Improving the Five-Point Bootstrap

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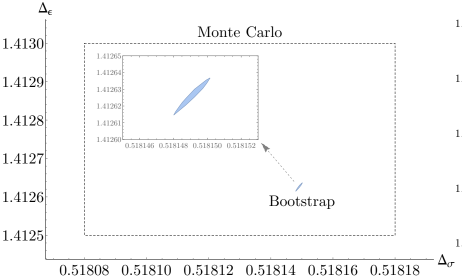
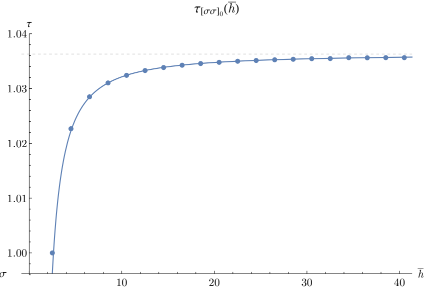
$50 + \epsilon$ Years of Conformal Bootstrap

Motivation

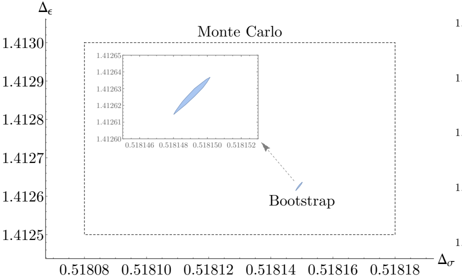


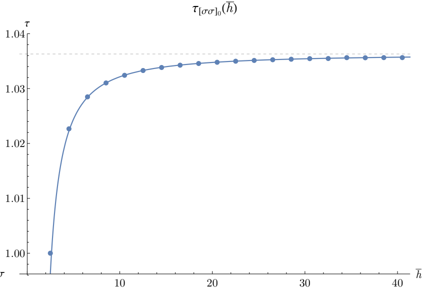


[Kos, DP, Simmons-Duffin, Vichi '16; Simmons-Duffin '16]

- The conformal bootstrap has had some surprising successes in computing low-lying CFT data in some theories
- This comes from applying crossing to 4-point functions involving scalars, and more recently fermions, currents, and stress tensors

Motivation

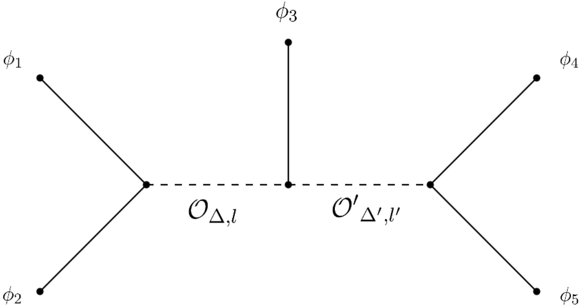




[Kos, DP, Simmons-Duffin, Vichi '16; Simmons-Duffin '16]

- However, some basic data is not so easy to access using this approach, e.g. 3-point couplings like $\langle C^{\ell=4}C^{\ell=4}\epsilon\rangle$ and $\langle C^{\ell=4}C^{\ell=4}T^{\ell=2}\rangle$
- In principle this data can be computed using 4-point functions like $\langle C^{\ell=4}C^{\ell=4}C^{\ell=4}C^{\ell=4}\rangle$, but it has 881 tensor structures!

The five-point bootstrap



- Recently we started exploring what can be extracted from CFT 5-point functions like $\langle\phi_1\phi_2\phi_3\phi_4\phi_5\rangle$ [DP, Prilepina, Tadic, May '23; Dec '23]
- It gives a convenient probe of 3-point functions w/ 2 spinning operators: $\langle\mathcal{O}_{\Delta,\ell}\,\phi\,\mathcal{O}'_{\Delta',\ell'}\rangle^{(n_{IJ})} \propto V_1^{\ell-n_{IJ}}V_3^{\ell'-n_{IJ}}H_{13}^{n_{IJ}} \quad (n_{IJ}=0,\ldots,\min(\ell,\ell'))$

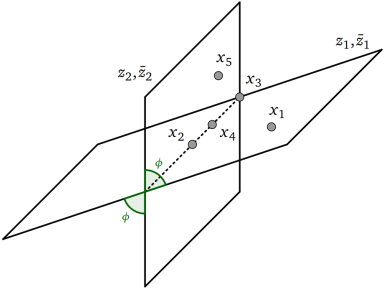
Five-point blocks

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\phi_5(x_5)\rangle = \sum_{(\mathcal{O}_{\Delta,\ell},\mathcal{O}'_{\Delta',\ell'})}\sum_{n_{IJ}=0}^{\min(\ell,\ell')}(\lambda_{\phi_1\phi_2\mathcal{O}_{\Delta,\ell}})(\lambda_{\phi_4\phi_5\mathcal{O}'_{\Delta',\ell'}})(\lambda^{n_{IJ}}_{\mathcal{O}_{\Delta,\ell}\phi_3\mathcal{O}'_{\Delta',\ell'}}) \times P(x_i)G^{(n_{IJ})}_{(\Delta,\ell,\Delta',\ell')}(u_1,v_1,u_2,v_2,w)$$

$$u_1=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\ v_1=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2},\ u_2=\frac{x_{23}^2x_{45}^2}{x_{24}^2x_{35}^2},\ v_2=\frac{x_{25}^2x_{34}^2}{x_{24}^2x_{35}^2},\ w=\frac{x_{15}^2x_{23}^2x_{34}^2}{x_{24}^2x_{13}^2x_{35}^2}$$

- Blocks with scalars exchanged can be computed as a series expansion [Rosenhaus '18; Parikh '19; Fortin, Ma, Skiba '19]
- Blocks with spins exchanged can be computed via a couple methods:
  - Recursion relations relating $\ell \to \ell - 1$ [DP, Prilepina '21]
  - Solving two quadratic Casimir equations order by order [Goncalves, Pereira, Zhou '19; DP, Prilepina, Tadic, May '23; Dec '23]

Cross ratios

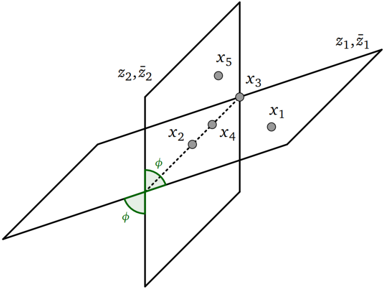


[Buric, Lacroix, Mann, Quintavalle, Schomerus '21]

