Fuzzy sphere regularization of 3D CFTs: the recent progress

PERIMETER INSTITUTE

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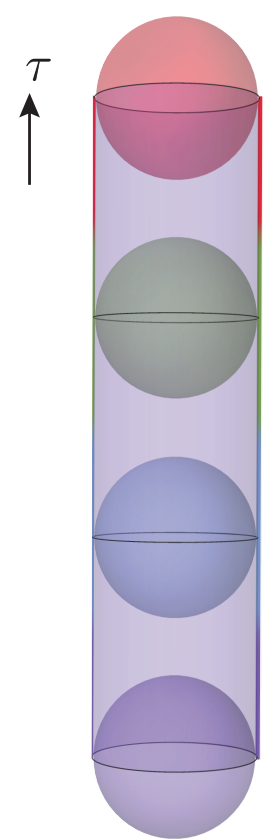
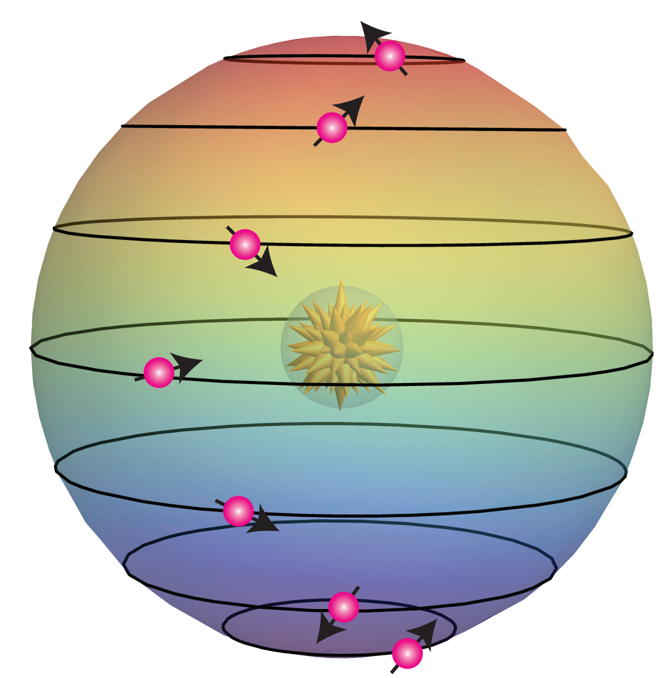
50+epsilon years of bootstrap
Pisa, Feb 2024

arXiv:2210.13482 (PRX 13, 021009);
arXiv:2303.08844 (PRL 131, 031601);
arXiv:2306.04681 (PRB 108, 235123);
arXiv:2306.16435
arXiv:2308.01903
arXiv:2401.00039
arXiv:2401.17362





Collaborators

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Wei Zhu

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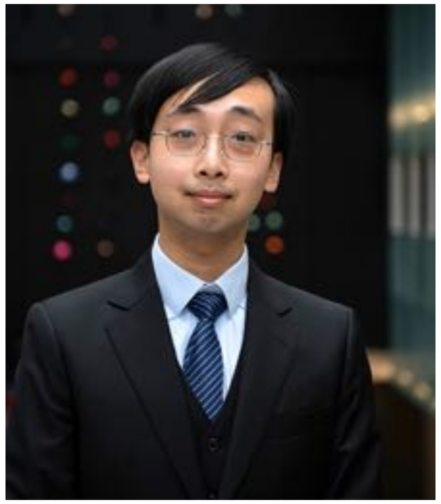

Zheng Zhou

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Outline

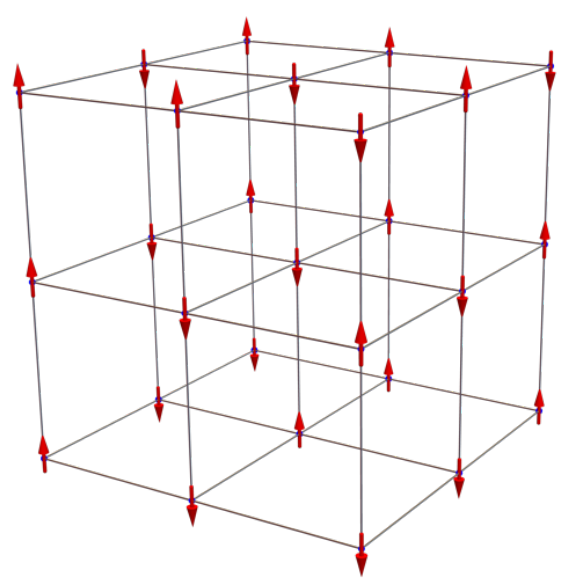
- Fuzzy sphere regularization
  - Why and how?
  - Radial quantization
  - Correlators, OPE coefficients, F-function
- Conformal defect

A condensed matter approach to CFTs

Study strongly interacting quantum mechanical models that realize CFTs.

Lattice models

Classical

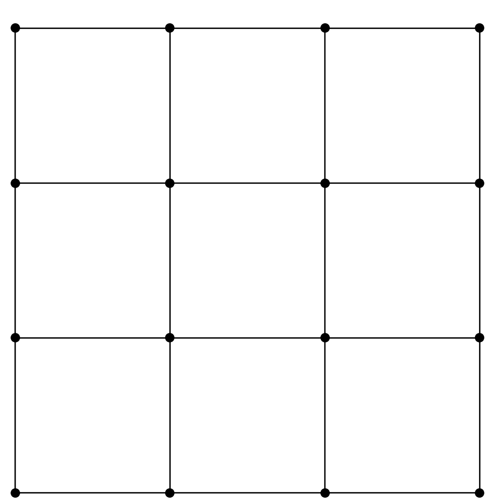


$$H = -\sum_{\langle ij \rangle} s_i \cdot s_j \qquad s_i = \pm 1$$

Ordered phase — Disordered phase → T

3D Ising CFT

Quantum



Spin-1/2 on each site $\begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix}$

$$H = -\sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$$

Spins point to +z or -z — Spins point to +x → h

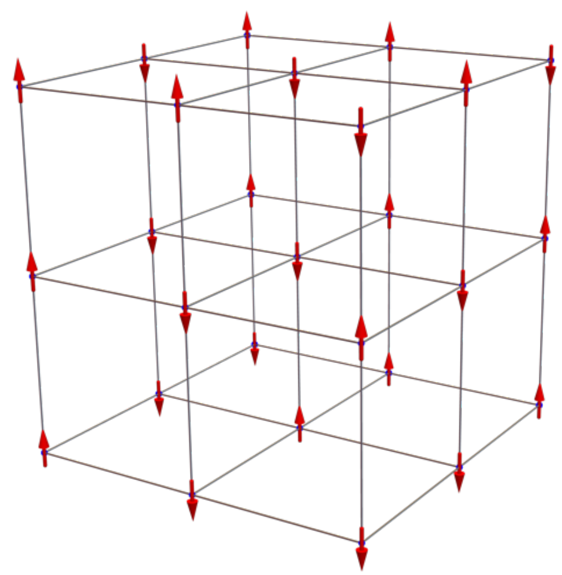
2+1D Ising CFT

A condensed matter approach to CFTs

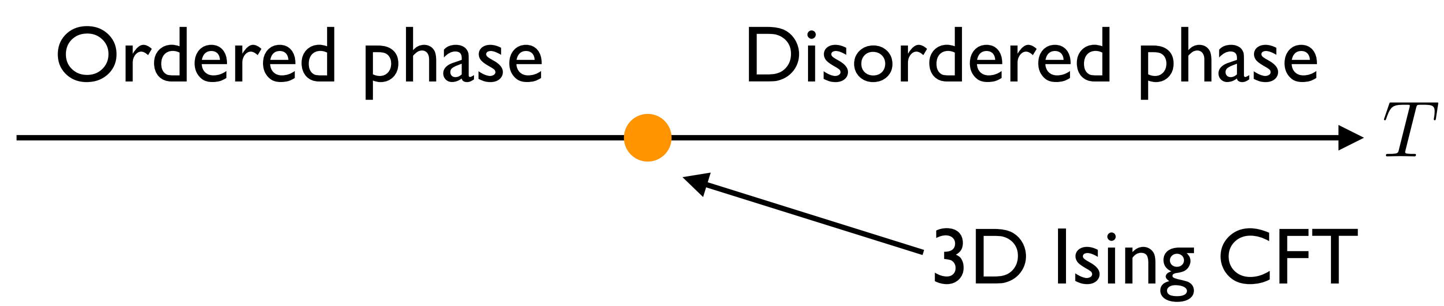
Study strongly interacting quantum mechanical models that realize CFTs.

Lattice models

Classical

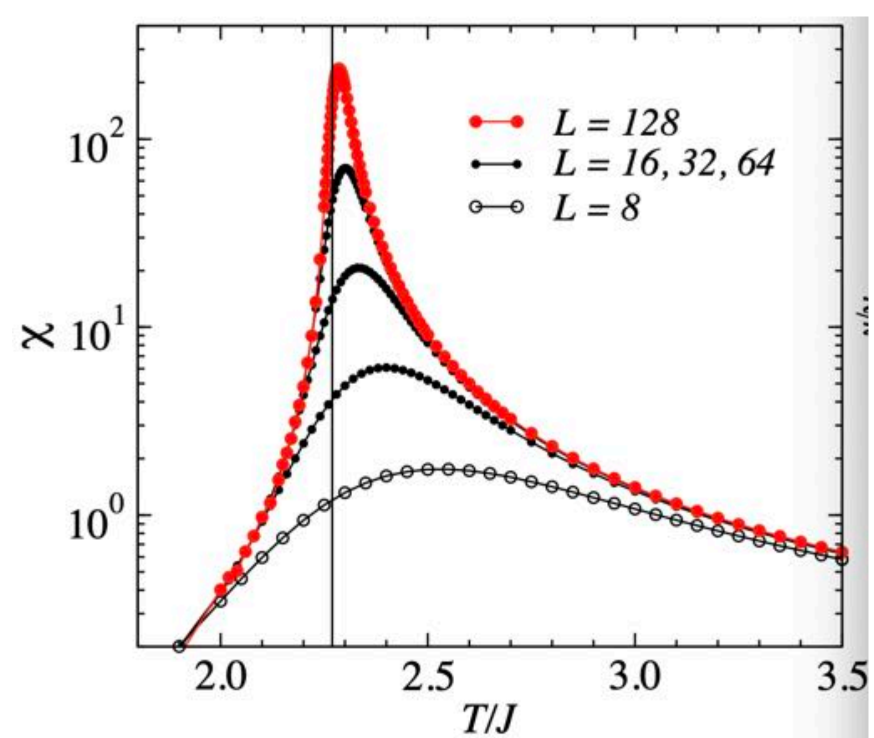


$$H = -\sum_{\langle ij \rangle} s_i \cdot s_j \qquad s_i = \pm 1$$



Extracting information:
finite scaling of observables

$$\chi \propto |T - T_c|^{-\gamma}$$



Lesson of conformal bootstrap: Leveraging conformal symmetry

Most lattice model studies are on the torus geometry where the consequence of conformal symmetry is poorly understood.

Example: 2-point correlator in d>2 dimensions

$\mathbb{R}^d$ $$\langle O(x=0)O(x)\rangle = x^{-2\Delta}$$

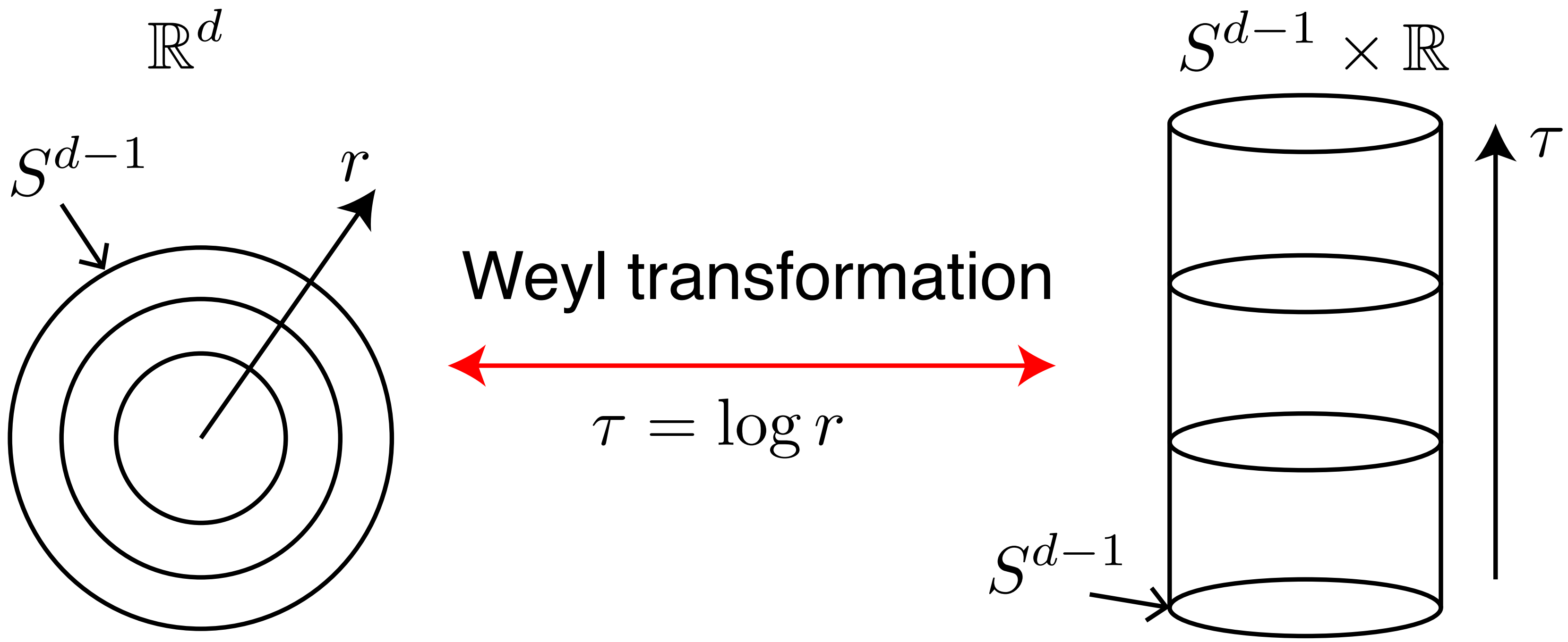
$$\uparrow \quad a \ll x \ll L$$

$T^d,\ T^{d-1}\times\mathbb{R}$ $$\langle O(x=0)O(x)\rangle = f(x,L) = \int_{\alpha=0}^{\infty} h(\alpha)x^{-2\Delta}(x/L)^{\alpha}$$

We need to study models on a conformally flat manifold $\mathbb{R}^d$, $S^{d-1}\times\mathbb{R}$, S^d!

State-operator correspondence

Radial quantization



Eigenstates of the quantum Hamiltonian defined on S^{d-1} are in one-to-one correspondence with CFT's scaling operators.

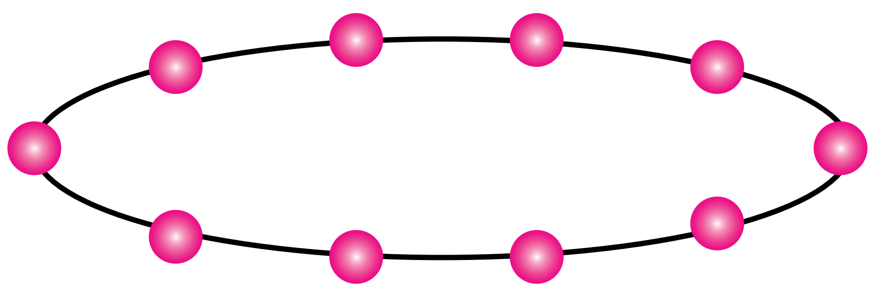
Energy gaps$\sim$scaling dimensions: $\delta E_n = E_n - E_0 = \frac{v}{R}\Delta_n$

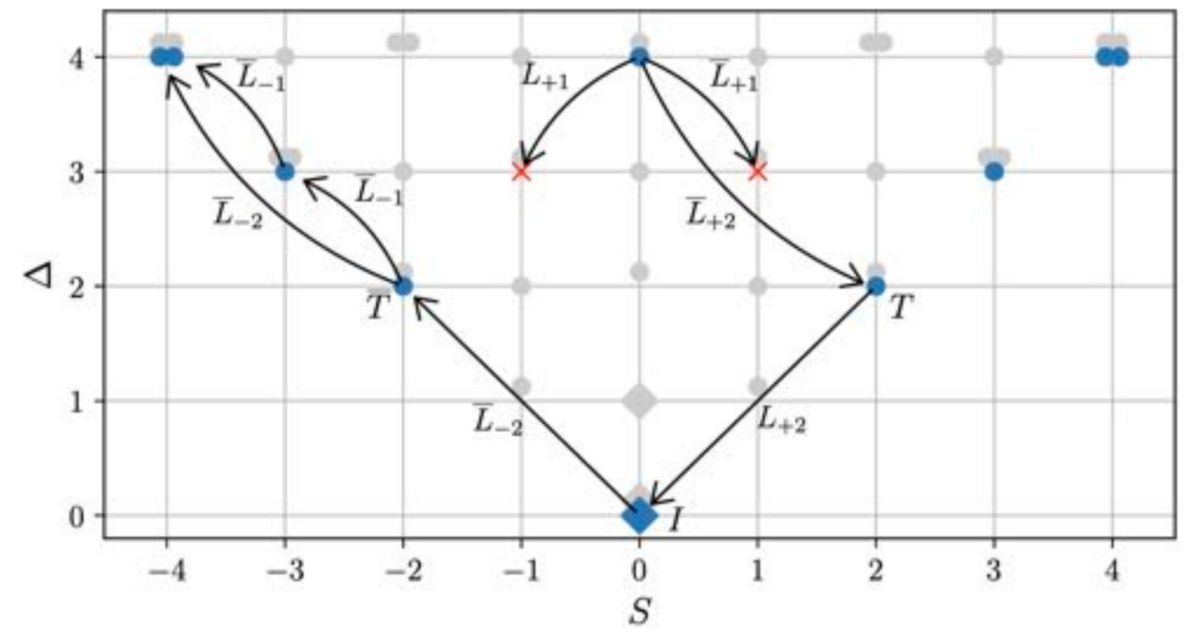
Radial quantization on a lattice

2D CFT: We can just study a quantum Hamiltonian on a circle.

