QCD on the lattice:

from New Physics to Extreme Conditions

M I C H E L E   P E P E

I N F N   Sez. Milano – Bicocca

Milan (I t a l y)

Introduction

- QCD: key features of Strong Interactions
- The lattice regularization of QCD
- Non-perturbative physics and Monte Carlo simulations
- Precision physics: the muon anomalous magnetic moment
- QCD in Extreme Conditions: screening masses and the Equation of State
- Conclusions and outlook

ORIGINAL RESULTS IN COLLABORATORATION WITH:

MATTIA DALLA BRIDA
CERN
Geneve (Switzerland)

LEONARDO GIUSTI
University of Milano – Bicocca
Milan (Italy)

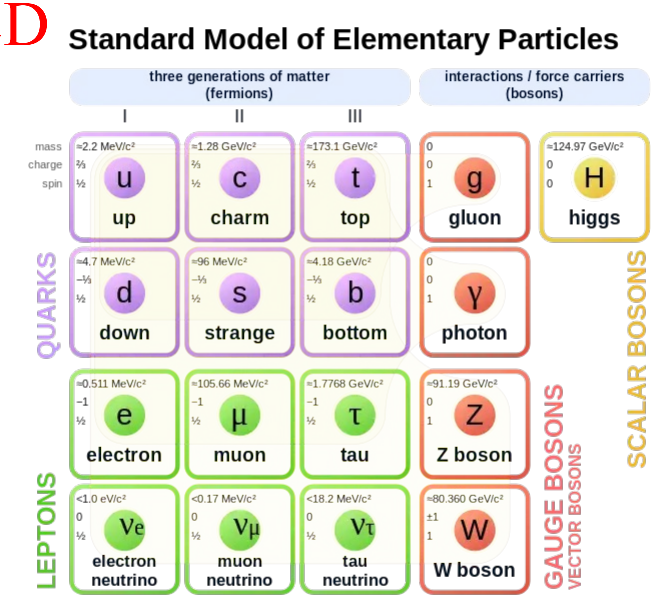
TIM HARRIS
ETH
Zürich (Switzerland)

MATTEO BRESCIANI
University of Milano – Bicocca
Milan (Italy)

DAVIDE LAUDICINA
University of Milano – Bicocca
Milan (Italy)

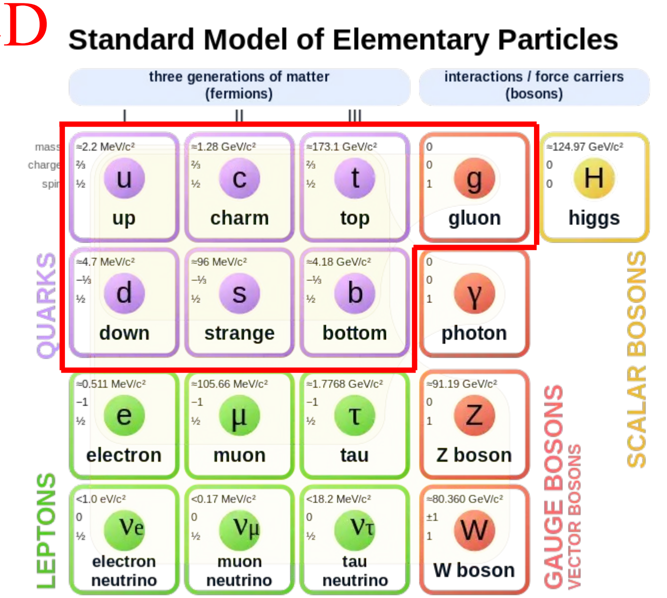
The Standard Model and QCD

- The Standard Model: matter fields (fermions) + interactions (gluons, $\gamma, Z, W^{\pm}$) + Higgs (scalar)



The Standard Model and QCD

- The Standard Model: matter fields (fermions) + interactions (gluons, $\gamma, Z, W^{\pm}$) + Higgs (scalar)
- The Quantum ChromoDynamics (QCD) is the fundamental quantum theory of Strong Interactions that involve quarks and gluons



$$\mathcal{L} = \frac{1}{2g_0^2}\mathrm{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right] + \overline{\psi}_f\left[\gamma_\mu D_\mu + m_0)\right]\psi_f$$

Fritsch, Gell-Mann, Leutwyler
PLB 47B (1973) 365

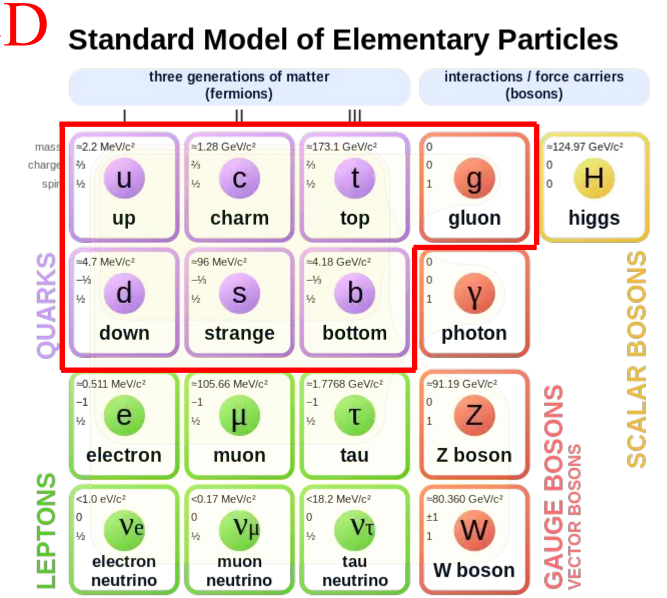
Gluon field: $A_\mu = A_\mu^a T^a$, T^a = 8 generators of the SU(3) group: $UU^\dagger = 1, \quad \det(U) = 1$

Quark, anti-quark fields: fermions $\psi_f^a, \quad \overline{\psi}_f^a$ with color and flavour d.o.f.

Field strength: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$; covariant derivative: $D_\mu = \partial_\mu - iA_\mu$

The Standard Model and QCD

- The Standard Model: matter fields (fermions) + interactions (gluons, $\gamma, Z, W^{\pm}$) + Higgs (scalar)
- The Quantum ChromoDynamics (QCD) is the fundamental quantum theory of Strong Interactions that involve quarks and gluons



$$\mathcal{L} = \frac{1}{2g_0^2}\mathrm{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right] + \overline{\psi}_f\left[\gamma_\mu D_\mu + m_0\right]\psi_f$$

Fritsch, Gell-Mann, Leutwyler PLB 47B (1973) 365

Gluon field: $A_\mu = A_\mu^a T^a$, T^a = 8 generators of the SU(3) group: $UU^\dagger = 1,\quad \det(U) = 1$

Quark, anti-quark fields: fermions $\psi_f^a, \quad \overline{\psi}_f^a$ with color and flavour d.o.f.

Field strength: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$; covariant derivative: $D_\mu = \partial_\mu - iA_\mu$

Pure glue sector: SU(3)Yang-Mills theory Dirac Operator

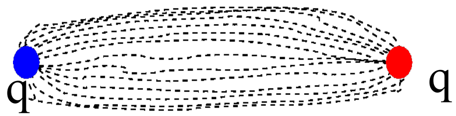
- Gauge symmetry: $\Omega(x) \in SU(3)$ local invariance under SU(3) transformations

$$A'_\mu(x) \to \Omega(x)A_\mu(x)\Omega^\dagger(x) + i(\partial_\mu\Omega(x)\Omega^\dagger(x) \qquad \psi'(x) \to \Omega(x)\psi(x) \qquad \overline{\psi}'(x) \to \overline{\psi}(x)\Omega^\dagger(x)$$

- Symmetry (massless quarks): $SU(6)_L \times SU(6)_R \times U(1)_B$

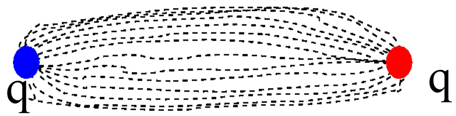
QCD: key features

- Color confinement: quarks/anti-quarks are not observed as isolated particles



QCD: key features

- Color confinement: quarks/anti-quarks are not observed as isolated particles



$$\sigma \simeq (\mathbf{0.4Gev})^{\mathbf{2}} \simeq \mathbf{10^5 N}$$

$$\sigma = \mathbf{M\,g}$$

$$\mathbf{M} \sim \mathbf{100}\ \text{people}$$

As strong as a cm-thick steel cable but 13 orders of magnitude thinner

The Yang-Mills cableway

- Asymptotic freedom: the gauge coupling g is not constant but becomes smaller and smaller as the distance between color sources decreases or as the energy μ goes large

$$\mu \frac{d}{d\mu} g = \beta(g) = -b_0 g^3 - b1 g^5 + \ldots$$

$$\Lambda = \mu \left[b_0 g^2(\mu) \right]^{-b_1/(2b_0^2)} \mathrm{e}^{-\frac{1}{2b_0 g^2(\mu)}} (1 + \ldots)$$

David J. Gross H. David Politzer Frank Wilczek

A dynamical scale Λ develops, breaking of scale invariance (massless quarks)

- Spontaneous breaking of chiral symmetry: mesons as Goldstone bosons

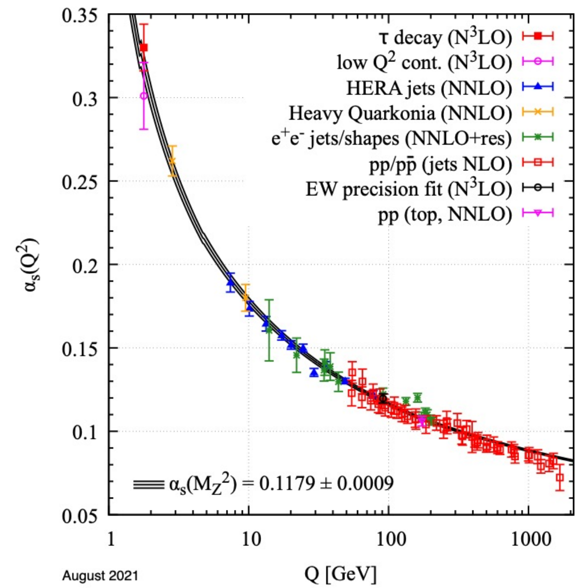
$$SU(3) \times SU(3) \to SU(3)$$

- From asymptotic freedom we have the running

$$\frac{1}{g^2(\mu)} = b_0 \log\left[\frac{\mu^2}{\Lambda^2}\right] + \frac{b_1}{b_0}\log\left[\log\left[\frac{\mu^2}{\Lambda^2}\right]\right] + \dots$$

For processes where the energy scale $\mu \gg \Lambda$

$g(\mu)$ is very small => perturbation theory



- Confinement comes from the dynamical appearance of an energy scale not present in the Lagrangian , no explanation in perturbation theory: $\Lambda \sim 0.2\,\mathrm{GeV}(\sim 1\,\mathrm{fm})$

From distances of the order of 1 fm or not too high energies QCD is non-perturbative

- Non-perturbative QCD plays a crucial role in many phenomena: Hadron physics, Nuclear Physics, Astrophysics (compact objects), Cosmology (early universe)

NOTE: having a small value for the gauge coupling $g(\mu)$ is a necessary condition for applying perturbation theory but it is not sufficient (see later)

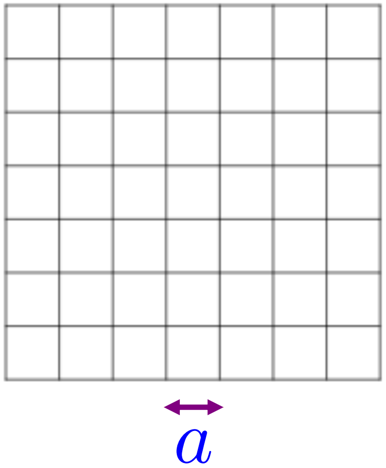
QCD on the lattice

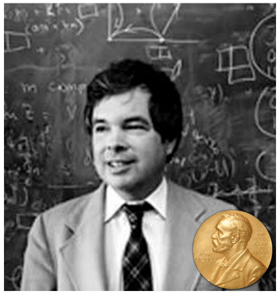
- The lattice regularization is the only currently known framework where QCD can be defined and investigated non-perturbatively.
- In 1974 Wilson proposed to regularize QCD by making continuous space-time discrete

  $g_{\mu\nu} = \delta_{\mu\nu}$: Euclidean metric

  a: lattice spacing

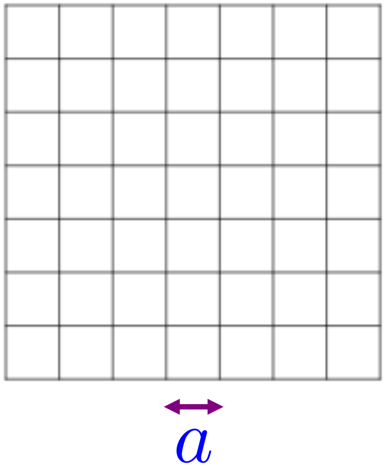
  a^{-1}: cutoff in momenta

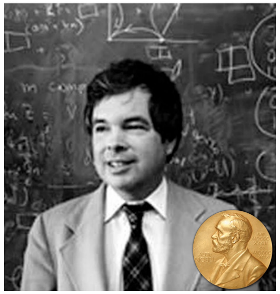




QCD on the lattice

- The lattice regularization is the only currently known framework where QCD can be defined and investigated non-perturbatively.





- In 1974 Wilson proposed to regularize QCD by making continuous space-time discrete

  $g_{\mu\nu} = \delta_{\mu\nu}$: Euclidean metric

  a: lattice spacing

  a^{-1}: cutoff in momenta

- In this framework quantum fields have to be properly defined:

QCD on the lattice

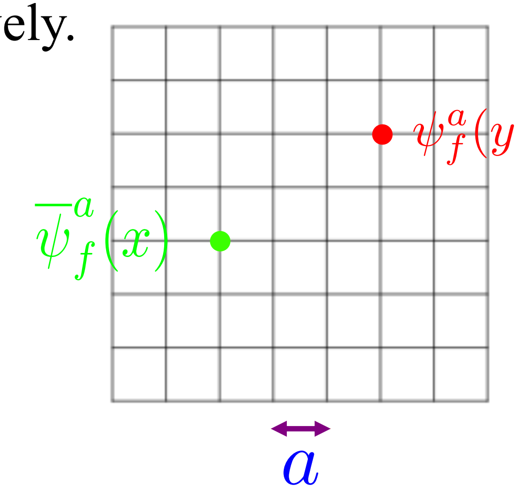
- The lattice regularization is the only currently known framework where QCD can be defined and investigated non-perturbatively.