- One can go to a conformal frame which puts $\{x_2, x_3, x_4\}$ at $\{0, \infty, 1\}$
- The position of x_1 on a plane is specified by $\{z_1, \overline{z}_1\}$, x_5 on a different plane by $\{z_2, \overline{z}_2\}$, and the angle between the planes $w_1 = \sin(\phi/2)^2$

$$u_1 = z_1\overline{z}_1 \,, \qquad v_1 = (1 - z_1)(1 - \overline{z}_1) \,,$$
$$u_2 = z_2\overline{z}_2 \,, \qquad v_2 = (1 - z_2)(1 - \overline{z}_2) \,,$$
$$w = w_1(z_1 - \overline{z}_1)(z_2 - \overline{z}_2) + (1 - z_1 - z_2)(1 - \overline{z}_1 - \overline{z}_2)$$

Cross ratios



We've found it useful to define a set of "radial coordinates" on each plane: (4pt radial coords: [Pappadopulo, Rychkov, Espin, Rattazzi '12; Hogervorst, Rychkov' 13])

$$z_i = \frac{4\rho_i}{(1+\rho_i)^2}, \qquad \rho_i = r_i e^{i\theta_i}\,, \qquad \eta_i = \cos\theta_i\,, \qquad i = 1,2\,,$$

$$R = \sqrt{r_1 r_2}\,, \qquad r = \sqrt{\frac{r_1}{r_2}}\,, \qquad \hat{w} = \left(\frac{1}{2} - w_1\right)\sqrt{(1-\eta_1^2)(1-\eta_2^2)}$$

Five-point blocks

In these coordinates the blocks have a nice expansion:

$$G^{(n_{IJ})}_{(\Delta,\ell,\Delta',\ell')}(R,r,\eta_1,\eta_2,\hat{w}) = \sum_{n=0}^{\infty} R^{\Delta+\Delta'+n} \sum_{m} \sum_{j_1,j_2} \sum_{k=0}^{\min(j_1,j_2)} c\left(\frac{n+m}{2},\frac{n-m}{2},j_1,j_2,k\right) r^{\Delta-\Delta'+m} \eta_1^{j_1-k} \eta_2^{j_2-k} \hat{w}^k$$

$$m \in [-n,-n+2,\ldots,n-2,n]$$

$$j_1 \in \left[\frac{n+m}{2}+\ell, \frac{n+m}{2}+\ell-2,\ldots,\mathrm{Mod}\left(\frac{n+m}{2}+\ell,2\right)\right]$$

$$j_2 \in \left[\frac{n-m}{2}+\ell', \frac{n-m}{2}+\ell'-2,\ldots,\mathrm{Mod}\left(\frac{n-m}{2}+\ell',2\right)\right]$$

The power of R gives total exchanged dimension and there is a single ∞-sum

Five-point blocks

The blocks satisfy two quadratic Casimir equations $\mathcal{D}_{12}^2 G = \mathcal{D}_{45}^2 G = 0$, giving two recursion relations for the c-coefficients:

$$\sum_{\{\hat{m}_1,\hat{m}_2,\hat{j}_1,\hat{j}_2,\hat{k}\}\in\mathcal{S}_j} q_j(\hat{m}_1,\hat{m}_2,\hat{j}_1,\hat{j}_2,\hat{k}) \, c\left(\frac{n+m}{2}+\hat{m}_1, \frac{n-m}{2}+\hat{m}_2, j_1+\hat{j}_1, j_2+\hat{j}_2, k+\hat{k}\right) = 0$$

- They have 499 terms but can be easily solved in e.g. Mathematica
- To relate to structure n_{IJ} labeling block have boundary conditions:

$$c(0,0,\ell,\ell',k) = (-1)^{\ell+\ell'+k+n_{IJ}} 2^{k+2(\Delta+\Delta')} \binom{n_{IJ}}{k}$$

Mean-field theory

- One application is to expand a known 5-point function in blocks and read off OPE coefficients
- E.g., we can expand the MFT correlator $\langle \phi\phi\phi^2\phi\phi \rangle$:

$$\langle \phi(x_1)\phi(x_2)\phi^2(x_3)\phi(x_4)\phi(x_5) \rangle = \left(\frac{x_{24}}{x_{12}x_{23}x_{34}x_{45}} \right)^{2\Delta} \times$$

$$\sqrt{2} \left((u_1)^\Delta + (u_2)^\Delta + (u_1 u_2)^\Delta + \left(\frac{u_1 u_2}{v_1} \right)^\Delta + \left(\frac{u_1 u_2}{v_2} \right)^\Delta + \left(\frac{u_1 u_2}{w} \right)^\Delta \right)$$

and read off the product of OPE coefficients

$$P^{n_{IJ}}_{n,\ell,n',\ell'} \equiv \lambda_{\phi\phi[\phi,\phi]_{n,\ell}} \lambda_{\phi\phi[\phi,\phi]_{n',\ell'}} \lambda^{n_{IJ}}_{[\phi,\phi]_{n,\ell}[\phi,\phi]_{0,0}[\phi,\phi]_{n',\ell'}}$$

(Here $[\phi,\phi]_{n,\ell} \sim \phi\partial^{\mu_1\cdots\mu_\ell}\partial^{2n}\phi$) are double-twist operators)

Mean-field theory

- The coefficients with leading twists ($n = n' = 0$) were computed in [Antunes, Costa, Goncalves, Vilas Boas '22]
- We were able to extract the general formula [DP, Prilepina, Tadic, May '23]:

$$
P^{n_{IJ}}_{n,\ell,n',\ell'} =
\frac{(-1)^{n_{IJ}} 2^{\frac{5}{2}-n_{IJ}} (\ell - n_{IJ}+1)_{n_{IJ}} (\ell' - n_{IJ}+1)_{n_{IJ}} (\Delta)_{\frac{\ell}{2}+n} (\Delta)_{\frac{\ell'}{2}+n'}}{\ell! \ell'! n! n'! n_{IJ}! (\ell+\nu+1)(\ell'+\nu+1)(\ell+\nu+2)_{n-1}(\ell'+\nu+2)_{n'-1}}
\frac{(\Delta-\nu)_n (\Delta-\nu)_{n'} (\Delta-\nu)_{n+n'}}{\left(\frac{\ell-1}{2}+n+\Delta\right)_{\frac{\ell}{2}} \left(\frac{\ell'-1}{2}+n'+\Delta\right)_{\frac{\ell'}{2}} (n+2\Delta-2\nu-1)_n (n'+2\Delta-2\nu-1)_{n'}}
\frac{(\Delta)_{\ell+n+n'} (\Delta)_{\ell'+n+n'}}{(\ell+n+2\Delta-\nu-1)_n (\ell'+n'+2\Delta-\nu-1)_{n'} (\Delta)_{n+n'+n_{IJ}}}
$$

