A new parton shower based on the small-x evolution equation

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- YS, Shu-yi Wei and Jian Zhou, Phys.Rev.D 107, 016017 (2023).
- YS, Shu-yi Wei and Jian Zhou, Phys.Rev.D 108, 096025 (2023).
- Collaboration with Wei-yao Ke, Xin-nian Wang and Jian Zhou, working in progress.

QCD evolution 2024

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Outline

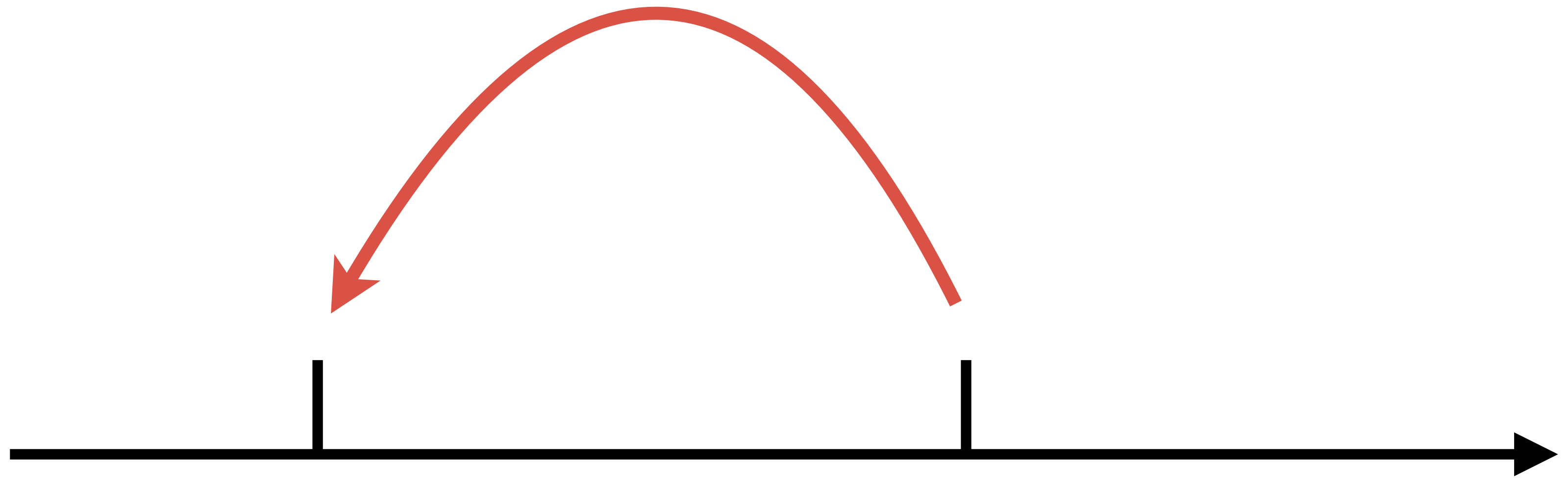
1) Introduction
2) Parton shower algorithm based on the Gribov-Levin-Ryskin evolution equation
3) Summary