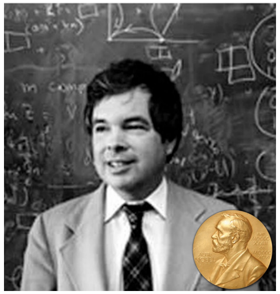
- In 1974 Wilson proposed to regularize QCD by making continuous space-time discrete

  $g_{\mu\nu} = \delta_{\mu\nu}$: Euclidean metric

  a: lattice spacing

  a^{-1}: cutoff in momenta



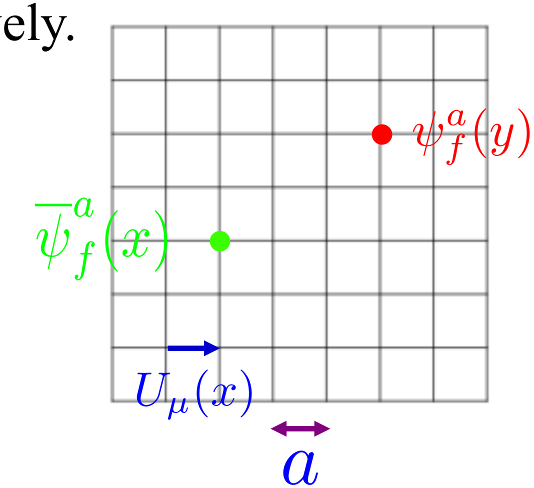


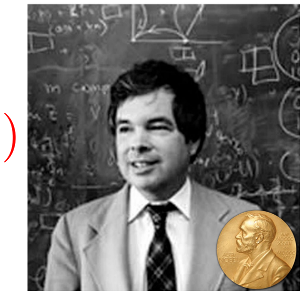
- In this framework quantum fields have to be properly defined:

$\psi^a_f(x)$, $\overline{\psi}^a_f(x)$: quarks and ant-quarks are defined in the lattice sites

QCD on the lattice

- The lattice regularization is the only currently known framework where QCD can be defined and investigated non-perturbatively.





- In 1974 Wilson proposed to regularize QCD by making continuous space-time discrete

$g_{\mu\nu} = \delta_{\mu\nu}$: Euclidean metric

a: lattice spacing

a^{-1}: cutoff in momenta

- In this framework quantum fields have to be properly defined:

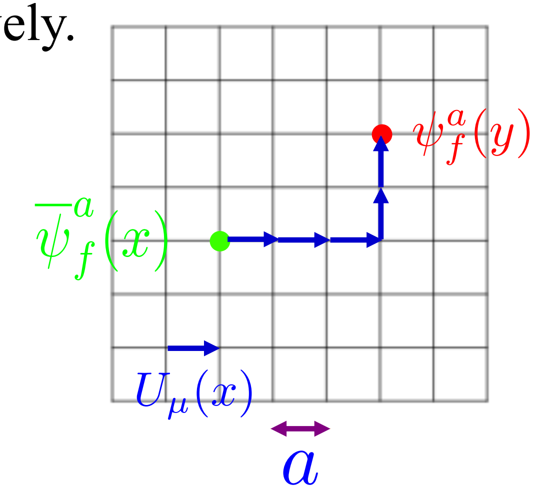
$\psi_f^a(x)$, $\overline{\psi}_f^a(x)$: quarks and anti-quarks are defined in the lattice sites

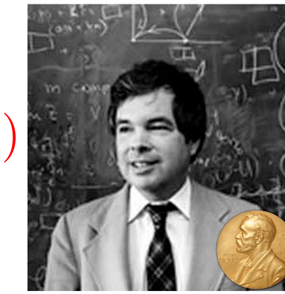
$A_\mu(x)$ connection between different points in space-time $\quad G(x,y) = \exp\left[i \int_{\mathcal{C}_{xy}} A \cdot ds\right] \in$ group SU(3)

the link field: $\quad U_\mu(x) = \exp\left[i \int_x^{x+a\hat{\mu}} A_\mu(x) dx_\mu\right] \in SU(3)$

QCD on the lattice

- The lattice regularization is the only currently known framework where QCD can be defined and investigated non-perturbatively.





- In 1974 Wilson proposed to regularize QCD by making continuous space-time discrete

  $g_{\mu\nu} = \delta_{\mu\nu}$: Euclidean metric

  a: lattice spacing

  a^{-1}: cutoff in momenta

- In this framework quantum fields have to be properly defined:

$\psi_f^a(x)$, $\overline{\psi}_f^a(x)$: quarks and anti-quarks are defined in the lattice sites

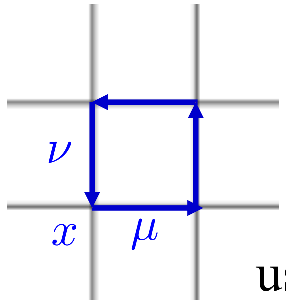
$A_\mu(x)$ connection between different points in space-time $\quad G(x,y) = \exp\left[i\int_{\mathcal{C}_{xy}} A\cdot ds\right] \in$ group SU(3)

the link field: $\quad U_\mu(x) = \exp\left[i\int_x^{x+a\hat{\mu}} A_\mu(x)dx_\mu\right] \in SU(3)$

- Matter fields at different lattice points are connected by a path-ordered products of links

The action: the gauge sector

- The plaquette



$$U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+a\hat{\mu})U_\nu(x+a\hat{\nu})^\dagger U_\nu(x)^\dagger = \exp\left[ia^2F_{\mu\nu}(x) + \mathcal{O}(a^3)\right]$$

$$S_G = \frac{2}{g_0^2}\sum_x\sum_{\mu<\nu}\mathrm{Re\ Tr}\left[1 - U_{\mu\nu}(x)\right] \xrightarrow[a\to 0]{} \frac{a^4}{2g_0^2}\sum_x\sum_{\mu\nu}\mathrm{Tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right]$$

usually rewritten as $S_G = \beta\sum_x\sum_{\mu<\nu}\mathrm{Re\ Tr}\left\{\frac{1}{3}\left[1 - U_{\mu\nu}(x)\right]\right\}; \qquad \beta = \frac{6}{g_0^2}$ Wilson action

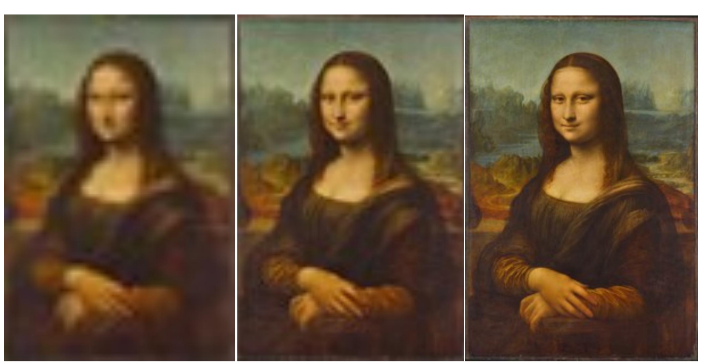
- The Yang-Mills theory is defined and SU(3)-gauge-invariant: $U'_\mu(x) = \Omega(x)U_\mu(x)\Omega(x+a\hat{\mu})$

- From asymptotic freedom $a(g) = \frac{1}{\Lambda}\left[b_0g^2\right]^{-b_1/(2b_0^2)}\,\mathrm{e}^{-\frac{1}{2b_0g^2}}$

the gauge coupling is the handle to change the lattice spacing

We finally have to remove the regulator: the continuum limit

$a \to 0$ ⟶ vanishing gauge coupling



- The physics must stay the same: dimensionful quantities $\xi = 1/m$ diverge in lattice units ξ/a in the continuum limit ➔ gaussian fixed point

The action: the quark sector

- Putting fermions on the lattice is not straightforward: Wilson fermions, staggered fermions, variants, …
- Wilson fermions: fully theoretically solid

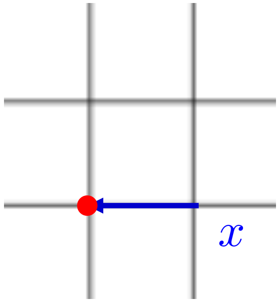
$$S_F = a^4 \sum_x \overline{\psi}(x)(D + m_0)\psi(x), \qquad D = D_W$$

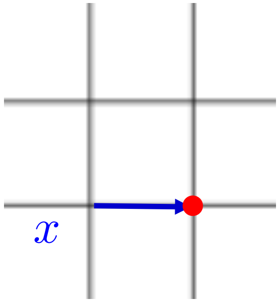
$$D_W = \frac{1}{2}\left\{\gamma_\mu(\nabla^*_\mu + \nabla_\mu) - a\nabla^*_\mu \nabla_\mu\right\}$$

Dirac-Wilson operator

$$a\nabla^*_\mu \psi(x) = \psi(x) - U^\dagger_\mu(x - a\hat{\mu})\psi(x - a\hat{\mu})$$

$$a\nabla_\mu \psi(x) = U_\mu(x)\psi(x + a\hat{\mu}) - \psi(x)$$





The action: the quark sector

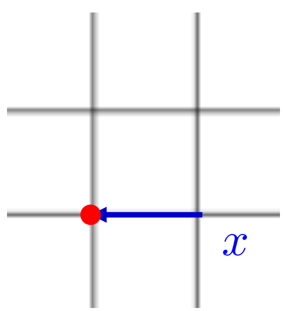
- Putting fermions on the lattice is not straightforward: Wilson fermions, staggered fermions, variants, …
- Wilson fermions: fully theoretically solid

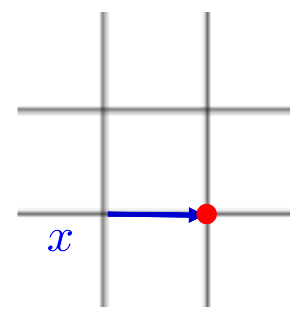
$$S_F = a^4 \sum_x \overline{\psi}(x)(D + m_0)\psi(x), \qquad D = D_W + a\,D_{SW}$$

Dirac-Wilson operator

$$D_W = \frac{1}{2}\left\{\gamma_\mu(\nabla^*_\mu + \nabla_\mu) - a\nabla^*_\mu \nabla_\mu\right\}$$

$$a\nabla^*_\mu \psi(x) = \psi(x) - U^\dagger_\mu(x - a\hat{\mu})\psi(x - a\hat{\mu}) \qquad a\nabla_\mu \psi(x) = U_\mu(x)\psi(x + a\hat{\mu}) - \psi(x)$$





- Lattice discretization artifacts are $\mathcal{O}(a)$ and they can be reduced to $\mathcal{O}(a^2)$
- The lattice action is gauge-invariant: $\psi'(x) \to \Omega(x)\psi(x) \qquad \overline{\psi}'(x) \to \overline{\psi}(x)\Omega^\dagger(x)$
- $\beta = \dfrac{6}{g_0^2}$ handle on a, m_0 modifies the quark masses: lines of constant physics

The non-perturbative computations

- Lattice QCD: continuum infinity ➔ countable infinity of d.o.f. ➔ numerical methods

$$S = S_G + S_F \qquad\qquad Z = \int \mathcal{D}U_\mu \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; \mathrm{e}^{-S}$$

- Fermions on computers: not simply implementable, possible but very inefficient

$$\int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; \mathrm{e}^{-\overline{\psi}\, D_F \psi} = \det(D_F) \in \mathcal{R} \qquad\qquad \gamma_5 D_F \gamma_5 = D_F^\dagger$$

$$S_{eff} = S_G - \sum_f \log\det(D_F^f) \qquad\qquad Z = \int \mathcal{D}U_\mu \; \mathrm{e}^{-S_{eff}}$$

- for an observable we rewrite

$$\langle O \rangle = \frac{1}{Z}\int \mathcal{D}U_\mu \, \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; O[U_\mu,\psi,\overline{\psi}] \; \mathrm{e}^{-S} = \frac{1}{Z}\int \mathcal{D}U_\mu \, \frac{\int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; O[U_\mu,\psi,\overline{\psi}] \; \mathrm{e}^{-S_F}}{\int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; \mathrm{e}^{-S_F}} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \; \mathrm{e}^{-S_F} \; \mathrm{e}^{-S_G} = \frac{1}{Z}\int \mathcal{D}U_\mu \, \langle O \rangle_G \; \mathrm{e}^{-S_{eff}}$$

- QCD functional integral in terms only of the gauge field

- Crucial issue: if $\mathrm{e}^{-S_{eff}}$ is positive ➔ it can be viewed as a probability distribution

- For f=s,c,b,t $\det(D_F^f) > 0$, light quarks degenerate $\det(D_F^u)\det(D_F^d) = \det(D_F^l)^2 > 0$

- technical note: fermionic det's are then rewritten as local actions of auxiliary bosonic fields (pseudofermions). S_{eff} : action of fields with local interactions

The Monte Carlo simulations

- QCD functional integral is the partition function of a statistical system with positive probability distribution

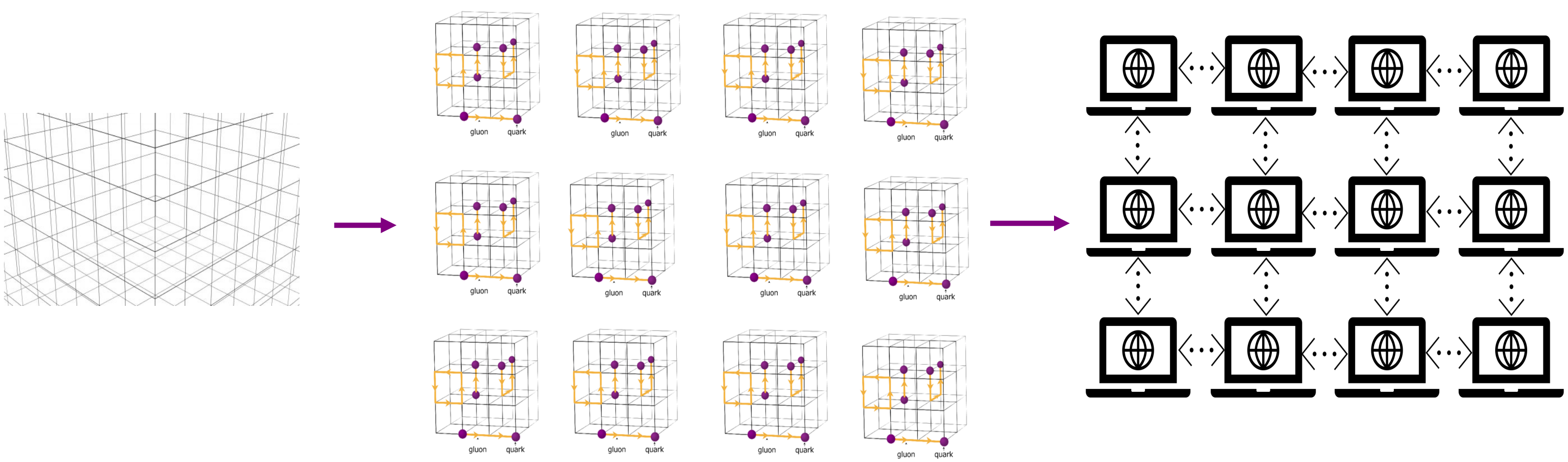
$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}U_\mu \, \langle O \rangle_G \, \mathrm{e}^{-S_{eff}} = \int \mathcal{D}U_\mu \, \langle O \rangle_G \, P[U_\mu] \qquad P[U_\mu] > 0, \qquad \int \mathcal{D}U_\mu \, P[U_\mu] = 1$$

- We have to perform integrals in many variables: Monte Carlo sampling

- The path-integral is computed by generating an ensamble of configurations of the gauge field (Markov chains) $U_\mu^{(1)}, U_\mu^{(2)}, U_\mu^{(3)}, \ldots U_\mu^{(N)}$ with probability $P[U_\mu]$

$$\langle O \rangle = \frac{1}{N} \sum_{n=1}^{N} O_n[U_\mu^{(k)}]$$

- Large number of integration variables: can be of order 10^{10}
- Computationally very demanding: parallel computing



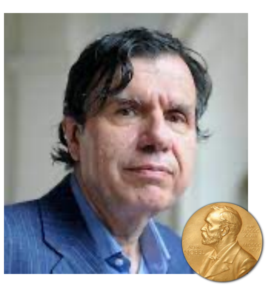
High Performance Computing (HPC): Leonardo

- Non-perturbative studies of QCD on the lattice: driving force for the development of supercomputers

Il rack che ospita la versione attuale di APE100. Al momento sono state realizzate sei schede, per una potenza complessiva di 2,4 Gflops.