Most conformal data can be extracted.

Cardy

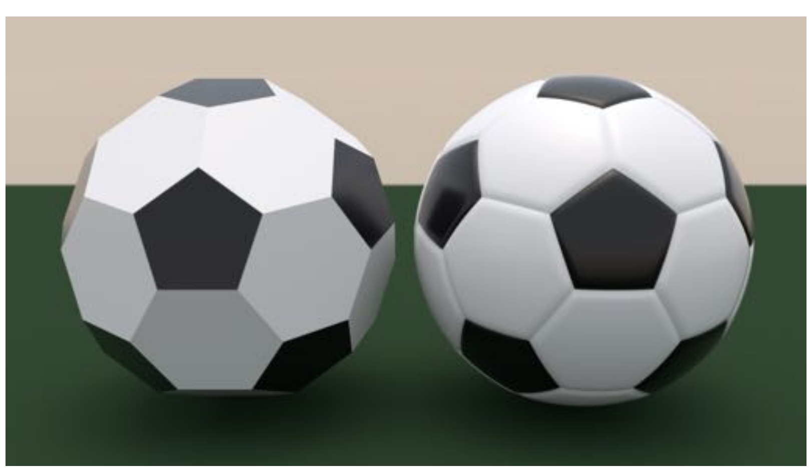


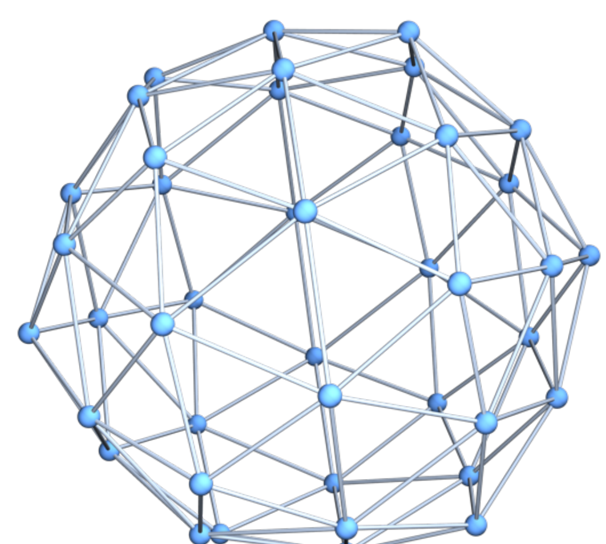


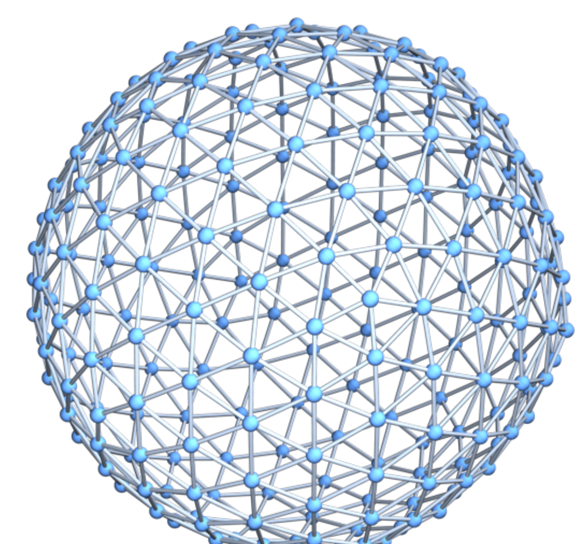
Milsted and Vidal 2017

3D CFT: We need to put a quantum Hamiltonian on a two-sphere.

But a regular lattice won't fit since two-sphere has a curvature…

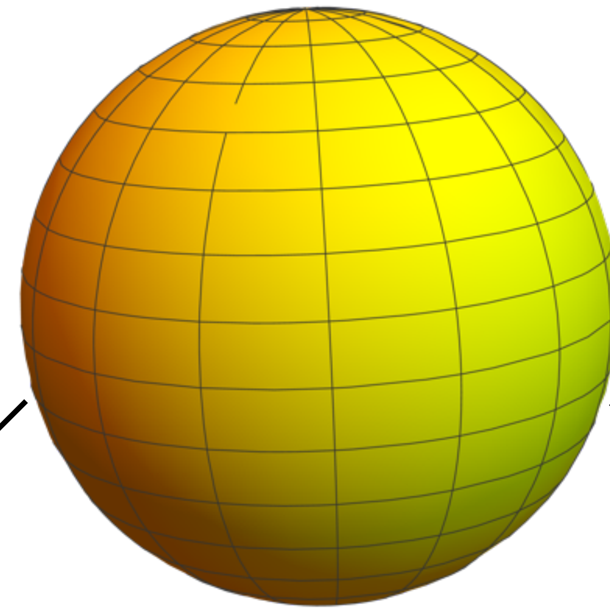






Our recipe: make it fuzzy!

Sphere is a curved space.



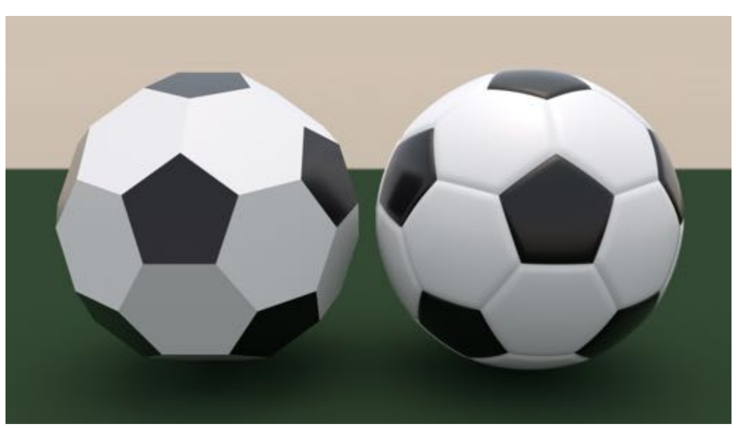
Discretize

Fuzzify

Lowest Landau
level projection

Haldane 1983

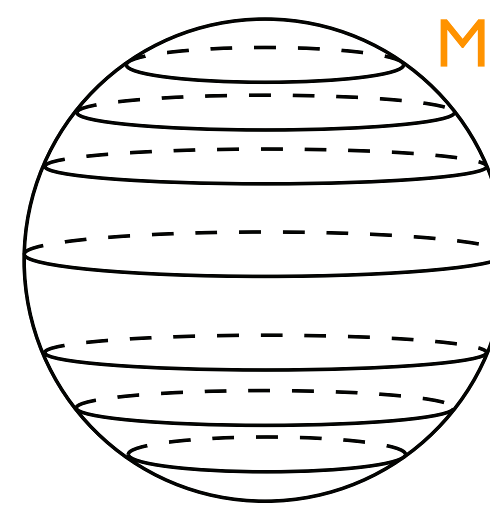
Spherical tiling



Spherical rotation
is broken badly.

fuzzy (non-commutative) sphere

Madore 1992

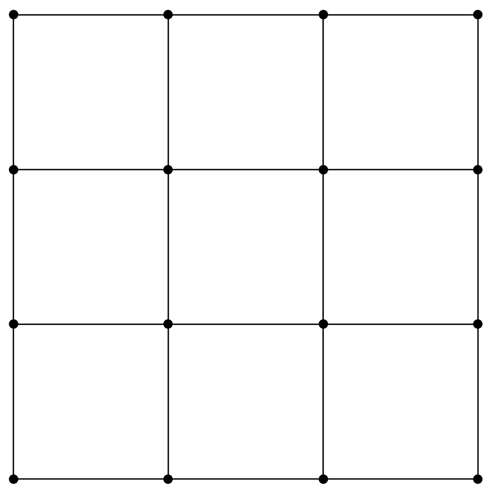


Spherical rotation
is kept exactly.

Fuzzy sphere regularization of 3D CFTs

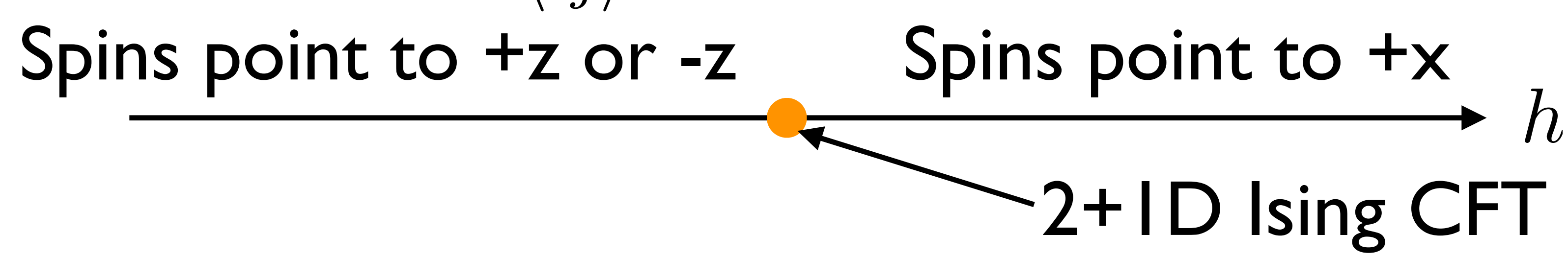
Quantum mechanical model realizations of 2+1D CFTs.

Lattice model

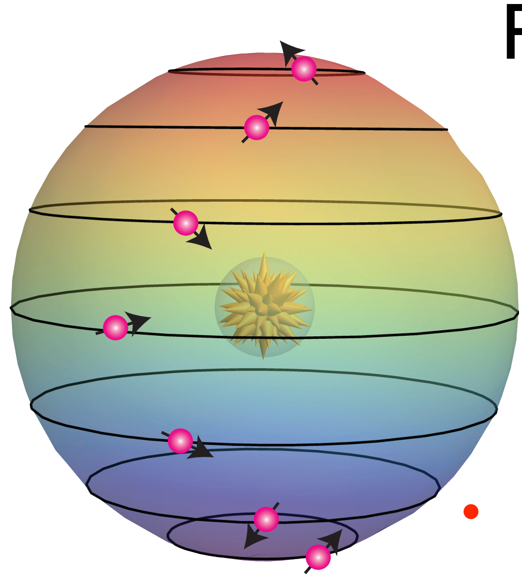


Spin-1/2 on each site $\begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix}$ $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$H = -\sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$$



Fuzzy sphere model



Particles moving on sphere in the presence of a monopole.

$$H = \frac{1}{2m} \sum_{i=1}^{N_e} (\vec{p}_i + \vec{A}(\vec{x}_i))^2 + \sum_{i,j=1}^{N_e} U(\vec{x}_i - \vec{x}_j)$$

Kinetic term | Interaction term

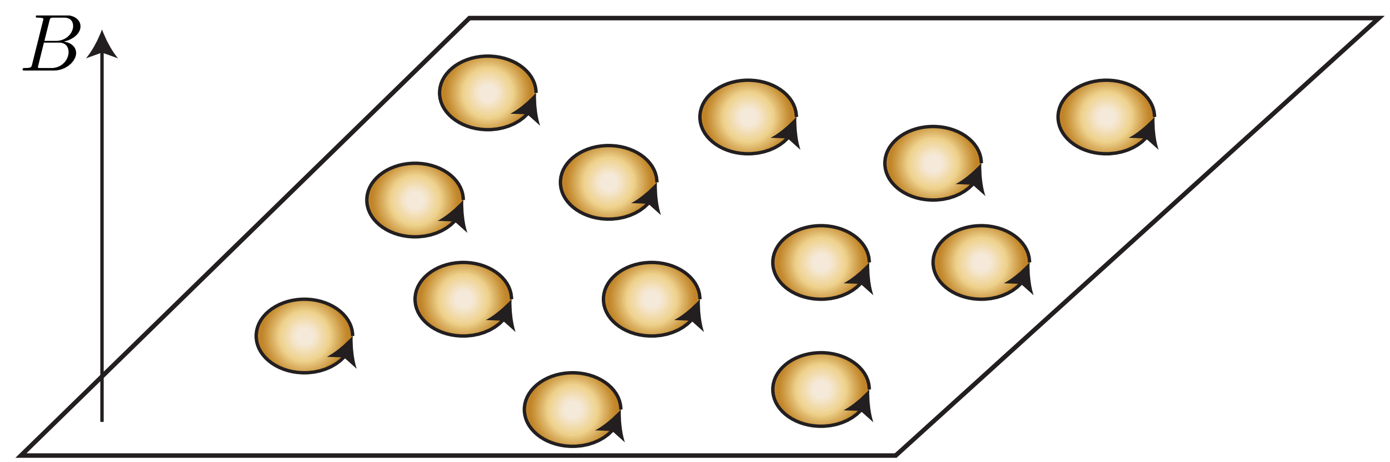
- The model is local if interactions are local.
- 2+1D CFTs can be realized by tuning the interaction form.

Landau level and non-commutative geometry

Particles moving in a strong magnetic field leads to non-commutative geometry!

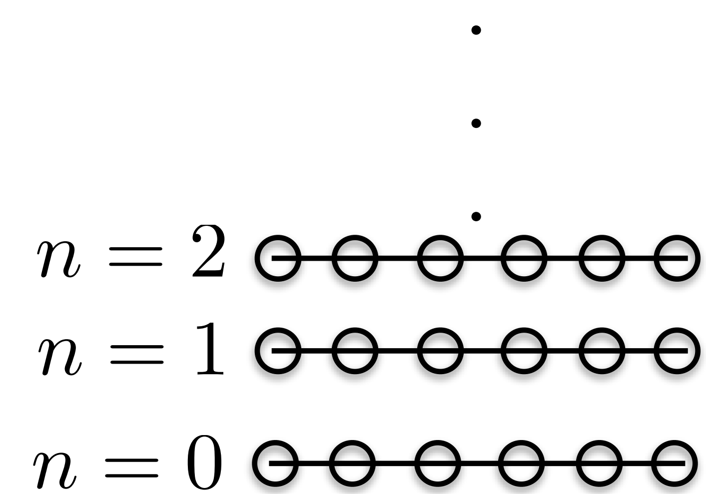
$$\mathcal{L} = \frac{M}{2}\dot{\vec{x}}^2 - \dot{\vec{x}} \cdot \vec{A}$$

$$A_i = -\frac{B}{2}\epsilon_{ij}x^j$$



Landau level: single particle states in the presence of magnetic field.

- Quantized energy: $E_n = \frac{B}{M}(n + 1/2)$
- Complete flat.
- Massive degeneracy at each level: $\frac{B\mathcal{A}}{2\pi}$



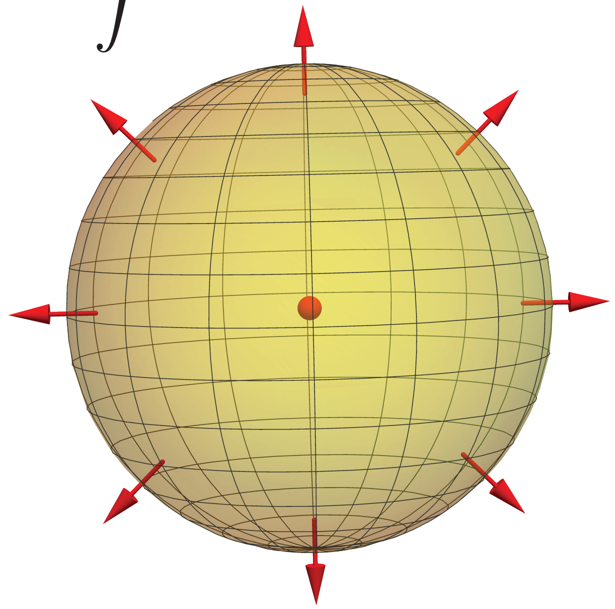
Restrict/Project to the lowest Landau level:

$$\mathcal{L}_0 = -\dot{\vec{x}} \cdot \vec{A} = \frac{B}{2}\epsilon_{ij}\dot{x}^i x^j \Rightarrow [x^i, x^j] = \frac{i}{B}\epsilon^{ij}$$

Spherical Landau levels

$$\int \vec{B} \cdot d\vec{r} = 4\pi \cdot s$$

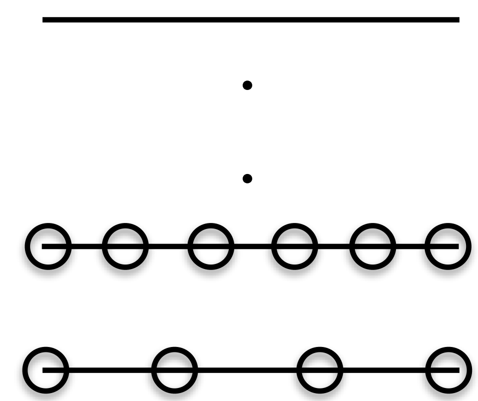
Electrons moving under a magnetic monopole.