Parton shower algorithms in M.C. event generator

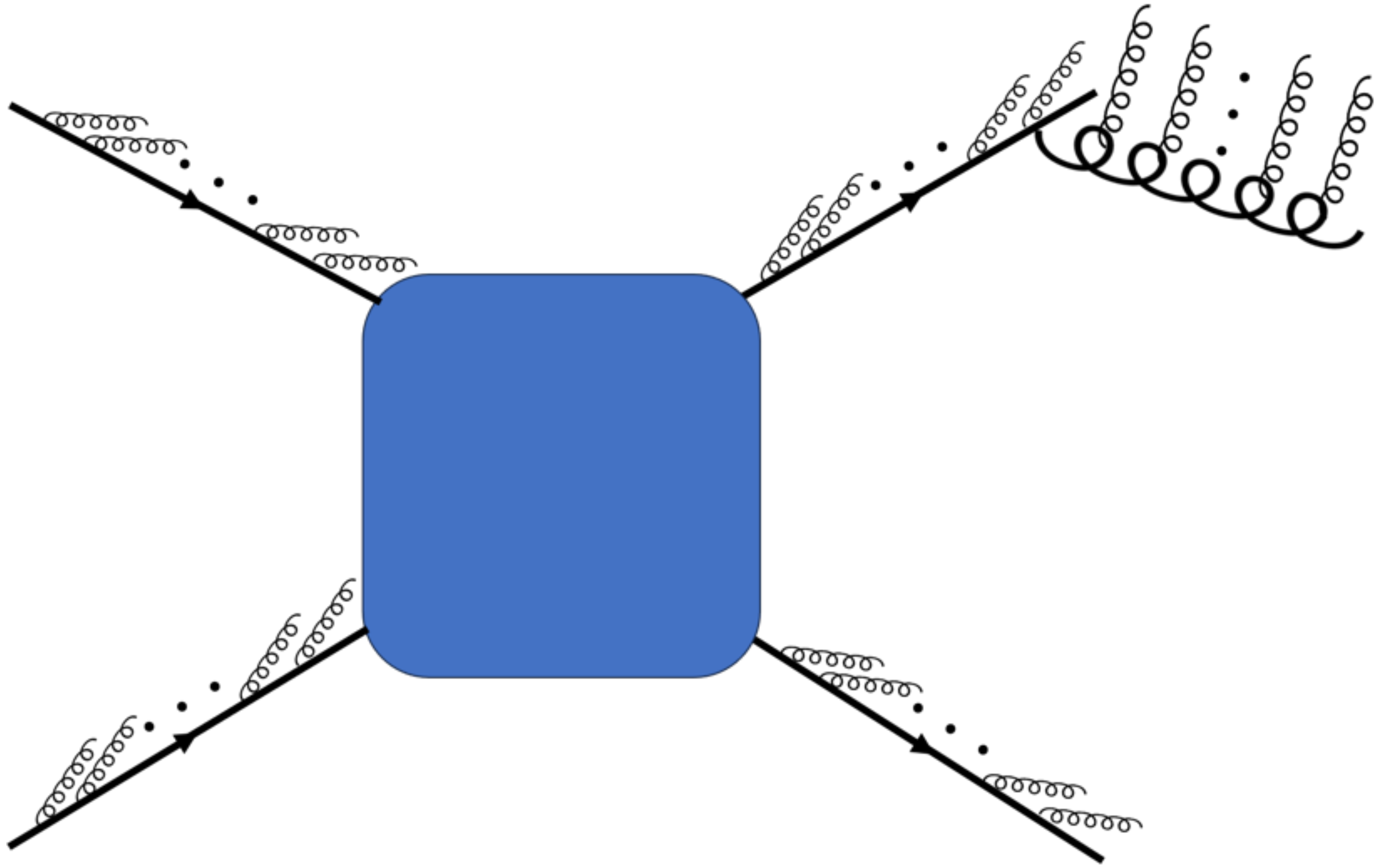
Sudakov form factor

$$\Delta_a(t,t') = \exp\left\{ -\sum_{b\in\{q,g\}} \int_t^{t'} \frac{\mathrm{d}\bar{t}}{\bar{t}} \int_{z_{\min}}^{z_{\max}} \mathrm{d}z\, \frac{\alpha_s}{2\pi} \frac{1}{2} P_{ab}(z) \right\}$$

Parton shower



Hadronization scale **Hard scale**



Parton shower: a model for simulating the radiation behavior of quarks and gluons.

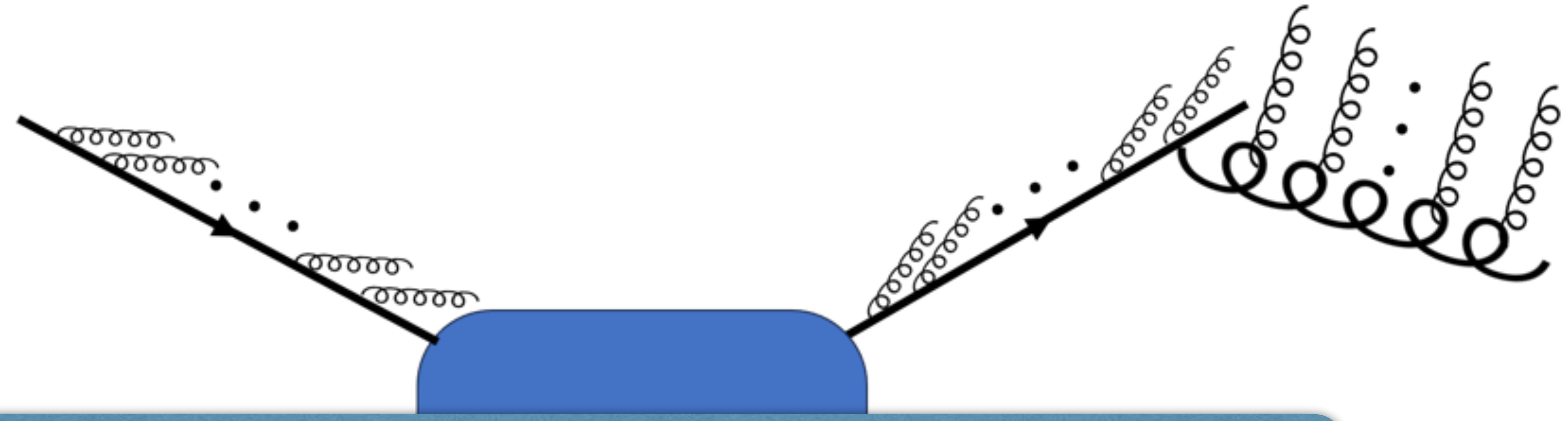
The evolution from hard scale to hadronization scale based on DGLAP/CCFM.

The same physics as resummation

Parton shower algorithms in M.C. event generator

Sudakov form factor

$$\Delta_a(t,t') = \exp\left\{ -\sum_{b\in\{q,g\}} \int_t^{t'} \frac{\mathrm{d}\bar{t}}{\bar{t}} \int_{z_{\min}}^{z_{\max}} \mathrm{d}z\, \frac{\alpha_s}{2\pi}\, \frac{1}{2}\, P_{ab}(z) \right\}$$



