrac{1}{J(J-1)} - \tfrac{11}{12}\right)$$

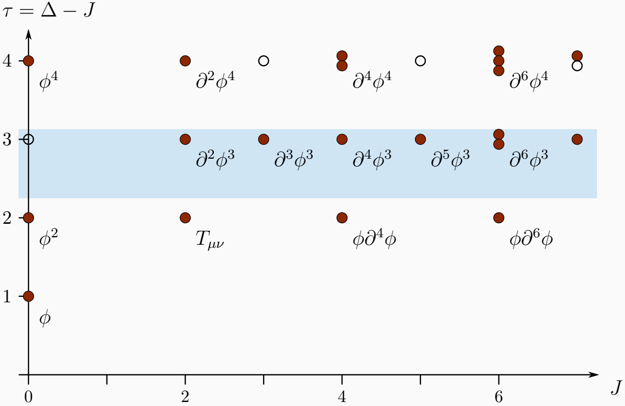
Regge trajectories for Wilson–Fisher



Twist-3 operators



Twist-3 operators



Interlude on twist-3 operators

Basis at spin J: $\mathcal{O} \sim \sum_{j_1,j_2} c_{j_1,j_2} \partial^{j_1}\phi \partial^{j_2}\phi \partial^{J-j_1-j_2}\phi$

J	2	3	4	5	6	7	8	9	10	11	12	13	14
# primaries	1	1	1	1	2	1	2	2	2	2	3	2	3

- At $J = 2, 3, 4, \ldots$: one operator with $\gamma^{(1)} = \frac{1}{3} + \frac{2(-1)^J}{3(J+1)}$ **[Kehrein et al 1993]**
- At $J = 6, 8, 9, 10, 11, 2 \times 12, \ldots$: additional operators with $\gamma^{(1)} = 0$

Only first set of operators appear at leading order in $\phi \times \phi^2$

Lorentzian inversion formula for $\langle \phi\phi^2\phi^2\phi \rangle$ **[Bertucci, JH, McPeak 2022]**

$$\gamma^{\neq 0}_{3,J} = \gamma_\phi + \gamma_{\phi^2} + \frac{2(-1)^J}{3(J+1)}\varepsilon - \frac{4\varepsilon^2}{9(J+3)(J-1)}\left(S_1(J) - \frac{J^3+2J^2+2J+3}{(J+1)^3}\right) + (-1)^J \frac{9(J+1)(3J^2+6J-1)S_1(J) - 38J^3 - 69J^2 + 38J - 75}{81(J+3)(J+1)^2(J-1)}\varepsilon^2 + O(\varepsilon^3).$$