Five-point numerical bootstrap

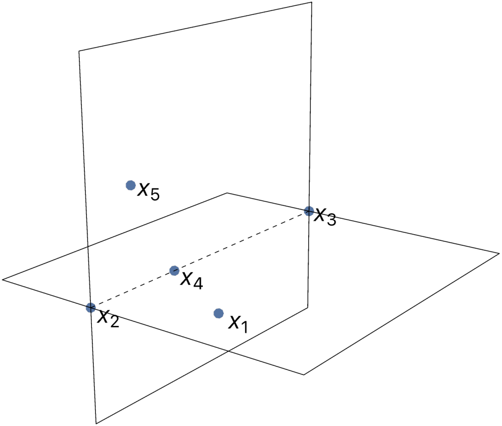
We can also try the numerical bootstrap [DP, Prilepina, Tadic, May '23]:

- We expand $\langle \sigma\sigma\epsilon\sigma\sigma \rangle$ in the $(12)(45)$ and $(14)(25)$ OPEs
- After separating the $(\mathbb{1}, \epsilon) + (\epsilon, \mathbb{1})$ contributions, we get a sum rule:

$$\sum_{\mathcal{O},\mathcal{O}' \neq \mathbb{1}} \frac{\lambda_{\sigma\sigma\mathcal{O}} \lambda^{n_{IJ}}_{\mathcal{O}\epsilon\mathcal{O}'} \lambda_{\sigma\sigma\mathcal{O}'} \left(PG_{\mathcal{O},\mathcal{O}'} - \tilde{P}\tilde{G}_{\mathcal{O},\mathcal{O}'} \right)}{\lambda_{\sigma\sigma\epsilon} \left(\tilde{P}\tilde{G}_{\mathbb{1},\epsilon} + \tilde{P}\tilde{G}_{\epsilon,\mathbb{1}} - PG_{\mathbb{1},\epsilon} - PG_{\epsilon,\mathbb{1}} \right)} = 1$$

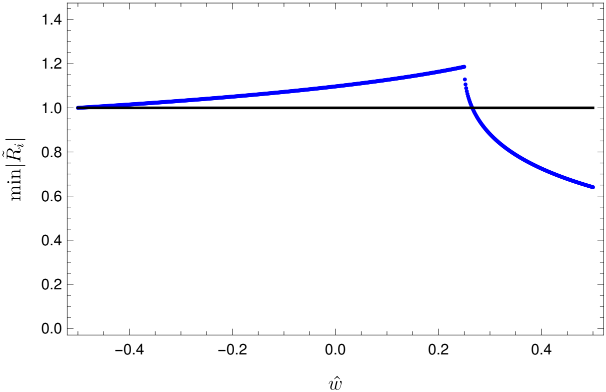
- We truncate to include operators with $\Delta \leq \Delta_{\text{cutoff}}$, and fix exchanged dimensions and $\lambda_{\sigma\sigma\mathcal{O}}$ coefficients to their known values
- Then we choose a set of derivatives of the sum rule and search for the $\lambda^{n_{IJ}}_{\mathcal{O}\epsilon\mathcal{O}'}$ coefficients that make these equations closest to being satisfied (by minimizing a cost function)

Five-point configuration



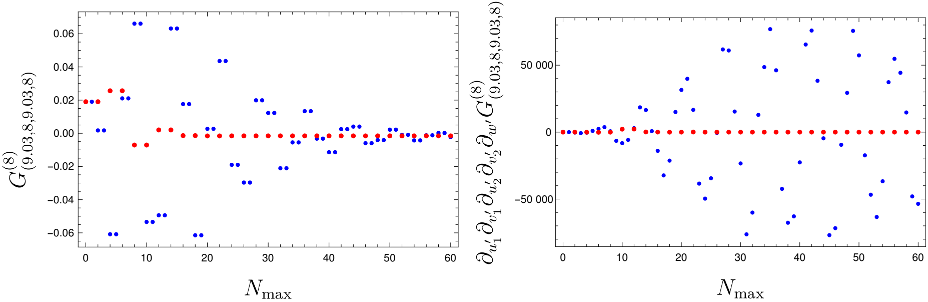
- Expand around configuration: $\phi = \pi/2$ and $x_{12}^2 = x_{14}^2 = x_{52}^2 = x_{54}^2 = 1$
- In radial coordinates it is: $R = 2 - \sqrt{3}, r = 1, \eta_i = \hat{w} = 0$

Five-point block convergence



- The radius of convergence in R is controlled by the singularity $x_{15}^2 = 0$
- In general it is a complicated function of $\{r, \eta_1, \eta_2, \hat{w}\}$ (determined by the smallest root of an 8th order polynomial)
- Above shows $\tilde{R} = R_{\mathsf{max}}/(2-\sqrt{3})$ vs. $\hat{w}$ at $r = 1, \eta_i = 0$

Five-point block convergence



$$\ell = \ell' = n_{IJ} = 8,\ \Delta_{12} = \Delta_{45} = 0,\ \Delta_3 = 1.413,\ \Delta = \Delta' = 9.03$$
$$R = 2 - \sqrt{3},\ r = 1,\ \eta_1 = \eta_2 = \hat{w} = 0.$$

- Blocks/derivatives eventually converge but first show big oscillations
- It is very helpful to accelerate the convergence of the series using a Padé approximant: $G_{\text{Padé}} \equiv \left[\frac{N_{\max}}{2} / \frac{N_{\max}}{2}\right]_G (R)$