Hadroniz

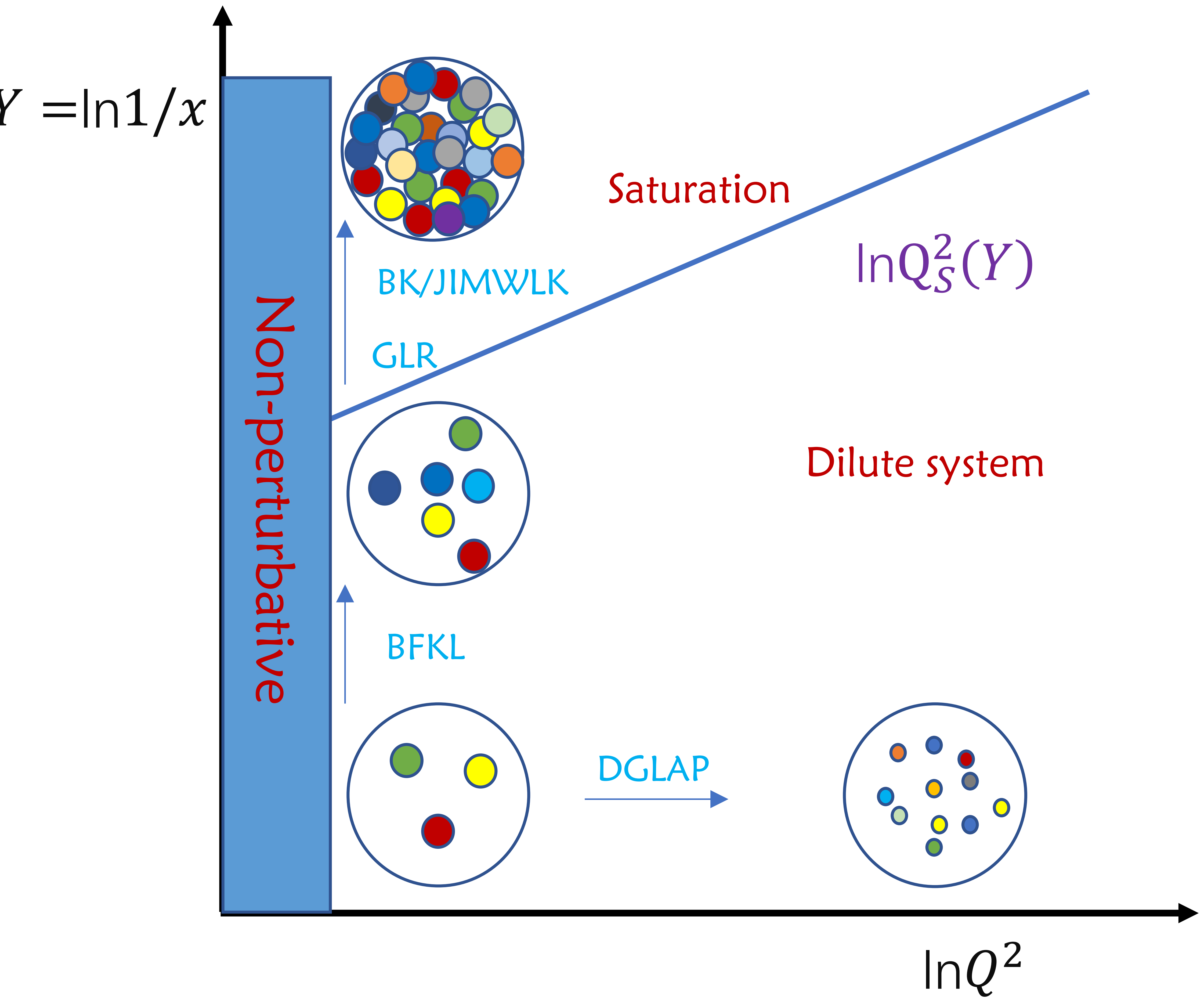
Can we use the parton shower to study the small-x physics?

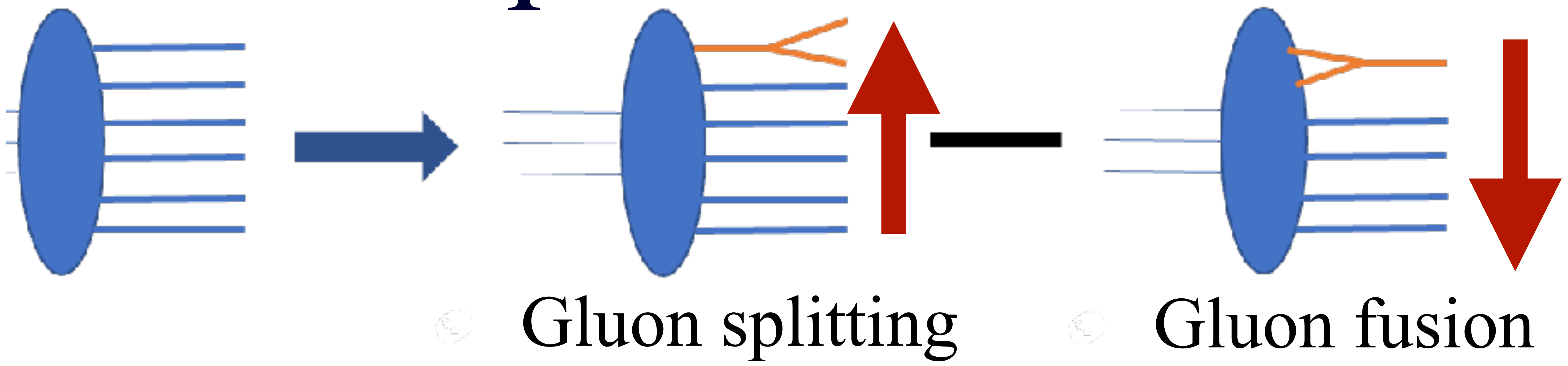
Parton shower: a model for simulating the radiation behavior of quarks and gluons.