APE100: 40 years ago (top)

Leonardo: installation in progress at Cineca, 4th top

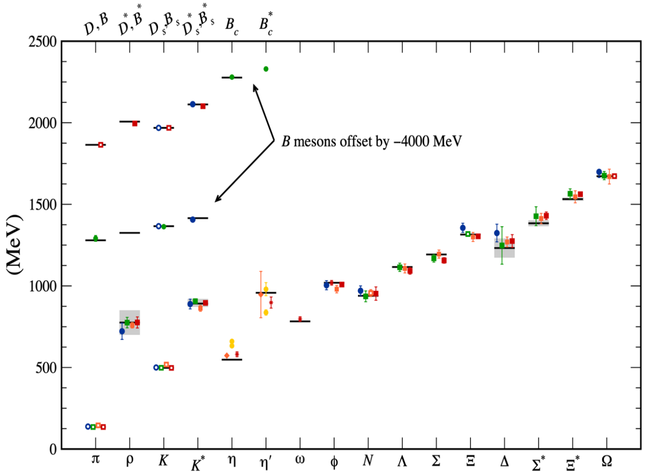


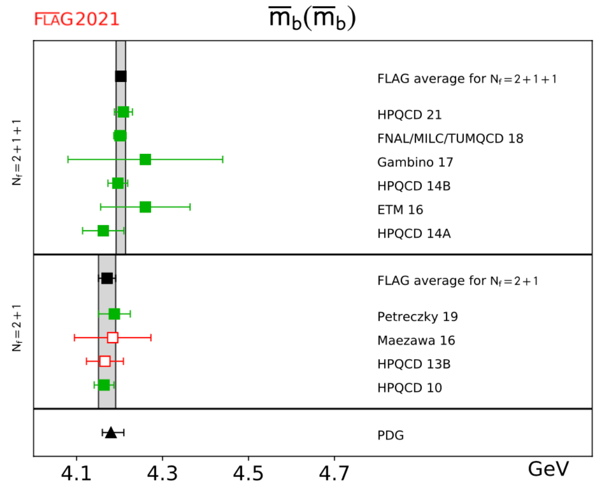
2.4 Gflop = 2.4 10^9 operations/sec

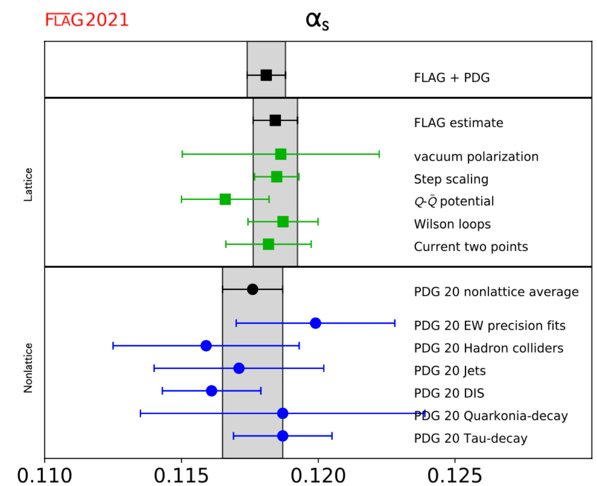
250 Pflop = 2.5 10^{17} operations/sec

Essential improvement of algorithms

- Computations of QCD on the lattice have accomplished a lot interesting results: both theoretically and experimentally



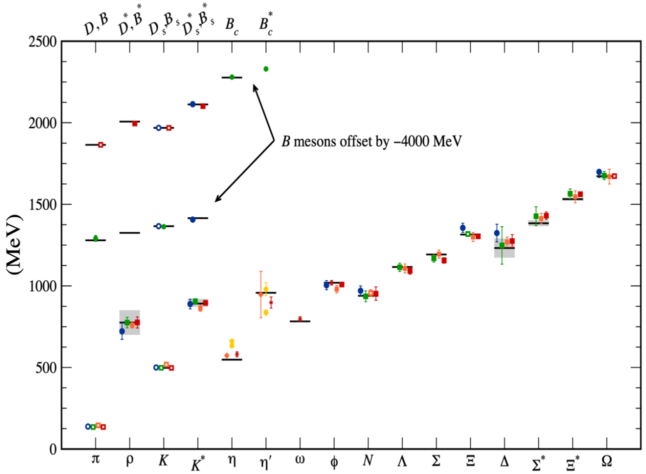


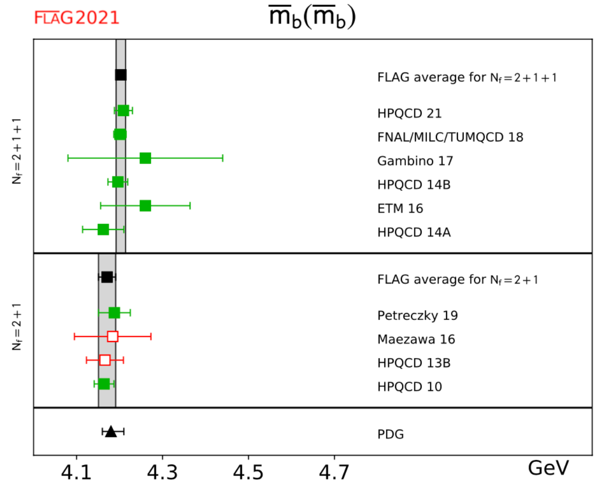


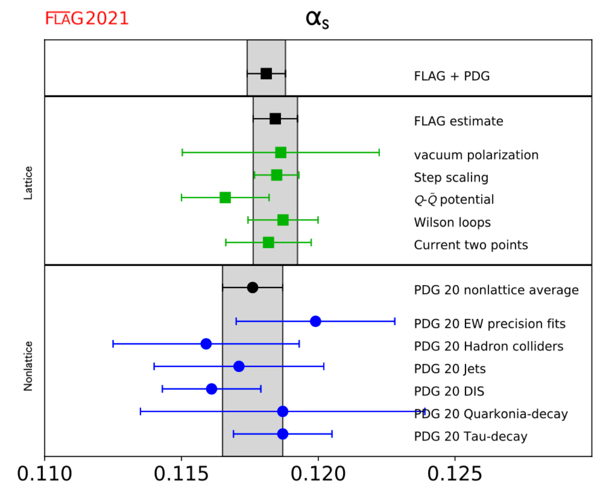
- Non-perturbative computations of QCD span a wide range of activities

- Quark masses
- CKM matrix
- α_s
- (g_μ-2): muon anomalous magnetic moment
- Flavour physics
- Structure and features of Baryons and Mesons (dipole moments, proton decay, …)
- Neutrino – Nucleon interaction
- Axions
- Nuclear physics
- Quark-gluon plasma: QCD under Extreme Conditions
- BSM physics
- …

- Computations of QCD on the lattice have accomplished a lot interesting results: both theoretically and experimentally





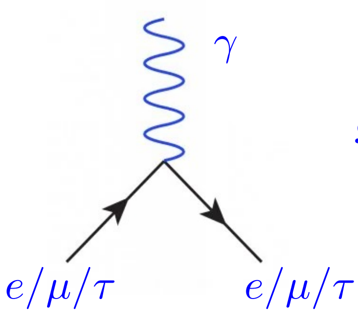


- Non-perturbative computations of QCD span a wide range of activities

- Quark masses
- CKM matrix
- α_s
- $(g_\mu$-2): muon anomalous magnetic moment
- Flavour physics
- Structure and features of Baryons and Mesons (dipole moments, proton decay, …)
- Neutrino – Nucleon interaction
- Axions
- Nuclear physics
- Quark-gluon plasma: QCD under Extreme Conditions
- BSM physics
- …

Precision physics: the muon anomalous magnetic moment

- At classical level a spinning charged particle has a magnetic moment $\vec{\mu} = g\frac{e}{2m}\vec{S}$
- In QED an electron/muon/tau interact with the photon field

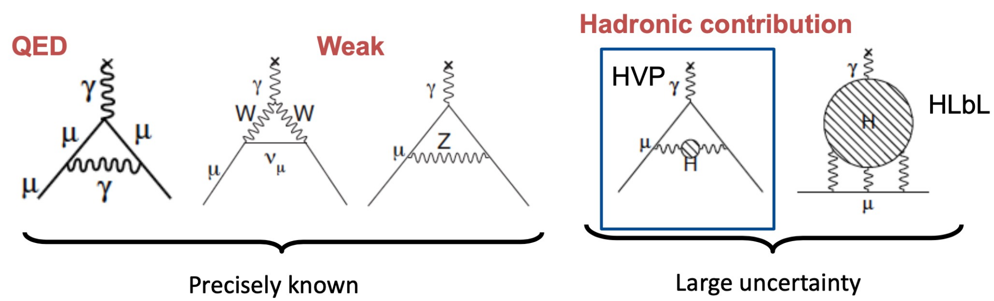


$g = 2 + \ldots$ anomaly: $a = \frac{(g-2)}{2}$

it can be measured very accurately and it is sensible to New Physics (m_{lepton}^2)

- In the Standard Model:

$$a_\mu^{SM} = a_\mu^{QED} + a_\mu^{Weak} + a_\mu^{Had}$$

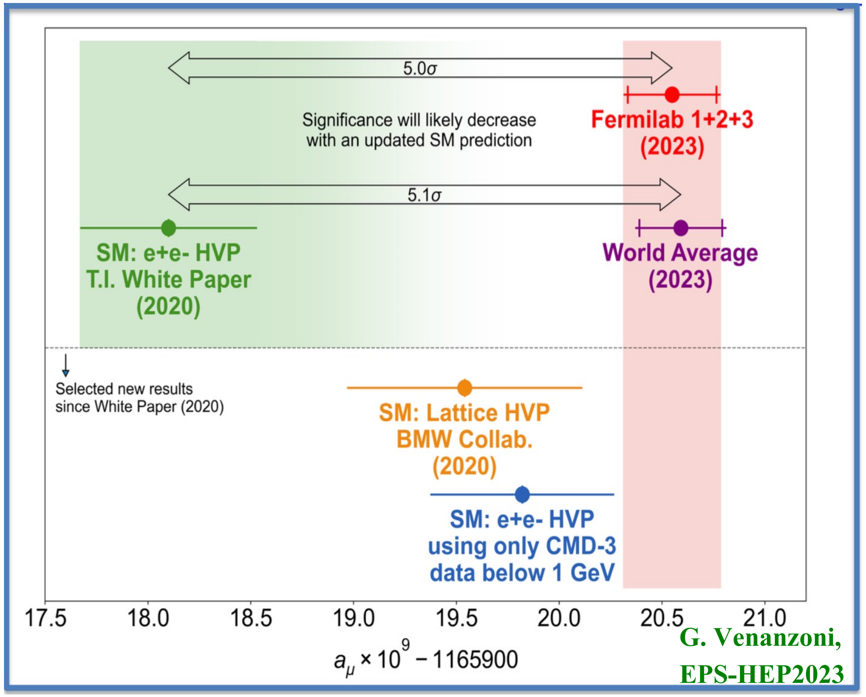


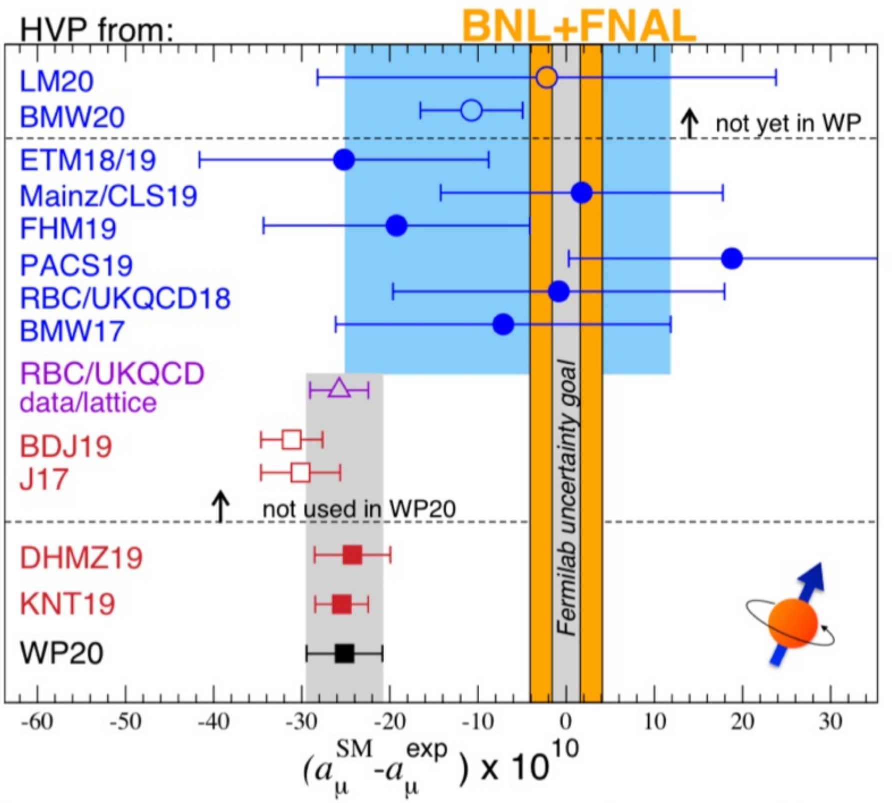
- The hadronic contribution is dominated by the Hadronic Vacuum Polarization and it is the leading order of the final theoretical uncertainty: we must be precise!