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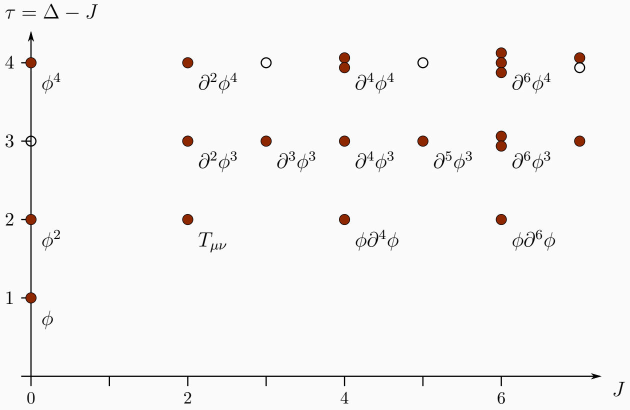
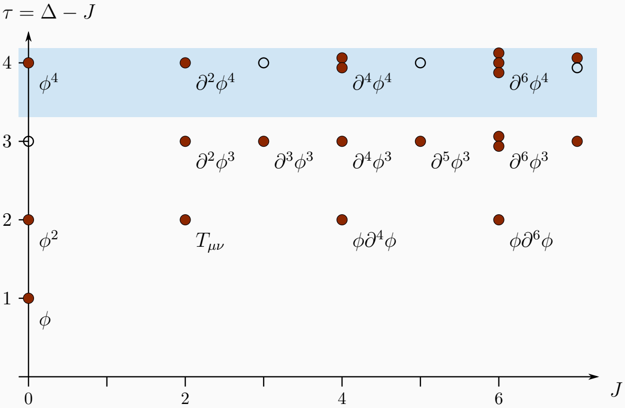
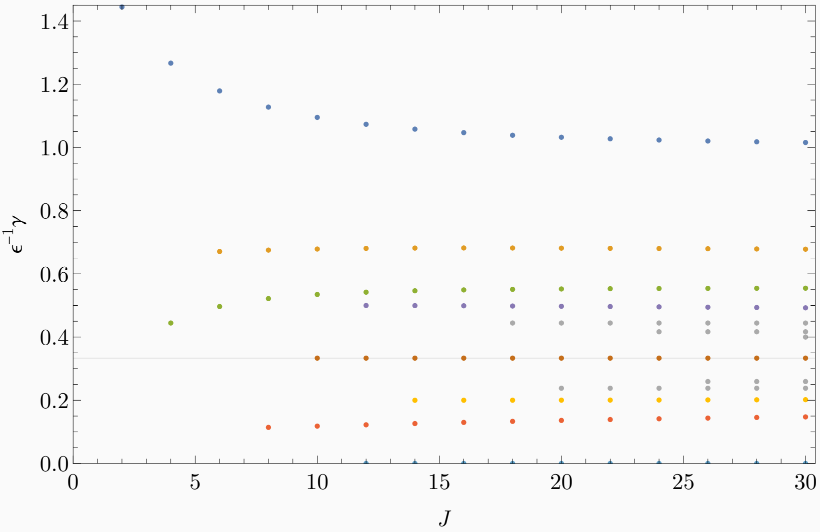
Twist-4 operators

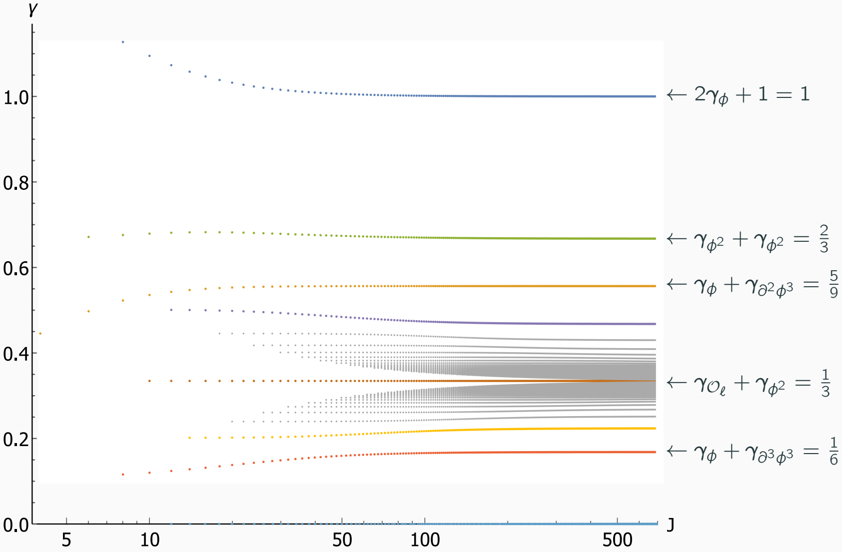


Twist-4 operators



Local twist-4 operators of even spin





Possible to identify $\frac{1}{(J+1)(J+2)}$ correction for each trajectory

Twist-4 light-ray operators

$$\mathbb{O}_J(x,z) = \prod_i \int_{-\infty}^{\infty} d\alpha_i (-\alpha_i)^{-\Delta_\phi} \underbrace{\psi(\alpha_1, \alpha_2, \ldots \alpha_n)}_{\text{wave function}} :\phi(x - \tfrac{z}{\alpha_1}) \cdots \phi(x - \tfrac{z}{\alpha_n}):$$