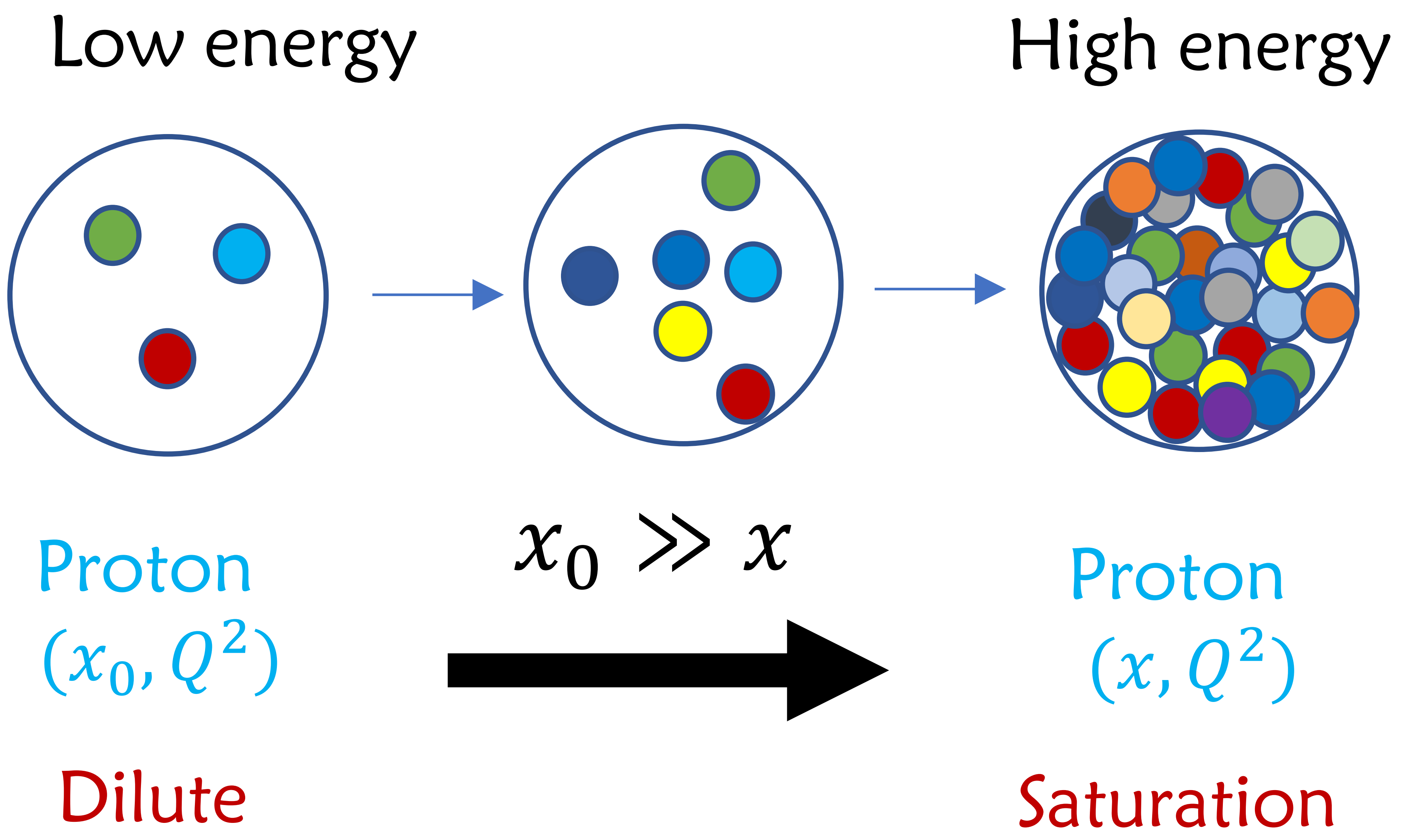
The evolution from hard scale to hadronization scale based on DGLAP/CCFM.