Single particle kinetic term

$$\frac{1}{2Mr^2}(\partial_\mu + iA_\mu)^2$$

Landau levels (LLs)



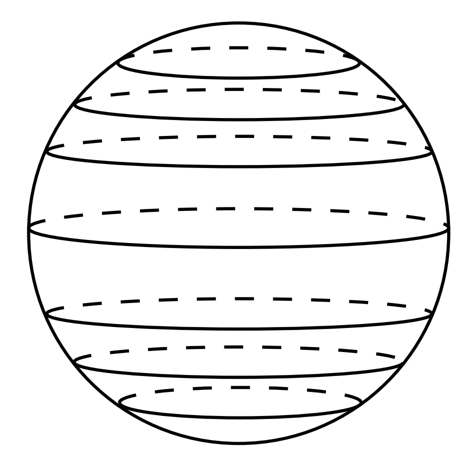
$n = 1$

$n = 0$

- Each LL has a level dependent quantized energy $\dfrac{n(n+1)+(2n+1)s}{2Mr^2}$
- The states (orbitals) in each LL form a spin-(n+s) SO(3) representation.
- The wavefunctions of each LL are monopole Harmonics $Y^{(s)}_{n+s,m}(\theta,\varphi)$.

Lowest LL wavefunction $\quad m = -s, -s+1, \cdots, s$

$$Y^{(s)}_{s,m}(\theta,\varphi) = \mathcal{N}_{s,m} e^{im\varphi} \cos^{s+m}\left(\frac{\theta}{2}\right) \sin^{s-m}\left(\frac{\theta}{2}\right)$$



Lowest Landau level (LLL) projection

$$H = \frac{1}{2Mr^2}\int d\Omega \psi^\dagger(\Omega)(\partial_\mu + iA_\mu)^2\psi(\Omega) + H_{int}$$

E.g. $H_{int} = U\int d\Omega\,(\psi^\dagger(\Omega)\partial_\mu\psi(\Omega))^2$

$$\{\psi^\dagger(\Omega_a), \psi(\Omega_b)\} = \delta(\Omega_{ab})$$
$$\{\psi(\Omega_a), \psi(\Omega_b)\} = 0$$

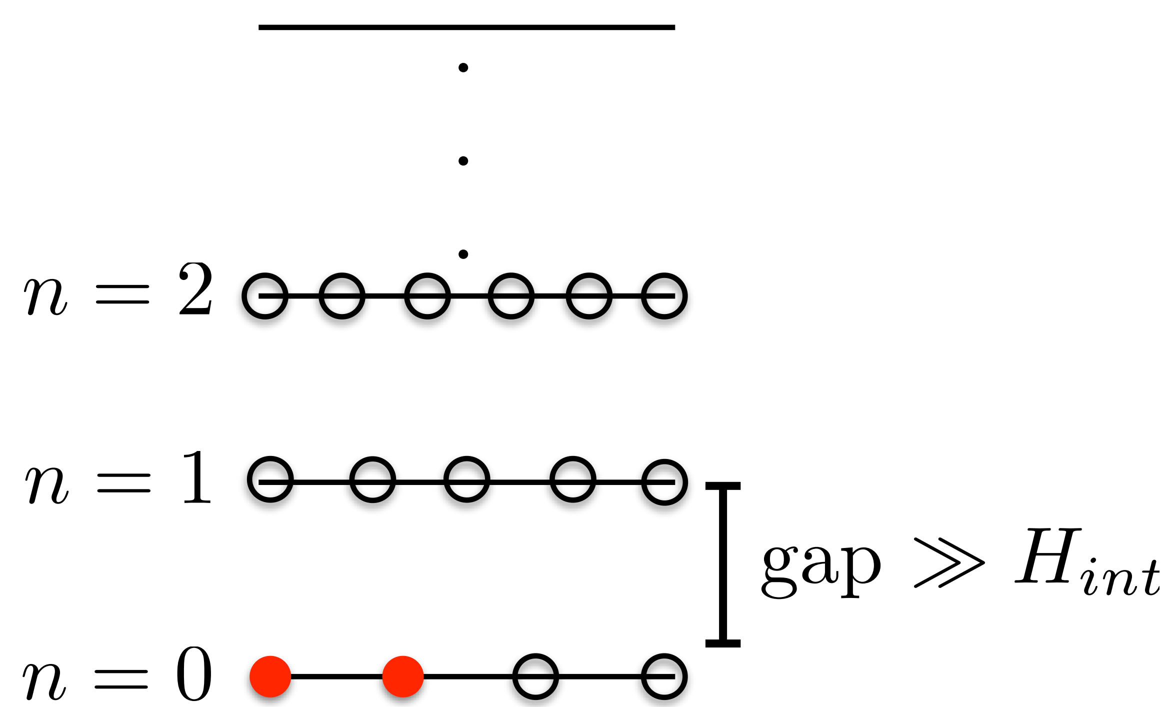
$$\psi^\dagger(\Omega) = \sum_{l=s}^{\infty}\sum_{m=-s}^{s} c^\dagger_{l,m} Y^{(s)}_{l,m}(\Omega)$$

$$\{c^\dagger_{l_a,m_a}, c_{l_b,m_b}\} = \delta_{l_a,l_b}\delta_{m_a,m_b}$$
$$\{c_{l_a,m_a}, c_{l_b,m_b}\} = 0$$

LLL projection

$$\psi^\dagger(\Omega) = \sum_{m=-s}^{s} c^\dagger_{s,m} Y^{(s)}_{s,m}(\Omega)$$

Landau levels



LLL projection and fuzzy sphere

On the LLL the sphere coordinates $x_1^2 + x_2^2 + x_3^2 = 1$ become:

$$(X_i)_{m_1,m_2} = \int d\Omega \, x_i(\Omega) \bar{Y}^{(s)}_{s,m_1}(\Omega) Y^{(s)}_{s,m_2}(\Omega)$$

$$[X_i, X_j] = \frac{1}{s+1} i\varepsilon_{ijk} X_k \qquad \sum_{i=1}^{3} X_i X_i = \frac{s}{s+1} \mathbf{1}_{2s+1}$$

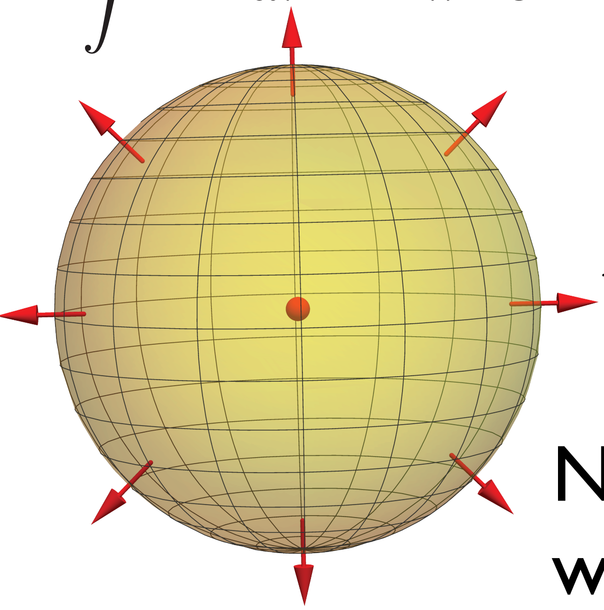
Fuzzy two-sphere: $[\hat{x}_i, \hat{x}_j] = i\varepsilon_{ijk}\hat{x}_k, \quad \sum_{i=1}^{3} \hat{x}_i \hat{x}_i = \mathrm{const} \cdot \mathbf{1}$

Madore 1992

$$Y^{(s)}_{s,m}(\theta, \varphi) = \mathcal{N}_{s,m} e^{im\varphi} \cos^{s+m}\left(\frac{\theta}{2}\right) \sin^{s-m}\left(\frac{\theta}{2}\right)$$

Fuzzy sphere model for the 2+1D Ising CFT

$\int \vec{B} \cdot d\vec{r} = 4\pi \cdot s$



$$H = \frac{1}{2Mr^2} \int d\Omega \psi^\dagger(\Omega)(\partial_\mu + iA_\mu)^2 \psi(\Omega) + H_{int}$$

$$H_{int} = -\int d\Omega_a d\Omega_b \, U(\Omega_a, \Omega_b) n^z(\Omega_a) n^z(\Omega_b) - h \int d\Omega \, n^x(\Omega)$$

Non-relativistic fermions with an isospin.

$$n^\alpha(\Omega) = (\hat{\psi}^\dagger_\uparrow(\Omega), \, \hat{\psi}^\dagger_\downarrow(\Omega)) \, \sigma^\alpha \begin{pmatrix} \hat{\psi}_\uparrow(\Omega) \\ \hat{\psi}_\downarrow(\Omega) \end{pmatrix}$$

$$U(\Omega_a, \Omega_b) = g_0 \, \delta(\Omega_{ab}) + g_1 \, \nabla^2 \delta(\Omega_{ab})$$

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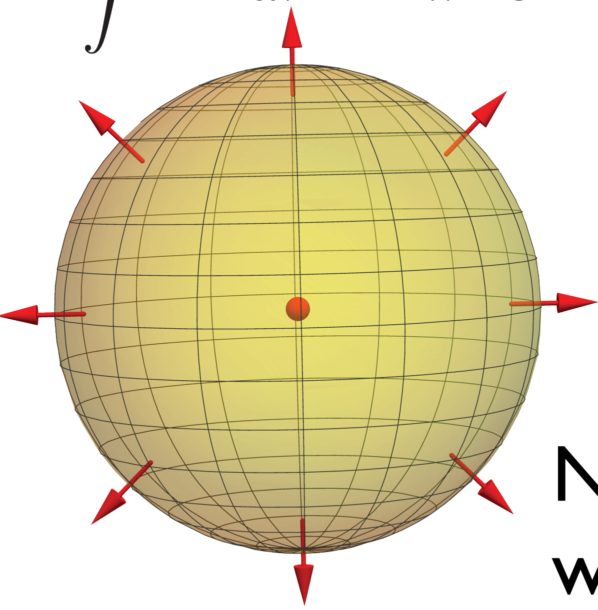

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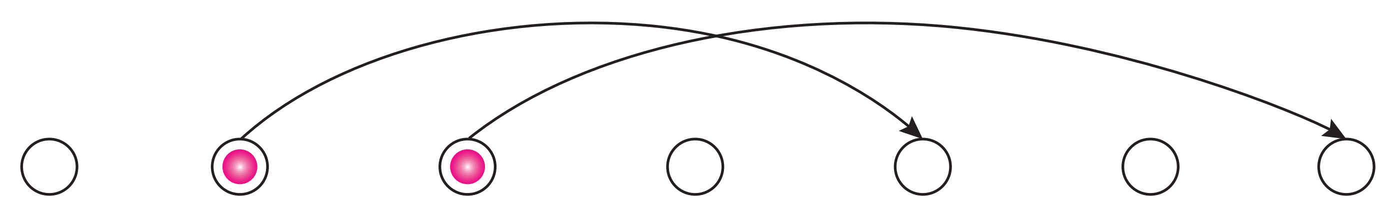
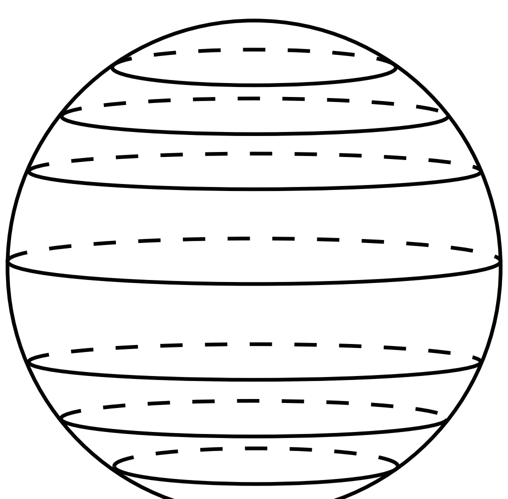
LLL projection

$$\psi^\dagger_\alpha(\vec{\Omega}) = \sum_{m=-s}^{s} c^\dagger_{m,\alpha} \, Y^{(s)}_{s,m}(\vec{\Omega})$$

$2s + 1$-site fermionic model

$m = -s, -s+1, \cdots, s$ spin-s rep of $SO(3)$





2-body term

$$\sum_{m=-s}^{s} c^\dagger_m c_m$$

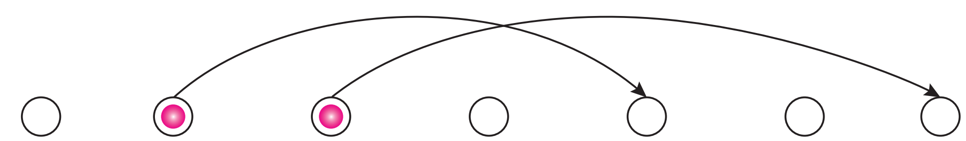
4-body interaction

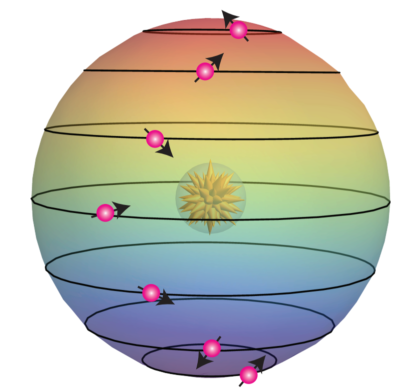
Haldane

$$V_l \sum_{m_1, m_2, m_3, m_4} F(m_1, m_2, m_3, m_4, s, l) c^\dagger_{m_1} c^\dagger_{m_2} c_{m_3} c_{m_4}$$

A closer look at the fuzzy sphere model

$2s+1$-site fermionic model





Many-body Hilbert space $\prod_{i=1}^{2s+1} c^\dagger_{m_i,\alpha_i}|0\rangle$

$m_i = -s, -s+1, \cdots, s$

spin-s rep of $SO(3)$

$\alpha_i = \uparrow, \downarrow$

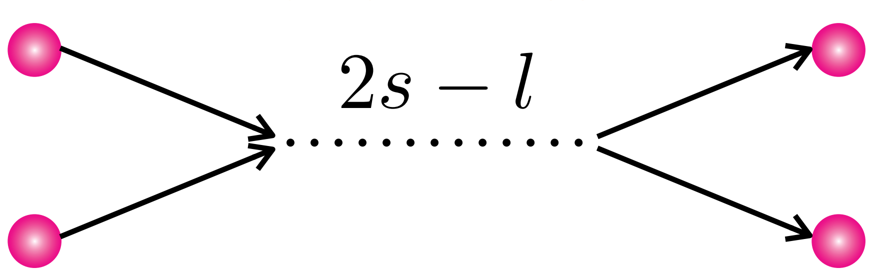
Hamiltonian for the 2+1D Ising model $\mathbf{c}^\dagger_m = (c^\dagger_{m,\uparrow}, c^\dagger_{m,\downarrow})$

$$H = -\sum_{m_{1,2},m=-s}^{s} V_{m_1,m_2,m_2-m,m_1+m}\left(\mathbf{c}^\dagger_{m_1}\sigma^z\mathbf{c}_{m_1+m}\right)\left(\mathbf{c}^\dagger_{m_2}\sigma^z\mathbf{c}_{m_2-m}\right) - h\sum_{m=-s}^{s}\mathbf{c}^\dagger_m\sigma^x\mathbf{c}_m$$

$$V_{m_1,m_2,m_3,m_3} = \sum_l V_l\,(4s-2l+1)\begin{pmatrix} s & s & 2s-l \\ m_1 & m_2 & -m_1-m_2\end{pmatrix}\begin{pmatrix} s & s & 2s-l \\ m_3 & m_4 & -m_3-m_4\end{pmatrix}$$

Haldane 1983

V_l



$$U(\Omega_a,\Omega_b) = g_0\,\delta(\Omega_{ab}) + g_1\,\nabla^2\delta(\Omega_{ab})$$

$$\Updownarrow$$

$$V_0 = \frac{1}{2}g_0 - \frac{1}{4}g_1, V_1 = \frac{1}{4}g_1$$

Phase diagram

$$H = \frac{1}{2Mr^2}\int d\Omega \psi^\dagger(\Omega)(\partial_\mu + iA_\mu)^2\psi(\Omega) + H_{int}$$

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$$U(\Omega_a,\Omega_b) = g_0\,\delta(\Omega_{ab}) + g_1\,\nabla^2\delta(\Omega_{ab})$$

Similar model was studied on the torus.