Properties of $\tilde{\psi}(\beta_1, \ldots \beta_n)$ (Fourier space)

- $\tilde{\psi}(\lambda\beta_1, \ldots, \lambda\beta_n) = \lambda^J \tilde{\psi}(\beta_1 \ldots \beta_n)$
- $\tilde{\psi}(\beta_i) = \tilde{\psi}(\sigma \circ \beta_i)$
- $\tilde{\psi}(-\beta_i) = \pm\tilde{\psi}(\beta_i)$
- $\tilde{\psi}(\beta_i)$ polynomial $\leftrightarrow \mathbb{O}_J = \mathbf{L}[\mathcal{O}_J]$

Twist-4 light-ray operators

Light-ray operators determined by wave function

$$\mathbb{O}_J \leftrightarrow \tilde{\psi}(\beta_1, \ldots, \beta_n)$$

Find $\tilde{\psi}$ by diagonalising one-loop dilatation operator [Derkachov & Manashov 1995]

$$H\tilde{\psi} = \gamma\tilde{\psi}$$

$$(H\tilde{\psi})(\beta_i) = \tfrac{\varepsilon}{3} \sum_{i<j} \int_0^1 dt \tilde{\psi}(\beta_1, \ldots, \underset{i}{t(\beta_i + \beta_j)}, \ldots, \underset{j}{(1-t)(\beta_i + \beta_j)}, \ldots, \beta_n)$$

Reduce to two-variable function

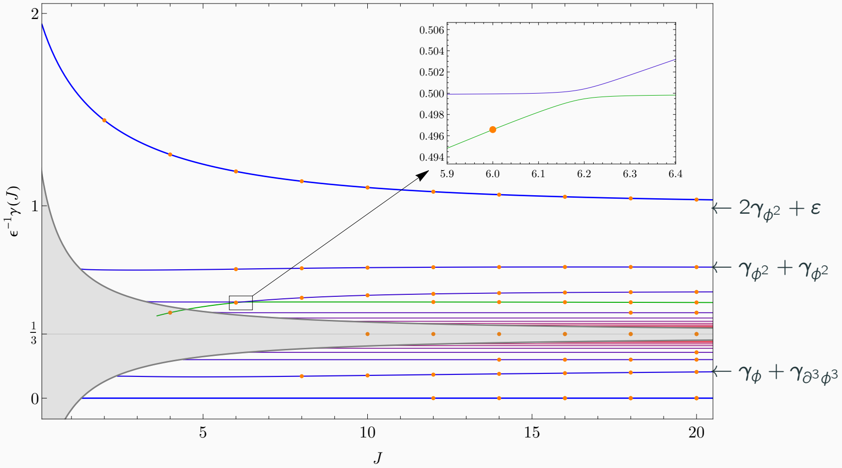
$$\tilde{\psi}(\beta_i) = \sum_{i<j} \Psi(\beta_i, \beta_j), \qquad \Psi \leftrightarrow (\Psi_1(x), \Psi_2(x)),\ x \in [0, 1]$$

$$H'\Psi_1(x) = \tfrac{1}{2} \int_0^1 dz\, \Psi_1(z) + x^{J+1} \int_x^1 dz\, z^{-J-2}\Psi_2(z) + \ldots, \qquad H'\Psi_2(x) = \ldots$$