The same physics as resummation

Small-x evolution equations





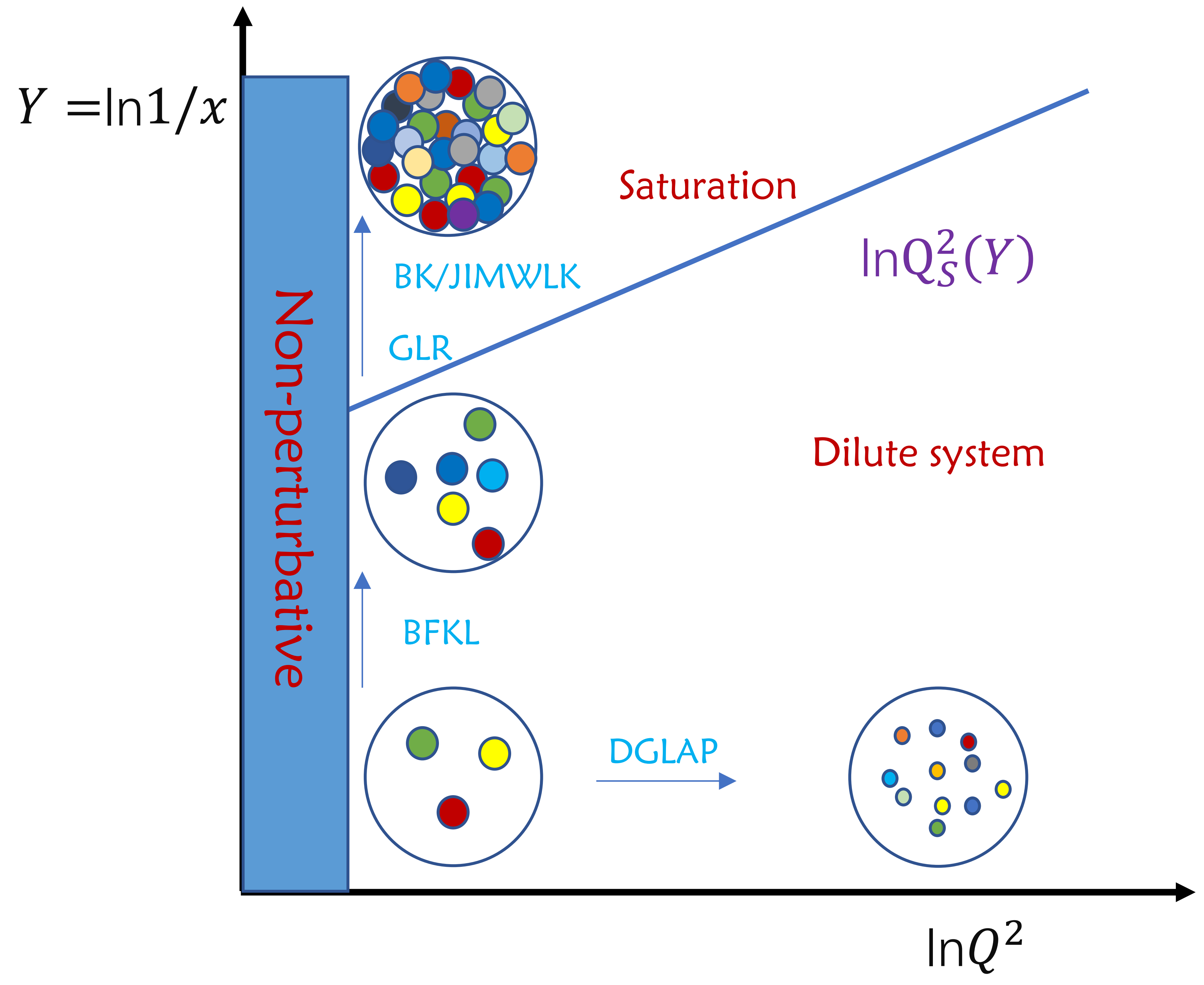


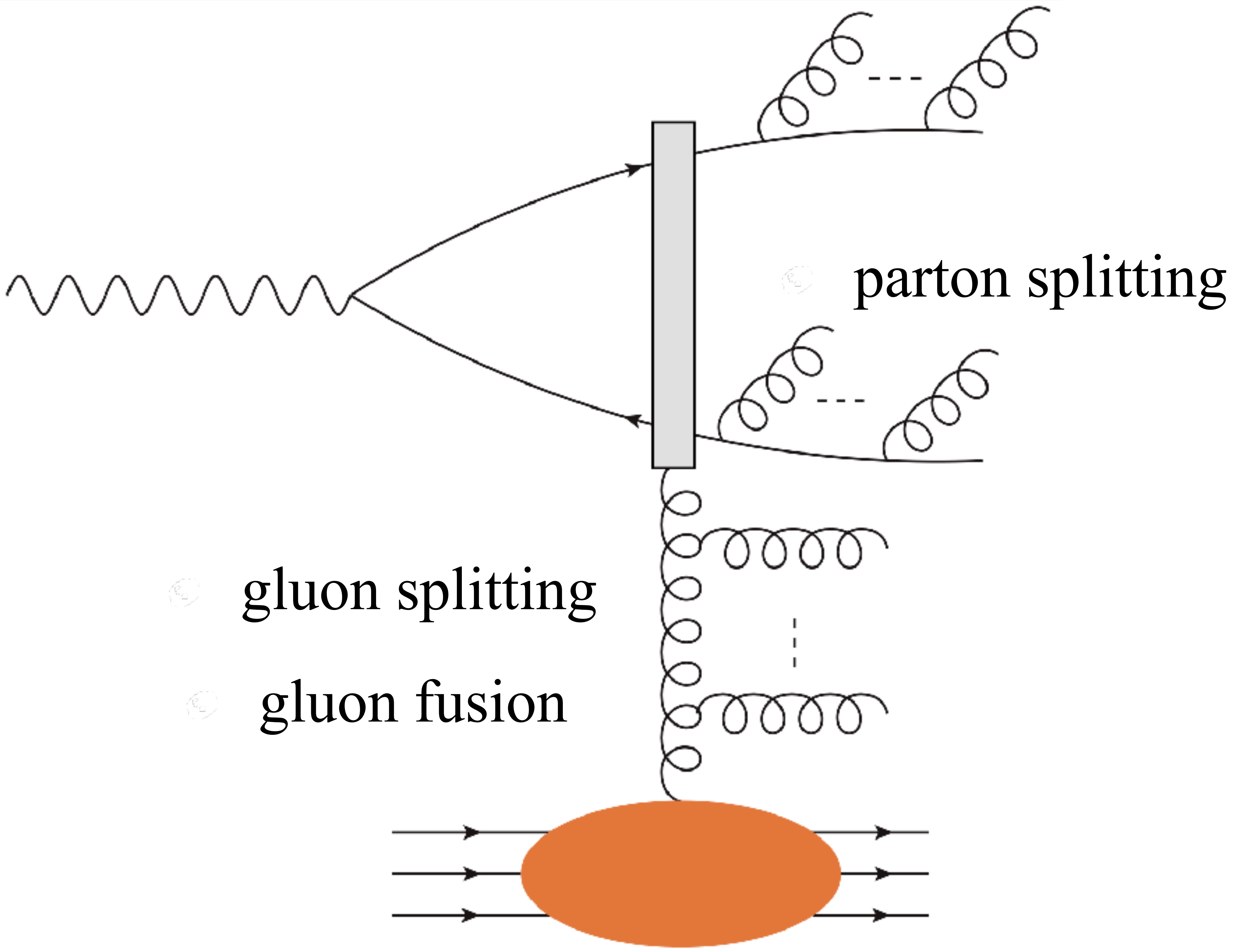
- Gluons rapidly increase as x decreases, gluons dominate in small x region.
- Using BFKL, and GLR/BK/JIMWLK equation instead of DGLAP equation.
- GLR/BK/JIMWLK equations are the non-linear evolution equations which describe gluons' non-linear evolution in the small-x region.

Dijet/Dihardon in the DIS

Full exclusive process

$$N_{\text{event}} = \mathscr{H}_{hard} \otimes \mathcal{N}(k_\perp) \otimes D(z) \otimes S_{\text{ISR}} \otimes S_{\text{FSR}} \otimes P_{\text{MPI}} \otimes P_{\text{decay}} \cdots$$





