Trace anomalies and the graviton-dilation amplitude

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$50 + \varepsilon$ **Years of Conformal Bootstrap**

Overview of the talk

Part 1: Introduction: QFTs, CFTs and trace anomalies

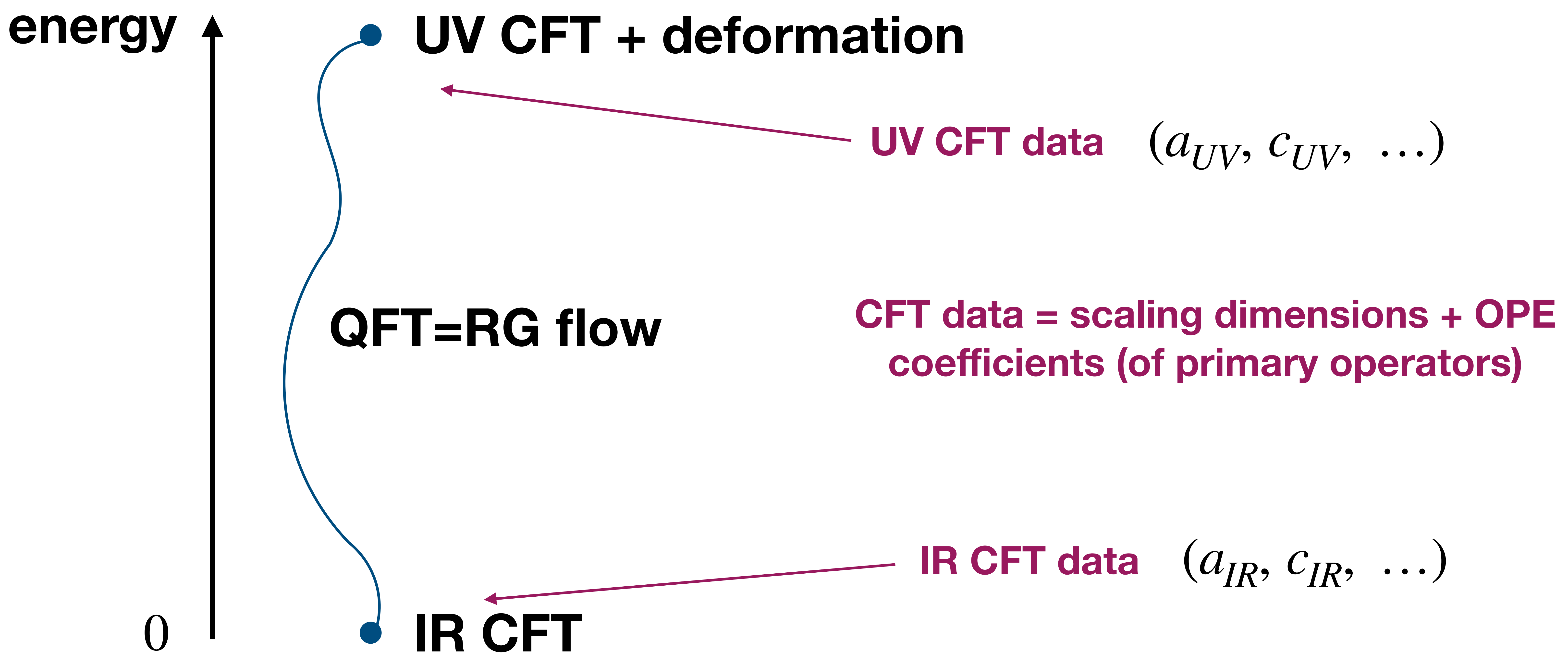
Part 2: Background field method for probing trace anomalies

Part 3: Test: free massive boson

Part 4: Graviton-dilaton amplitude

Part 5: Bootstrap applications

Part 1: QFTs and CFTs



$$\Delta a \equiv a_{UV} - a_{IR} \qquad \textit{and} \qquad \Delta c \equiv c_{UV} - c_{IR}$$

Part 1: trace anomalies in 4d

c-trace anomaly

[H. Osbor, A. Petkos; 1993]

$$\langle 0 | T^{\mu\nu}(x_1) T^{\rho\sigma}(x_2) | 0 \rangle_{CFT} = \frac{640}{\pi^2} \frac{c}{x_{12}^8} \mathbf{T}_0^{\mu\nu;\rho\sigma}$$

$$\langle 0 | T^{\mu\nu}(x_1) T^{\rho\sigma}(x_2) T^{\alpha\beta}(x_3) | 0 \rangle_{CFT} = \frac{1}{x_{12}^4 x_{23}^4 x_{31}^4} \left(\mathbb{A} \mathbf{T}_1^{\mu\nu;\rho\sigma;\alpha\beta} + \mathbb{B} \mathbf{T}_2^{\mu\nu;\rho\sigma;\alpha\beta} + \mathbb{C} \mathbf{T}_3^{\mu\nu;\rho\sigma;\alpha\beta} \right)$$

$$c \equiv \frac{\pi^4}{1920} \left(14\mathbb{A} - 2\mathbb{B} - 5\mathbb{C} \right)$$

$$a \equiv \frac{\pi^4}{5760} \left(9\mathbb{A} - 2\mathbb{B} - 10\mathbb{C} \right)$$