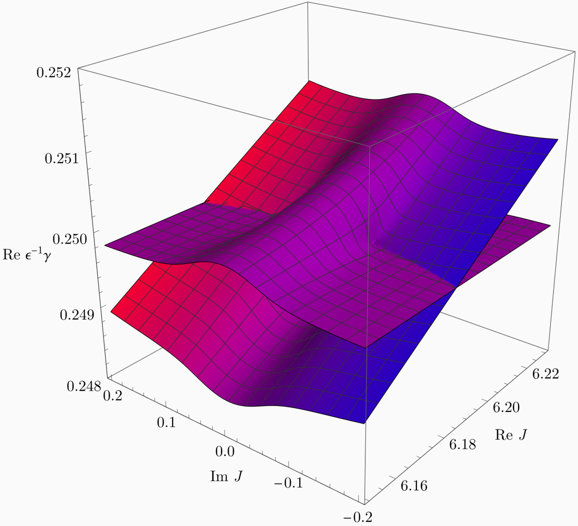
$\Rightarrow$ eigenvalues (γ) and eigenfunctions (Ψ_1 and Ψ_2)

Numerical solution by discretisation in x

Anomalous dimensions for $J \in \mathbb{C}$

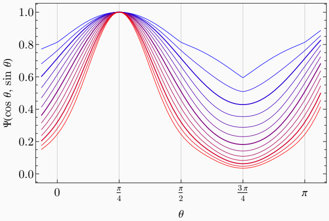


Anomalous dimensions for $J \in \mathbb{C}$

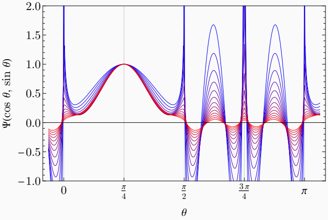


Wave functions

$\Psi(\cos\theta, \sin\theta)$



traj. 1: $\gamma \to 1$



traj. 3: $\gamma \to \gamma_\phi + \gamma_{\partial^2\phi^3}$

For traj. 1, spin runs from $J = 1$ to $J = 7$ in steps of 0.5; for traj. 3 from $J = 3$ to $J = 5.4$ in steps of 0.2. Colour blue $\to$ red as J increases. The thick lines: where Ψ is polynomial.

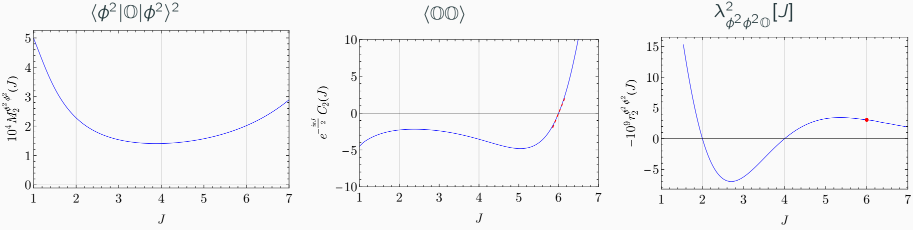
Resolution of mystery with missing local operators

Lorentzian inversion formula for $\langle\varphi\varphi\varphi\varphi\rangle$ (e.g. $\varphi = \phi^2$) $\Rightarrow$ light-ray operator in specific normalisation [Kravchuk & Simmons-Duffin 2018]

$$\mathbb{O}^{(0)}_{J,i} = 2 \operatorname*{res}_{\Delta=\Delta_i} \int d^d x_1 d^d x_2 K_{J,\Delta}(x_1, x_2, x, z)\varphi(x_1)\varphi(x_2), \qquad J \in \mathbb{C}$$

Define $\lambda^2_{\varphi\varphi\mathbb{O}}[J] = \frac{\langle 0|\varphi\mathbb{O}^{(0)}\varphi|0\rangle}{\langle 0|\varphi\mathbf{L}[\mathcal{O}]\varphi|0\rangle_0}$ and massage to normalisation-independent form:

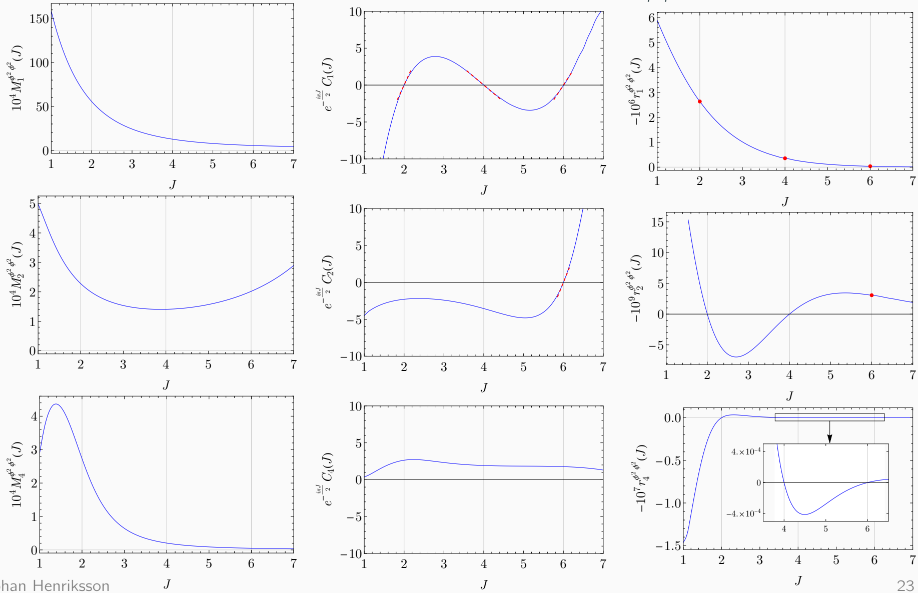
$$\lambda^2_{\varphi\varphi\mathbb{O}}[J] = \sin\left(\tfrac{\pi J}{2}\right) \underbrace{e^{-i\pi J}\left(\frac{\langle 0|\varphi\mathbb{O}\varphi|0\rangle}{\langle 0|\varphi\mathbf{L}[\mathcal{O}]\varphi|0\rangle_0}\right)^2}_{M} \underbrace{\frac{-2}{\pi} e^{i\pi J/2} \frac{\mathrm{vol}(SO(1,1))\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle_0}{\langle\mathbb{O}\mathbb{O}\rangle}}_{-1/e^{-i\pi J/2} C}$$



$$\langle\mathbb{O}_\psi\mathbb{O}_{\psi'}\rangle = \mathrm{vol}(SO(1,1))\langle\mathbf{L}[\mathcal{O}]\mathbf{L}[\mathcal{O}]\rangle_0 \langle\tilde{\psi}, \tilde{\psi}'\rangle$$

Different trajectories

$\langle \phi^2 | \mathbb{O} | \phi^2 \rangle^2$ $\quad\quad$ $\langle \mathbb{O}\mathbb{O} \rangle$ $\quad\quad$ $\lambda^2_{\phi^2\phi^2\mathbb{O}}[J] = \sin\left(\frac{\pi J}{2}\right) \frac{\langle \phi^2 | \mathbb{O} | \phi^2 \rangle^2}{\langle \mathbb{O}\mathbb{O} \rangle}$

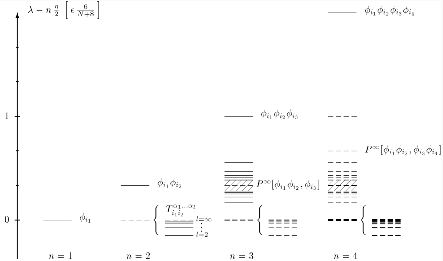


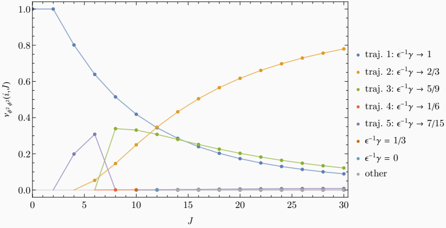
Double-twist structure

Full compatibility with:

- *Hierarchical structure* in perturbation theory **[Kehrein 1995; Derkachov & Manashov 1996]**
- Non-perturbative *twist additivity* **[Alday & Maldacena 2007; Fitzpatrick et al 2012; Komargodski & Zhiboedov 2012; Pal et al 2022]**

$$\mathcal{O}_{\tau=4,J} = [\mathcal{O}_{\tau_1,j_1}, \mathcal{O}_{\tau_2,j_2}]_{0,J} \qquad (\tau_1+\tau_2\approx 4), \qquad \text{or } [\phi,\phi]_{1,J}$$

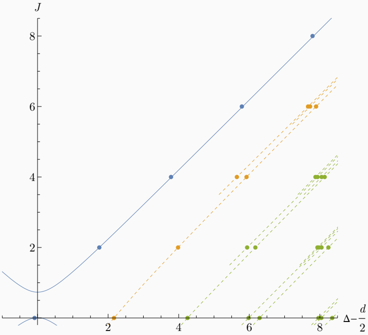




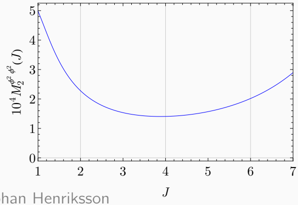
Relative OPE coefficients in $\phi^2 \times \phi^2$

Summary

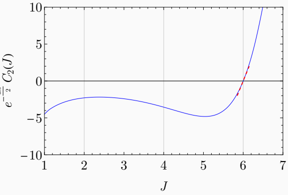
- Explicit construction of light-ray operators that interpolate the local operator spectrum
- Resolution of mystery
  - 3-point and 2-point functions remain smooth everywhere
  - 2-point function has zeros where there is a local operator
  - $\lambda_{\mathcal{O}_1\mathcal{O}_2\mathbb{O}}\lambda_{\mathcal{O}_3\mathcal{O}_4\mathbb{O}}[J]$ have zeros at $J \in 2\mathbb{Z}$ where there is no local operator
- We confirm hierarchical structure of double-twist operators



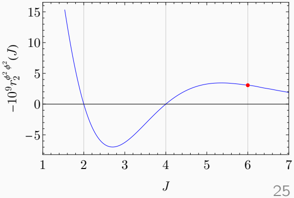
$\langle\phi^2|\mathbb{O}|\phi^2\rangle^2$



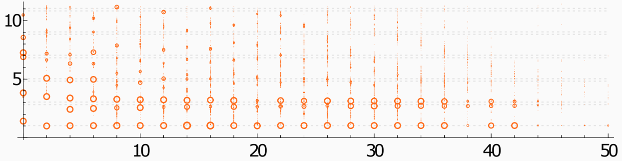
$\langle\mathbb{O}\mathbb{O}\rangle$



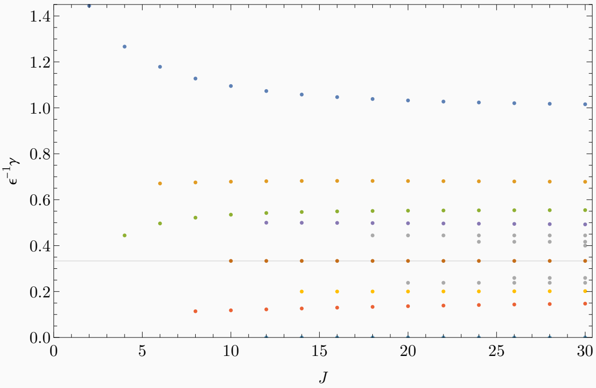
$\lambda^2_{\phi^2\phi^2\mathbb{O}}[J]$



Bonus slide: a challenge for the numerical bootstrap



[Simmons-Duffin 2016]



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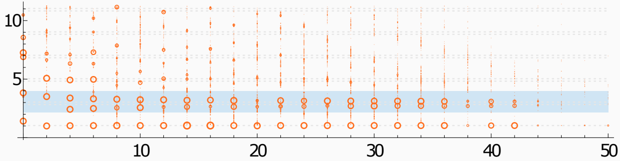

[Simmons-Duffin 2016]