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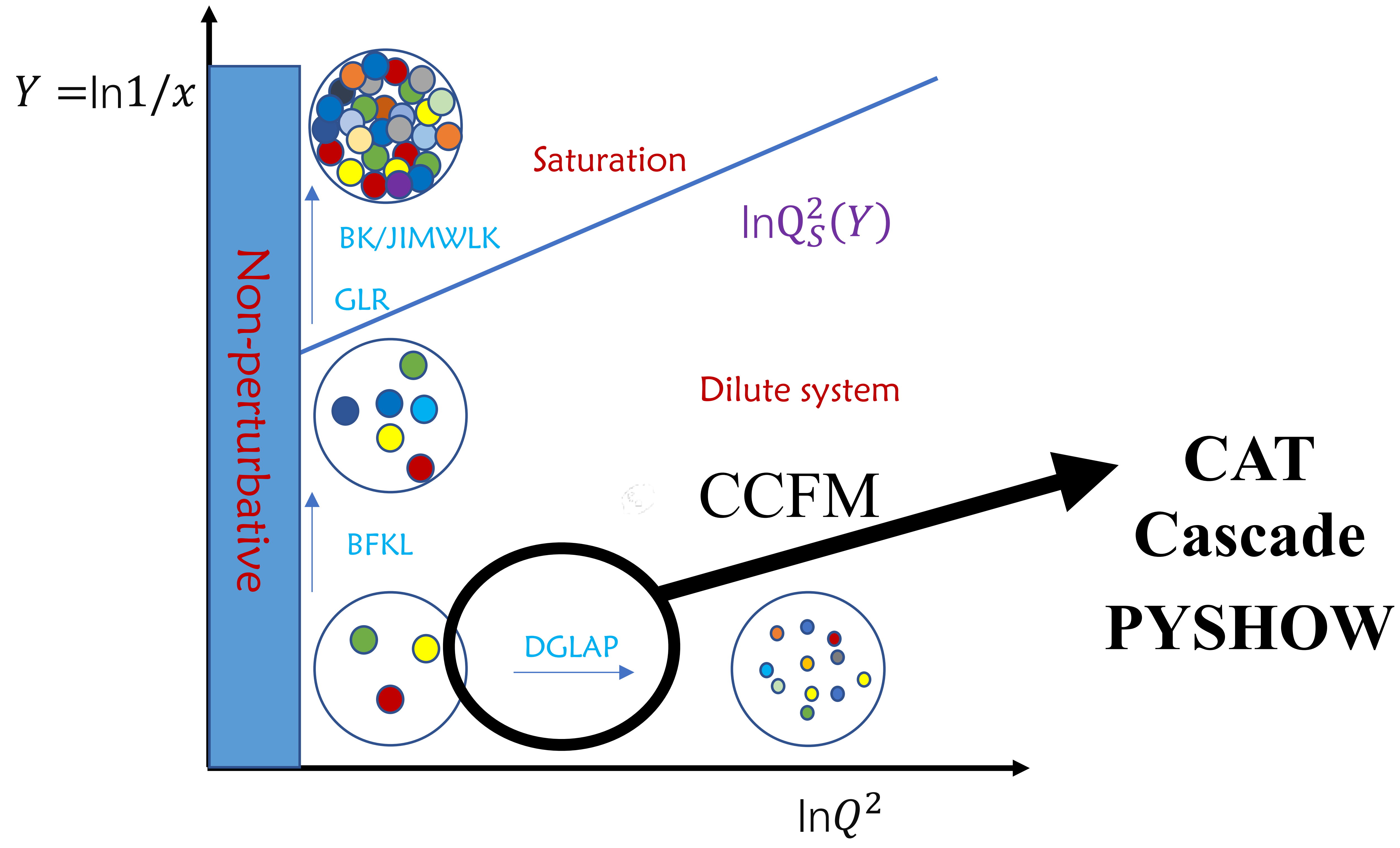