a-trace anomaly

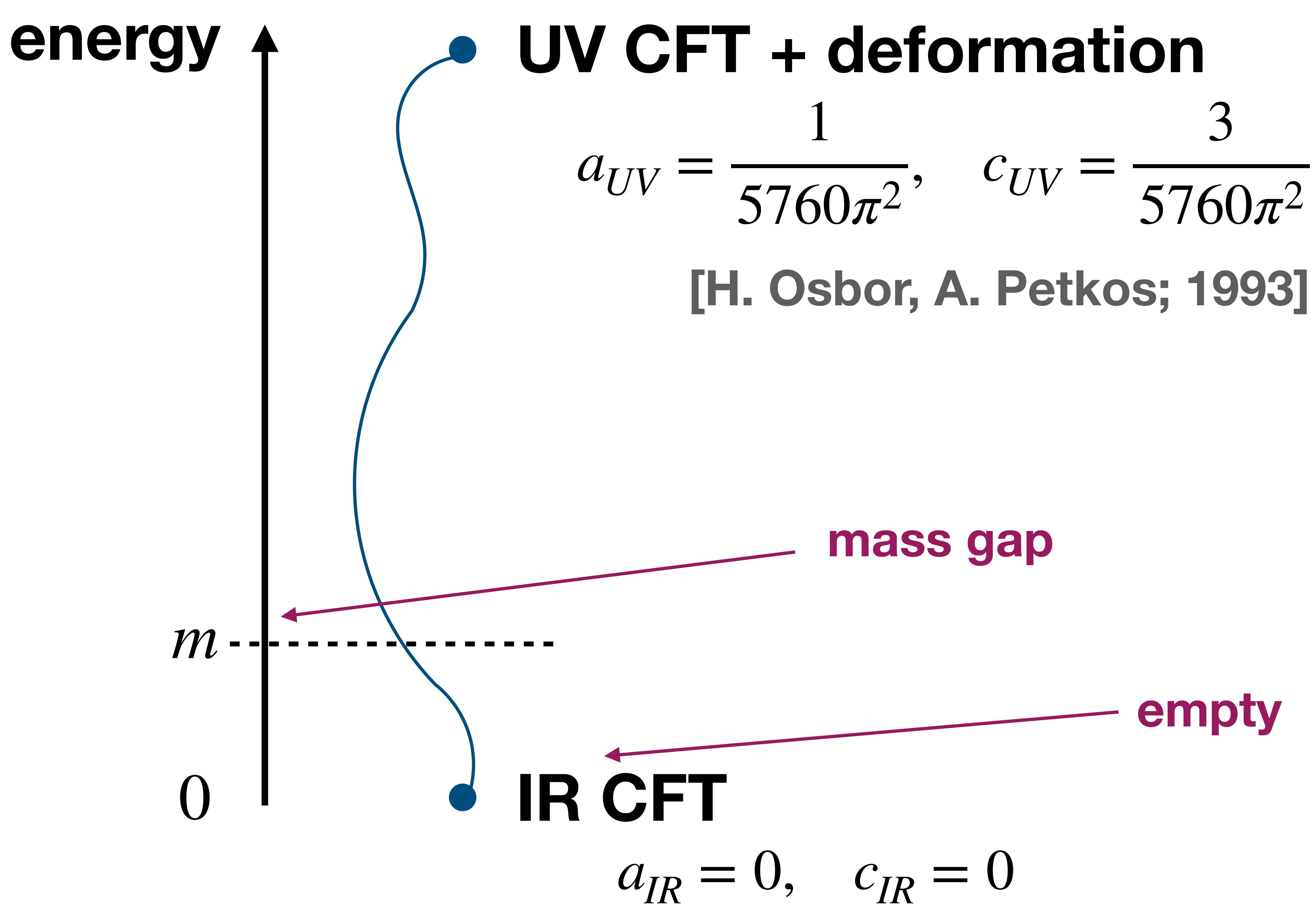
Tensor structures:

$$x_{ij}^\mu \equiv x_i^\mu - x_j^\nu$$

$$I^{\mu\nu}(x) \equiv \delta^{\mu\nu} - 2 \frac{x^\mu x^\nu}{x^2}$$

$$\mathbf{T}_0^{\mu\nu;\rho\sigma} \equiv \frac{1}{2} I^{\mu\rho}(x) I^{\nu\sigma}(x) + \frac{1}{2} I^{\mu\sigma}(x) I^{\nu\rho}(x) - \frac{1}{4} \eta^{\mu\nu} \eta^{\rho\sigma}$$

Part 1: example 1



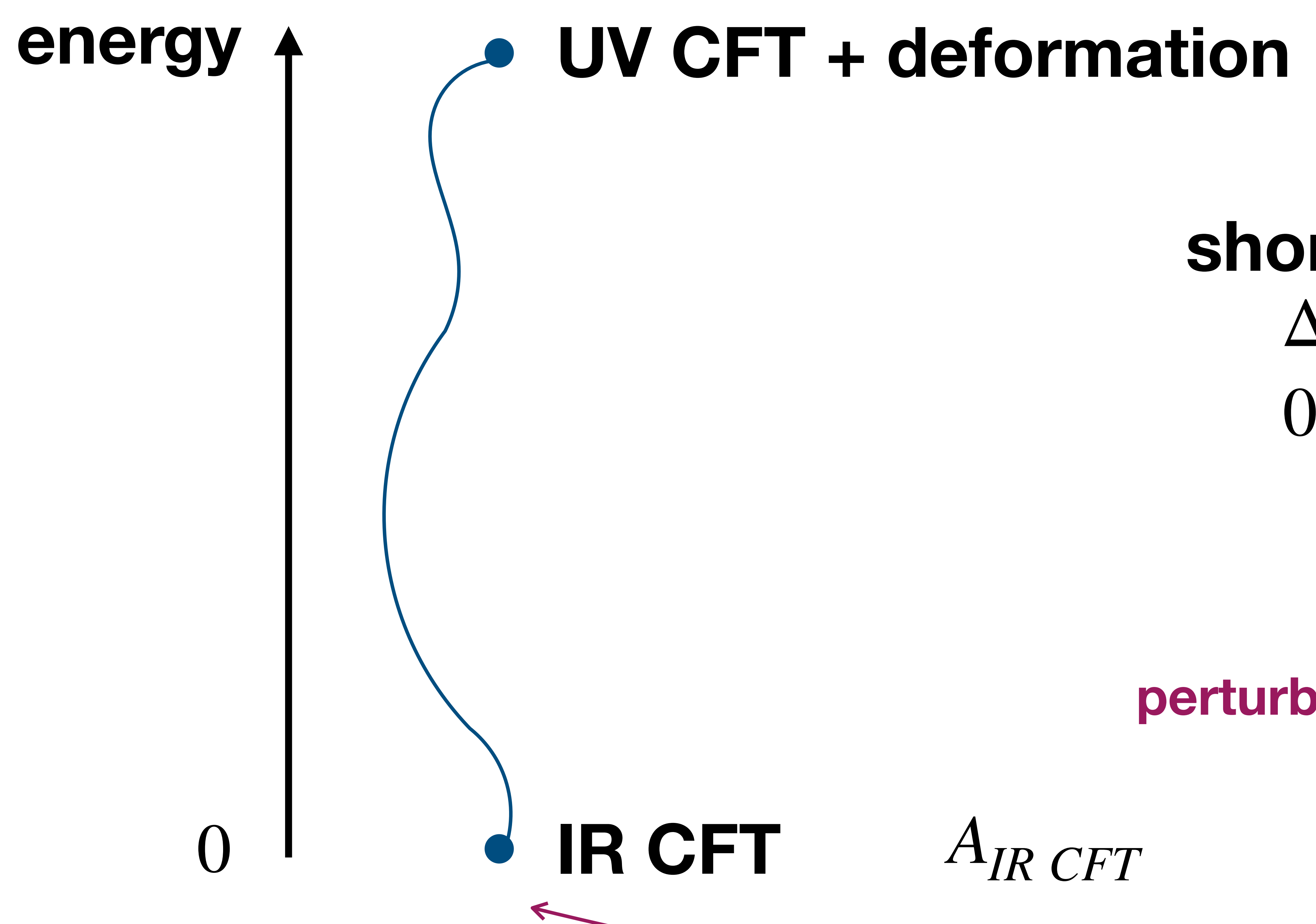
$$A_{QFT} = \int d^4x \left(-\frac{1}{2}\big(\partial\Phi(x)\big)^2 - \frac{1}{2}m^2\Phi(x)^2 \right)$$