Ippoliti, Mong, Assaad, Zaletel (2018)

Tuning parameter:

(g_0, g_1, h)

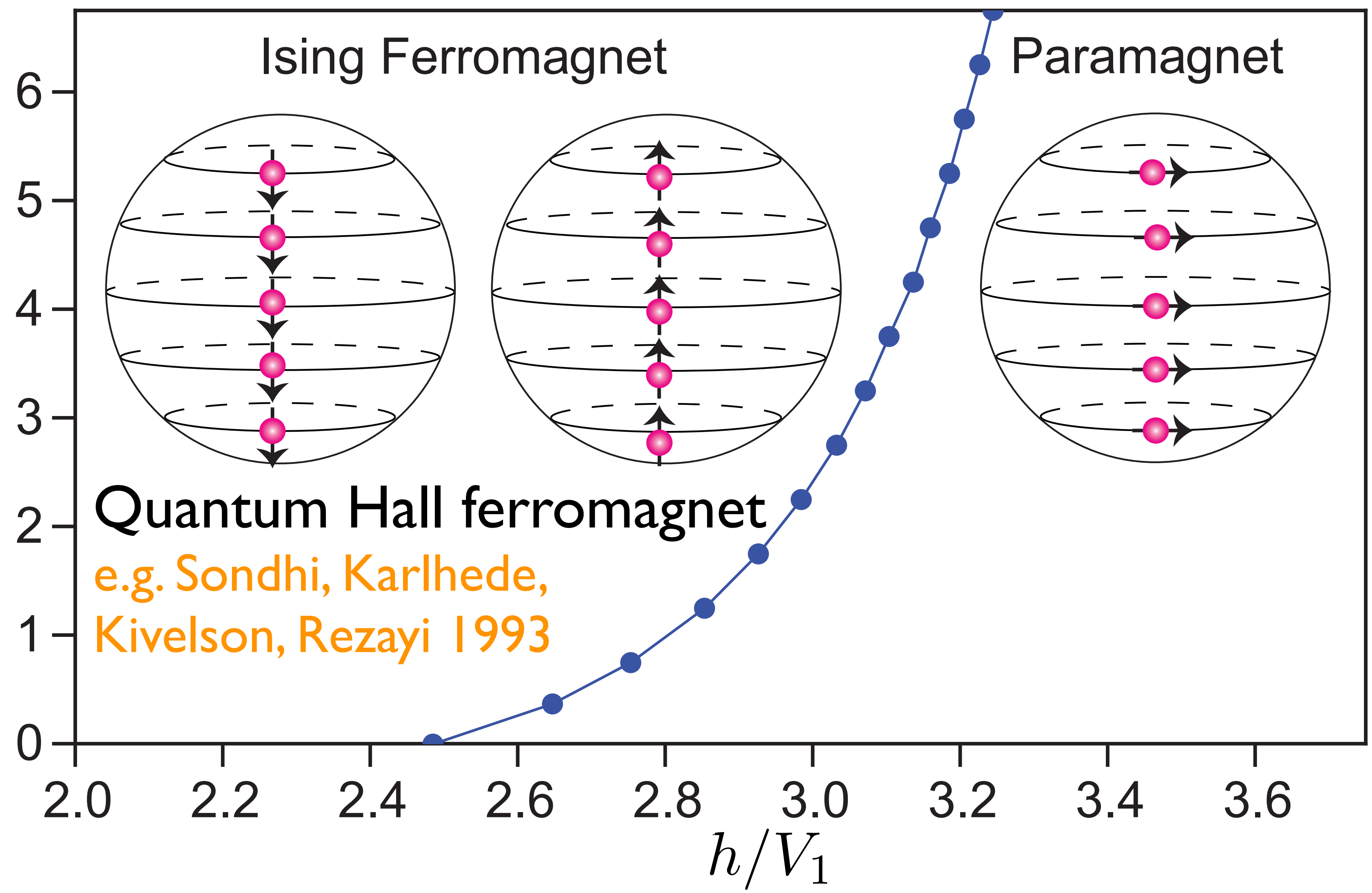
$\psi(\vec\Omega)^\dagger$

LLL projection

$$\sum_{m=-s}^{s} c^\dagger_m\, Y^{(s)}_{s,m}(\vec\Omega)$$

(V_0, V_1, h)

Haldane pseudo-potential



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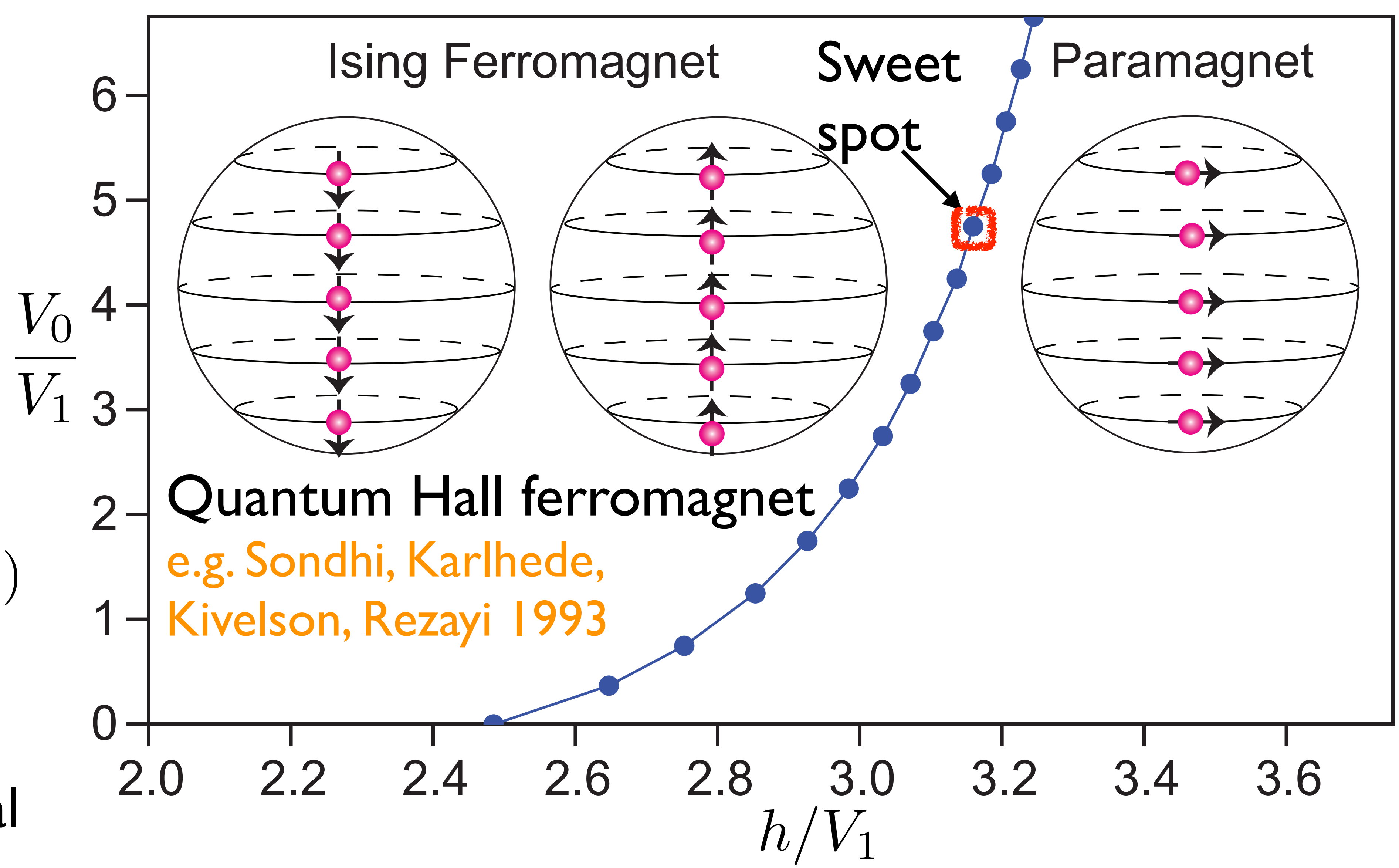

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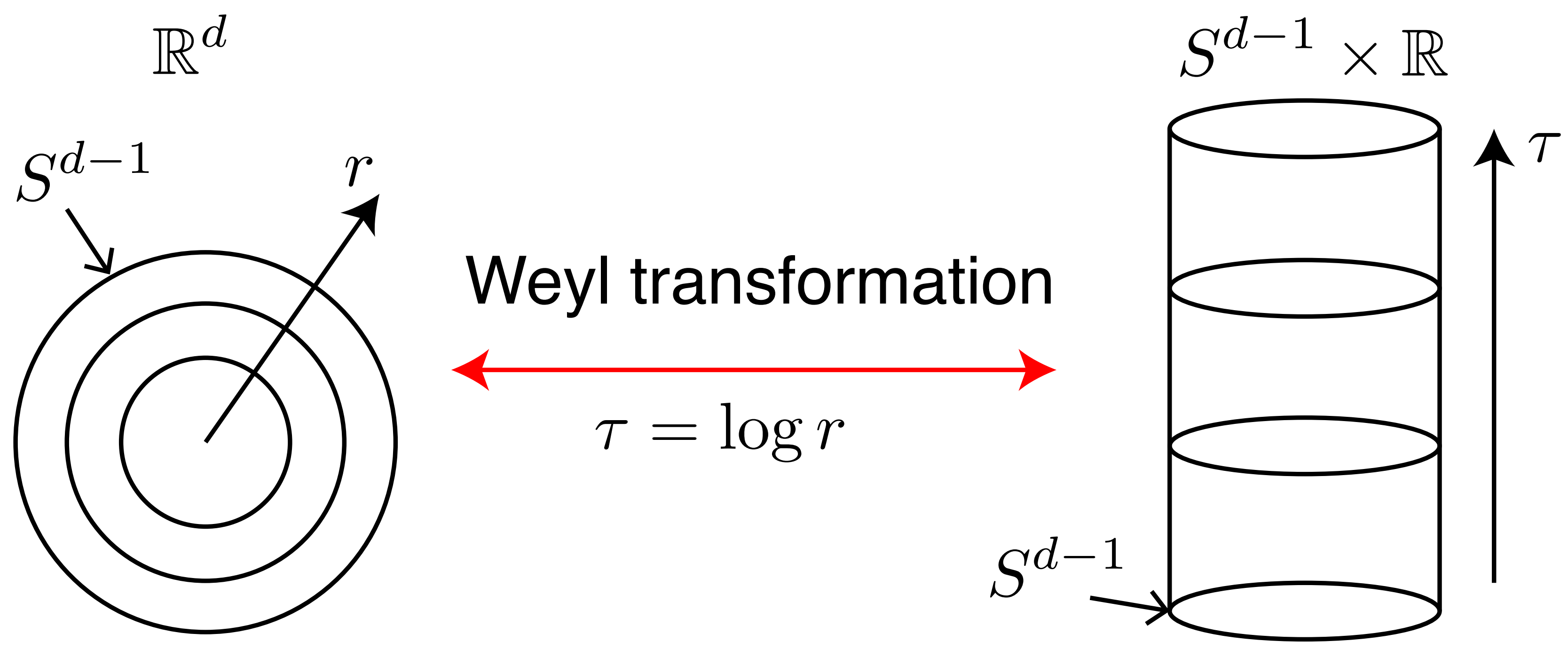
Radial quantization



Eigenstates of the quantum Hamiltonian defined on S^{d-1} are in one-to-one correspondence with CFT's scaling operators.

Energy gaps$\sim$scaling dimensions: $\delta E_n = E_n - E_0 = \frac{v}{R}\Delta_n$

Even 4 electrons work!!!

Gaps of ALL the excited states of the system with N=4 electrons.

6 primaries are found!!

	CB	4 spins	Errors
σ	0.518	0.530	2.3%
$\partial_{\mu_1}\sigma$	1.518	1.522	0.3%
$\Box\sigma$	2.518	2.427	3.6%
$\partial_{\mu_1}\partial_{\mu_2}\sigma$	2.518	2.428	3.6%
$\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\sigma$	3.518	2.847	20%
$\partial_{\mu_1}\Box\sigma$	3.518	3.291	6.5%
$\sigma_{\mu_1\mu_2}$	4.180	4.241	1.5%
$\sigma_{\mu_1\mu_2\mu_3}$	4.638	4.618	0.4%

	CB	4 spins	Errors
ϵ	1.413	1.382	2.2%
$\partial_{\mu_1}\epsilon$	2.413	2.337	3.1%
$T_{\mu_1\mu_2}$	3	3	NA
$\partial_{\mu_1}\partial_{\mu_2}\epsilon$	3.413	3.126	8.4%
$\Box\epsilon$	3.413	3.577	4.8%
$\partial_{\mu_3}T_{\mu_1\mu_2}$	4	3.663	8.4%
$\varepsilon_{\mu_2\rho\tau}\partial_\rho T_{\mu_1\mu_2}$	4	4.054	1.4%
ϵ'	3.830	4.019	4.9%
$\partial_{\mu_3}\partial_{\mu_4}T_{\mu_1\mu_2}$	5	4.856	2.9%

State-operator correspondence

- We identified 15 primary operators, the numerical errors of all primaries are within 1.6%.
- We looked at 70 lowest lying states with L<5, all of them match theoretical expectations with small errors~3%.

	CB	16 spins	Error
σ	0.518	0.524	1.2%
σ'	5.291	5.303	0.2%
$\sigma_{\mu_1\mu_2}$	4.180	4.214	0.8%
$\sigma'_{\mu_1\mu_2}$	6.987	7.048	0.9%
$\sigma_{\mu_1\mu_2\mu_3}$	4.638	4.609	0.6%
$\sigma_{\mu_1\mu_2\mu_3\mu_4}$	6.113	6.069	0.7%
σ^{P-}	NA	11.19	–

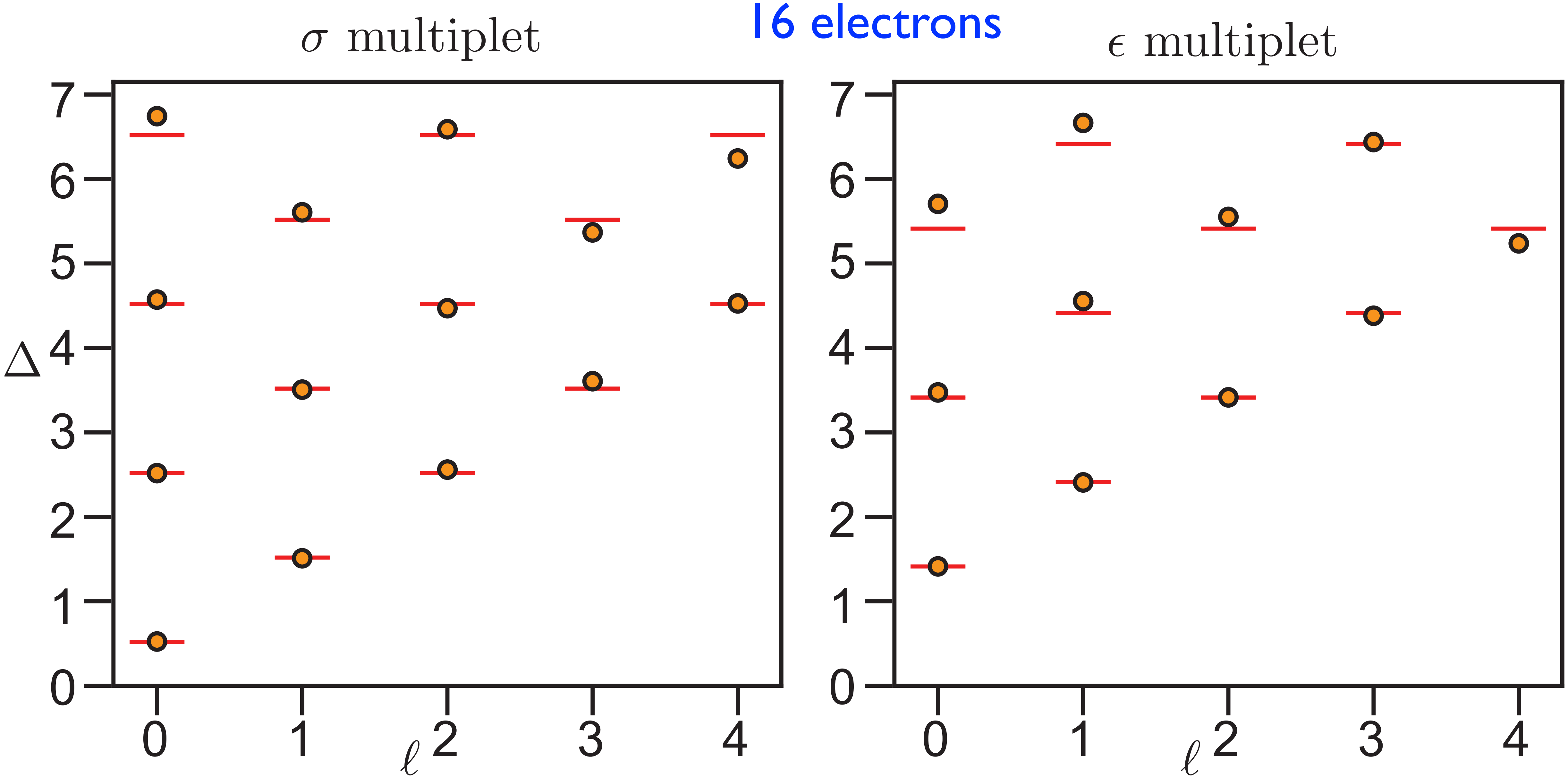
	CB	16 spins	Error
ϵ	1.413	1.414	0.07%
ϵ'	3.830	3.838	0.2%
ϵ''	6.896	6.908	0.2%
$T_{\mu\nu}$	3	3	–
$T'_{\mu\nu}$	5.509	5.583	1.3%
$\epsilon_{\mu_1\mu_2\mu_3\mu_4}$	5.023	5.103	1.6%
$\epsilon'_{\mu_1\mu_2\mu_3\mu_4}$	6.421	6.347	1.2%
ϵ^{P-}	≤ 11.2	10.01	–

Bootstrap data from Simmons-Duffin, 2017

State-operator correspondence

descendents: $\partial_{\mu_1} \cdots \partial_{\mu_j} \Box^n O, \quad n, j \geq 0 \qquad (\Delta + 2n + j, j)$

16 electrons



Outline

- Fuzzy sphere regularization
  - Why and how?
  - Radial quantization
  - Correlators, OPE coefficients, F-function
- Conformal defect