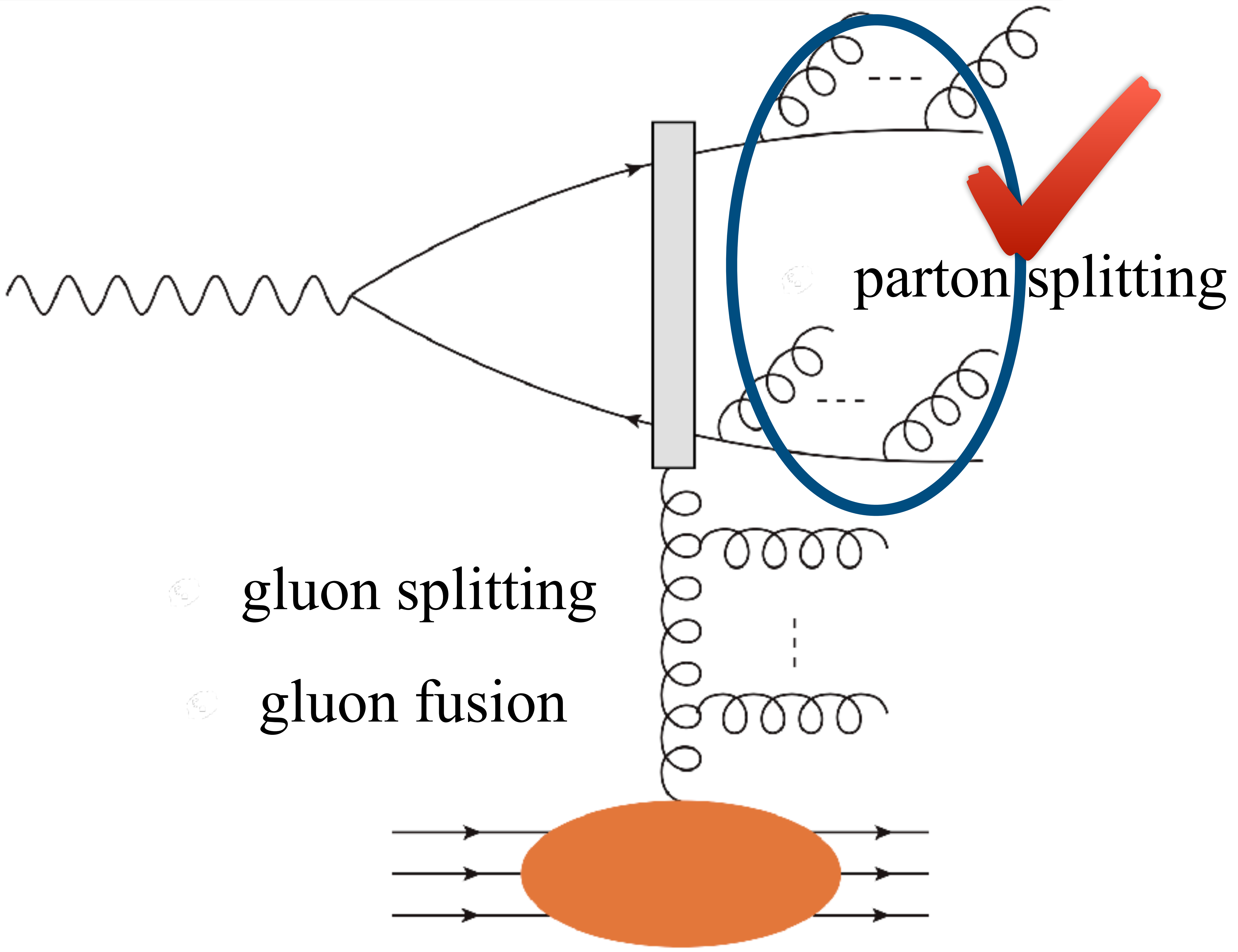

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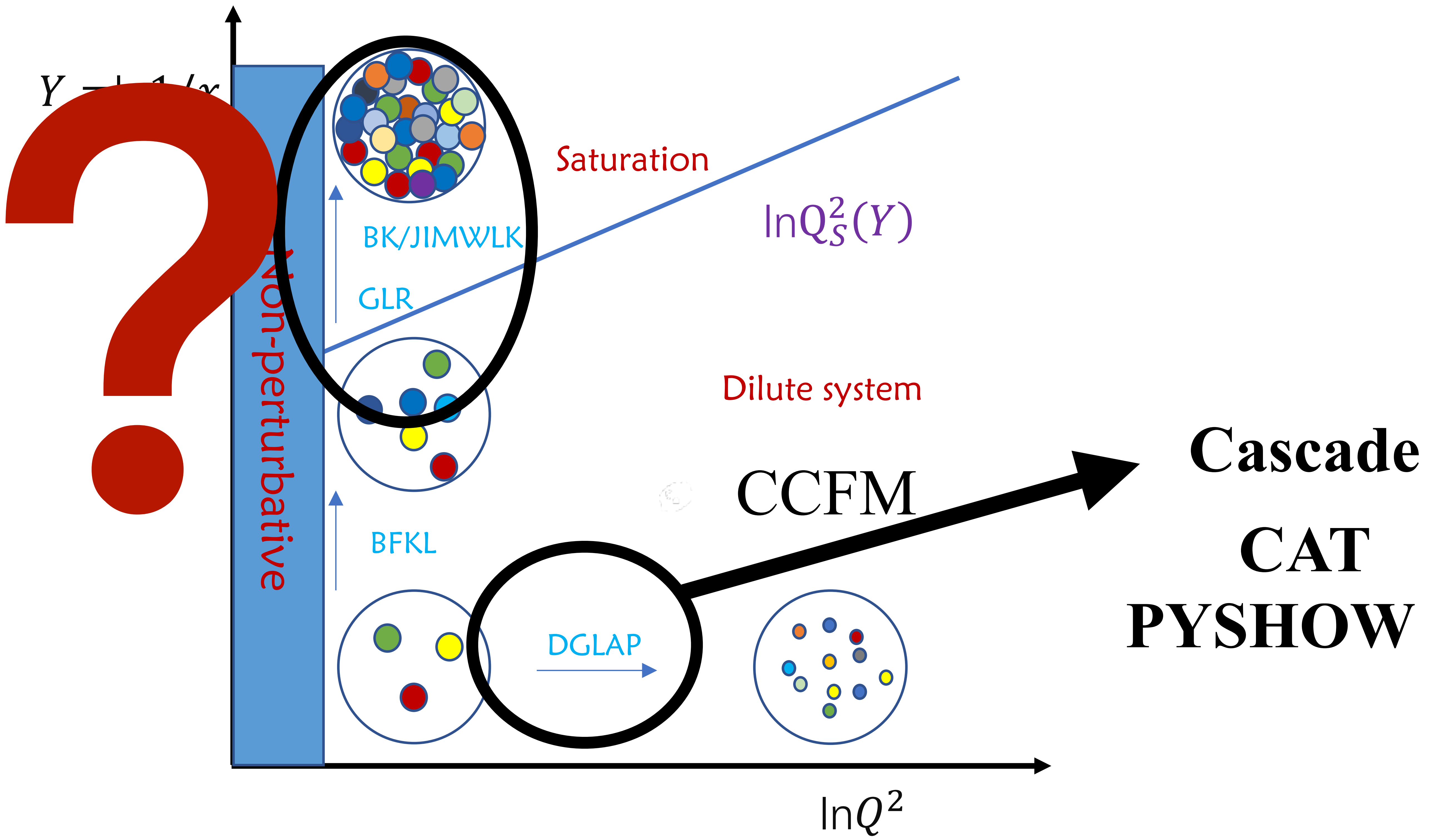