UV CFT: $-\frac{1}{2}\big(\partial\Phi(x)\big)^2$

deformation: $-\frac{1}{2}m^2\Phi(x)^2$

Part 1: example 2

[DK, B. Sahoo; in progress]



$A_{IR\ CFT}$

$$A_{QFT} = A_{UV\ CFT} + \lambda M^{d-\Delta} \int d^d x \mathcal{O}_\Delta(x)$$

UV CFT data

deformation

short flow if

$$\Delta = d - \delta$$

$$0 < \delta \ll 1$$

perturbative parameter

OPE coefficient

$$\beta(\lambda) = -\delta\lambda + \frac{1}{2} C_{\mathcal{O}\mathcal{O}\mathcal{O}} \Omega_{d-1} \lambda^2 + O(\lambda^3)$$

$$\beta(\lambda_\star) = 0$$

$$\lambda_\star = \frac{2\delta}{C_{\mathcal{O}\mathcal{O}\mathcal{O}} \Omega_{d-1}}$$

Volume

Part 1: sum-rules

$$\Delta c = \frac{1}{2\,(d-1)} \int_0^\infty dr\, r^{2d-1} R^{abcd}(x) \langle 0 \,|\, T^{ab}(x) T^{cd}(0) \,|\, 0 \rangle_E,$$

[J. Cardy; 1988]

[A. Cappelli, D. Friedan, J. Latorre; 1991]

[DK; 2020]

$$R^{abcd}(x) \equiv (4-d^2)\frac{x^a x^b x^c x^d}{r^4} + \frac{d^2+d-2}{2}\,\frac{x^a x^b \delta^{cd} + x^c x^d \delta^{ab}}{r^2}$$

$$-\frac{x^a x^c \delta^{bd} + x^b x^c \delta^{ad} + x^a x^d \delta^{bc} + x^b x^d \delta^{ac}}{r^2} + (d+2)\,\delta^{ab}\delta^{cd} + \left(\delta^{ac}\delta^{bd} + \delta^{bc}\delta^{ad}\right).$$

$$d = 2: \qquad \Delta c \geq 0$$

Zamolodchikov's c-theorem

[A. Zamolodchikov; 1986]

$$d = 4: \qquad \Delta a \geq 0$$

a-theorem

[J. Cardy; 1988]

[H. Osborn; 1989]

[Z. Komargodski, A. Schwimmer; 2011]

[G. Mathys, T. Hartman; 2023]

Part 1: why sum-rules matter

1. Lead to universal constraints on QFTs

2. Play an important role in bootstrap studies

Part 2: background field method

$g_{\mu\nu}(x)$ ← curved (non-dynamical) background

$$Z[g_{\mu\nu}] \equiv \int [d\Phi]\, e^{iA[g_{\mu\nu}]}$$

action (UV CFT + deformation)

measure (UV d.o.f)

partition function

$$Z[g_{\mu\nu}] = e^{iW[g_{\mu\nu}]}$$

connected functional

Weyl transformation: Not a symmetry

Compensated QFT: $M \longrightarrow M(x) \equiv Me^{-\tau(x)}$ (compensator field) Weyl symmetric action if $\tau(x) \xrightarrow{W} \tau(x) + \sigma(x)$

Weyl anomaly:

$$\delta_\sigma W[g_{\mu\nu}] = \int d^4x\sqrt{-g}\,\sigma(x)\left(-a_{UV}E_4 + c_{UV}W^2\right)$$

$$\delta_\sigma W[g_{\mu\nu}] = \int d^4x\sqrt{-g}\,\sigma(x)\langle 0\,|\,T^\mu_\mu(x)\,|\,0\rangle_g^{UVCFT}$$

Part 2: background field method

$$A_{EFT}[\tau, g_{\mu\nu}, \Theta] = -\, a_{UV} \times A_a[\tau, g_{\mu\nu}] + c_{UV} \times A_c[\tau, g_{\mu\nu}] + A_{invariant}[\widehat{g}_{\mu\nu}]$$

$$A_a[\tau, g_{\mu\nu}] = \int d^4x \sqrt{-g} \left(\tau E_4 + 4 \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\mu \tau \partial_\nu \tau + 2(\partial\tau)^4 - 4(\partial\tau)^2 \Box \tau \right)$$

$$A_c[\tau, g_{\mu\nu}] = \int d^4x \sqrt{-g}\, \tau\, W^2$$

$$A_{invariant}[\widehat{g}_{\mu\nu}] = \int d^4x \sqrt{-\widehat{g}} \left(M^4 \lambda + M^2 r_0 \widehat{R} + r_1 \widehat{R}^2 + r_2 \widehat{W}^2 + r_3 \widehat{E}_4 + \dots \right)$$

[E.Fradkin, A. Tseytlin; 1984]

[A. Schwimmer, S. Theisen; 2011]

$$A_{EFT}[\tau, g_{\mu\nu}, \Theta] = A_{IR\ CFT}[\tau, g_{\mu\nu}, \Theta] - \Delta a \times A_a[\tau, g_{\mu\nu}] + \Delta c \times A_c[\tau, g_{\mu\nu}] + A_{invariant}[\widehat{g}_{\mu\nu}]$$
$$+ \sum_{1 \le \Delta \le 2} \lambda_\Delta \int d^4x \sqrt{-\widehat{g}}\; M^{2-\Delta} R(\widehat{g})\; \widehat{\mathcal{O}}_\Delta(x)$$

$$\Delta a \equiv a_{UV} - a_{IR} \qquad and \qquad \Delta c \equiv c_{UV} - c_{IR}$$

caused confusion

$$\widehat{g}_{\mu\nu}(x) \equiv e^{2\tau(x)} g_{\mu\nu}(x)$$
$$\widehat{\mathcal{O}}_\Delta(x) \equiv e^{\Delta\tau(x)} \mathcal{O}(x)$$

[Z. Komargodski, A. Schwimmer; 2011]

[M. Luty, J. Polchinski, R. Rattazzi; 2012]