- The HVP can be computed theoretically from the data of $e^+e^- \rightarrow$ hadron and, from first principles, from Monte Carlo calculations on the lattice

Comparison with the Standard Model

$$a_\mu^{exp} = 116\,592\,059\,(22) \times 10^{-11}\;[190\,\mathrm{ppb}]$$





- The situation is quite uncertain on the theory side

HVP from Lattice QCD

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty K(t, m_\mu)\, G(t)\, dt$$

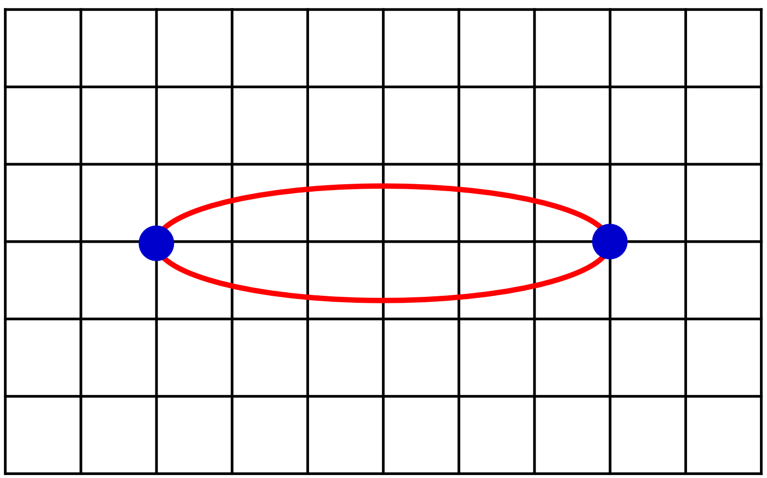
$\alpha, \; m_\mu$: electromagnetic coupling constant and the muon mass

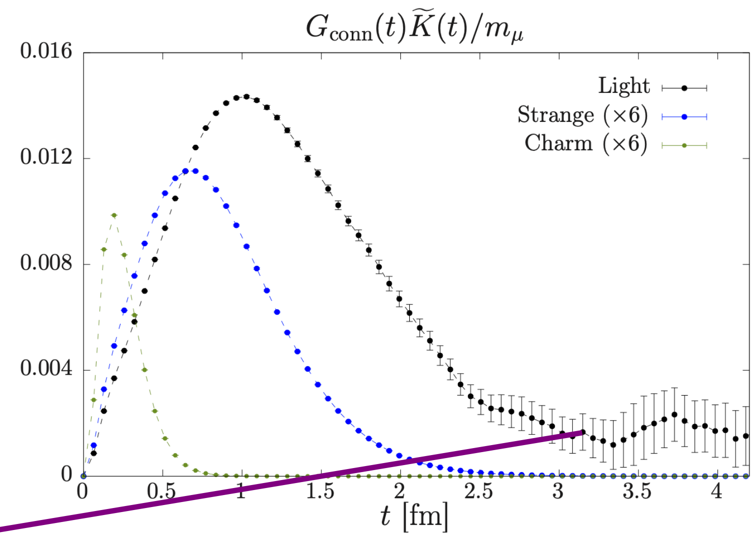
$K(t, m_\mu)$: known analytic function increasing as t^2 at large time

$$G(t) = \int d^3\vec{x}\, \langle J_k^{em}(\vec{x}, t) J_k^{em}(0) \rangle$$

$$J_k^{em} = i \sum_{f=1}^{N_f} q_i \overline{\psi} \gamma_k \psi$$

electric charge



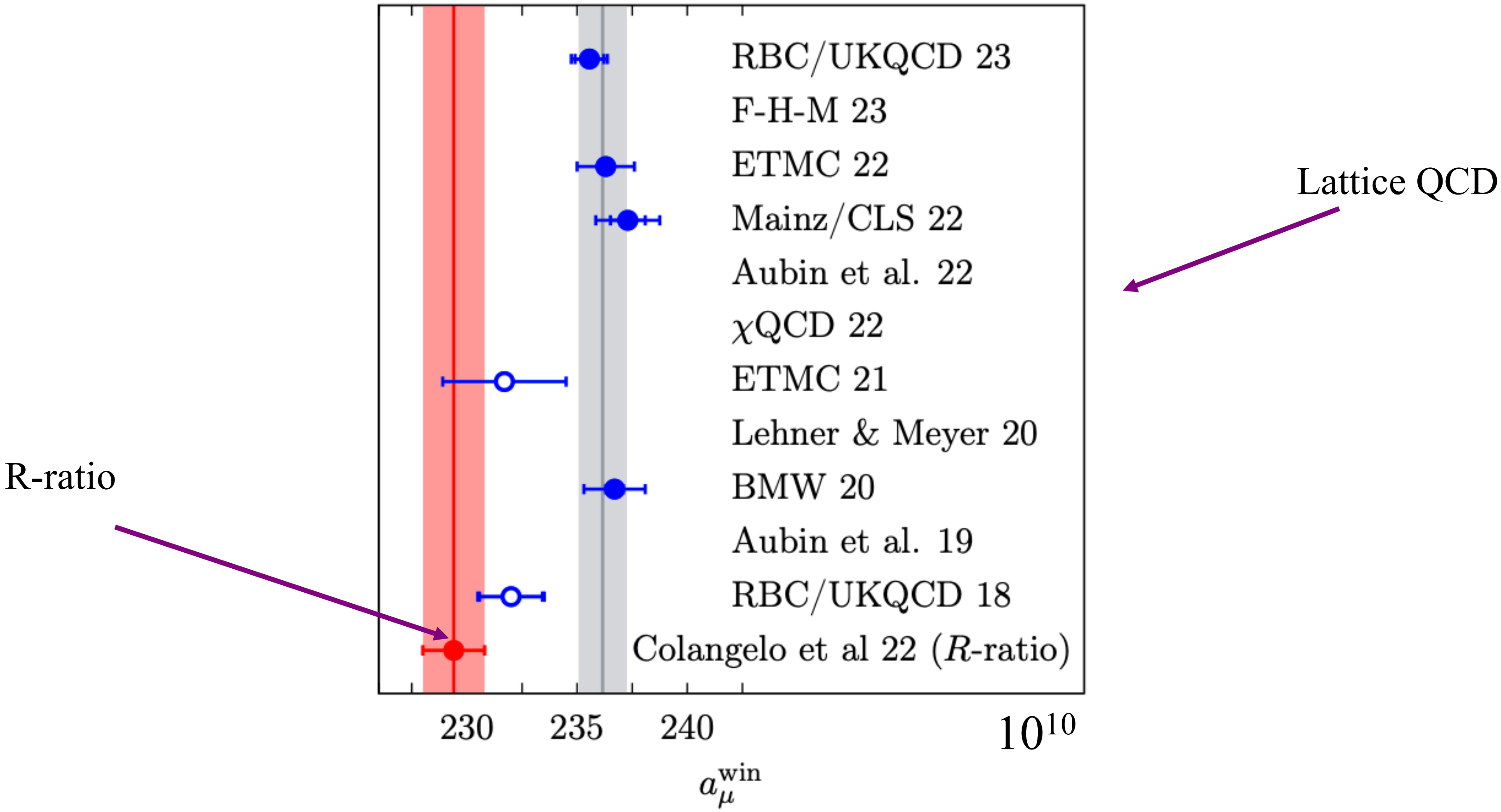


The signal/noise decreases exponentially with the time separation.

There are also disconnected contributions that are numerically very challenging

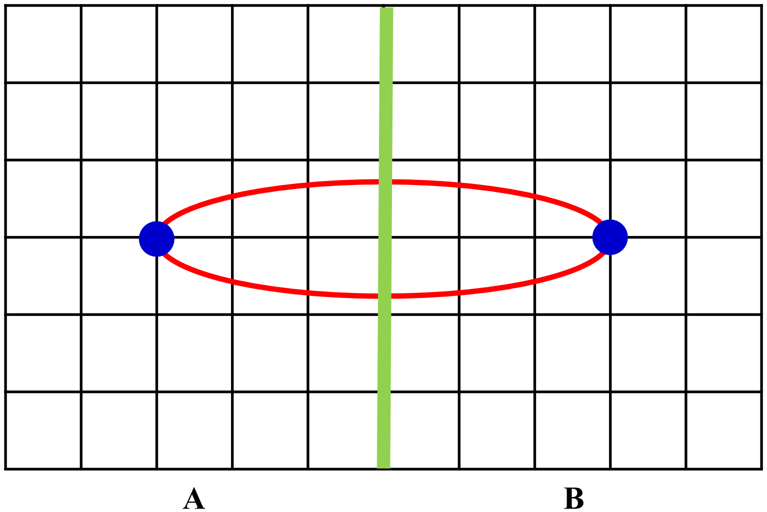
HVP from Lattice QCD: window observable

- A weight function is added to the integral to focus on the window 0.4 – 1 fm



There is a puzzling discrepancy, with the Lattice QCD computations is agreement

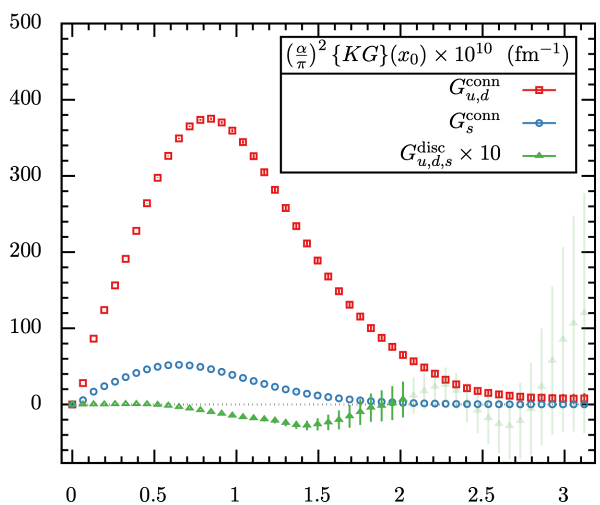
A call for an algorithmic improvement



M. Cè, L. Giusti, S.Schaefer
PRD 93 (2016) 094507
PRD 95 (2017) 034503

- Split the dynamics of the lattice into independent parts and combine them to improve the statistics with the same computational effort: N_{conf}^2

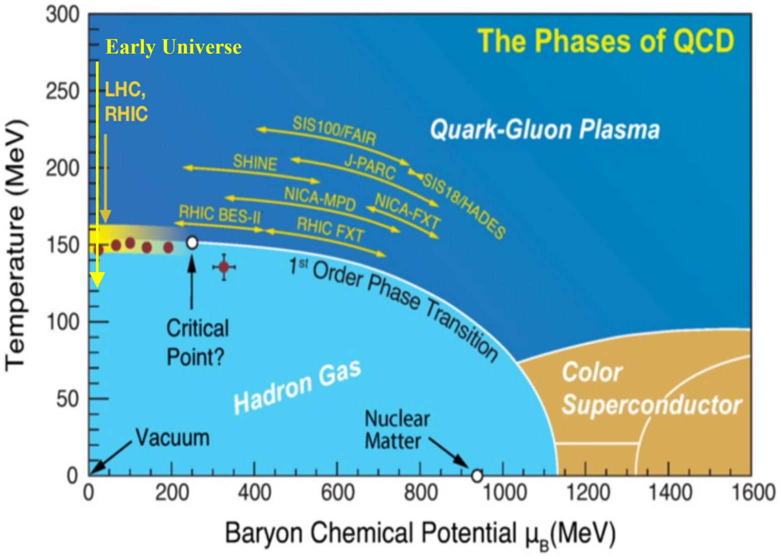
- Theoretically and algorithmically very difficult problem: factorization of the fermion determinant det(D) $\simeq$ det(D_A) det(D_B)

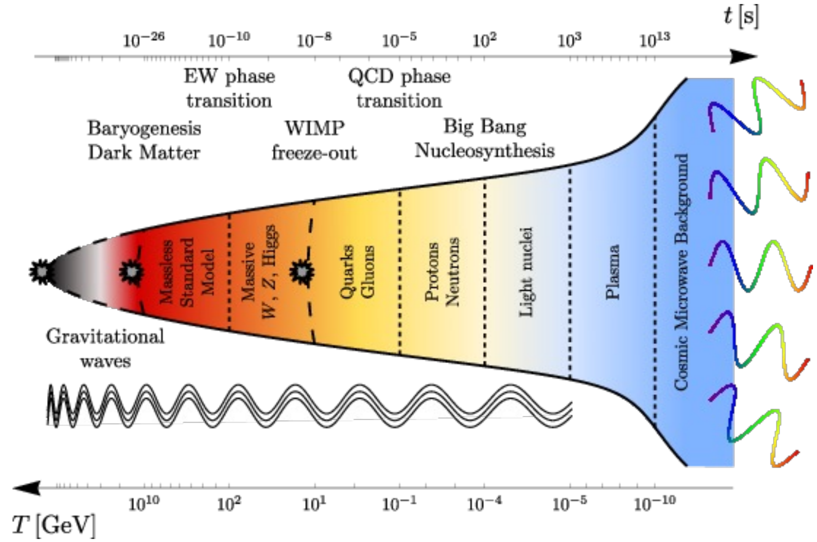


- The method has been applied in an exploratory lattice study ($m_\pi = 270MeV$) but it has turned out to work very efficiently
- Work in progress to improve efficiency to reach the physical point

M. Dalla Brida, L. Giusti, T. Harris, M. Pepe
PLB 816 (2021) 136191

QCD under Extreme Conditions





- QCD from elementary particle physics to extreme conditions (finite T and μ): intricate, collective, non-perturbative dynamics of Quark-Gluon Plasma emerges
- Relevant for heavy ion collisions, Nuclear Physics, AstroPhysics, Cosmology
- Monte Carlo simulations at finite density (net number of baryons) represent a formidable challenge:

$$S_F = \int_0^{1/T} dt \int d^3x\, \overline{\psi} \left[\gamma_\nu(\partial_\nu + iA_\nu) + \mu\gamma_4 + m\right]\psi = \int d^4x\, \overline{\psi} D(\mu)\psi$$

$$D^\dagger(\mu) = \gamma_5 D(-\mu^*)\gamma_5 \qquad \det(D^\dagger(\mu)) = \det(D(\mu))^* = \det(D(-\mu^*))$$

Dirac operator D is complex for real μ => no probabilistic interpretation

Imaginary μ or small values and then using Taylor expansion

Thermal QCD: from finite to High Temperatures

- Finite temperature: a new energy scale, T, influences QCD dynamics

Low T regime
essentially a gas of hadrons

Extremely high T regime
a plasma of free gluons and quarks

a lot of interesting physics takes place in-between:
chiral symmetry restoration, QGP, thermodynamics, early universe physics …

- Idea: when T goes large, the dynamics is ruled by a high-energy scale

asymptotic freedom: the gauge coupling g(T) is small ➔ attempt for a perturbative approach from the Stefan-Boltzmann limit

- Thermal QCD: temporal direction is compactified with size $\frac{1}{T}$ and when it becomes of the order or shorter of typical length scale of the system …

High Temperature Effective Field Theory

The theory becomes practically static: dimensional reduction

bosons: $p_0 = 2\pi n_0 T$

fermions: $p_0 = 2\pi(n_0 + \frac{1}{2})T$

Gauge non-zero modes are heavy and suppressed

Quarks are heavy and pick up a mass of order πT

- Gauge sector $S^g_{\mathrm{EFT}} = \frac{1}{g_E^2}\int d^3x\left[\frac{1}{2}\mathrm{Tr}\,(F_{ij}F_{ij}) + \mathrm{Tr}(D_jA_0)^2 + m_E^2\,\mathrm{Tr}\left(A_0^2\right)\right] + \ldots$

3d Yang-Mills theory with gauge coupling $g_E^2 = g^2T$.