3d free scalar with hard truncation

- We studied $\langle \phi\phi\phi^2\phi\phi \rangle$ and included the exchanged operators:

$$\{\mathbb{1}, \quad \phi^2, \quad T_{\mu\nu} \sim \phi\partial_\mu\partial_\nu\phi, \quad C_{\mu\nu\rho\sigma} \sim \phi\partial_\mu\partial_\nu\partial_\rho\partial_\sigma\phi\}$$

- Treat $\{\Delta_\phi, \lambda_{T\phi^2T}, \lambda_{T\phi^2C}, \lambda_{C\phi^2C}\}$ as unknowns, input other data
- Pick a set of derivatives of the sum rule $\mathcal{D}_i(\Delta_\phi, \lambda)$, minimize cost function $\sum w_i \left(\frac{\mathcal{D}_i(\Delta_\phi,\lambda)}{\mathcal{D}_i(\Delta_\phi,0)}\right)^2$ with randomly chosen weights $w_i \in [0,1]$
- Take $\mathcal{D}_i$ which gives $\Delta_\phi \sim 0.5$ with smallest deviation as w_i varied

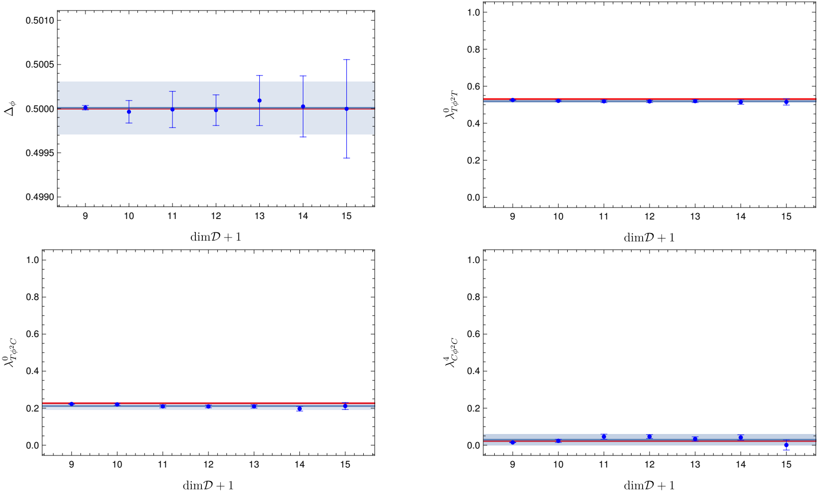
3d free scalar with hard truncation

Results:

	truncation	exact
Δ_ϕ	0.5000(3)	0.500000
$\lambda^0_{T\phi^2 T}$	0.52(1)	0.530330
$\lambda^0_{T\phi^2 C}$	0.21(2)	0.226428
$\lambda^4_{C\phi^2 C}$	0.03(2)	0.022097
$\lambda^3_{C\phi^2 C}$	-1.0(3)	-0.618718
$\lambda^2_{C\phi^2 C}$	5.1(9)	2.320194
$\lambda^1_{C\phi^2 C}$	2(2)	-1.546796
$\lambda^0_{C\phi^2 C}$	1.2(5)	0.096675

(Inputs: $\Delta_{\phi^2}, \Delta_T, \Delta_C, \lambda_{\phi\phi\phi^2}, \lambda_{\phi\phi T}, \lambda_{\phi\phi C}, \lambda_{\phi^2\phi^2\phi^2}, \lambda_{\phi^2\phi^2 T}, \lambda_{\phi^2\phi^2 C}$)

3d free scalar with hard truncation



3d Ising with hard truncation

- We studied $\langle\sigma\sigma\epsilon\sigma\sigma\rangle$ and included the exchanged operators:

$$\{\mathbb{1}, \quad \epsilon, \quad \epsilon', \quad T_{\mu\nu}, \quad C_{\mu\nu\rho\sigma}\}$$

- Treat $\{\Delta_\sigma, \lambda_{T\epsilon T}, \lambda_{T\epsilon C}, \lambda_{C\epsilon C}, \lambda_{\epsilon'\epsilon\epsilon'}, \lambda_{\epsilon'\epsilon C}\}$ as unknowns, input other data using best 4-point bootstrap results
- Pick a set of derivatives of the sum rule $\mathcal{D}_i(\Delta_\sigma, \lambda)$, minimize cost function $\sum w_i \left(\frac{\mathcal{D}_i(\Delta_\sigma,\lambda)}{\mathcal{D}_i(\Delta_\sigma,0)}\right)^2$ with randomly chosen weights $w_i \in [0,1]$
- Take $\mathcal{D}_i$ which gives $\Delta_\sigma \sim 0.51815$ with smallest deviation as w_i varied

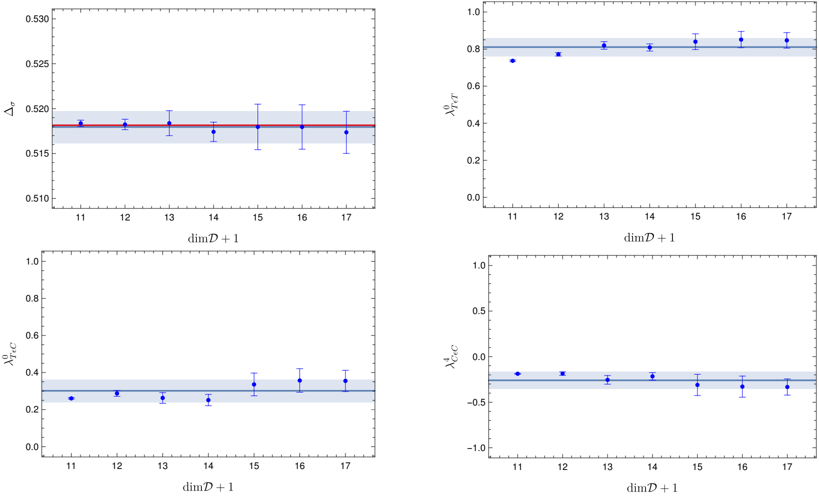
3d Ising with hard truncation

Results:

	truncation
Δ_σ	0.518(2)
$\lambda^0_{T\epsilon T}$	0.81(5)
$\lambda^0_{T\epsilon C}$	0.30(6)
$\lambda^4_{C\epsilon C}$	-0.3(1)
$\lambda^3_{C\epsilon C}$	-2(2)
$\lambda^2_{C\epsilon C}$	2(5)
$\lambda^1_{C\epsilon C}$	-5(11)
$\lambda^0_{C\epsilon C}$	-3(11)
$\lambda_{\epsilon'\epsilon\epsilon'}$	1(3)
$\lambda_{\epsilon'\epsilon C}$	0(2)