Let us continue with fuzzy journey

- We have explored the energy gaps of the states.
- A lot of information ready for exploration:

  A. Wave-functions of the states. $|\phi\rangle = \phi(\tau = -\infty)|\mathbb{I}\rangle$

  B. Operators.

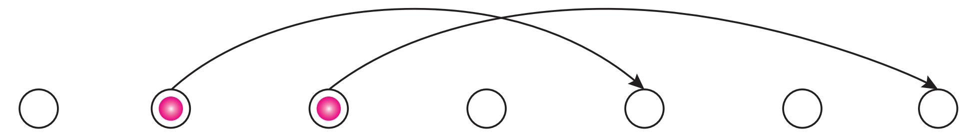
$$n^z(\tau,\Omega) \sim \alpha_\sigma \frac{\sigma(\tau,\Omega)}{R^{\Delta_\sigma}} + \alpha_{\partial_\mu \sigma} \frac{\partial_\mu \sigma(\tau,\Omega)}{R^{\Delta_\sigma+1}} + \cdots + \alpha_{\sigma_{\mu\nu}} \frac{\sigma_{\mu\nu}(\tau,\Omega)}{R^{\Delta_{\sigma_{\mu\nu}}}} + \cdots$$

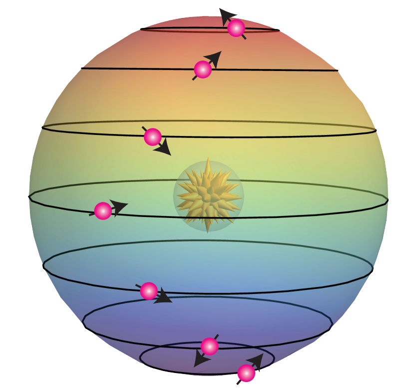
$$n^x(\tau,\Omega) \sim \alpha_{\mathbb{I}} + \alpha_\epsilon \frac{\epsilon(\tau,\Omega)}{R^{\Delta_\epsilon}} + \cdots + \alpha_{\sigma_{\mu\nu}} \frac{T\mu\nu(\tau,\Omega)}{R^3} + \cdots$$

From orbital space to real space

All the computations are done in the orbital space

$2s+1$-site fermionic model





Many-body Hilbert space $\prod_{i=1}^{2s+1} c^\dagger_{m_i,\alpha_i}|0\rangle$

$m_i = -s, -s+1, \cdots, s$

spin-s rep of $SO(3)$

$\alpha_i = \uparrow, \downarrow$

Real space is continuous (NOT discrete like lattice model)!

$$\psi^\dagger_a(\theta,\varphi) = \sum_{m=-s}^{s} c^\dagger_{m,a} Y^{(s)}_{s,m}(\theta,\varphi)$$

$$Y^{(s)}_{s,m}(\theta,\varphi) = \mathcal{N}_{s,m} e^{im\varphi} \cos^{s+m}\left(\frac{\theta}{2}\right) \sin^{s-m}\left(\frac{\theta}{2}\right)$$

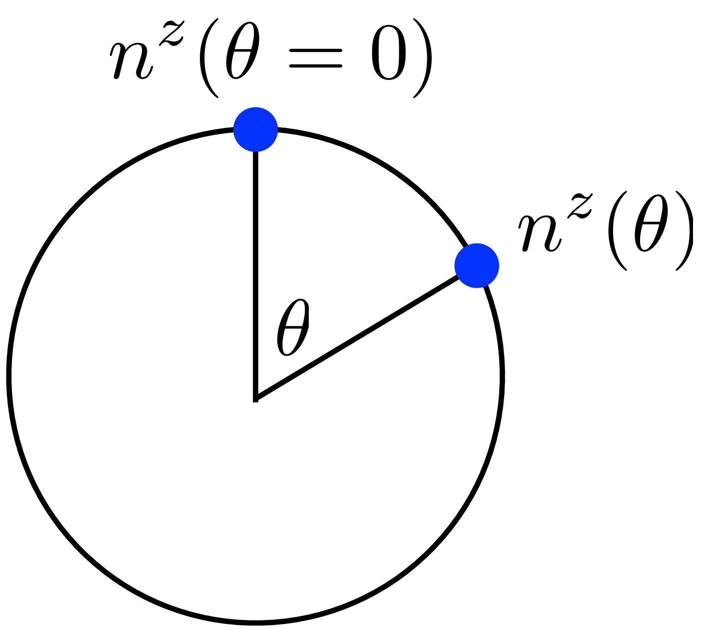
Any observable can be computed in real space!

Numerical data of 2-point correlator

We get a function defined in the continuum:

$$0.6941 + 0.3724\cos\theta + 0.2840\cos^2\theta + 0.2091\cos^3\theta + \cdots$$

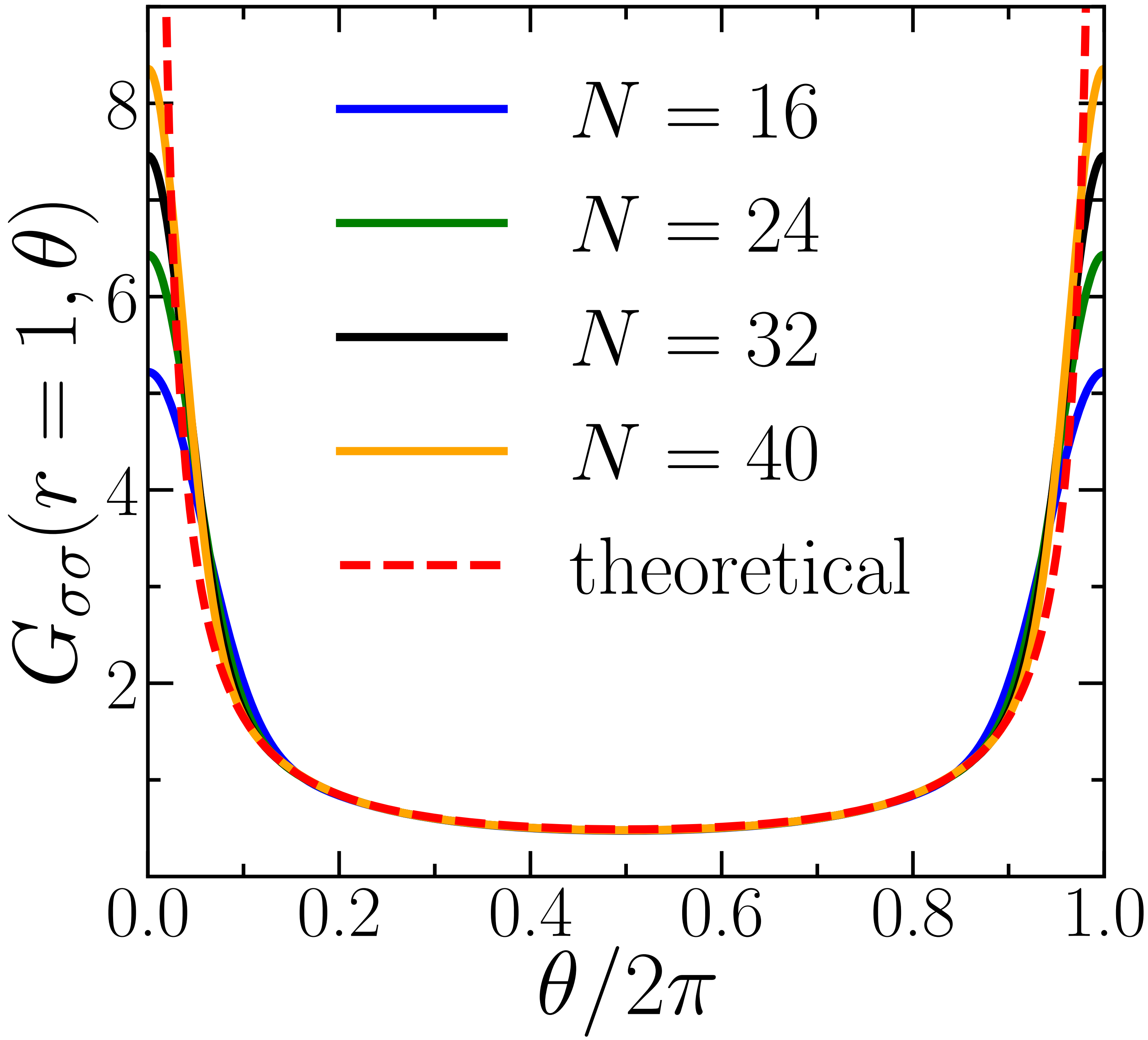
Han, Hu, Zhu, YCH, arXiv: 2306.04681



$$\langle n^z n^z \rangle = \frac{\langle 0|n^z(\theta=0)n^z(\theta)|0\rangle}{\langle \sigma|n^z(\theta=0)|0\rangle^2}$$

CFT prediction: $\frac{1}{(2\sin(\theta/2))^{2\Delta}}$

$\Delta \approx 0.518149$

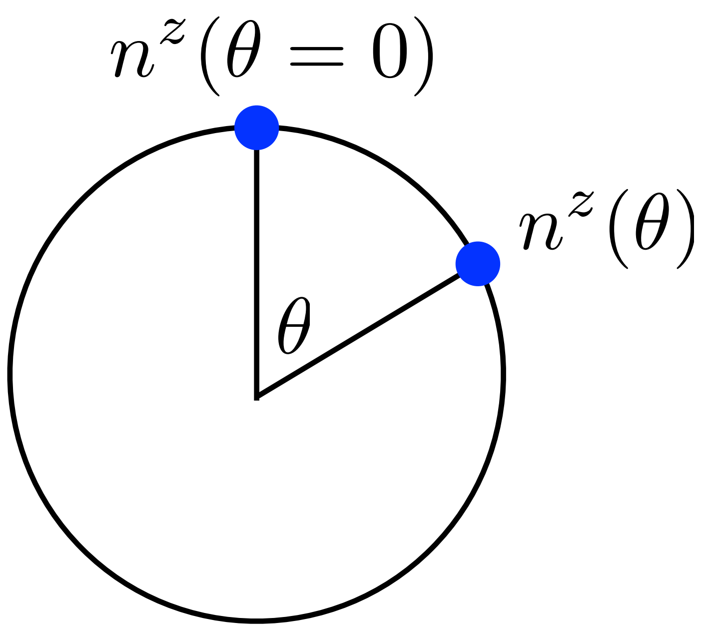


Operators and their correlators are sharp, continuous and conformal!

Four-point correlator

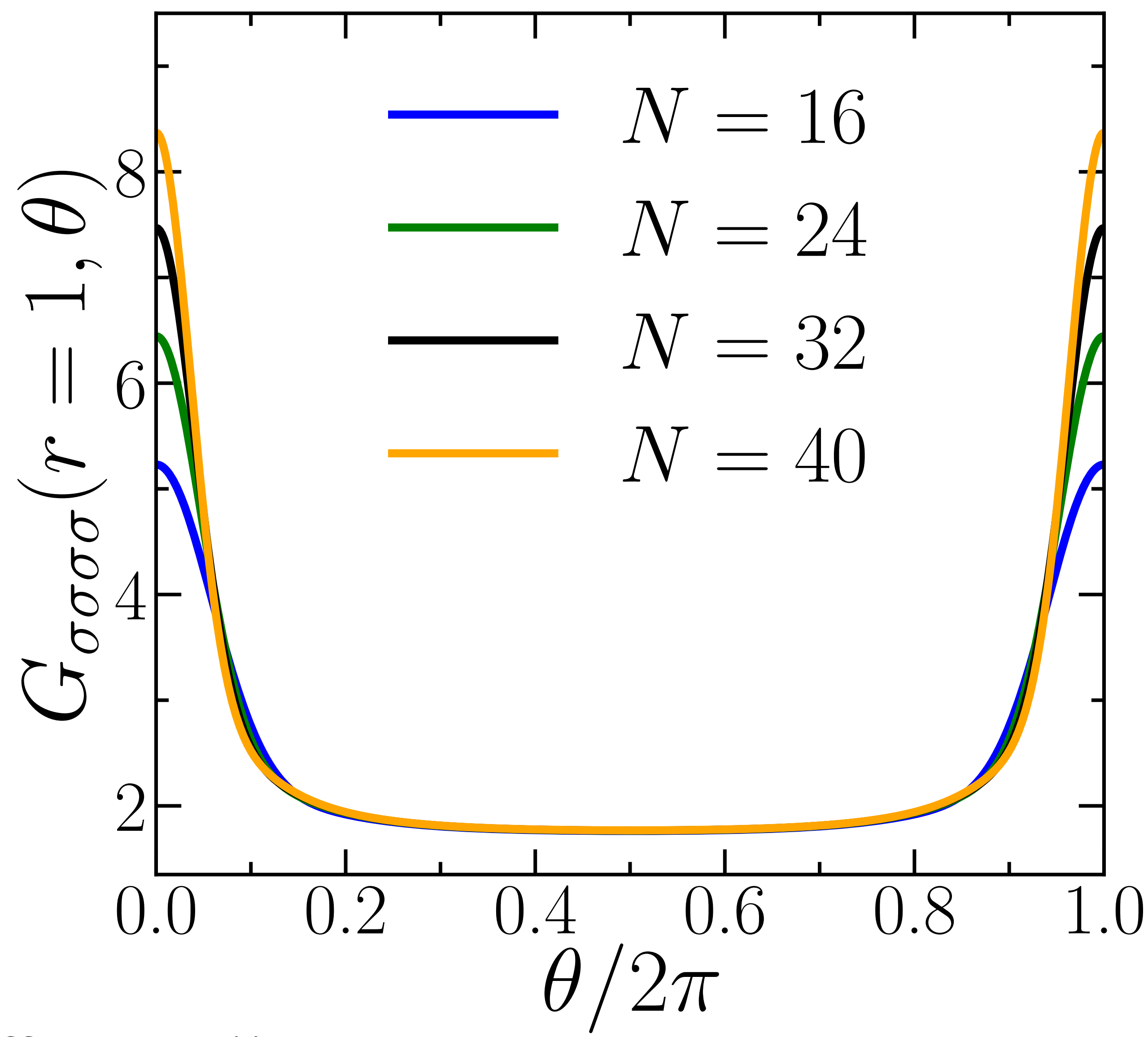
Han, Hu, Zhu, YCH, arXiv: 2306.04681

$$1.846 + 0.171\cos\theta + 0.152\cos^2\theta$$
$$+0.109\cos^3\theta + 0.109\cos^4\theta + \cdots$$



$$G(z = e^{i\theta}, \bar{z} = e^{-i\theta})$$

$$\frac{\langle\sigma|n^z(\theta=0)n^z(\theta)|\sigma\rangle}{\langle\sigma|n^z(\theta=0)|0\rangle^2}$$



0.06% difference!!

	Bootstrap	$N = 40$	$N = 32$	$N = 24$	$N = 16$
$\theta = \pi$	1.76855	1.76742	1.76671	1.76549	1.76244
$\theta = \pi/3$	2.049	2.03921	2.03495	2.02470	2.01212

OPE coefficients

$$\frac{\langle\sigma|n^z(\vec{\Omega})|\epsilon\rangle}{\langle\sigma|n^z(\vec{\Omega})|0\rangle} = f_{\sigma\sigma\epsilon} + \frac{a}{R^2} + \cdots$$

Hu, YCH, Zhu, arXiv:2303.08844 (PRL)