3d scalar field in the adjoint rep. with mass $m_E \sim gT$.

- At very high T the adjoint scalar field A_0 decouples and one is left with a 3d Yang-Mills theory: non-perturbative effects are important even if $g^2(T)$ is small

High Temperature Effective Field Theory

- Quarks are at the lowest Matsubara mode $m_q = \pi T$

$$S^q_{EFT} = \int d^3x \Big\{ i\chi^\dagger \Big[m_q - g_E A_0 + D_3 - \frac{1}{2m_q}\Big(D_k^2 + \frac{g_E}{4i}[\sigma_k, \sigma_l] F_{kl} \Big) \Big] \chi$$
$$+ i\phi^\dagger \Big[m_q - g_E A_0 - D_3 - \frac{1}{2m_q}\Big(D_k^2 + \frac{g_{\mathrm{E}}}{4i}[\sigma_k, \sigma_l] F_{kl} \Big) \Big] \phi \Big\} + \ldots$$

spin-dependent

- At high T a hierarchy of 3 energy scales shows up

$$\pi T \gg gT \gg g^2 T$$

and non-perturbative effects can be relevant

Thermal QCD: non-perturbative approach

- Perturbatively based approaches have been widely used to have information on the behaviour of QCD at finite temperature: thermodynamics, screening masses, …

E. Braaten, L. Yaffe, L. McLerran,
R. Pisarski, S. Huang, M. Lissia,
J._P. Blaziot, E. Iancu, M. Laine, …

- Method of choice: Lattice QCD, from first principles and fully non-perturbative

BUT

Equation of State

G. Boyd et al.
NPB 469 (1996) 419

$$\frac{T_{\mu\mu}}{T^4} = T\frac{d}{dT}\left(\frac{p}{T^4}\right)$$

successful but it needs subtraction
at two scales: 0 and T, T/2 and T

A. Bazavov et al.
PRD 97 (2018) 014510

T_{max} about 2 Gev

Lines of Constant Physics

parameters unknown to perform
Monte Carlo simulations: $a \leftrightarrow g_0$.

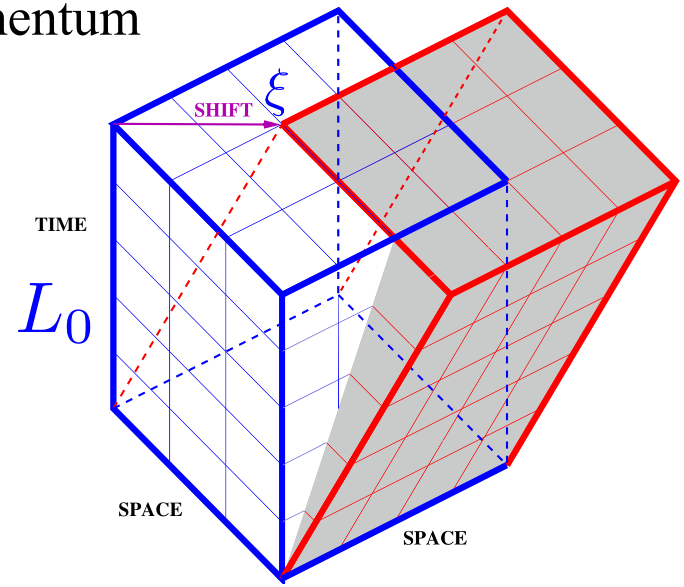
Screening masses

A. Bazavov et al.
PRD 100 (2019) 094510

T_{max} about 1 Gev
(no cont. limit up to 2.5 GeV)

! Quite limited range in temperature, especially taking into account the 1/log expected dependence of the gauge coupling on the temperature.

QCD at High Temperature: the moving frame



Thermodynamics: trace anomaly of the energy-momentum tensor $T_{\mu\nu}$ needs subtraction at two energy scales

L. Giusti and H. Meyer, PRL 2011, JHEP 2011 and 2013

L. Giusti and M. Pepe, PRL 2014, PRD 2015, PLB 2017

M. Dalla Brida, L. Giusti and M. Pepe, JHEP 2020

- Thermal theory in a moving reference frame

$$Z(L_0, \xi) = \mathrm{Tr}\left[e^{-L_0(H-\xi_k T_{0k})}\right]$$

That corresponds to introducing a shift ξ when closing the p.b.c. along the temporal direction

By Lorentz invariance the free-energy is given by

$$f\left(L_0\sqrt{1+\xi^2}\right) = -\lim_{V\to\infty} \frac{1}{L_0 V} \log Z(L_0, V, \vec{\xi})$$

the temperature of the system is given by $T = \dfrac{1}{L_0\sqrt{1+\xi^2}}$

- theoretical and numerical challenges are overcome

$$U_\mu(x_0 + L_0, \mathbf{x}) = U_\mu(x_0, \mathbf{x} - L_0\xi)$$

$$\psi(x_0 + L_0, \mathbf{x}) = -\psi(x_0, \mathbf{x} - L_0\xi)$$

$$\bar{\psi}(x_0 + L_0, \mathbf{x}) = -\bar{\psi}(x_0, \mathbf{x} - L_0\xi)$$

Not necessary to compute screening masses, Crucial for thermodynamics, no need to work at two different temperatures

Mesonic screening masses

- They characterize the behaviour of spatial 2-point functions

$$C_{\mathcal{O}}(x_3) = \int dx_0 dx_1 dx_2 \, \langle \mathcal{O}^a(x) \mathcal{O}^a(0) \rangle \sim \mathrm{e}^{-m_{\mathcal{O}} x_3}$$

in which the fermionic bilinear operators are

$$\mathcal{O}^a(x) = \overline{\psi}(x) \Gamma_{\mathcal{O}} T^a \psi(x) \qquad \text{where} \qquad \Gamma_{\mathcal{O}} = \{ 1\!\!1, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5 \}$$

Flavour non-singlet mesons: T^a are the generators of $SU(N_f)$

- response of the system to the insertion of $\mathcal{O}^a$
- restoration of chiral symmetry
- numerically simple to compute non-perturbatively on the lattice
- comparison with EFT: computed at 1-loop order in high-T perturbation theory

$$m_{PT} = 2\pi T \, (1 + 0.032739961 \, g^2)$$

! No dependence on $\Gamma_{\mathcal{O}}$

M. Laine and M. Vepsäläinen, JHEP 02 (2004) 004

T.H. Hansson and I. Zahed, NPB 374 (1992) 277

The numerical study

M. Dalla Brida, L. Giusti, T. Harris, D. Laudicina, M. Pepe
JHEP 04 (2022) 034

- QCD on the lattice with $N_f = 3$ quarks in the chiral limit

- reduced lattice artifacts: $O(a)$ - improved Wilson fermions

- continuum limit extrapolation: $L_0/a = 4, 6, 8, 10$

- large spatial volumes to have finite volume effects under control: $LT \sim 20 - 50$

- shifted boundary conditions

- 12 values of the temperature in the range $1.167 - 164.6$ GeV

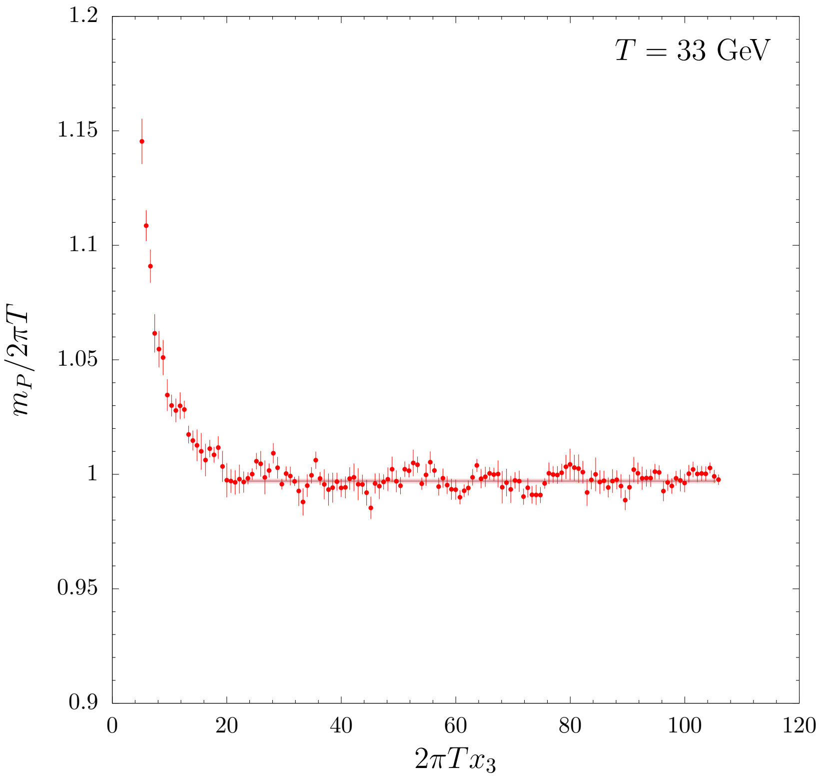
T	$T(\mathrm{GeV})$
T_0	164.6(5.6)
T_1	82.3(2.8)
T_2	51.4(1.7)
T_3	32.8(1.0)
T_4	20.63(63)
T_5	12.77(37)
T_6	8.03(22)
T_7	4.91(13)
T_8	3.040(78)
T_9	2.833(68)
T_{10}	1.821(39)
T_{11}	1.167(23)

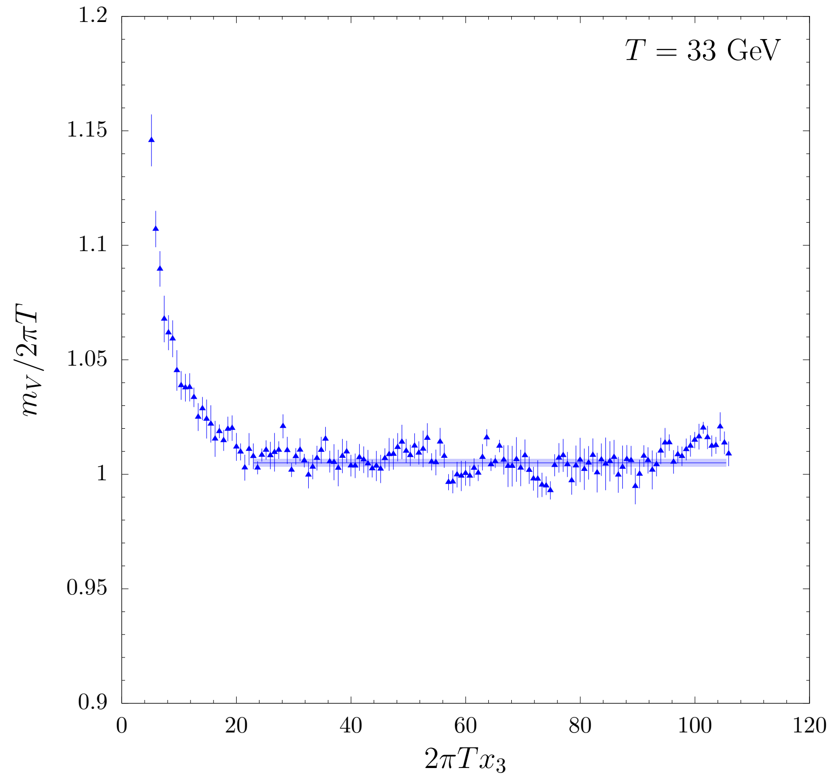
Measure of the screening masses

- Masses are obtained form the 2-point functions at nearby points

$$m_{\mathcal{O}} = \frac{1}{a}\,\mathrm{arcCosh}\left[\frac{C_{\mathcal{O}}(x_3+a)+C_{\mathcal{O}}(x_3-a)}{2C_{\mathcal{O}}(x_3)}\right]$$