(Inputs: $\Delta_\epsilon, \Delta_{\epsilon'}, \Delta_T, \Delta_C, \lambda_{\sigma\sigma\epsilon}, \lambda_{\epsilon\epsilon\epsilon}, \lambda_{\sigma\sigma\epsilon'}, \lambda_{\epsilon\epsilon\epsilon'}, \lambda_{\sigma\sigma T}, \lambda_{\epsilon\epsilon T}, \lambda_{\sigma\sigma C}, \lambda_{\epsilon\epsilon C}$)

3d Ising with hard truncation



Disconnnected-correlator improvement

- These results can be improved further by introducing an approximation to the truncated contributions [Li, Dec '23; DP, Prilepina, Tadic, Dec '23]
- E.g. for the 4-point function, we can write:

$$\langle \sigma\sigma\sigma\sigma \rangle \quad = \langle \sigma\sigma\sigma\sigma \rangle_{MFT} + \frac{1}{x_{12}^{2\Delta_\sigma} x_{34}^{2\Delta_\sigma}} \sum_{\mathcal{O}}^{\Delta_{\text{max}}} (P_{\mathcal{O}} g_{\Delta,\ell} - P_{\mathcal{O}}^{MFT} g_{\Delta_{MFT},\ell})$$

  where we use $\langle \sigma\sigma\sigma\sigma \rangle_{MFT} = \langle \sigma\sigma \rangle \langle \sigma\sigma \rangle + (\text{perm.})$ and subtract and replace a finite number of MFT contributions
- Crossing symmetry then gives a sum rule

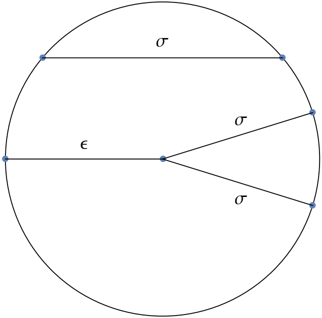
$$0 = \sum_{\mathcal{O}}^{\Delta_{\text{max}}} (P_{\mathcal{O}} F_{\Delta,\ell} - P_{\mathcal{O}}^{MFT} F_{\Delta_{MFT},\ell})$$

Disconnnected-correlator improvement for 4pt bootstrap

	no MFT	with MFT	Numerical Bootstrap
Δ_σ	0.514(5)	0.5182(4)	0.5181489(10)
$\mathcal{P}[\epsilon]$	1.15(4)	1.106(5)	1.106396(9)
$\mathcal{P}[\epsilon']$	-0.010(8)	0.003(2)	0.002810(6)
$\mathcal{P}[T_{\mu\nu}]$	0.33(5)	0.422(2)	0.425463(1)
$\mathcal{P}[C_{\mu\nu\rho\sigma}]$	0.115(9)	0.0768(5)	0.0763(1)

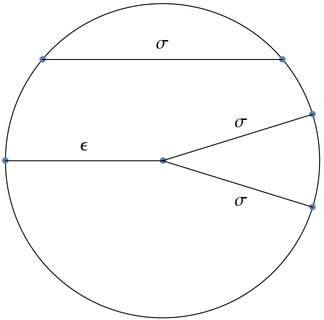
- Here we fixed scaling dimensions of $\{\epsilon, \epsilon', T_{\mu\nu}, C_{\mu\nu\rho\sigma}\}$ to their known values and computed OPE coefficients using the improved truncation
- As before, minimized cost function after selecting "optimal" derivative set and estimated errors by varying random weights of each constraint

Disconnnected correlator for 5pt function



- For a 5-point function like $\langle\sigma\sigma\epsilon\sigma\sigma\rangle$ it is not a good idea to use a MFT correlator like $\langle\sigma\sigma\sigma^2\sigma\sigma\rangle_{MFT}$, since σ^2 is very different from ϵ.
- Instead, one can use $\langle\sigma\sigma\epsilon\sigma\sigma\rangle_d = \langle\sigma\sigma\rangle\langle\epsilon\sigma\sigma\rangle + (\text{perm.})$, which in an AdS dual is the leading contribution of a bulk 3-point interaction

Disconnnected correlator for 5pt function



$$\langle \sigma(x_1)\sigma(x_2)\epsilon(x_3)\sigma(x_4)\sigma(x_5)\rangle_d = \frac{\lambda_{\sigma\sigma\epsilon}}{x_{12}^{2\Delta_\sigma} x_{45}^{2\Delta_\sigma} x_{34}^{\Delta_\epsilon}} \left(\frac{x_{24}}{x_{23}}\right)^{\Delta_\epsilon} \times$$

$$\left({u_1}^{\frac{\Delta_\epsilon}{2}} + {u_2}^{\frac{\Delta_\epsilon}{2}} + \left(\frac{u_1 u_2}{v_1 v_2}\right)^{\Delta_\sigma} \left({v_1}^{\frac{\Delta_\epsilon}{2}} + {v_2}^{\frac{\Delta_\epsilon}{2}}\right) + \left(\frac{u_1 u_2}{w}\right)^{\Delta_\sigma} \left(w^{\frac{\Delta_\epsilon}{2}} + 1\right) \right)$$