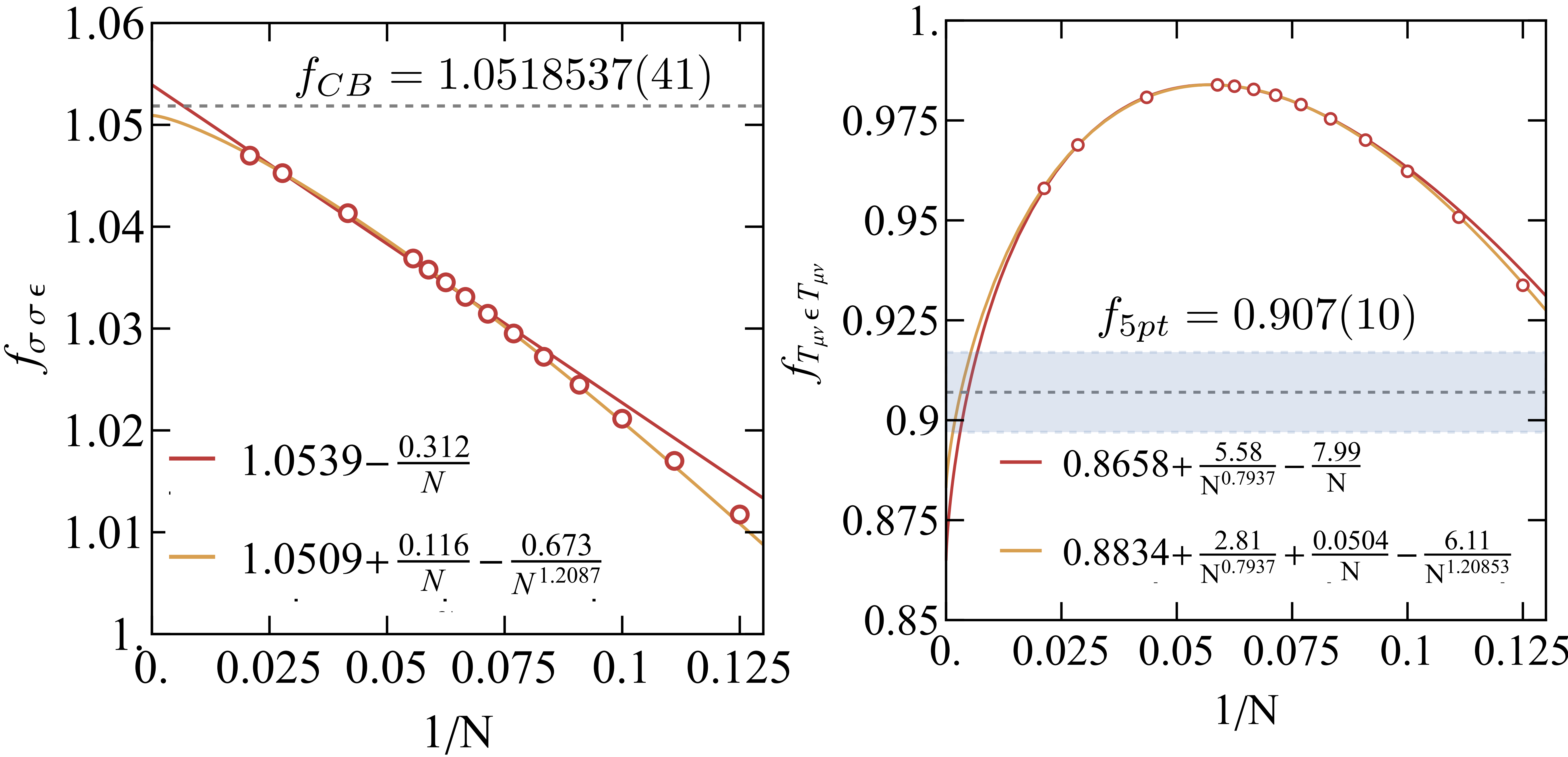
Operators	Spin	Z_2	$f_{\alpha\beta\gamma}$ (Fuzzy Sphere)	$f_{\alpha\beta\gamma}$ (Bootstrap)
σ	0	$-$	$f_{\sigma\sigma\epsilon} \approx 1.0539(18)$	$f_{\sigma\sigma\epsilon} \approx 1.0519$
ϵ	0	$+$	$f_{\epsilon\epsilon\epsilon} \approx 1.5441(23)$	$f_{\epsilon\epsilon\epsilon} \approx 1.5324$
ϵ'	0	$+$	$f_{\sigma\sigma\epsilon'} \approx 0.0529(16)$	$f_{\sigma\sigma\epsilon'} \approx 0.0530$
			$f_{\epsilon\epsilon\epsilon'} \approx 1.566(68)$	$f_{\epsilon\epsilon\epsilon'} \approx 1.5360$
σ'	0	$-$	$f_{\sigma'\sigma\epsilon} \approx 0.0515(42)$	$f_{\sigma'\sigma\epsilon} \approx 0.0572$
			$f_{\sigma'\sigma\epsilon'} \approx 1.294(51)$	NA
			$f_{\sigma'\epsilon\sigma'} \approx 2.98(13)$	NA
$T_{\mu\nu}$	2	$+$	$f_{\sigma\sigma T} \approx 0.3248(35)$	$f_{\sigma\sigma T} \approx 0.3261$
			$f_{\sigma'\sigma T} \approx -0.00007(96)$	$f_{\sigma'\sigma T} = 0$
			$f_{\epsilon\epsilon T} \approx 0.8951(35)$	$f_{\epsilon\epsilon T} \approx 0.8892$
			$f_{T\epsilon T} \approx 0.8658(69)$	$f_{T\epsilon T} \approx 0.765(47), 0.907(10)$
$\sigma_{\mu\nu}$	2	$-$	$f_{\sigma\epsilon\sigma_{\mu\nu}} \approx 0.400(33)$	$f_{\sigma\epsilon\sigma_{\mu\nu}} \approx 0.3892$
			$f_{\sigma\epsilon'\sigma_{\mu\nu}} \approx 0.18256(69)$	NA

OPE coefficients

Hu, YCH, Zhu, arXiv:2303.08844 (PRL)

$$\frac{\langle\sigma|n^z(\vec{\Omega})|\epsilon\rangle}{\langle\sigma|n^z(\vec{\Omega})|0\rangle} = f_{\sigma\sigma\epsilon} + \frac{a}{N} + \cdots$$

$$\frac{\langle T_{\mu\nu}|O(\vec{\Omega})|T_{\mu\nu}\rangle}{\langle\epsilon|O(\vec{\Omega})|0\rangle} = f_{TT\epsilon} + \frac{a}{N^{(3-\Delta_\epsilon)/2}} + \frac{b}{N} + \cdots$$



RG monotonic theorem

- RG monotonic theorem states that there exist some measure of degrees of freedom that monotonically decreases under RG flow.

- 2D CFT: c-theorem. Zamolodchikov 1986

- 4D CFT: a-theorem. Cardy 1988; Jack, Osborn 1990; Komargodski & Schwimmer 2011

- 3D CFT: F-theorem. Casini, Huerta 2012, Jafferis 2010; Myers, Sinha 2010; Jafferis, Klebanov, Pufu, Safdi 2011;…

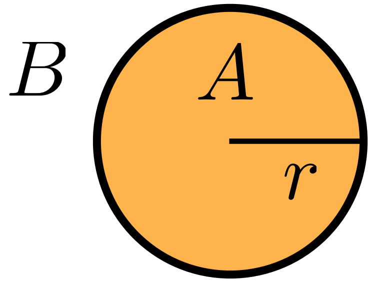
Irreversibility of RG in 3D: F-Theorem

No conformal anomaly in odd space-time dimensions!

- Partition function on 3-sphere $\log Z_{S^3} \sim \alpha_1 r^3 + \alpha_2 r - F$

Jafferis 2010; Jafferis, Klebanov, Pufu, Safdi 2011;…

- Entanglement entropy



$$S_A = \alpha r - F$$

Myers, Sinha 2010; Casini, Huerta, Myers, 2011; Liu, Mezei 2012

F-theorem: $F_{UV} > F_{IR}$

Casini, Huerta 2012

F is a non-local quantity

- In 2D and 4D, c and a are conformal anomalies. So they can be computed by correlation functions.

$$\langle T_{\mu\nu}(x_1) T_{\rho\eta}(x_2) \rangle \qquad \langle T_{\mu\nu}(x_1) T_{\rho\eta}(x_2) T_{\sigma\tau}(x_3) \rangle$$

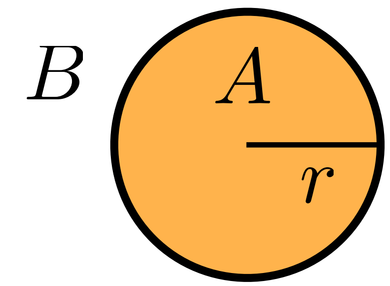
- In odd dimensions (e.g. 3D), there is no conformal anomaly. F is a non-local quantity encoded in either the 3-sphere partition function or entanglement entropy.

	2D	3D	4D
Free fermion	$c = 1$	$F = \frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2}$	$a = \frac{11}{180(8\pi)^2}$
Free scalar	NA	$F = \frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2}$	$a = \frac{1}{90(8\pi)^2}$
Ising CFT	$c = 1/2$	Not known	NA

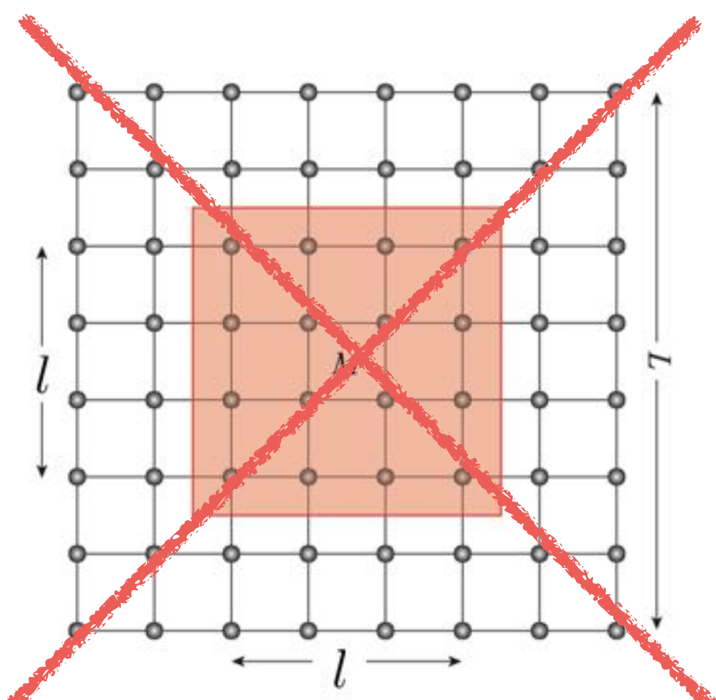
Extracting F-function via Entanglement entropy

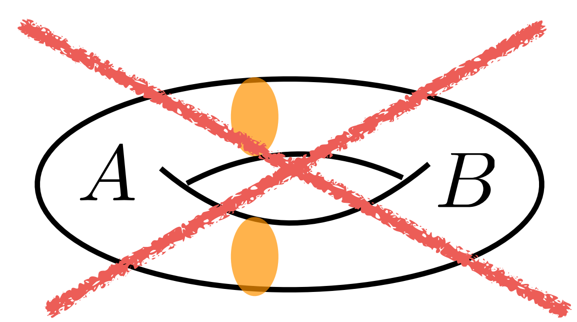
For a 2+1D CFT defined on $R^2 \times R$

$$S_A = -Tr(\rho_A \ln \rho_A) = \alpha \frac{r}{\delta} - F$$



- A smooth entanglement cut (no sharp corners).
- The entanglement cut should be much smaller than the system size if the system is not conformally equivalent to R^3.
- Entanglement entropy NOT Renyi entropy.

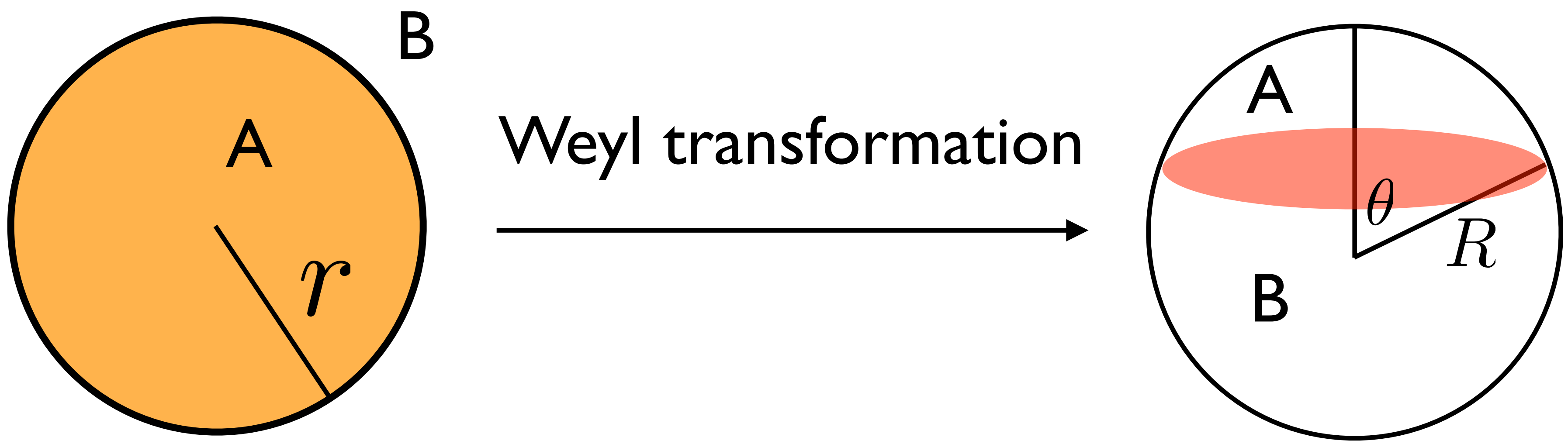




Monte Carlo

F-function from sphere entanglement entropy

$R^2 \times R$ $\qquad\qquad$ $S^2 \times R$



$$S_A = \alpha \frac{r}{\delta} - F$$

$$S_A = \alpha \frac{R}{\delta} \sin\theta - F$$

$$F(R, \theta_0) = (\tan\theta \partial_\theta - 1) S_A|_{R,\theta_0}$$

Myers, Sinha 2010; Casini, Huerta, Myers, 2011; Liu, Mezei 2012

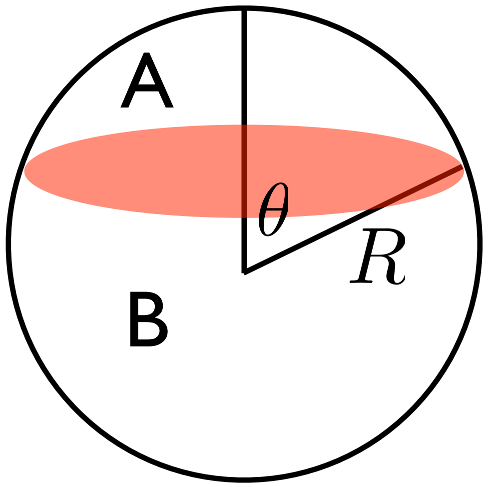
Banerjee, Nakaguchi, Nishioka 2015

The real-space entanglement entropy can be computed on the fuzzy sphere.

Dubail, Read, Rezayi 2012; Sterdyniak, Chandran, Regnault, Bernevig, and Parsa Bonderson 2012

F-function of 3D Ising CFT

Hu, Zhu, YCH, arXiv:2401.17362



$$S_A(\theta) = -Tr(\rho_A \ln \rho_A)$$
$$= \alpha \frac{R}{\delta} \sin\theta - F$$

$$F = (\tan\theta \partial_\theta - 1) S_A|_{\theta=\pi/2}$$
$$F(R) = F_{Ising} + aR^{-\omega} + O(R^{-1})$$
$$R/\delta \sim \sqrt{N} = \sqrt{2s+1}$$

Our estimates on fuzzy sphere:

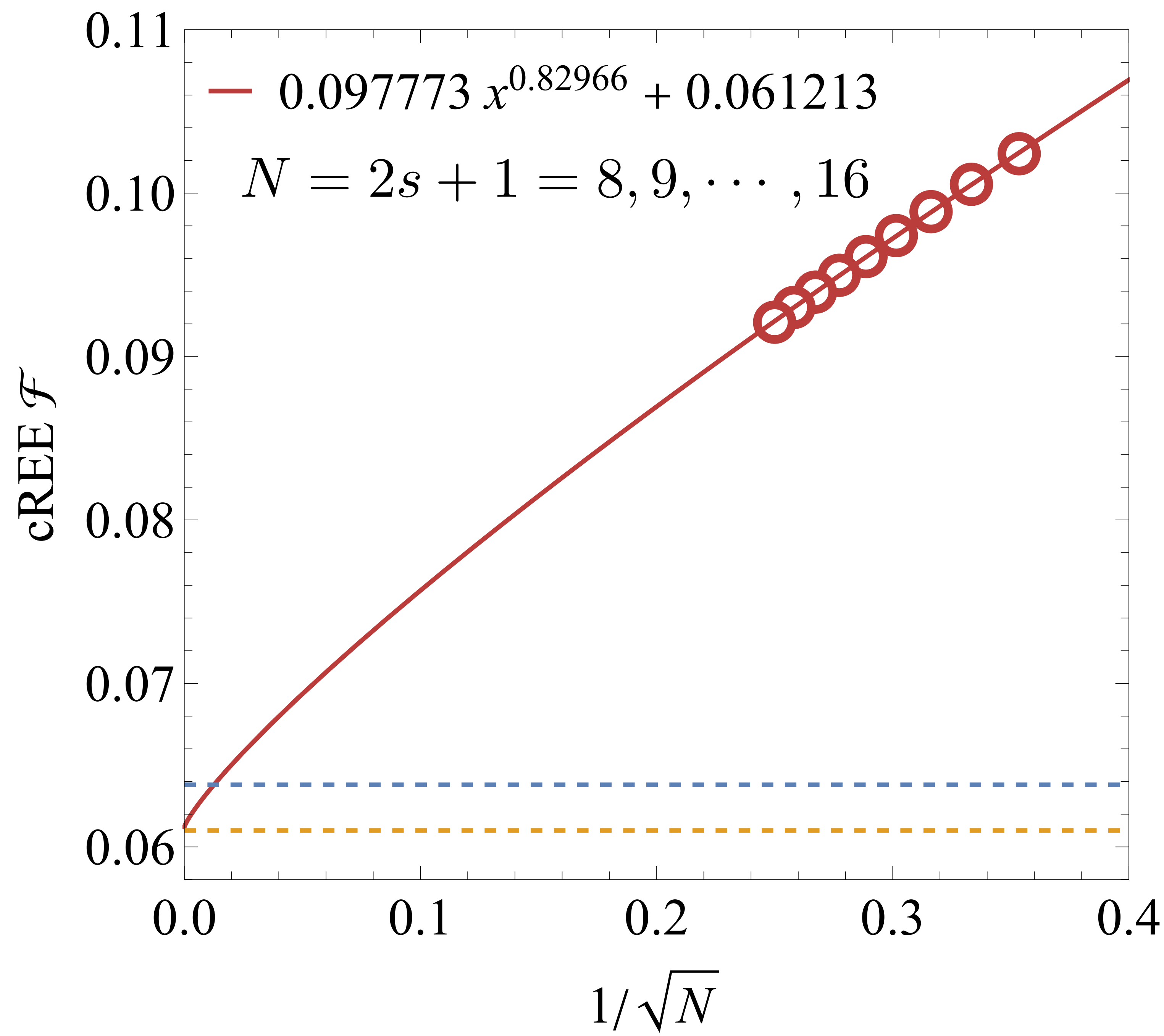
$$F_{Ising} \approx 0.0612(5)$$

$$F_{scalar} = \frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2} \approx 0.0638$$