Part 2: dilaton-graviton vertices

$$e^{-\tau(x)} \equiv 1 - \frac{\varphi(x)}{\sqrt{2}f}$$

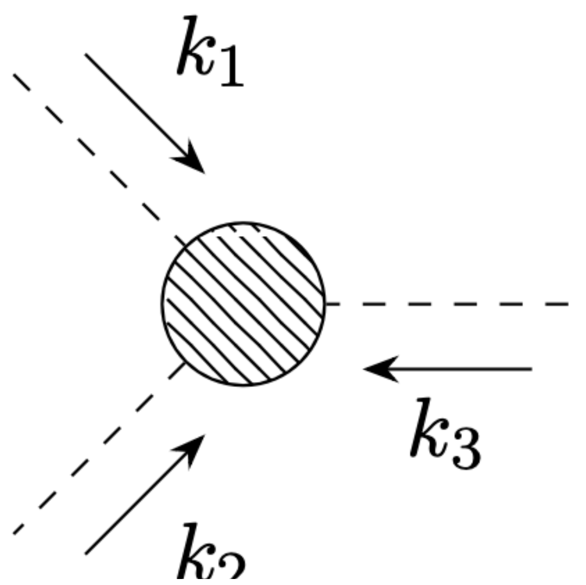
dilaton field

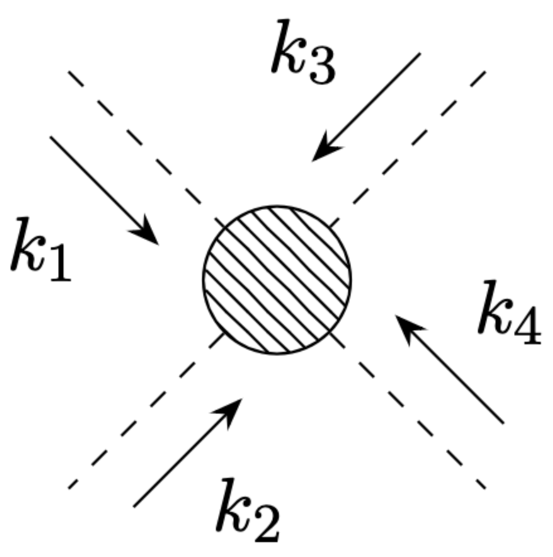
graviton field

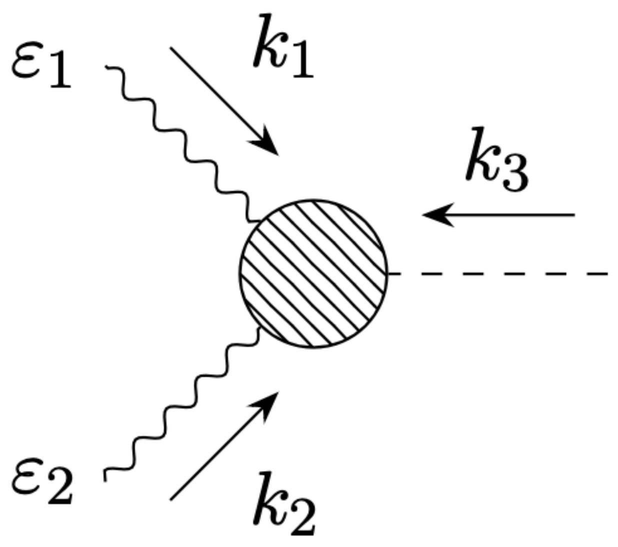
$$g_{\mu\nu}(x) \equiv \eta_{\mu\nu} + 2\kappa\, h_{\mu\nu}(x)$$

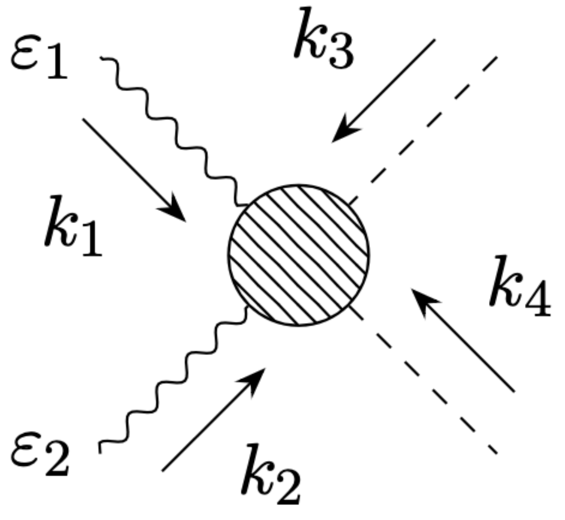
graviton-dilation vertex

$$(2\pi)^4\delta^{(4)}(p_1 + \ldots + p_m + q_1 + \ldots + q_n) \times V^{\mu_1\nu_1,\ldots,\mu_m\nu_m}_{(h\ldots h\varphi\ldots\varphi)}(p_1, \ldots, p_m, q_1, \ldots, q_n) \equiv \left. \frac{i\,\delta^{m+n} A_{EFT}[\tau, g_{\mu\nu}]}{\delta h_{\mu_1\nu_1}(p_1)\ldots\delta h_{\mu_m\nu_m}(p_m)\delta\varphi(q_1)\ldots\delta\varphi(q_n)} \right|_{h,\varphi=0}$$

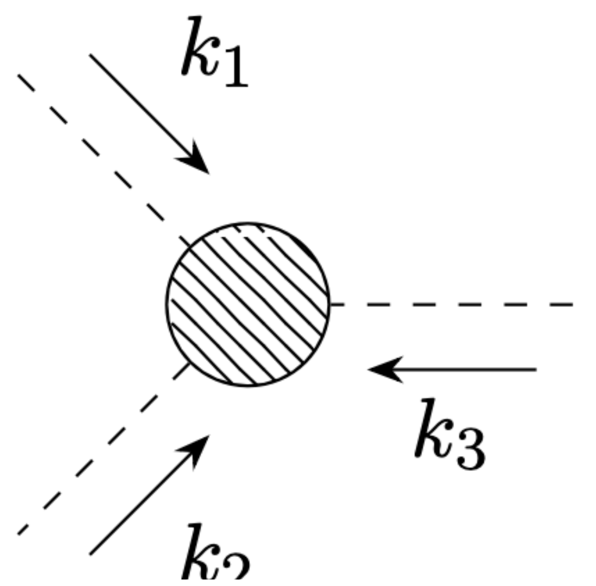
$V_{(\varphi\varphi\varphi)}(k_1, k_2, k_3) =$ 