spatial size
L/a=288



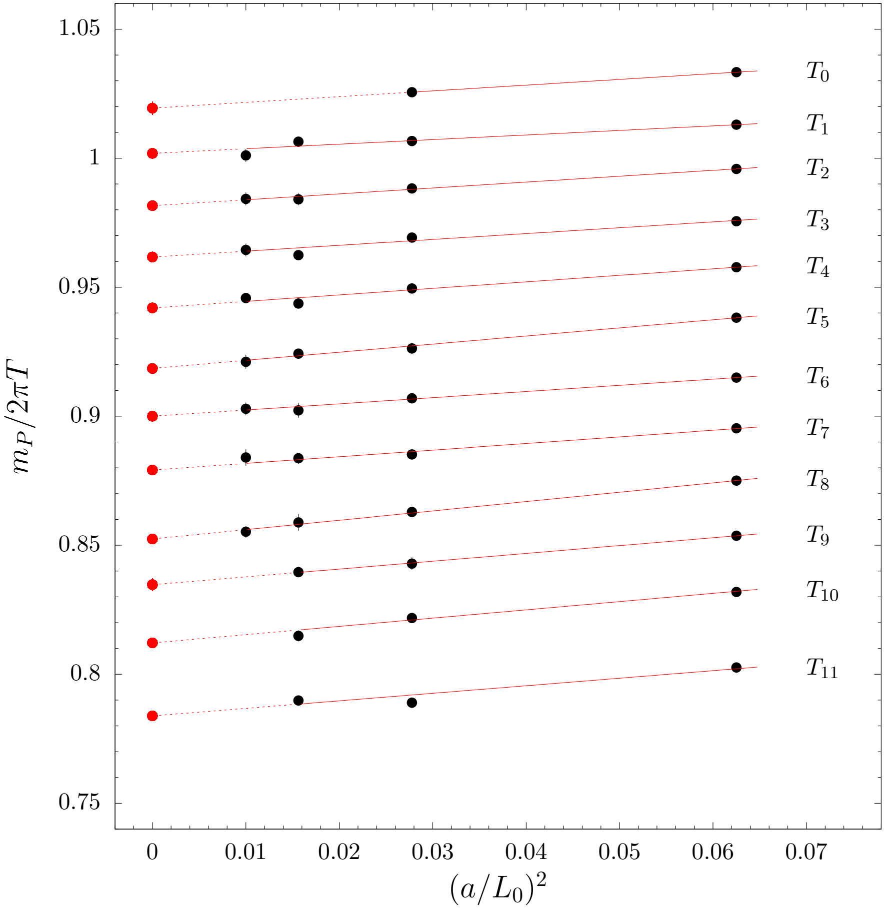


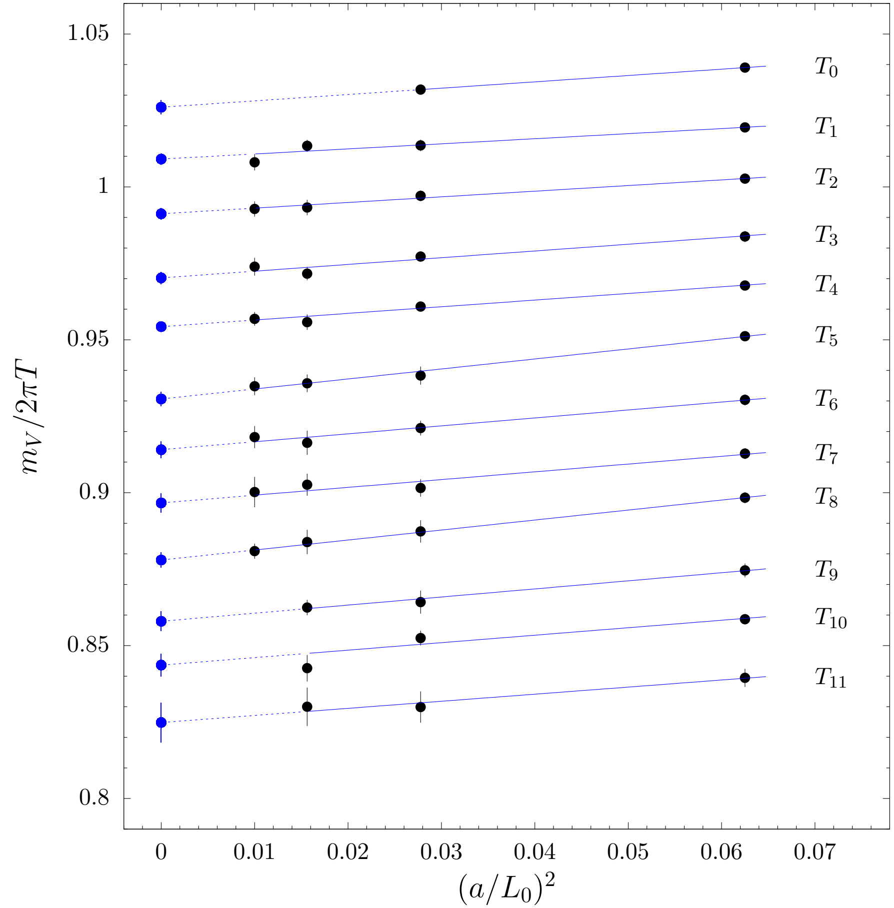
Long plateaux allow to estimate the masses with high accuracy

No contamination from excited states

Continuum Limit: pseudoscalar and vector masses

- Masses are measured, at fixed physical temperature T, for several values of the lattice spacing $L_0/a = 4, 6, 8, 10$ and then extrapolated to the continuum limit
- The procedure has been repeated at the 12 physical temperatures





data of T_i shifted down by 0.02 i

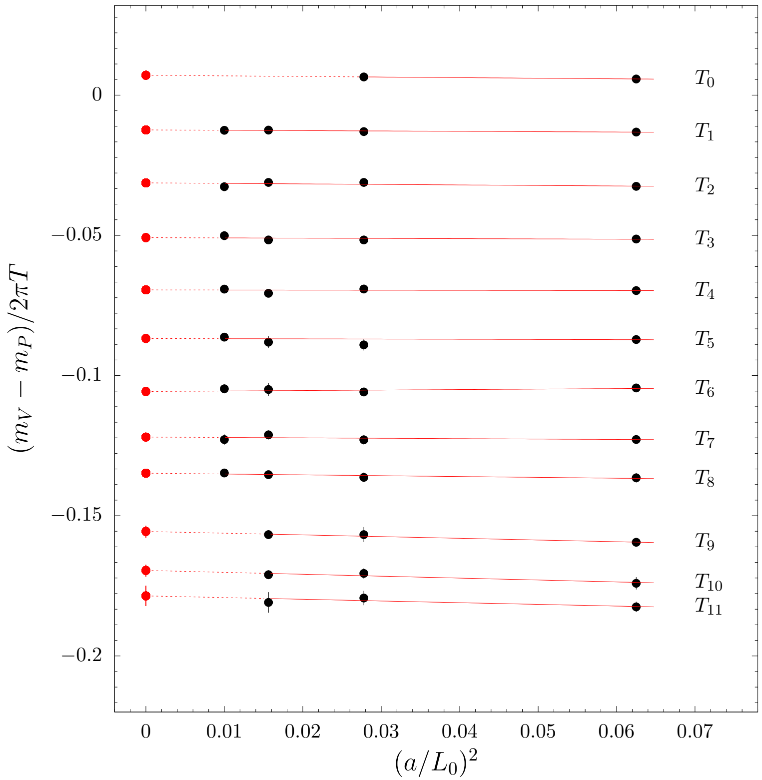
a few ‰ final accuracy

Remark: spin effects

- Mass-splitting between P and V is due to spin effects
- Such effects do not show up at 1-loop order in High-T PT

V. Koch et al. PRD 46 (1992) 3169, PRD 47 (1993) 2157

T.H. Hansson and I. Zahed, NPB 374 (1992) 277



data of T_i shifted down by 0.02 i

Evidence of a mass-splitting between Pseudoscalar and Vector screening masses

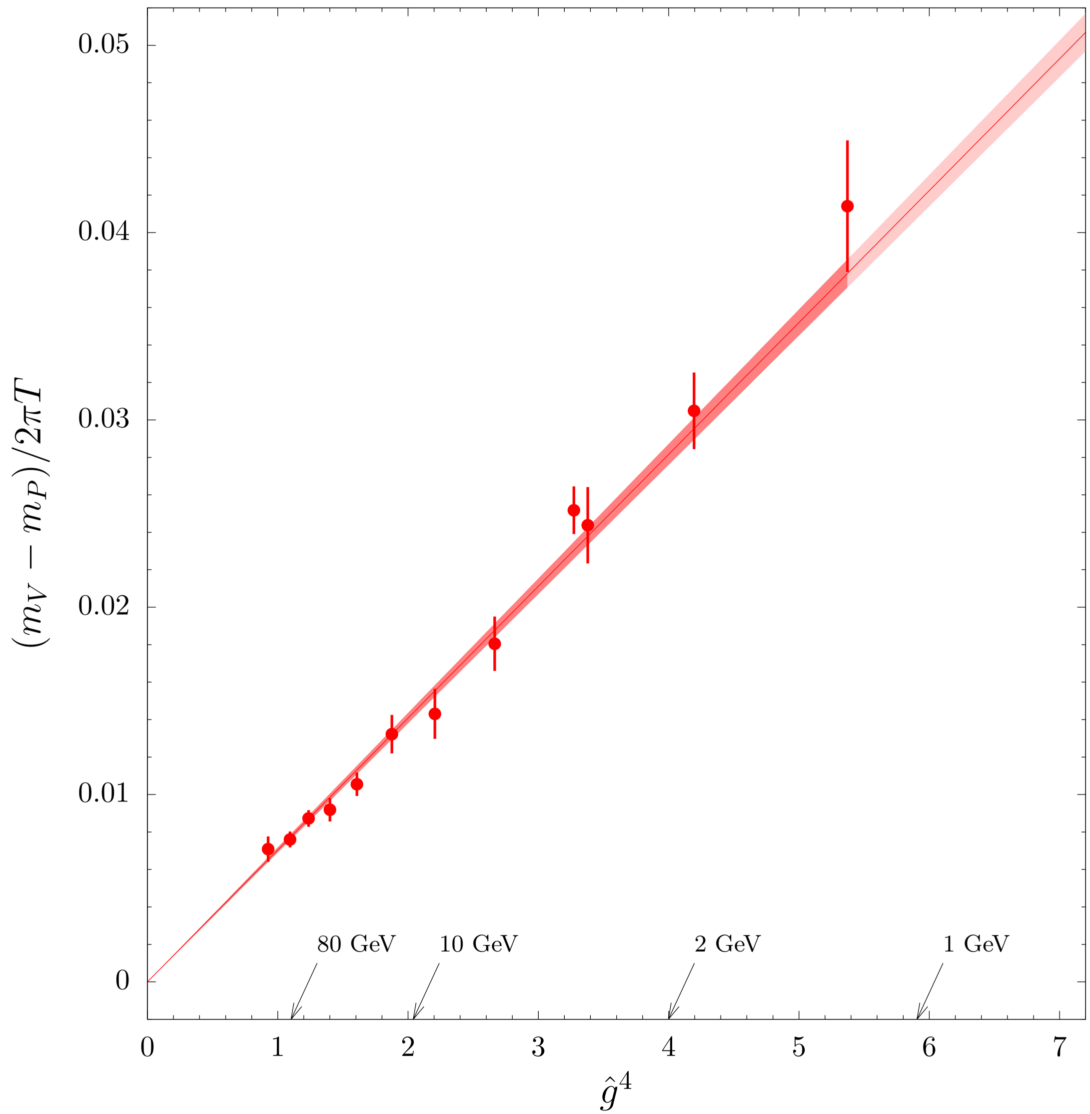
- Barely visible lattice artifacts
- Check1. Exclude $L_0/a = 4$ in the fits: excellent agreement in the CL values
- Check2. Fits with $(\frac{a}{L_0})^2 \ln(\frac{a}{L_0})$ and $(\frac{a}{L_0})^3$ but the coefficients are compatible with 0

The mass splitting

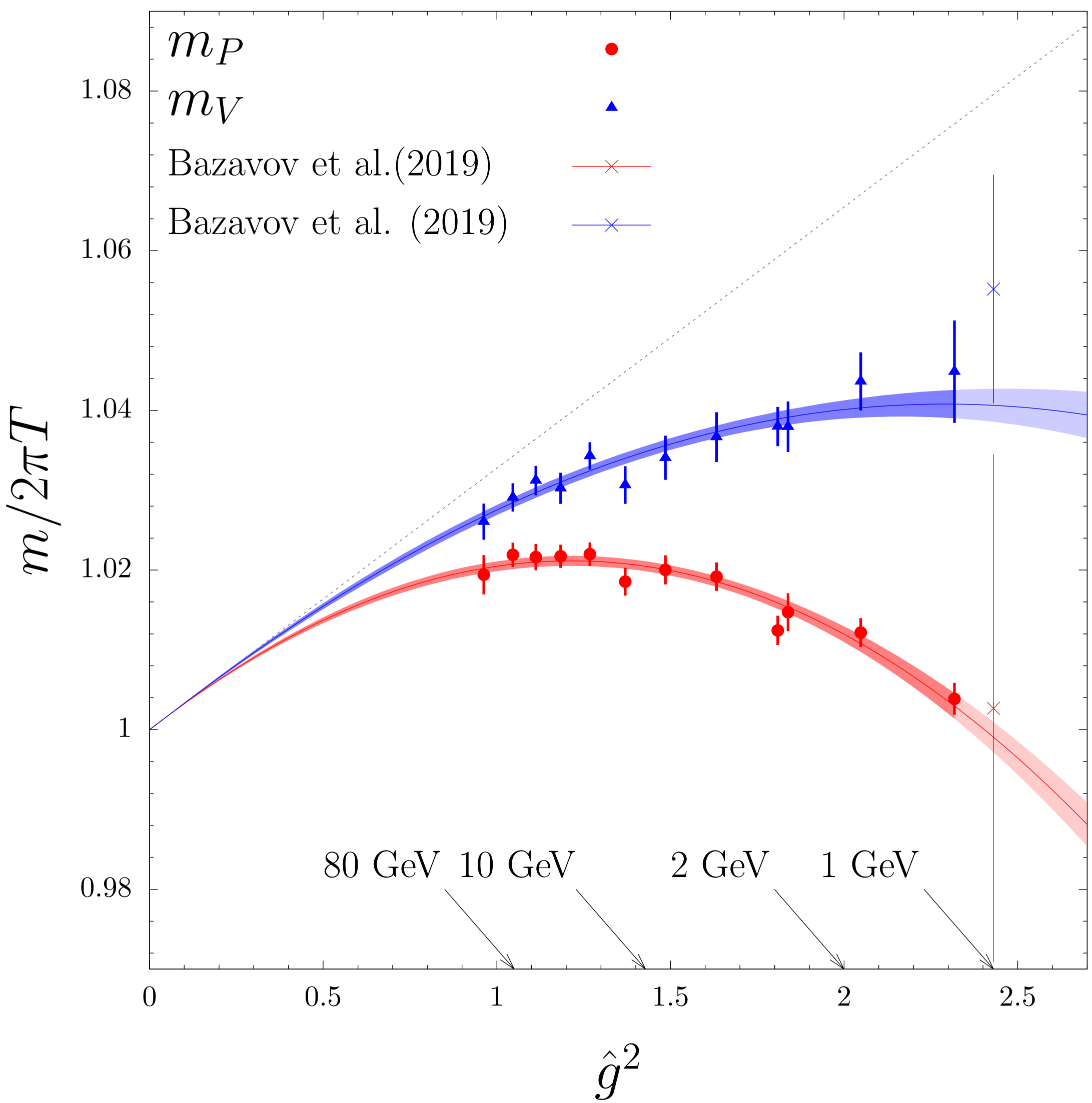
- No prediction for such a term in 1-loop PT