ϵ expansion
Giobmbi, Klebanov 2015;
Fei, Giombi, Klebanov, Tarnopolsky 2015

$$F_{Ising} \approx 0.957 F_{scalar} \approx 0.0610$$

$$F_{Ising} \approx 0.979 F_{scalar} \approx 0.0622$$



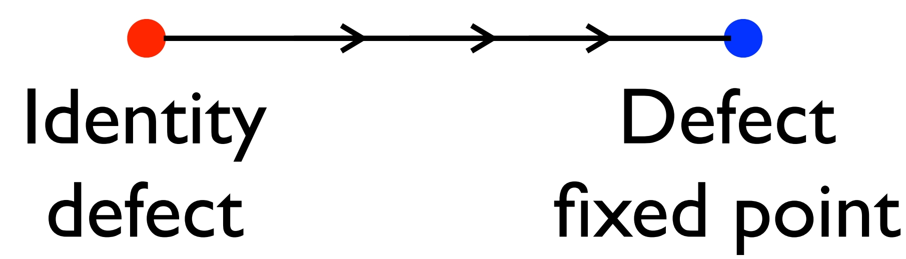
Outline

- Fuzzy sphere regularization
  - Why and how?
  - Radial quantization
  - Correlators, OPE coefficients, F-function
- Conformal defect

Conformal defect

$$S = S_{CFT} + h_d \int d^p r \, \mathcal{O}(r)$$ p=1: Line defect; p=2: Plane defect

$\Delta_{\mathcal{O}} < p$:relevant perturbation on the defect



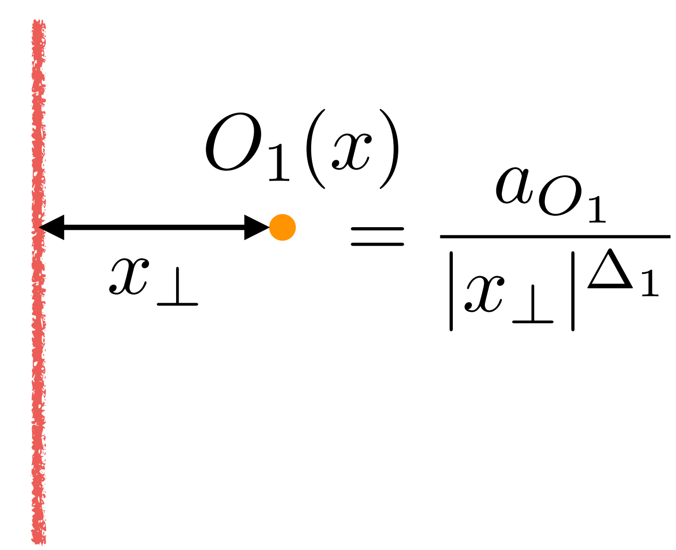
e.g. Magnetic line defect of 3D Ising

Cuomo, Komargodski & Mezei 2022

Bulk conformal symmetry → Defect conformal symmetry

$$SO(d+1,1) \to SO(p+1,1) \times SO(d-p)$$

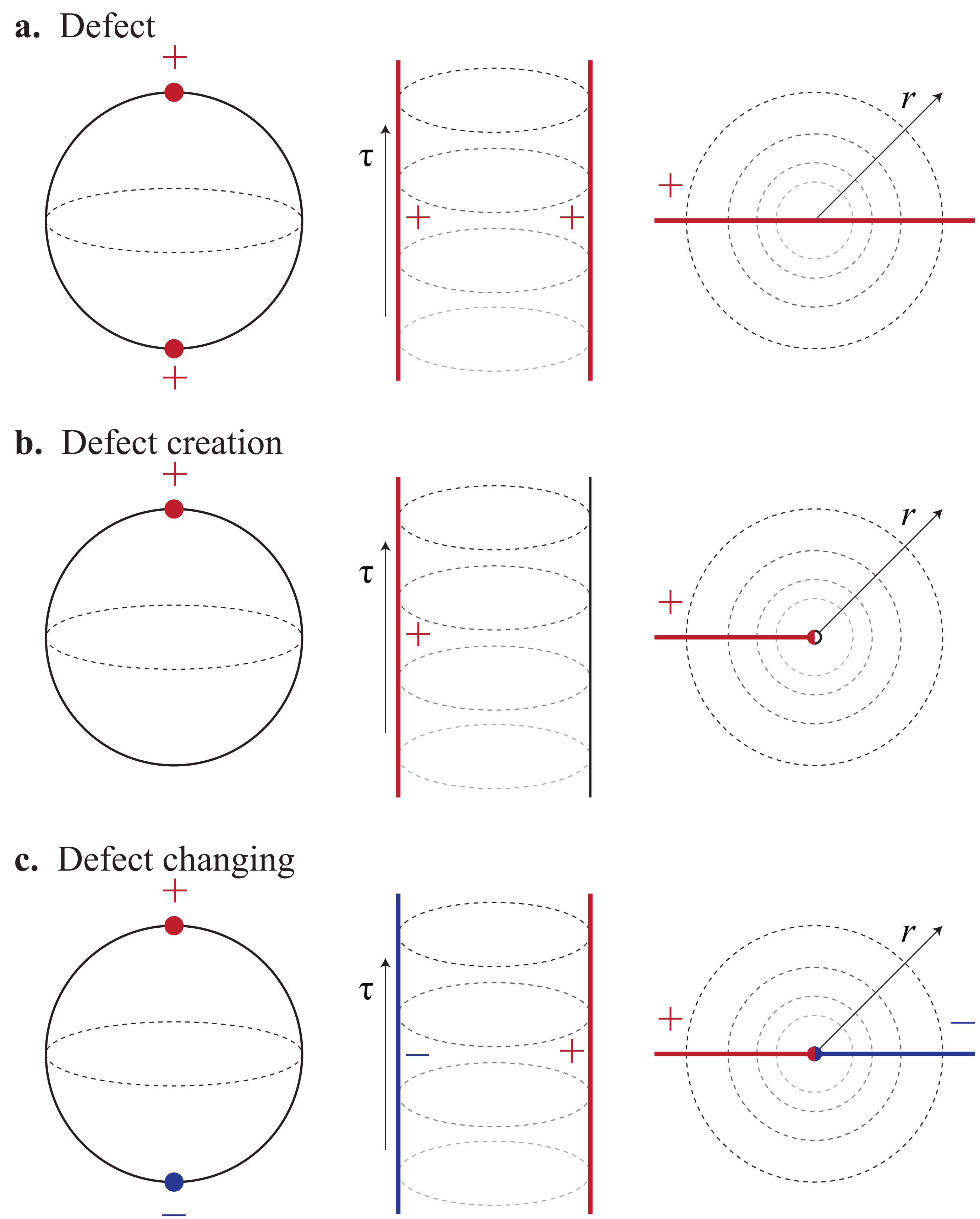
- New operators living on the defect.
- Non-trivial interplay between bulk and defect.



Solving conformal defect using fuzzy sphere

Hu, YCH, Zhu, arXiv:2308.01903
Zhou, Gaiotto, YCH, Zou, arXiv:2401.00039

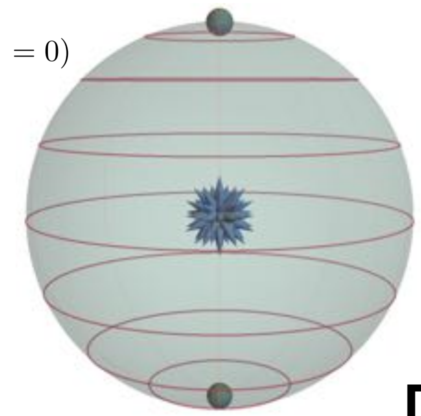
State-operator correspondence
still works for conformal defects.



Magnetic line defect of 3D Ising:

$$H = H_{\text{bulk}} + H_{\text{defect}}$$
$$H_{\text{defect}} = -h_N n^z(N) - h_S n^z(S)$$

State-operator correspondence



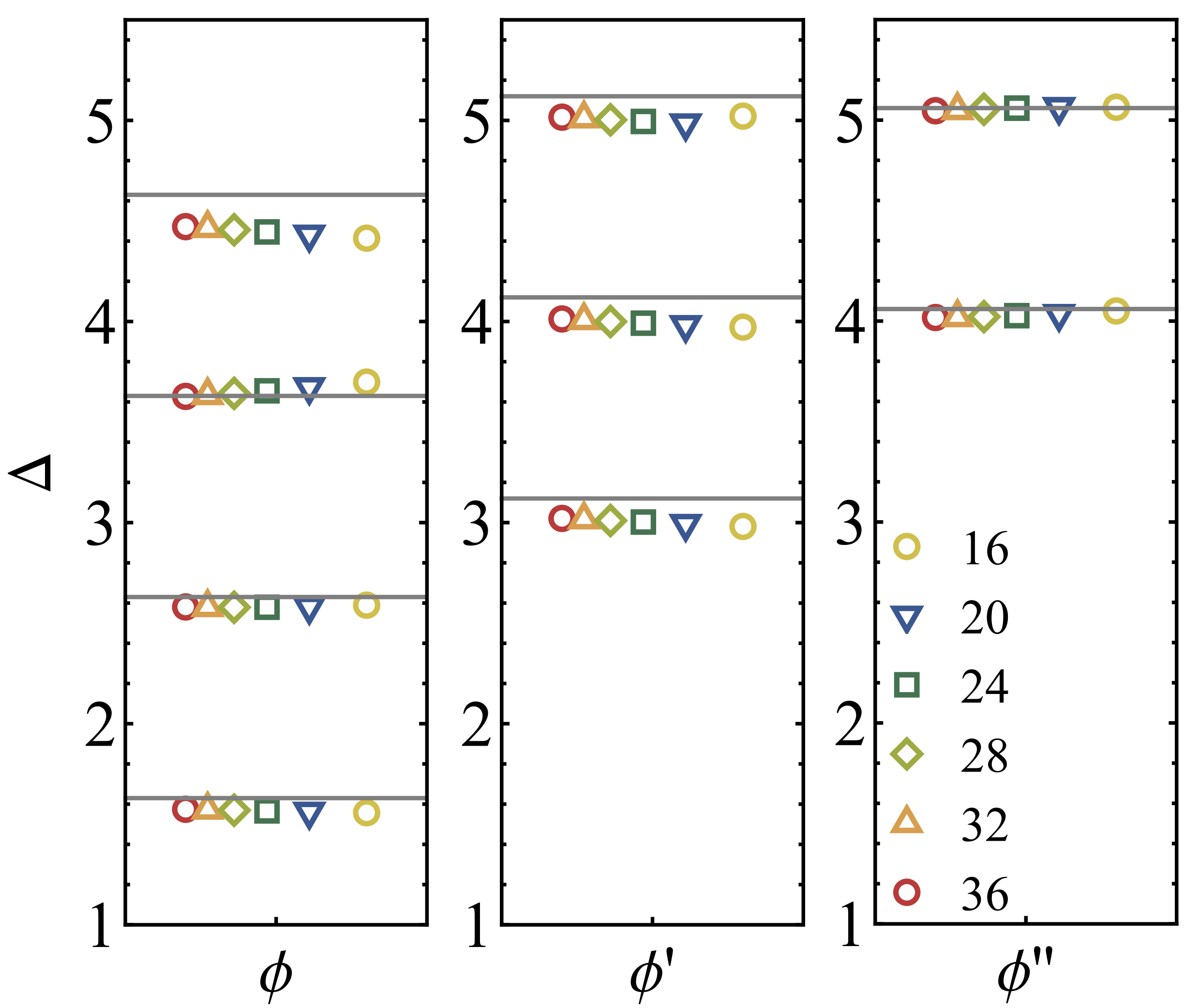
$$H = H_{\text{bulk}} + H_{\text{defect}}$$