$V_{(\varphi\varphi\varphi\varphi)}(k_1, k_2, k_3, k_4) =$ 

$V_{(hh\varphi)}(k_1, k_2, k_3; \varepsilon_1, \varepsilon_2) =$ 

$V_{(hh\varphi\varphi)}(k_1, k_2, k_3, k_4; \varepsilon_1, \varepsilon_2) =$ 

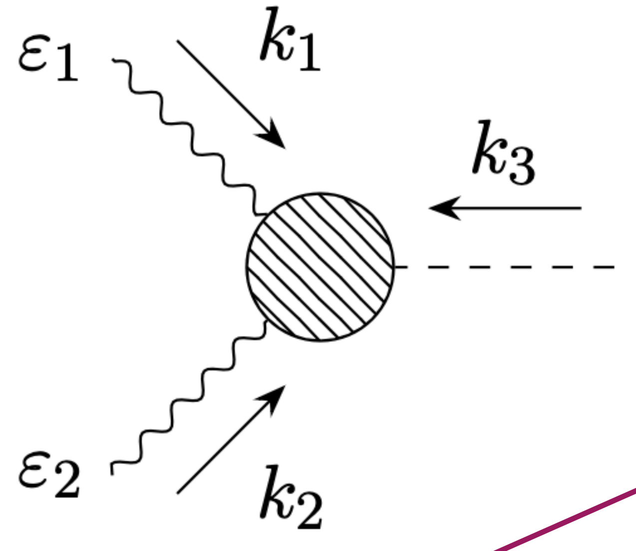
Part 2: dilaton-graviton vertices



$$= \frac{i\sqrt{2}}{f^3}\left(\Delta a \left(\left(k_1^2\right)^2 + \left(k_2^2\right)^2 + \left(k_3^2\right)^2\right) + 2(18r_1 - \Delta a)\left(k_1^2 k_2^2 + k_2^2 k_3^2 + k_3^2 k_1^2\right) + \ldots\right)$$

$$(\varepsilon_1 \,.\, \varepsilon_2) \equiv \varepsilon_{1\mu\nu}\varepsilon_2^{\mu\nu}$$
$$(k_i \,.\, \varepsilon_j \,.\, k_k) \equiv k_{i\mu}\varepsilon_j^{\mu\nu}k_{k\nu}$$
$$(k_i \,.\, \varepsilon_1 \,.\, \varepsilon_2 \,.\, k_j) \equiv k_{i\mu}\varepsilon_1^{\mu\rho}\varepsilon_{2\rho\nu}k_j^{\nu}$$

tensor structures:



$$= f_1(k_1, k_2) \times (\varepsilon_1 \,.\, \varepsilon_2) + f_2(k_1, k_2) \times (k_1 \,.\, \varepsilon_2 \,.\, k_1)(k_2 \,.\, \varepsilon_1 \,.\, k_2) + f_3(k_1, k_2) \times (k_1 \,.\, \varepsilon_2 \,.\, \varepsilon_1 \,.\, k_2)$$

$$f_1(k_1, k_2) = \frac{4i\kappa^2}{\sqrt{2}f}\left(2(-\Delta a + \Delta c + 18r_1)(k_1 \,.\, k_2)^2 + (2\Delta a - \Delta c + 24r_1)k_1^2 k_2^2 + 12r_1(k_1^4 + k_2^4) + 42r_1(k_1 \,.\, k_2)(k_1^2 + k_2^2) + \ldots\right)$$

$$f_2(k_1, k_2) = \frac{8i\kappa^2}{\sqrt{2}f}\left(-\Delta a + \Delta c + \ldots\right)$$

$$f_3(k_1, k_2) = \frac{8i\kappa^2}{\sqrt{2}f}\left(2(\Delta a - \Delta c - 6r_1)(k_1 \,.\, k_2) - 6r_1(k_1^2 + k_2^2) + \ldots\right)$$

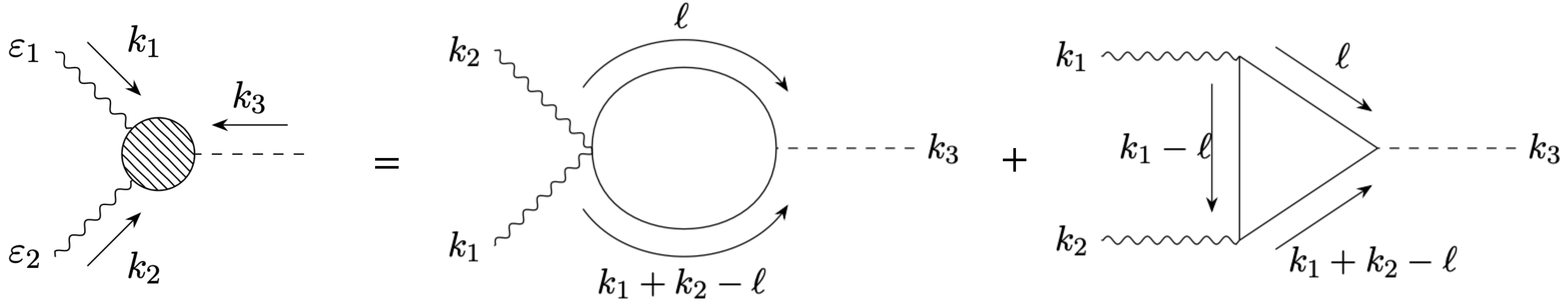
Part 3: example

$$\left(e^{-\tau(x)} \equiv 1 - \frac{\varphi(x)}{\sqrt{2}f}\right)$$

$$A_{free\ scalar}[\Phi] \equiv \int d^4x \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi(x)\partial_\nu\Phi(x) - \frac{1}{2}m^2\Phi^2(x)\right)$$

compensator field

$$A^{compensated}_{free\ scalar}[\Phi,\varphi,h] \equiv \int d^dx\,\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}m^2e^{-2\tau}\Phi^2 - \frac{d-2}{8(d-1)}R\Phi^2\right)$$



$$f_1(k_1,k_2) = \frac{i\kappa^2}{1440\sqrt{2}\pi^2 f}\left(2(k_1^2)^2 + 2(k_2^2)^2 + 10(k_1\,.\,k_2)^2 + 7k_1\,.\,k_2(k_1^2+k_2^2) + 3k_1^2k_2^2\right)$$

$$f_2(k_1,k_2) = +\frac{i\kappa^2}{360\sqrt{2}\pi^2 f}$$

$$f_3(k_1,k_2) = -\frac{i\kappa^2}{720\sqrt{2}\pi^2 f}\left(k_1^2 + k_2^2 + 6(k_1\,.\,k_2)\right)$$

$$\Delta a = \frac{1}{5760\pi^2}, \qquad \Delta c = 3\Delta a, \qquad r_1 = \frac{\Delta a}{6}$$