Disconnnected correlator for 5pt function

- Decomposing in terms of conformal blocks, one finds $(\mathbb{1}, \epsilon)$ exchange as well as all the expected $[\sigma,\sigma]_{n,\ell} \sim \sigma\partial^{2n}\partial^{\ell}\sigma$ double-twist contributions

$$\langle\sigma\sigma\epsilon\sigma\sigma\rangle_d = P(x_i)\Big(\lambda_{\sigma\sigma\epsilon} G^{(0)}_{(0,0,\Delta_\epsilon,0)} + \lambda_{\sigma\sigma\epsilon} G^{(0)}_{(\Delta_\epsilon,0,0,0)} + \sum_{n,\ell,n',\ell',n_{IJ}} \mathcal{P}(n,\ell,n',\ell',n_{IJ}) G^{(n_{IJ})}_{(2\Delta_\sigma+2n+\ell,\ell,2\Delta_\sigma+2n'+\ell',\ell')}\Big),$$

with

$$\mathcal{P}(0,2,0,2,0) = \frac{\Delta_\sigma^2\Delta_\epsilon^2\left(\Delta_\epsilon+2\right)^2\lambda_{\sigma\sigma\epsilon}}{4\left(2\Delta_\sigma+1\right)^2},$$

$$\mathcal{P}(0,2,0,4,0) = \frac{\Delta_\sigma^2\left(\Delta_\sigma+1\right)\Delta_\epsilon^2\left(\Delta_\epsilon+2\right)^2\left(\Delta_\epsilon+4\right)\left(\Delta_\epsilon+6\right)\lambda_{\sigma\sigma\epsilon}}{96\left(2\Delta_\sigma+1\right)\left(2\Delta_\sigma+3\right)\left(2\Delta_\sigma+5\right)},$$

etc.

Disconnnected correlator for 5pt function

	$\langle\sigma\sigma\epsilon\sigma\sigma\rangle_d$
$\lambda^2_{[\sigma,\sigma]_{0,2}\epsilon[\sigma,\sigma]_{0,2}}$	-0.353885
$\lambda^1_{[\sigma,\sigma]_{0,2}\epsilon[\sigma,\sigma]_{0,2}}$	-2.892385
$\lambda^0_{[\sigma,\sigma]_{0,2}\epsilon[\sigma,\sigma]_{0,2}}$	0.988418
$\lambda^2_{[\sigma,\sigma]_{0,2}\epsilon[\sigma,\sigma]_{0,4}}$	0.580279
$\lambda^1_{[\sigma,\sigma]_{0,2}\epsilon[\sigma,\sigma]_{0,4}}$	-3.100420
$\lambda^0_{[\sigma,\sigma]_{0,2}\epsilon[\sigma,\sigma]_{0,4}}$	0.484382
$\lambda^4_{[\sigma,\sigma]_{0,4}\epsilon[\sigma,\sigma]_{0,4}}$	-0.439644
$\lambda^3_{[\sigma,\sigma]_{0,4}\epsilon[\sigma,\sigma]_{0,4}}$	-0.231194
$\lambda^2_{[\sigma,\sigma]_{0,4}\epsilon[\sigma,\sigma]_{0,4}}$	3.849048
$\lambda^1_{[\sigma,\sigma]_{0,4}\epsilon[\sigma,\sigma]_{0,4}}$	-3.276278
$\lambda^0_{[\sigma,\sigma]_{0,4}\epsilon[\sigma,\sigma]_{0,4}}$	0.237375

	$\langle\sigma\sigma\epsilon\sigma\sigma\rangle_d$
$\lambda^2_{[\sigma,\sigma]_{0,2}\epsilon[\sigma,\sigma]_{0,6}}$	1.034441
$\lambda^1_{[\sigma,\sigma]_{0,2}\epsilon[\sigma,\sigma]_{0,6}}$	-2.351395
$\lambda^0_{[\sigma,\sigma]_{0,2}\epsilon[\sigma,\sigma]_{0,6}}$	0.238792
$\lambda^4_{[\sigma,\sigma]_{0,4}\epsilon[\sigma,\sigma]_{0,6}}$	-0.426110
$\lambda^3_{[\sigma,\sigma]_{0,4}\epsilon[\sigma,\sigma]_{0,6}}$	-1.729458
$\lambda^2_{[\sigma,\sigma]_{0,4}\epsilon[\sigma,\sigma]_{0,6}}$	5.210505
$\lambda^1_{[\sigma,\sigma]_{0,4}\epsilon[\sigma,\sigma]_{0,6}}$	-2.475238
$\lambda^0_{[\sigma,\sigma]_{0,4}\epsilon[\sigma,\sigma]_{0,6}}$	0.117022
$\lambda^6_{[\sigma,\sigma]_{0,6}\epsilon[\sigma,\sigma]_{0,6}}$	-0.318259
$\lambda^5_{[\sigma,\sigma]_{0,6}\epsilon[\sigma,\sigma]_{0,6}}$	0.419036
$\lambda^4_{[\sigma,\sigma]_{0,6}\epsilon[\sigma,\sigma]_{0,6}}$	0.882017

- Plugging in $\{\Delta_\sigma, \Delta_\epsilon, \lambda_{\sigma\sigma\epsilon}\}$, we can get an approximation to the unknown 3d Ising data (inversion of $(\mathbb{1},\epsilon)+(\epsilon,\mathbb{1})$ exchange)

5pt bootstrap for 3d Ising with disc. improvement (DI)

- Now we can use it as the starting point for an improved 5pt truncation:

$$0 = \sum_{(\mathcal{O}_{\Delta,\ell},\mathcal{O'}_{\Delta',\ell'})\in\mathcal{S}} \sum_{n_{IJ}=0}^{\min(\ell,\ell')} \lambda_{\sigma\sigma\mathcal{O}_{\Delta,\ell}} \lambda_{\sigma\sigma\mathcal{O'}_{\Delta',\ell'}} \lambda^{n_{IJ}}_{\mathcal{O}_{\Delta,\ell}\epsilon\mathcal{O'}_{\Delta',\ell'}} \mathcal{F}^{n_{IJ}}_{\Delta,\ell,\Delta',\ell'} -$$

$$\sum_{([\sigma,\sigma]_{n,\ell},[\sigma,\sigma]_{n',\ell'})\in\mathcal{S}_d} \sum_{n_{IJ}=0}^{\min(\ell,\ell')} \mathcal{P}(n,\ell,n',\ell',n_{IJ}) \mathcal{F}^{n_{IJ}}_{2\Delta^d_\sigma+2n+\ell,\ell,2\Delta^d_\sigma+2n'+\ell',\ell'}$$

- $\mathcal{S}$ contains all pairs from $\{\epsilon, \epsilon', T_{\mu\nu}, C_{\mu\nu\rho\sigma}\}$
- $\mathcal{S}_d$ contains all pairs from $\{[\sigma,\sigma]_{0,0}, [\sigma,\sigma]_{1,0}, [\sigma,\sigma]_{0,2}, [\sigma,\sigma]_{0,4}\}$
- All data except $\{\Delta_\sigma, \lambda^0_{T\epsilon T}, \lambda^0_{T\epsilon C}, \lambda^{n_{IJ}=0\ldots4}_{C\epsilon C}, \lambda_{\epsilon'\epsilon C}\}$ fixed to known values