$$H_{\text{defect}} = -h_N n^z(N) - h_S n^z(S)$$

Defect conformal symmetry: $SO(2,1)$

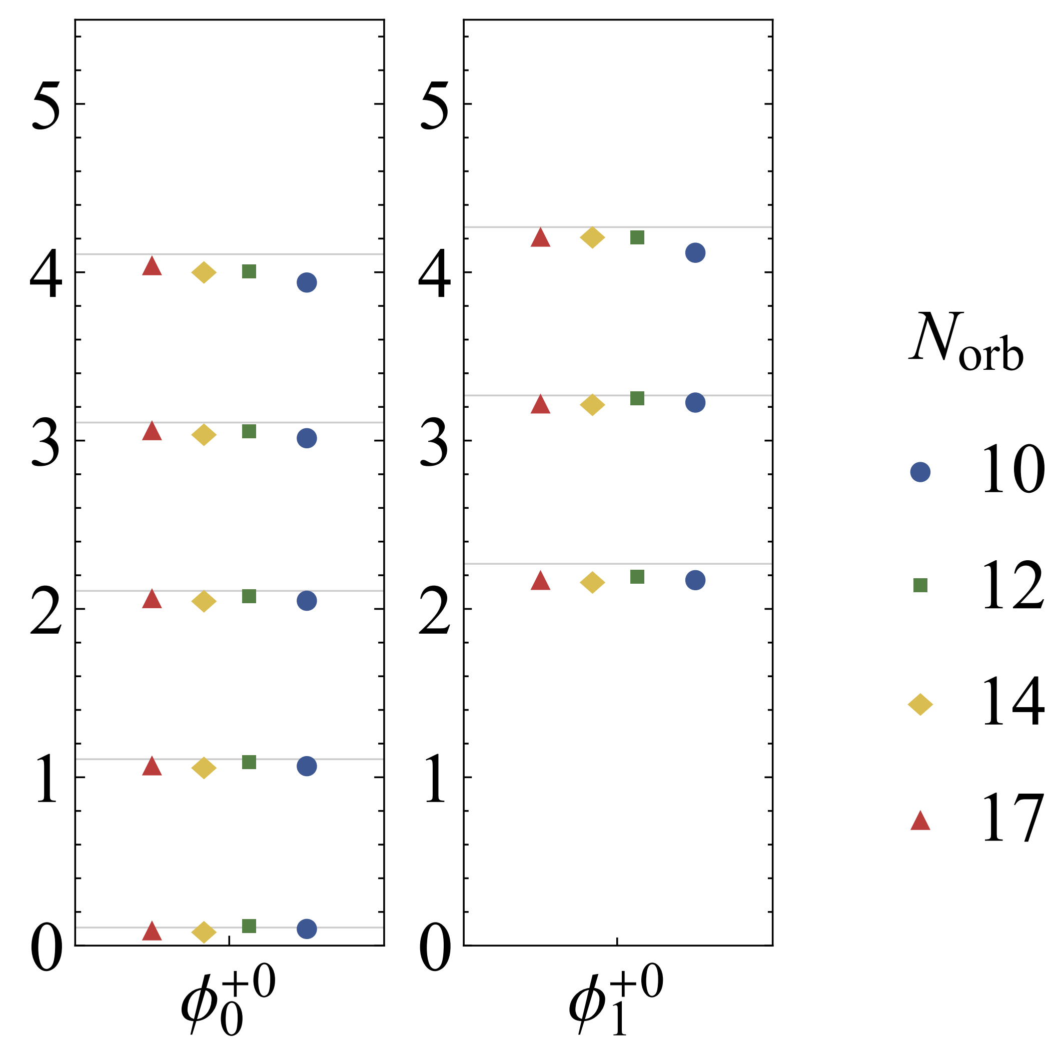
Hu, YCH, Zhu, arXiv:2308.01903

Defect operators $h_N = h_S > 0$



Zhou, Gaiotto, YCH, Zou, arXiv:2401.00039

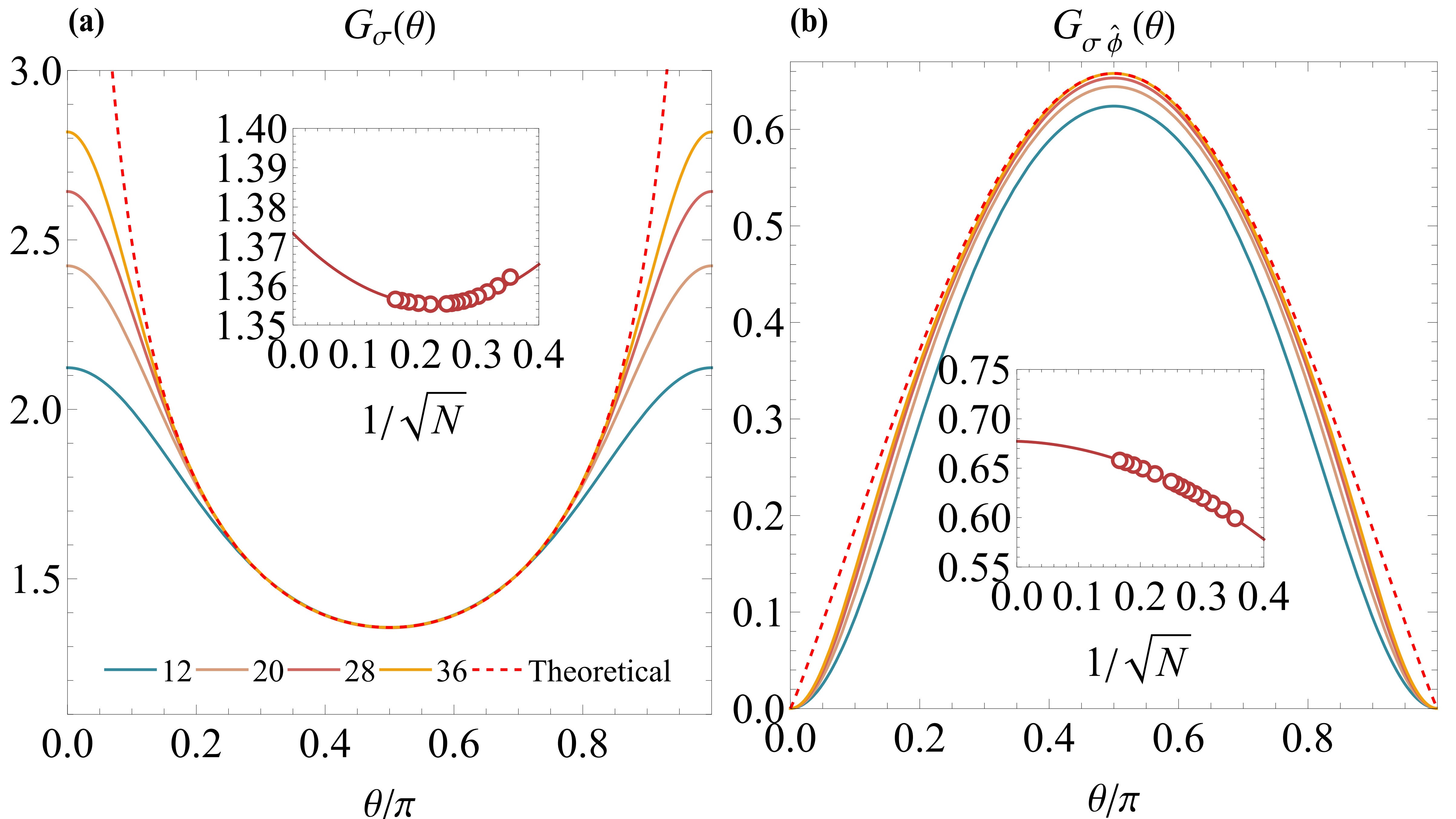
Defect creation $h_N > 0, h_S = 0$



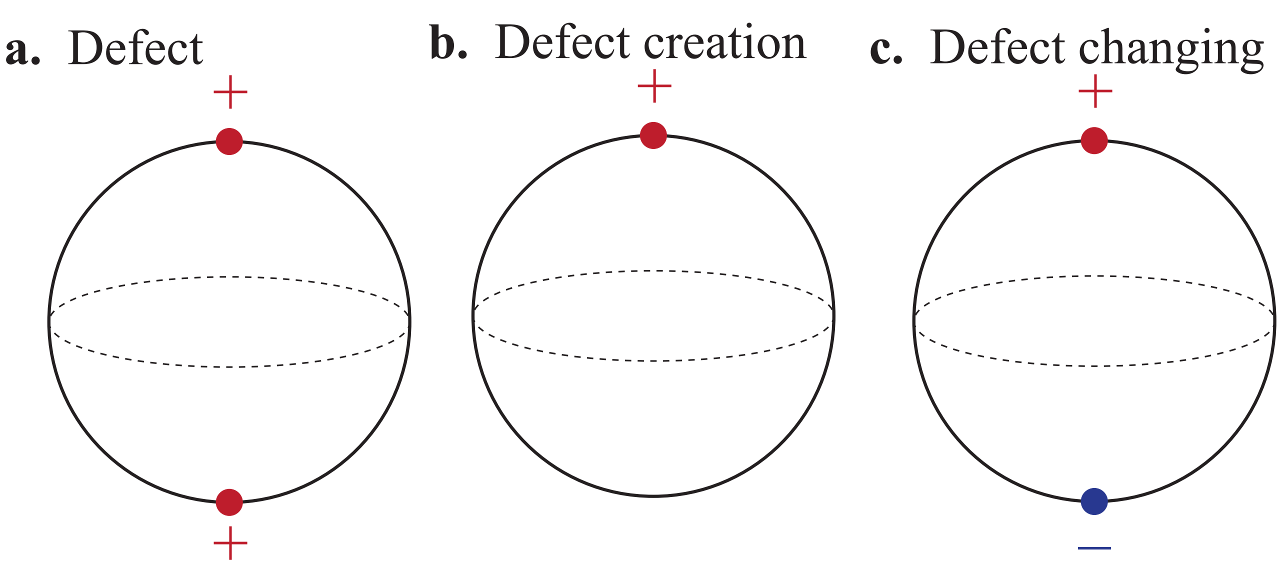
Correlator

$$\langle \hat{\mathbf{1}}_{def}|\sigma|\hat{\mathbf{1}}_{def}\rangle = \frac{a_\sigma}{(\sin\theta)^{\Delta_\sigma}} \qquad \langle \hat{\mathbf{1}}_{def}|\sigma|\hat{\phi}\rangle = \frac{b_{\sigma\hat{\phi}}}{(\sin\theta)^{\Delta_\sigma - \Delta_{\hat{\phi}}}}$$

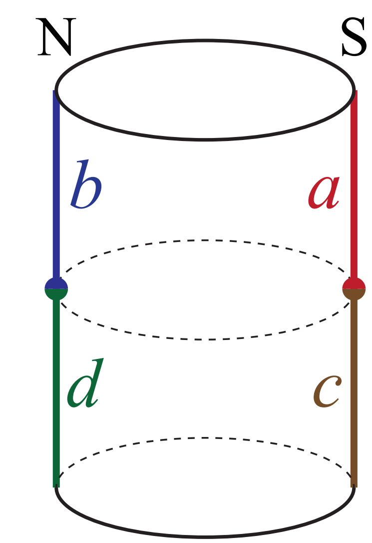
$$\Delta_\sigma \approx 0.518149 \quad a_\sigma = 1.37(1) \qquad \Delta_{\hat{\phi}} = 1.63(6) \quad b_{\sigma\hat{\phi}} = 0.68(1)$$



Conformal data encoded in the Wave-function overlap



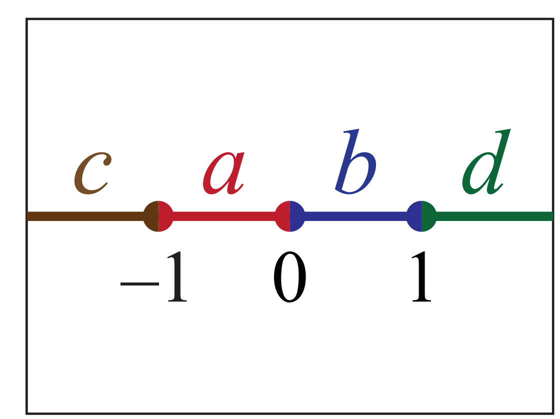
$$A_{\alpha\beta}^{abcd} = \langle \phi_\beta^{cd} | \phi_\alpha^{ab} \rangle$$



2D CFT Zou 2022

Zhou, Gaiotto, YCH, Zou, arXiv:2401.00039

Weyl transformation



$$M_0^{ca} M_0^{bd} \left(\frac{1}{R}\right)^{\Delta_0^{ca}+\Delta_0^{bd}} e^{-(\delta_{ab}\log g_a + \delta_{cd}\log g_c)/2} \langle \phi_0^{ca}(-1)\phi_\alpha^{ab}(0)\phi_0^{bd}(1)\phi_\beta^{dc}(\infty)\rangle$$

g-function

g-function: $g = \frac{Z_{\mathrm{dCFT}}}{Z_{\mathrm{CFT}}}$ Monotonic under RG flow of defect

- 2D bulk: Affleck & Ludwig 1991; Friedan & Konechny 2004
- 3D bulk: Cuomo, Komargodski, Raviv-Moshe 2022
- General dim bulk: Casini, Landea, Torroba 2023

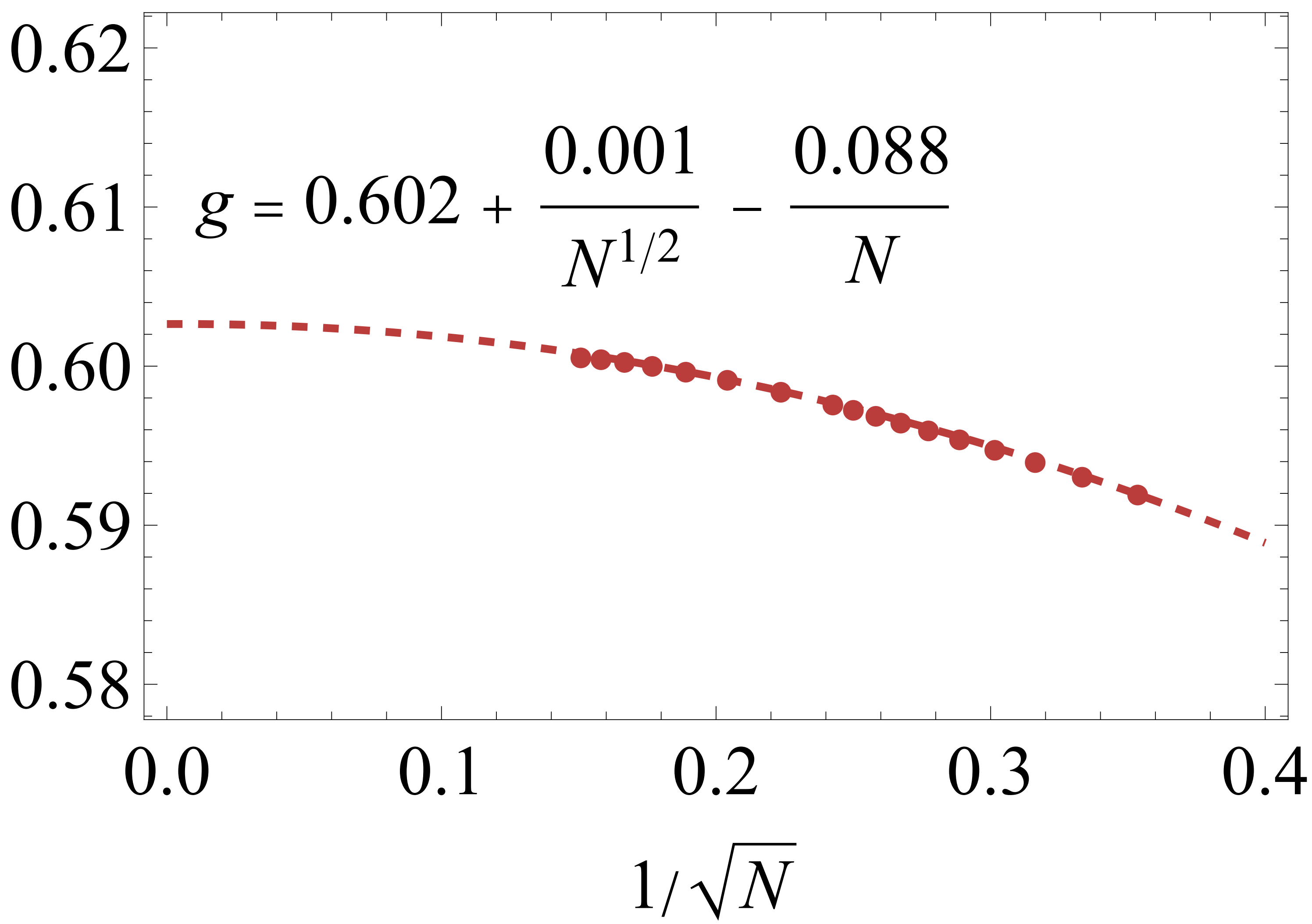
$$g_a = \left(\frac{A_{00}^{a000}}{A_{00}^{a0aa}} \right)^2$$

Our estimates: $g = 0.602(2)$

Zhou, Gaiotto, YCH, Zou, arXiv:2401.00039

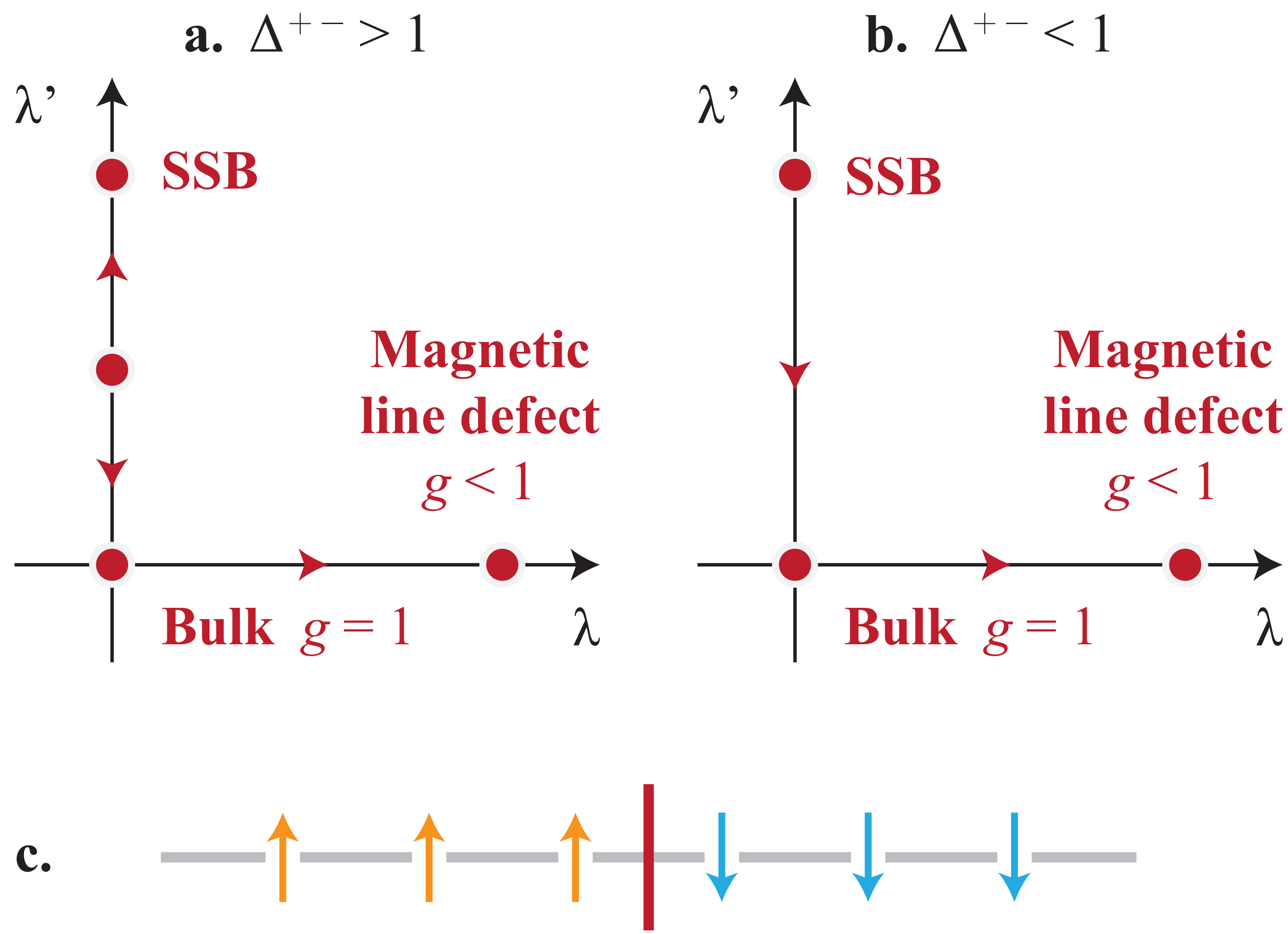
ϵ expansion: $g = 0.57 + O(\epsilon^2)$

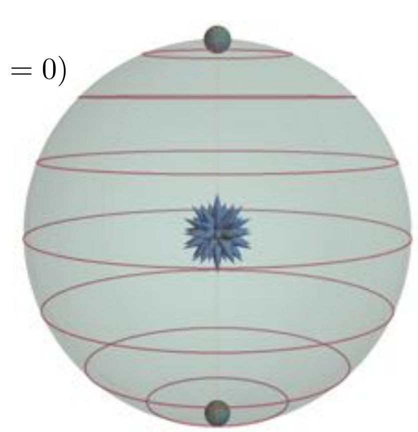
Cuomo, Komargodski & Mezei 2022



Can we have spontaneous Z2 breaking defect?

No spontaneous symmetry breaking in 1D,
but what if it couples to a gapless bulk?





Defect changing operator

$$h_N > 0, h_S < 0$$

$$\Delta^{+-} = 0.84(4) < 1$$

It is relevant, so no stable SSB.

A lot to explore in this fuzzy world

- Critical gauge theories: QED3, QCD3, Chern-Simons matter theories, etc.
- 2+1D CFT at finite temperature, Cardy formula
- Conformal defect
- Entanglement
- Non-equilibrium dynamics, quantum chaos
- Complex fixed point, complex CFT
- Landscape of CFTs, new CFTs
- Higher dimensional generalizations
- …

Summary

Thank you!

- We proposed a new scheme called fuzzy sphere regularization to study 3D CFTs by making use of the quantum Hall physics and non-commutative geometry.
- A major surprise is that it miraculously works for a very small system size, i.e. N=4~16 spins.
- A wealth of information (e.g. operator spectrum, OPE coefficients, F-function) as well as different CFTs (e.g. Wilson-Fisher, critical gauge theories, defect CFTs) can be computed efficiently in this scheme.
- A lot to explore in the future, e.g. the connection between non-commutative geometry, CFTs and QFTs.

Let's explore the fuzzy world!