Part 4: graviton-dilation amplitude

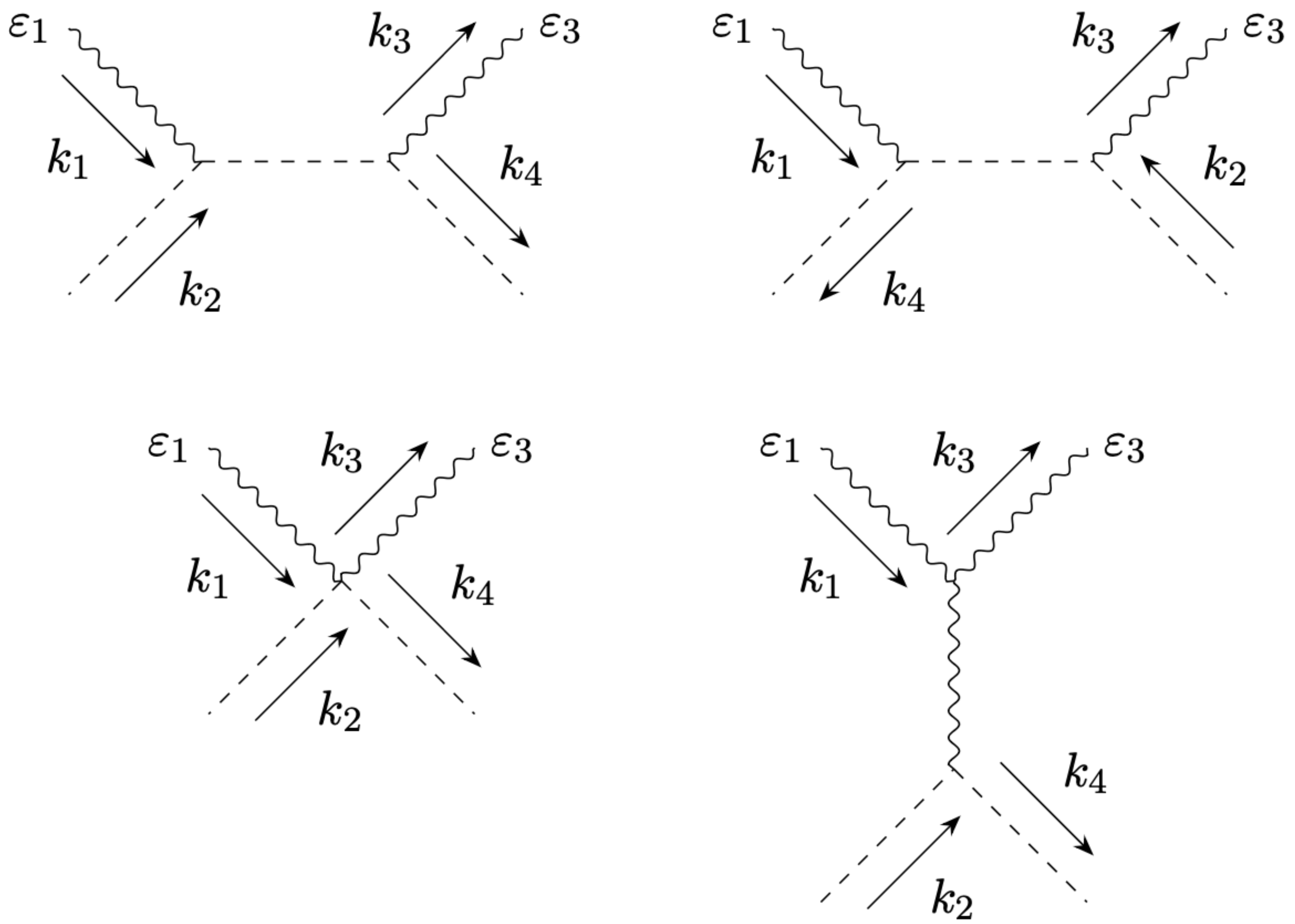
$$A = A_{EFT} + A^{\varphi}_{kinetic} + A^{h}_{kinetic}$$

$$(\kappa \to 0,\ f \to \infty,\ \kappa^{-1} \gg f)$$

$$A^{\varphi}_{kinetic} = -\frac{\bar{f}^2}{6}\int d^4x\sqrt{-\hat{g}}\ \widehat{R}\,, \qquad -\frac{\bar{f}^2}{6} \equiv -\frac{f^2}{6} - M^2 r_0$$

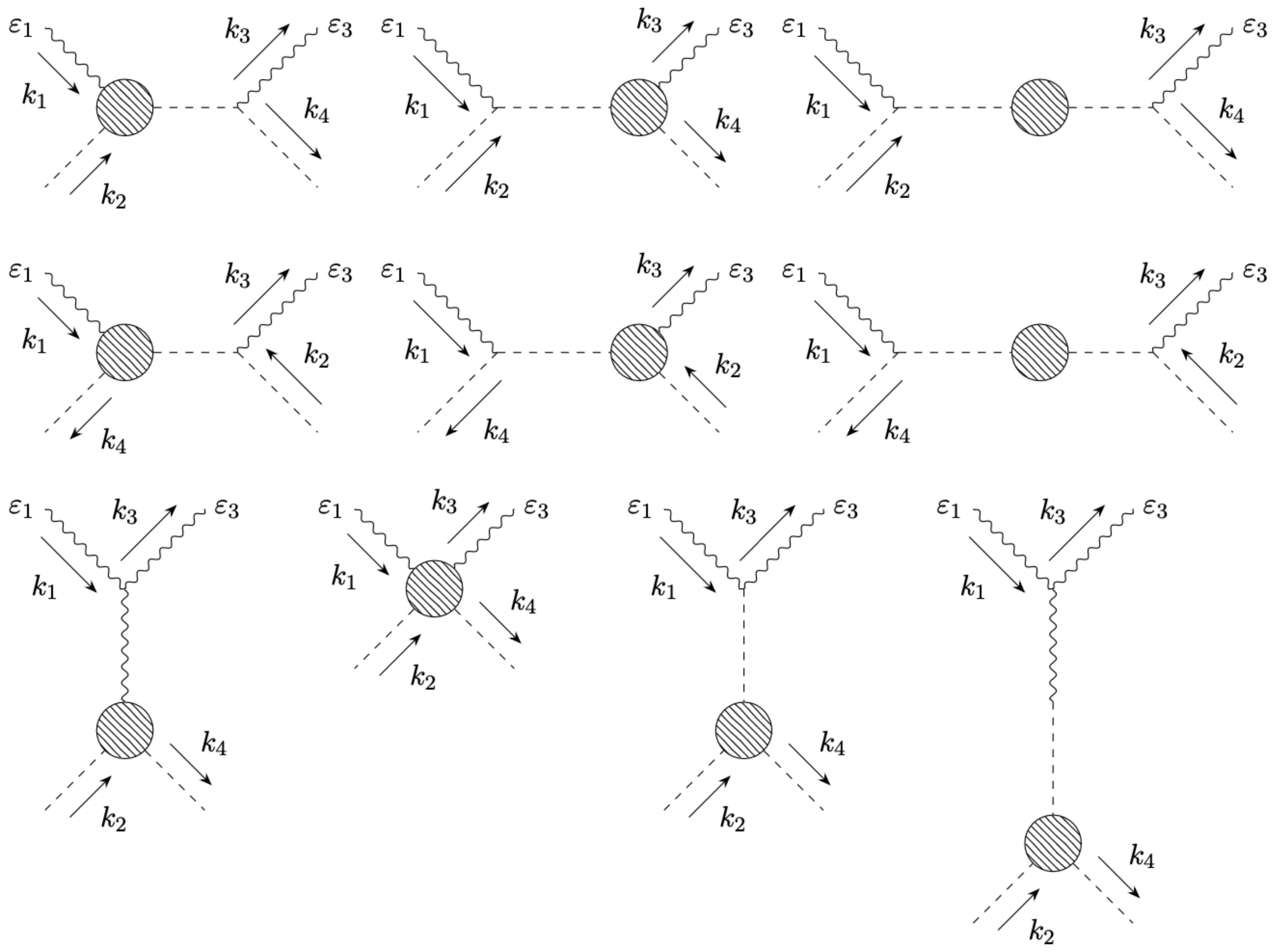
$$A^{h}_{kinetic} = \frac{1}{2\bar{\kappa}^2}\int d^4x\sqrt{-g}\ R, \qquad \frac{1}{2\bar{\kappa}^2} \equiv \frac{1}{2\kappa^2} + \frac{f^2}{6}\,.$$