Non-perturbative data look well fit by a $\hat{g}^4(T)$ term over two order of magnitude in T

mass-splitting clearly visible up to the Electro-Weak scale



The temperature dependence of mesonic screening masses



- A few % away from the $T \to \infty$ limit
- Mass-splitting visible up to T ~165 GeV
- Masses not compatible with 1-loop PT up to 165 GeV
- Terms of order $\hat{g}^4$ partially compensate for m_V
- At T ~1 GeV, m_V deviates from $2\pi T$ only for spin effects.

$$\frac{1}{\hat{g}^2(T)} \equiv \frac{9}{8\pi^2} \ln \frac{2\pi T}{\Lambda_{\overline{\rm MS}}} + \frac{4}{9\pi^2} \ln \left(2 \ln \frac{2\pi T}{\Lambda_{\overline{\rm MS}}} \right) ,$$

$\Lambda_{\overline{\rm MS}} = 341$ MeV

Baryonic screening masses

(work in final stage)

- Practically no results, numerically challenging: old results at low T, finite a

S.A. Gottlieb et al., PRL 59 (1987) 1881

- We have the nucleon and anti-nucleon operators:

$$N(x) = \epsilon_{abc}\left[u_a^T(x) C\gamma_5 d_b(x)\right] d_c(x)$$

$$\overline{N}(x) = \epsilon_{abc}\,\overline{d}_c(x)\left[\overline{d}_b(x) C\gamma_5 \overline{u}_a^T(x)\right]$$

$$C = \gamma_0\gamma_2$$

- Baryonic screening masses characterize the behaviour of spatial 2-point functions

$$C_{N^\pm}(x_3) = \int dx_0 dx_1 dx_2\, \mathrm{e}^{i\frac{x_0}{L_0}\pi} \langle P_\pm N(x)\overline{N}(0)\rangle \sim \mathrm{e}^{-m_{N^\pm} x_3}$$

$P_\pm = \dfrac{1\pm\gamma_3}{2}$: projectors on the positive/negative x_3 – parity states

- Thermal QCD: periodicity in x_0 fermions: $p_0 = 2\pi(n_0 + \frac{1}{2})T$

$\mathrm{e}^{i\frac{x_0}{L_0}\pi}$: N is a fermion, anti-periodic b.c.; projection on the lowest mode

- Due to shifted b.c.

$$C_{N^\pm}(x_3) = \int dx_0 dx_1 dx_2\, \mathrm{e}^{i\frac{x_0+\xi x_1}{L_0}\gamma^2\pi} \langle P_\pm N(x)\overline{N}(0)\rangle \sim \mathrm{e}^{-m_{N^\pm} x_3} \qquad \frac{1}{\gamma} = \sqrt{1+\xi^2}$$

The numerical study

- We have considered the same tempeartures as for mesonic screening masses $1.167 - 164.6$ GeV

- QCD on the lattice with $N_f = 3$ quarks in the chiral limit

- reduced lattice artifacts: $O(a)$ - improved Wilson fermions

- large spatial volumes to have finite volume effects under control: $LT \sim 20 - 50$

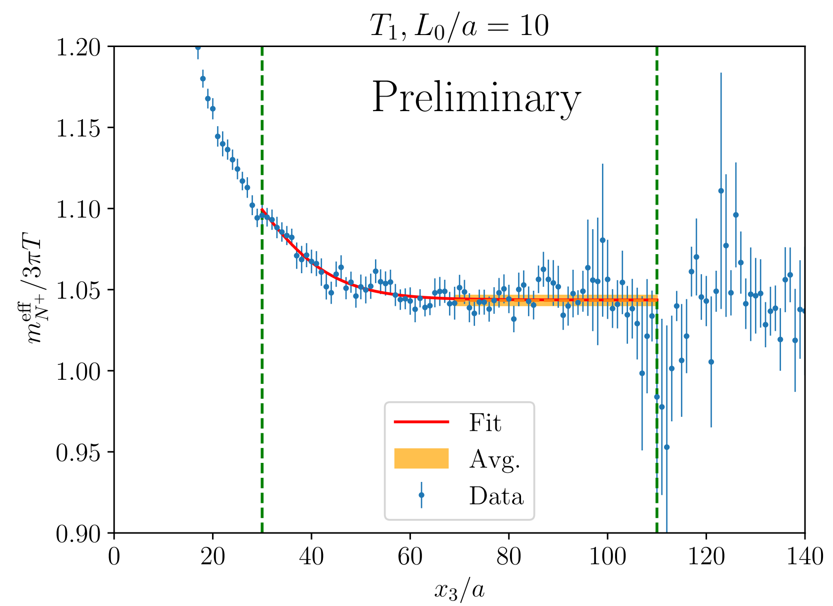
- Screening masses are obtained using:

$$m_{N^\pm} = -\frac{1}{a}\log\frac{C_{N^\pm}(x_3+a)}{C_{N^\pm}(x_3)}$$

In the limit T →∞ (Stefan-Boltzmann limit):

$$m_{N^\pm} \to 3\pi T$$

T	T(GeV)
T_0	164.6(5.6)
T_1	82.3(2.8)
T_2	51.4(1.7)
T_3	32.8(1.0)
T_4	20.63(63)
T_5	12.77(37)
T_6	8.03(22)
T_7	4.91(13)
T_8	3.040(78)
T_9	2.833(68)
T_{10}	1.821(39)
T_{11}	1.167(23)



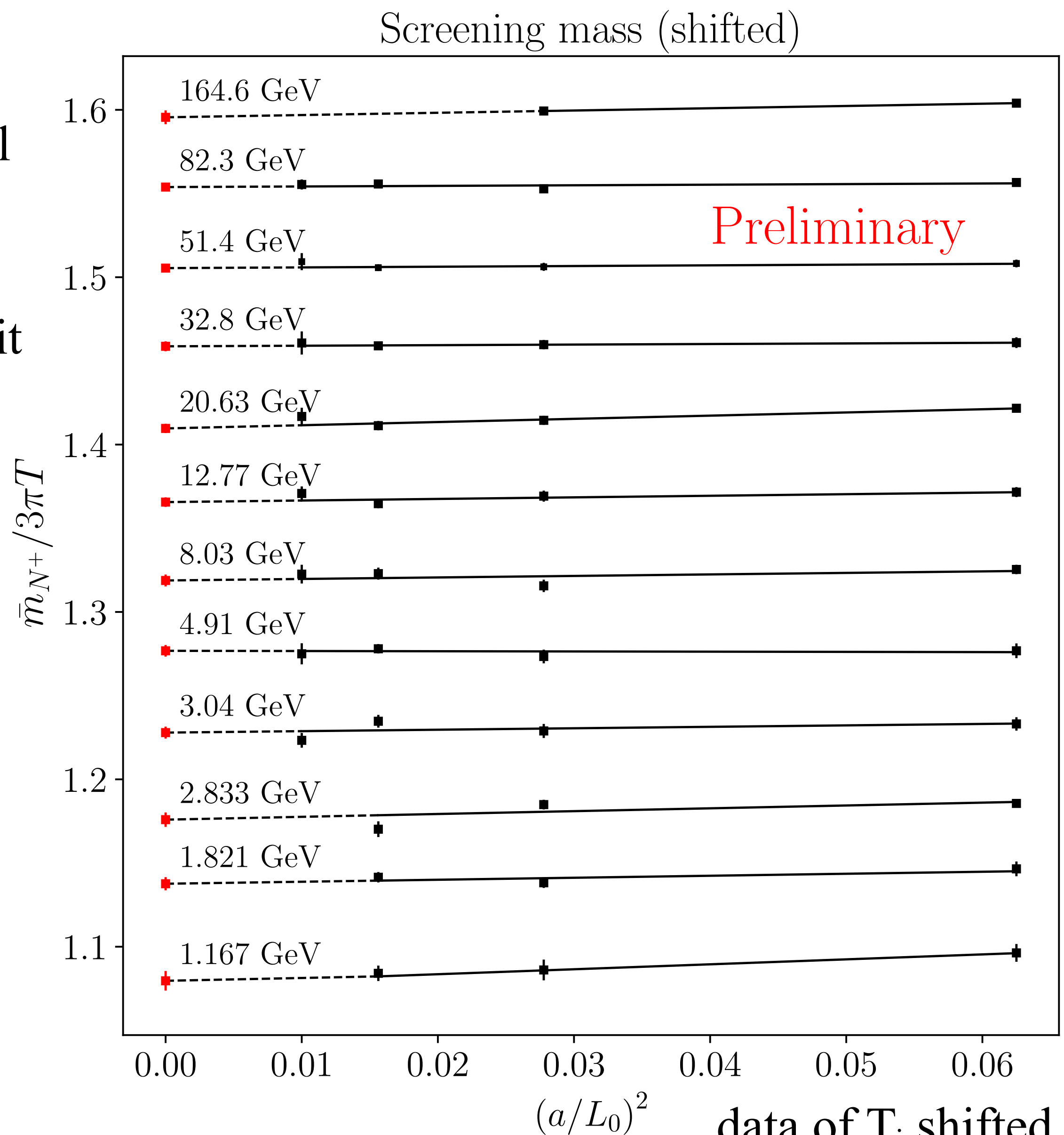
Continuum Limit: baryonic screening mass

- Masses are measured, at fixed physical temperature T, for several values of the lattice spacing $L_0/a = 4, 6, 8, 10$ and then extrapolated to the continuum limit

- The procedure has been repeated at the 12 physical temperatures

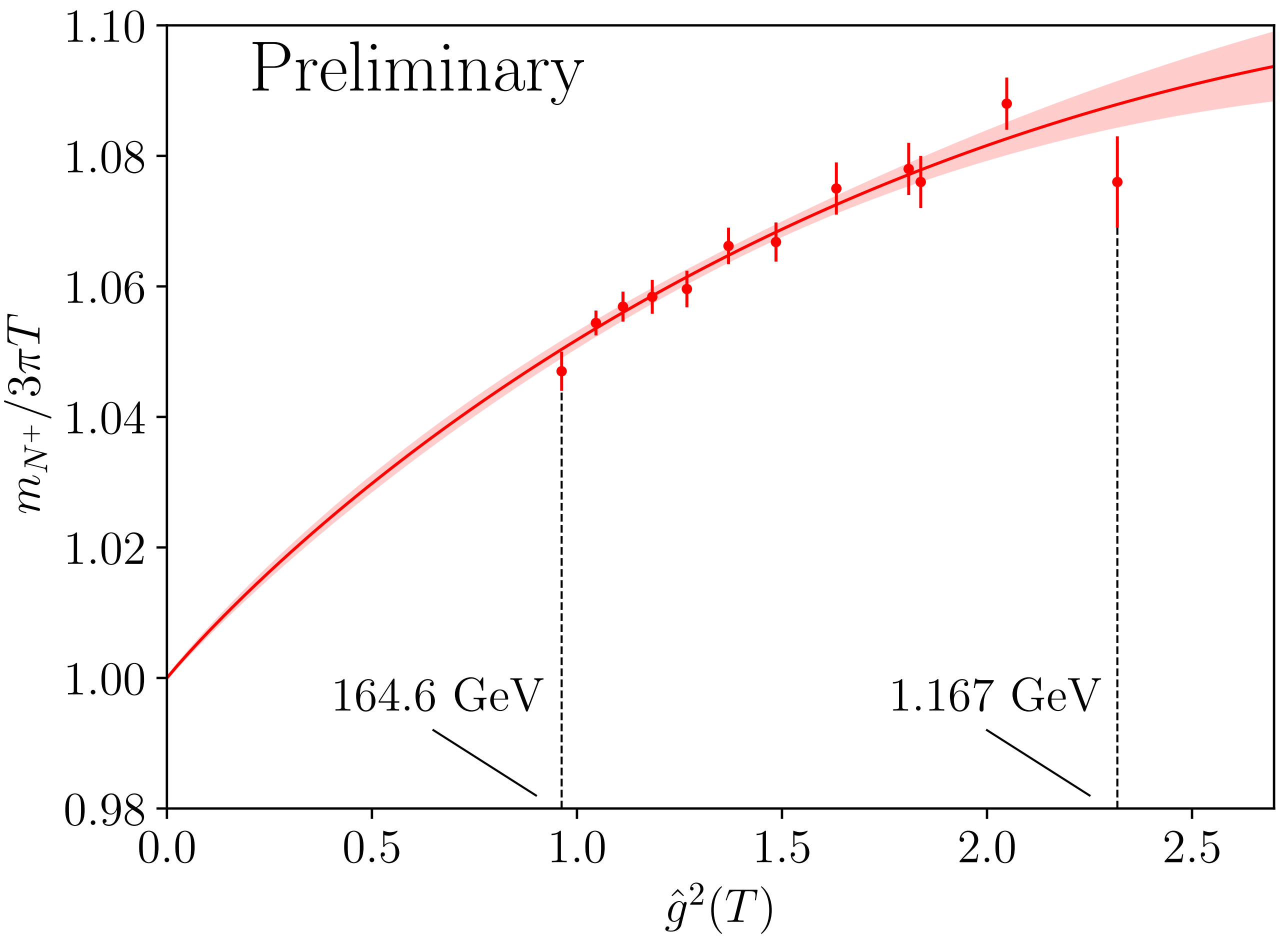
Sub-percent final accuracy, in most cases 0.5 %

- The splitting $(m_{N^+} - m_{N^-})$ is compatible with 0 at all temperatures and for all lattice spacings (chiral symmetry restoration)



data of T_i shifted up by 0.05 (11-i)

The temperature dependence of baryonic screening mass



- Bulk of the screening mass comes from the free theory but…
- A 4-8 % away from the $T \to \infty$ limit up to very high T
- Red line is a fit to a cubic polynomial

$$\frac{1}{\hat{g}^2(T)} \equiv \frac{9}{8\pi^2} \ln \frac{2\pi T}{\Lambda_{\overline{\mathrm{MS}}}} + \frac{4}{9\pi^2} \ln\left(2 \ln \frac{2\pi T}{\Lambda_{\overline{\mathrm{MS}}}}\right) ,$$

$$\Lambda_{\overline{\mathrm{MS}}} = 341 \text{ MeV}$$

The Equation of State

- The energy-momentum tensor $T_{\mu\nu}$□is a very important quantity for a QFT: it is connected to the translation, rotation and dilatation symmetries.