5pt bootstrap for 3d Ising with disc. improvement (DI)

	$\mathcal{S}$, no DI	$\mathcal{S}$, with DI
Δ_σ	0.518(2)	0.5181(6)
$\lambda^0_{T\epsilon T}$	0.81(5)	0.96(1)
$\lambda^0_{T\epsilon C}$	0.30(6)	0.48(3)
$\lambda^4_{C\epsilon C}$	-0.3(1)	-0.27(2)
$\lambda^3_{C\epsilon C}$	-2(2)	-0.5(4)
$\lambda^2_{C\epsilon C}$	2(5)	0.1(9)
$\lambda^1_{C\epsilon C}$	-5(11)	-10(1)
$\lambda^0_{C\epsilon C}$	-3(11)	-4(1)
$\lambda_{\epsilon'\epsilon C}$	0(2)	0.9(4)

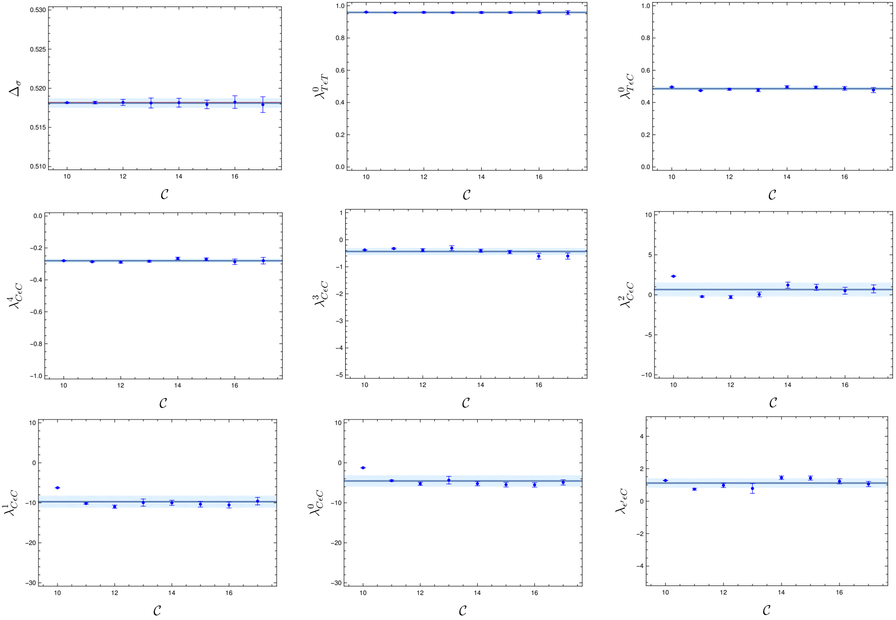
- Results significantly more constrained
- See noticable upward shift in $\lambda^0_{T\epsilon T}$ and $\lambda^0_{T\epsilon C}$

5pt bootstrap for 3d Ising with disc. improvement (DI)

	$\mathcal{S}$, no DI	$\mathcal{S}$, with DI	$+(\epsilon, S_{\mu\nu\rho\sigma\alpha\delta})$	$+(\epsilon, \mathcal{E}_{\mu_1\ldots\mu_8})$
Δ_σ	0.518(2)	0.5181(6)	0.5181(6)	0.5181(7)
$\lambda^0_{T\epsilon T}$	0.81(5)	0.96(1)	0.959(8)	0.958(7)
$\lambda^0_{T\epsilon C}$	0.30(6)	0.48(3)	0.48(2)	0.48(2)
$\lambda^4_{C\epsilon C}$	-0.3(1)	-0.27(2)	-0.28(2)	-0.28(2)
$\lambda^3_{C\epsilon C}$	-2(2)	-0.5(4)	-0.4(2)	-0.4(2)
$\lambda^2_{C\epsilon C}$	2(5)	0.1(9)	0.3(6)	0.6(9)
$\lambda^1_{C\epsilon C}$	-5(11)	-10(1)	-10.3(7)	-10(2)
$\lambda^0_{C\epsilon C}$	-3(11)	-4(1)	-4.9(6)	-4(2)
$\lambda_{\epsilon'\epsilon C}$	0(2)	0.9(4)	1.0(3)	1.1(3)

- Results appear stable against adding spin-6 and spin-8 contributions

5pt bootstrap for 3d Ising with DI



3d Ising

- Let us focus on our determination $\lambda^0_{T\epsilon T} \simeq 0.958(7)$
- This coefficient was bounded by [Cordova, Maldacena, Turiaci '17] to satisfy:

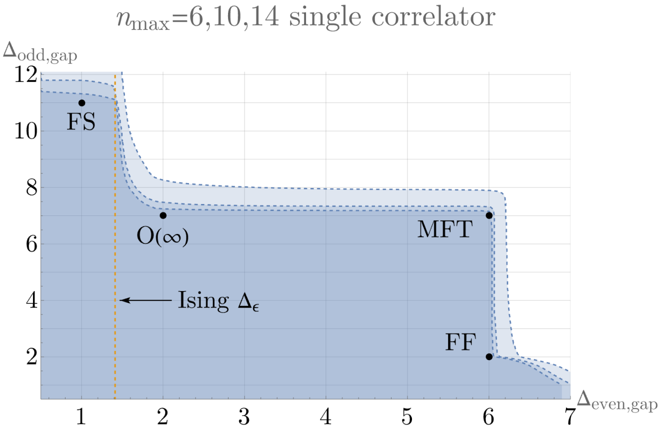
$$|\lambda^0_{T\epsilon T}| \leq 0.981(2)$$

- Also recently computed using the "fuzzy sphere" method [He, He, Zhu '23]:

$$\lambda^0_{T\epsilon T} \simeq 0.8057(65)$$

- This coefficient can also be probed from the stress tensor bootstrap...