$O\left(\kappa^2\right)$:



Part 4: graviton-dilation amplitude

$O\left(\frac{\kappa^2}{f^2}\right)$:



Part 4: graviton-dilation amplitude

Scattering amplitude:

$$\mathcal{T}_{h\varphi\longrightarrow h\varphi}(k_1,k_2,k_3,k_4;\varepsilon_1,\varepsilon_3)=\kappa^2\frac{su}{t^3}\,\mathbf{T}_2+\frac{\kappa^2}{f^2}(\Delta c-\Delta a)\,\mathbf{T}_1$$

Tensor structures:

$$\mathbf{T}_1\equiv(H_1)^{\mu\nu,\rho\sigma}(H_3)_{\mu\nu,\rho\sigma}$$

$$\mathbf{T}_2\equiv(H_1)^{\mu\nu,\rho\sigma}(H_3)_{\mu\nu,\rho\sigma}+\left(\frac{16}{su}\right)k_2^{\alpha_1}k_3^{\alpha_2}(H_1)^{\mu\nu}{}_{\alpha_1\alpha_2}k_1^{\beta_1}k_2^{\beta_2}(H_3)_{\mu\nu,\beta_1\beta_2}$$
$$+\left(\frac{8}{su}\right)^2\left(k_2^{\alpha_1}k_3^{\alpha_2}k_2^{\alpha_3}k_3^{\alpha_4}(H_1)_{\alpha_1\alpha_2,\alpha_3\alpha_4}\right)\left(k_1^{\beta_1}k_2^{\beta_2}k_1^{\beta_3}k_2^{\beta_4}(H_3)_{\beta_1\beta_2,\beta_3\beta_4}\right)$$

Basic building block:

$$(H_i)^{\mu\nu,\rho\sigma}\equiv k_i^\mu k_i^\rho\varepsilon_i^{\nu\sigma}(k_i)-k_i^\mu k_i^\sigma\varepsilon_i^{\nu\rho}(k_i)-k_i^\nu k_i^\rho\varepsilon_i^{\mu\sigma}(k_i)+k_i^\nu k_i^\sigma\varepsilon_i^{\mu\rho}(k_i)$$

Linearised gauge transformations:

$$\varepsilon_i^{\mu\nu}(k_i)\to\varepsilon_i^{\mu\nu}(k_i)+\chi^\mu k_i^\nu\ +\ \chi^\nu k_i^\mu$$

Center of mass amplitude:

$$\mathcal{T}_{+2}^{+2}(s,t,u)=\mathcal{T}_{-2}^{-2}(s,t,u)=\kappa^2\frac{su}{t}$$

$$\mathcal{T}_{+2}^{-2}(s,t,u)=\mathcal{T}_{-2}^{+2}(s,t,u)=\frac{\kappa^2}{f^2}(\Delta c-\Delta a)\,t^2$$

Summary: part 2 - part 4

$$\Delta a = \lim_{f\to\infty} f^4 \int_{m^2}^{\infty} \frac{ds}{\pi} \frac{\textbf{Im}\, \mathcal{T}(s,0,-s)}{s^3}$$

[Z. Komargodski, A. Schwimmer; 2011]