- The expectation values of its matrix elements are directly related to physical quantities like pressure, entropy and energy density, transport coefficients.

- Non-perturbative physics: $T_{\mu\nu}$□□□□□□□□ □□□□□□□□

S. Caracciolo et al.,
NPB 375 (1992) 195;
Ann. Phys. 197 (1990) 119

<u>Continuum</u>

$$T_{\mu\nu}$$

two-index symmetric tensor: {10} rep of SO(4)

<u>Lattice</u>

$$T^R_{\mu\nu} = Z\, T^{[6]}_{\mu\nu} + z\, T^{[3]}_{\mu\nu} + \beta\, T^{[1]}_{\mu\nu}$$

{10} ➜ {6} + {3} + {1} of H(4)

- Z and z: ren. constants whose computation is the non-perturbative definition of $T_{\mu\nu}$□

L. Giusti and M. Pepe,
PRD 2015

M. Dalla Brida, L. Giusti
and M. Pepe, JHEP 2020

$$\langle T^{R,[6]}_{0k}\rangle_\xi = Z_G(g_0^2)\, \langle T^{G,[6]}_{\mu\nu}\rangle_\xi + Z_F(g_0^2)\, \langle T^{F,[6]}_{\mu\nu}\rangle_\xi = -\frac{\partial}{\partial \xi_k} f[\xi] = \frac{1}{L_0 V}\frac{\partial}{\partial \xi_k}\log \mathcal{Z}[\xi]$$

$$Z_G(g_0^2)\, \langle T^{G,[6]}_{\mu\nu}\rangle_\xi + Z_F(g_0^2)\, \langle T^{F,[6]}_{\mu\nu}\rangle_\xi = \frac{\xi_k}{1-\xi_k^2}\Big\{ z_G(g_0^2)\langle T^{G,[3]}_{00} - T^{G,[3]}_{kk}\rangle_\xi + z_F(g_0^2)\langle T^{F,[3]}_{00} - T^{F,[3]}_{kk}\rangle_\xi \Big\}$$

- EoS is usually obtained from the direct measurement of the trace anomaly: needs subtraction at two energy scales e.g. 0 and T (very different!)

G. Boyd et al.,
Nucl. Phys. B469 (1996) 419

QFT in a moving frame

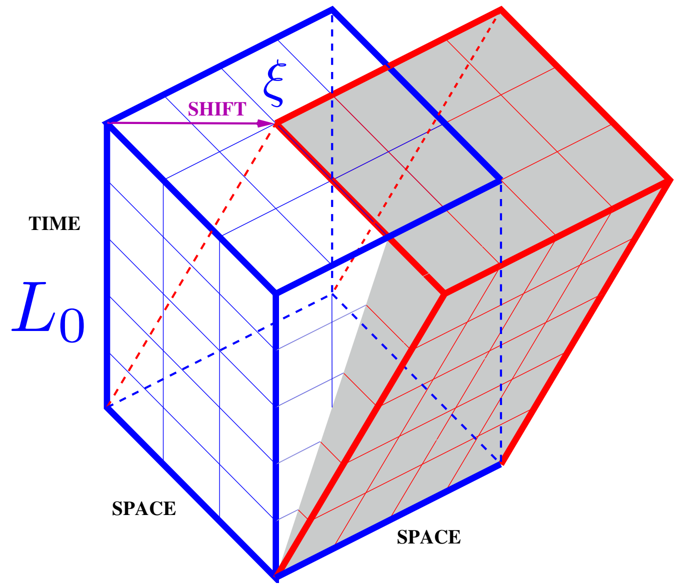
- The entropy density is the primary observable

$$\frac{s(T)}{T^3} = \frac{L_0^4(1+\xi^2)^3}{\xi_k}\left[Z_G\langle T_{0k}^{G,[6]}\rangle_\xi + Z_F\langle T_{0k}^{F,[6]}\rangle_\xi\right]$$

s = entropy density; e = energy density;
p = pressure; T = temperature

$$p(T) - p(T_0) = \int_{T_0}^{T} s(T)dT$$

$$\varepsilon(T) = Ts(T) - p(T) \qquad \text{or} \qquad \varepsilon(T) - \varepsilon(T_0) = \int_{s_0}^{s} T(s)ds$$



- The energy-momentum tensor on the lattice:

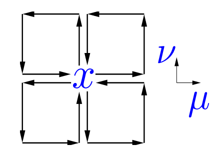
gluonic part

$$T^G_{\mu\nu}(x) = \frac{1}{g_0^2}F^a_{\mu\rho}(x)F^a_{\nu\rho}(x) - \delta_{\mu\nu}\mathcal{L}^G(x) \qquad F^a_{\mu\nu}(x) = 2\,\mathrm{Tr}[F_{\mu\nu}(x)T^a]$$

$$\mathcal{L}^G(x) = \frac{1}{4g_0^2}F^a_{\rho\sigma}(x)F^a_{\rho\sigma}(x)$$

$$F_{\mu\nu}(x) = \frac{1}{8}[Q_{\mu\nu}(x) - Q_{\nu\mu}(x)]$$

$$Q_{\mu\nu}(x) = \sum$$



fermionic part

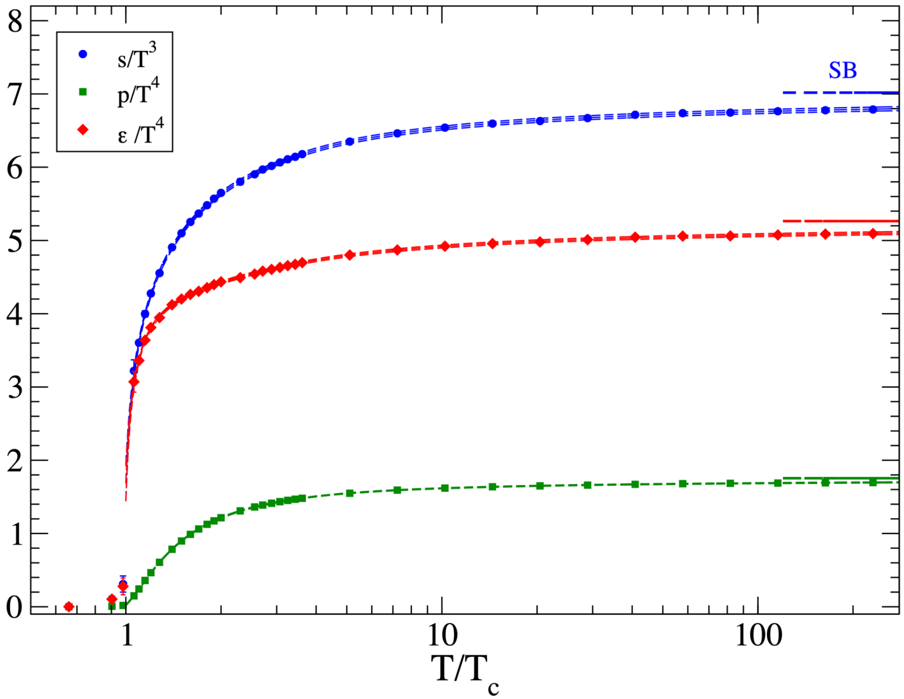
$$T^F_{\mu\nu}(x) = \frac{1}{2}\left\{\overline{\psi}(x)\gamma_\mu\overleftrightarrow{\nabla}_\nu\psi(x) + \overline{\psi}(x)\gamma_\nu\overleftrightarrow{\nabla}_\mu\psi(x)\right\} - \delta_{\mu\nu}\mathcal{L}^F(x)$$

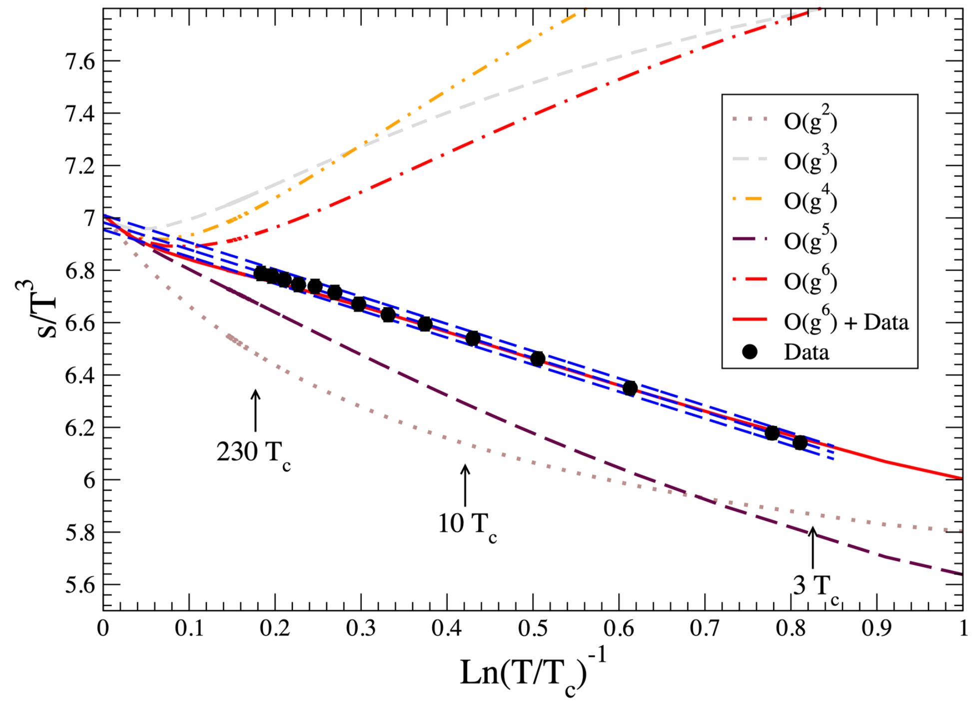
$$\mathcal{L}^F(x) = \sum_\mu \overline{\psi}(x)\left\{\gamma_\mu\overleftrightarrow{\nabla}_\mu + M_0\right\}\psi(x)$$

The Equation of State of SU(3) Yang-Mills theory

L. Giusti and M. Pepe, PLB 2017

- $s(T)/T^3$ is extrapolated to the continuum limit in an independent way at every T
- 35 temperatures in the range 0.6 T_c- 230 T_c have been considered
- at every temperature, numerical simulations on lattices with temporal extension $L_0 = 5, 6, 7, 8$ and sometimes $L_0 = 3, 4, 10$; $21 < L/L_0 < 26$





Final accuracy of 0.5% in the continuum limit

High-T perturbative computation is not reliable: non-perturbative physics is relevant

$$p_{pert} = \frac{8\pi^2}{45} T^4 \left[\sum_{i=0}^{6} p_i g^i \right]$$

K. Kajantie et al., PRD 67 (2003) 105008.

The Equation of State of QCD (work in progress)

- QCD EoS: up to T~ 1-2 GeV using staggered quarks

  up to T~ 600 MeV using Wilson quarks

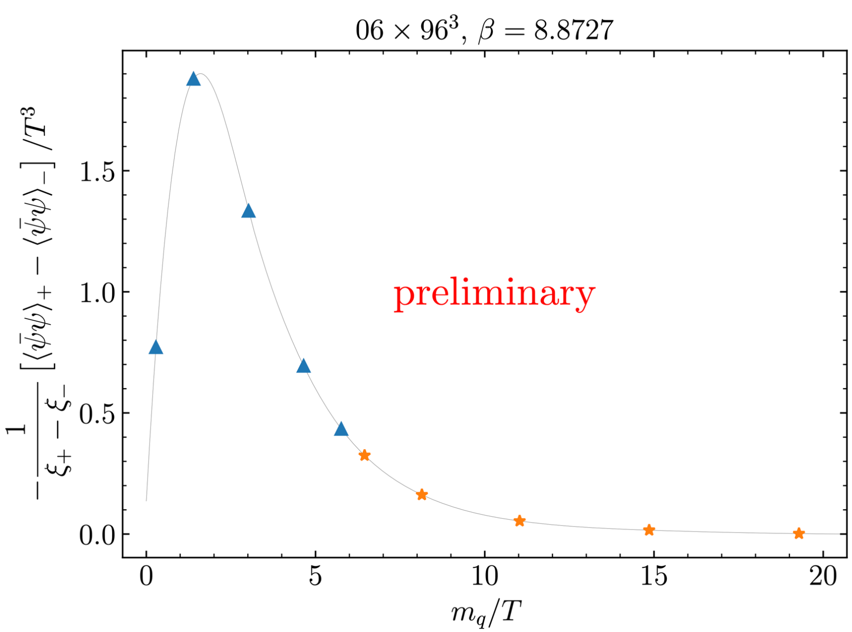
numerically very demanding

Budapest-Wuppertal Collab., 2014
A. Bazavov, P. Petreczky, J. Weber, PRD 2018
HOT-QCD Collab., PRD 2014
MILC Collab., PRD 2014
WHOT-QCD Collab., PRD 2018
tmFT Collab., PRD 2015

- Moving frame: we are coving a range in temperature between 1-80 GeV

$$\frac{s(T)}{T^3} = \frac{L_0^4(1+\xi^2)^3}{\xi_k}\langle T_{0k}^R\rangle_\xi \qquad \langle T_{0k}^R\rangle_\xi = -\frac{\partial}{\partial\xi_k}f[\xi] = \frac{1}{L_0 V}\frac{\partial}{\partial\xi_k}\log \mathcal{Z}[\xi]$$

$$\frac{\partial}{\partial\xi_k}f[\xi] = \frac{\partial}{\partial\xi_k}(f[\xi]-f_{YM}[\xi]+f_{YM}[\xi]) = \frac{\partial}{\partial\xi_k}\int_\infty^0 \frac{\partial}{\partial m_q}f[\xi]dm_q + \frac{\partial}{\partial\xi_k}f_{YM}[\xi] = \frac{\partial}{\partial\xi_k}\int_\infty^0 \langle\overline{\psi}\psi\rangle_\xi dm_q + \frac{\partial}{\partial\xi_k}f_{YM}$$



- Gaussian quadrature with a few points
- Final accuracy 0.5-1 % in the continuum limit
- QCD in a completely unexplored regime, comparison with high-T perturbation theory

Conclusions

- The lattice regularization is the tool that allows us to study the non-perturbative dynamics of QCD from first-principles.

- The increase of computer power and algorithmic development provide very precise predictions that can compared to experiments

- The accurate computation of the HVP is crucial to clarify the current discrepancy between theory and experiments for the muon anomalous magnetic moment

- First non-perturbative results of QCD at temperatures between 1 and 165 GeV: a range never explored so far.

- Computation of mesonic screening masses, evidence for a mass-splitting P-V, 1-loop PT not reliable. Results also for baryonic screening masses.

- The Equation of State: results for the pure gauge sector and QCD.