$\langle TTTT \rangle$ bootstrap in 3d

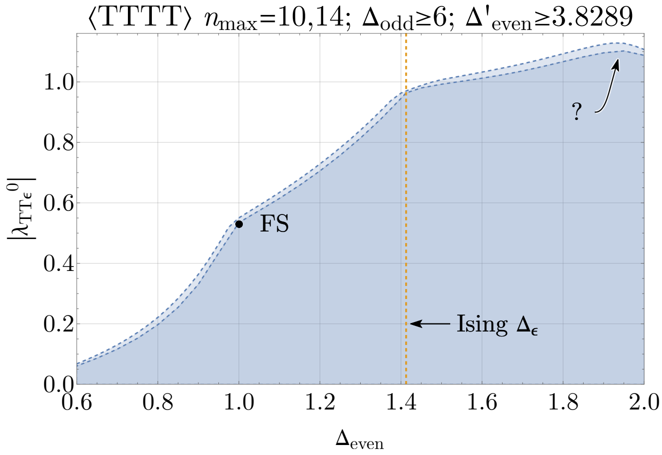


[Dymarsky, Kos, Kravchuk, Poland, Simmons-Duffin '17]

[Chang, Dommes, Erramilli, Homrich, Kravchuk, Liu, Mitchell, Poland, Simmons-Duffin]

- From stress-tensor 4-point functions one can get general bounds on parity-even and parity-odd scalar gaps, producing map of CFT landscape

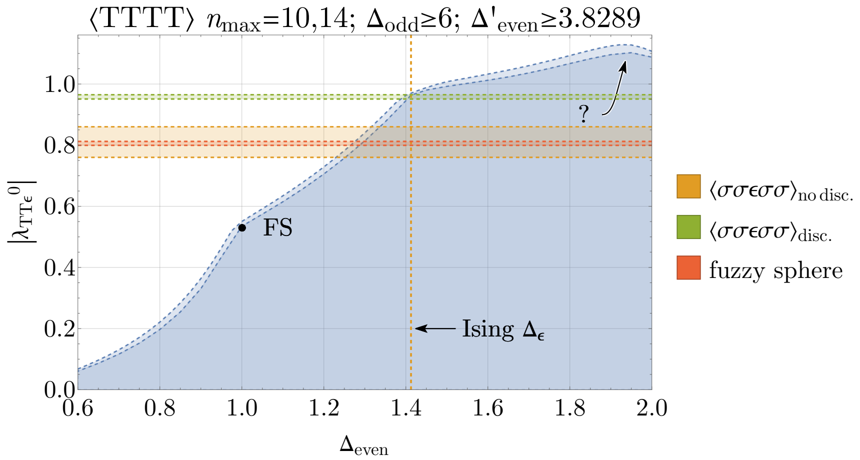
$\langle TTTT \rangle$ bootstrap in 3d



[Chang, Dommes, Erramilli, Homrich, Kravchuk, Liu, Mitchell, Poland, Simmons-Duffin]

- We have computed bounds on the $\langle TT\epsilon \rangle$ coefficient after assuming gaps compatible with 3d Ising

$\langle TTTT \rangle$ bootstrap in 3d



[Chang, Dommes, Erramilli, Homrich, Kravchuk, Liu, Mitchell, Poland, Simmons-Duffin]

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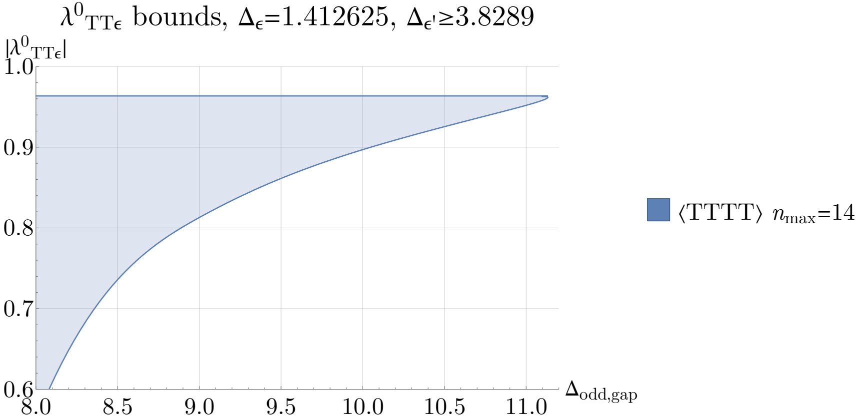

[Chang, Dommes, Erramilli, Homrich, Kravchuk, Liu, Mitchell, Poland, Simmons-Duffin]

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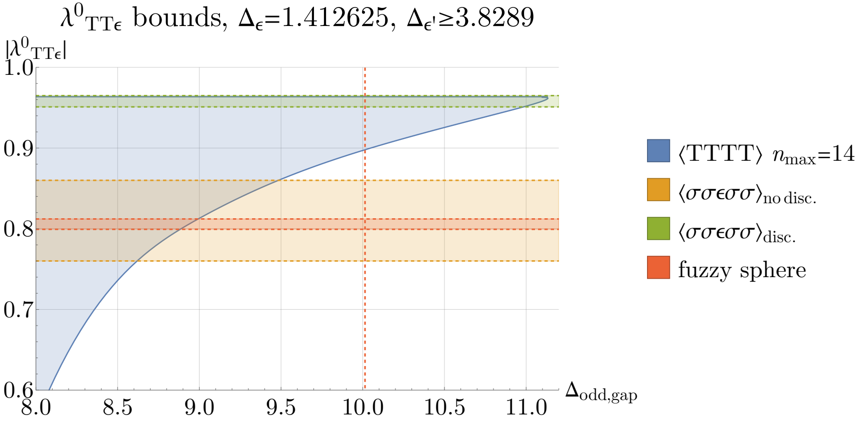

[Chang, Dommes, Erramilli, Homrich, Kravchuk, Liu, Mitchell, Poland, Simmons-Duffin]

- Expect definitive results from $\{T, \sigma, \epsilon\}$ mixed system (stay tuned...)

Lessons and Outlook

- The 5-point bootstrap works!
- We should explore other channels/correlators and extend to 6-points
- Blocks are under control, main bottleneck to adding more operators is handling the many unknown OPE coefficients
- Some low-lying observables (e.g. $\lambda_{TT\epsilon}$) are sensitive to hard truncations and greatly benefit from introducing a disconnected approximation
- Approximating the truncated spectrum may be useful more generally in other bootstrap problems where truncation methods are used