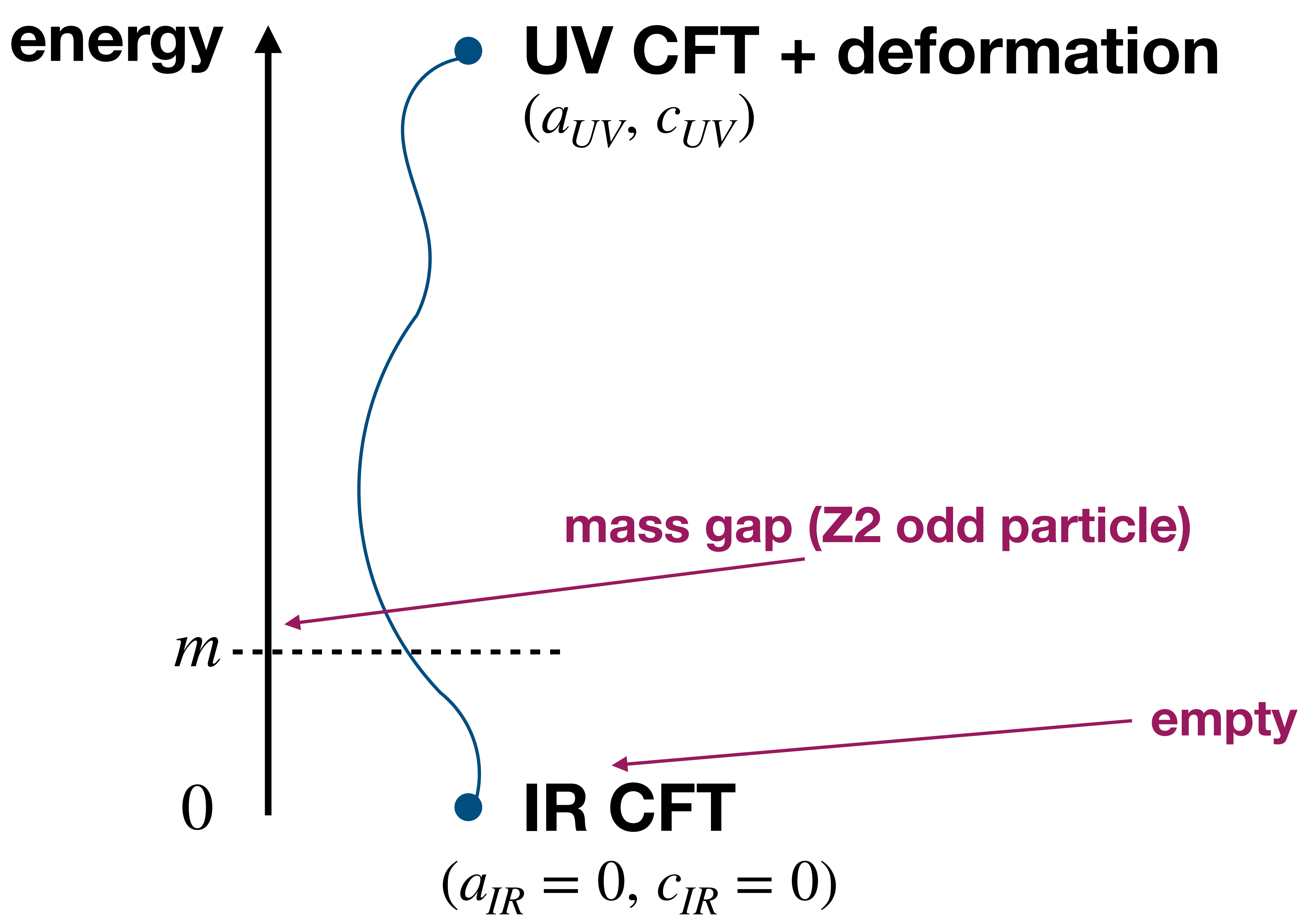
$$\Delta c - \Delta a = \lim_{f\to\infty} \lim_{\kappa\to 0} \frac{f^2}{\kappa^2} \int_{m^2}^{\infty} \frac{ds}{\pi} \frac{\textbf{Im}\, \partial_t^2 \mathcal{T}_{+2}^{-2}(s,0,-s)}{s}$$

new

(only states with spin=2,4,6,… contribute)

1. We used the background field method to study QFTs (dilatons and gravitons)
2. EFT of background fields is fixed by trace anomalies (Weyl anomaly matching)
3. Trace anomalies of QFTs are extracted from the vertices of background fields
4. We tested this technology in several examples (free scalar example presented)
5. We compactly package all the vertices into a dilaton-graviton amplitude

Part 5: bootstrap application



Scattering amplitude: $\mathcal{T}_{mm\to mm}(s,t,u)$

Observables:

$$\lambda_0 \equiv \frac{1}{32\pi}\mathcal{T}_{mm\to mm}(4m^2/3, 4m^2/3, 4m^2/3)$$

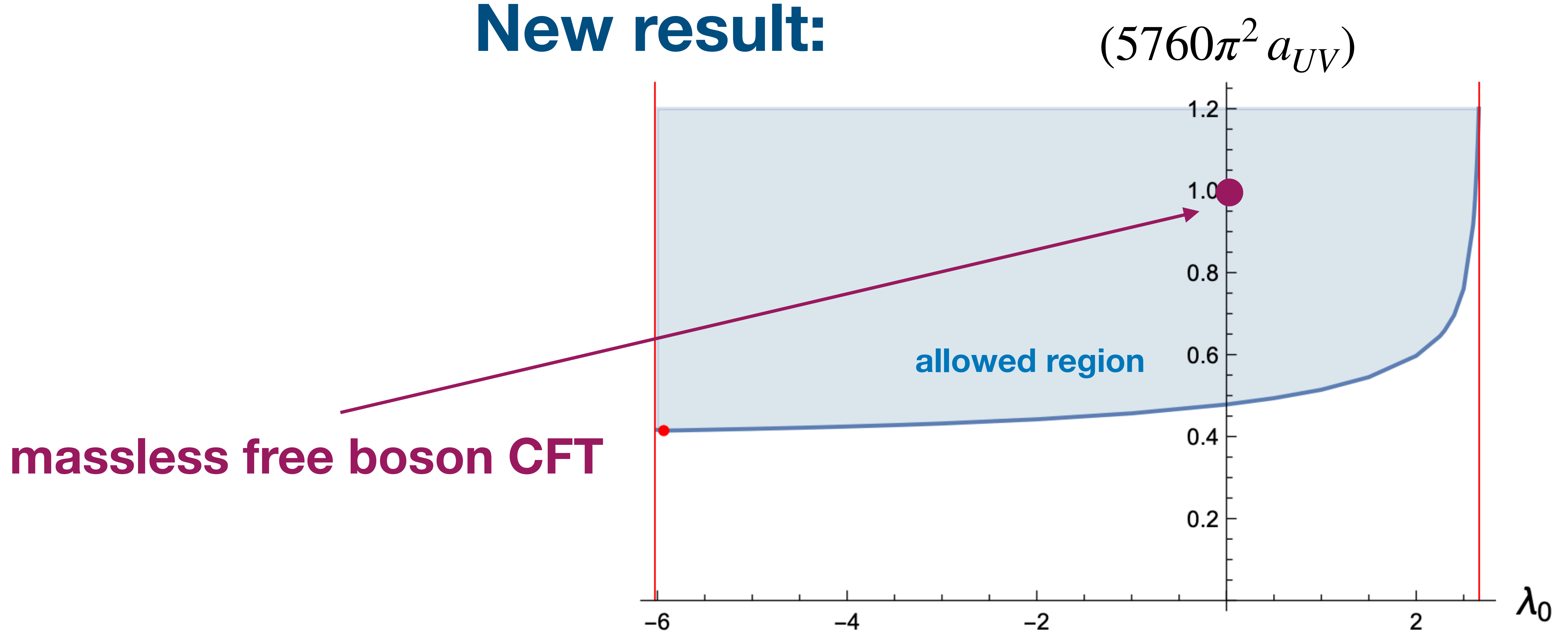
$$\lambda_2 \equiv \frac{1}{32\pi}m^4\partial_s^2\mathcal{T}_{mm\to mm}(4m^2/3, 4m^2/3, 4m^2/3)$$

S-matrix bootstrap bounds:

$$-6.0253 \leq \lambda_0 \leq +2.6613$$
$$0 \leq \lambda_2 \leq +2.2568$$

[DK, J. Marucha, B. Sahoo, J. Penedones; 2022]

New result:



Thank you!