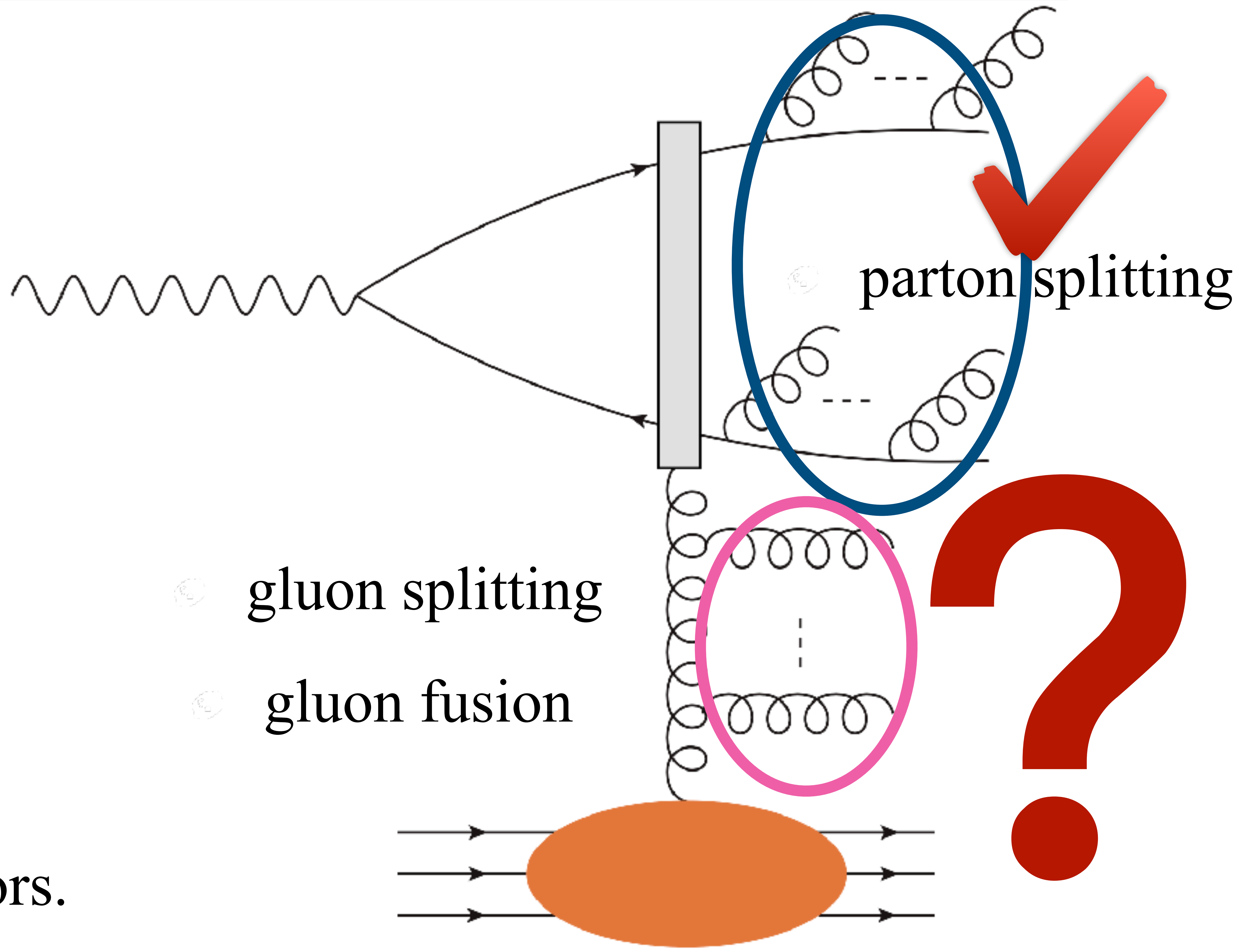

- gluon fusion effect is absent in all existing generators.
- Developing a P.S. algorithm based on the small-x nonlinear evolution equation is important.

2) A new parton shower algorithm based on the Gribov-Levin-Ryskin (GLR) equation

YS, S. Y. Wei and J. Zhou, Phys.Rev.D 107, 016017 (2023).

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GLR evolution Equation

- The GLR equation [Gribov, Levin, Ryskin, PR, 83] Gluon fusion $2 \rightarrow 1$

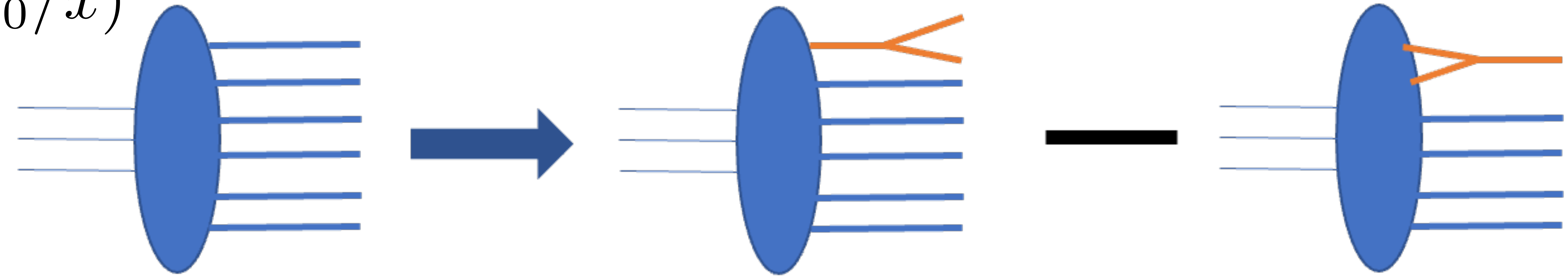
$$\frac{\partial G(\eta, k_\perp)}{\partial \eta} = \frac{\bar{\alpha}_s}{\pi}\left[\int \frac{d^2 l_\perp}{l_\perp^2} G(\eta, k_\perp + l_\perp) - \int_0^{k_\perp} \frac{d^2 l_\perp}{l_\perp^2} G(\eta, k_\perp)\right] - g_{\mathrm{TPV}} \frac{\alpha_s^2}{S_\perp (8\pi)^2} G^2(\eta, k_\perp)$$

with the dipole gluon distribution

$$G(\eta, k_\perp) = \frac{k_\perp^2 N_c}{2\pi^2 \alpha} S_\perp \int \frac{d^2 r_\perp}{(2\pi)^2} e^{-ik_\perp \cdot r_\perp} \frac{1}{N_c} \langle \mathrm{Tr}\left[U^\dagger(0) U(r_\perp)\right] \rangle$$

$$g_{\mathrm{TPV}} = 8(2\pi)^4$$

$$\eta = \ln\left(x_0/x\right)$$



- GLR equation is the non-linear evolution equation that describes the gluon diffusion process.

GLR evolution Equation

- The GLR equation [Gribov, Levin, Ryskin, PR, 83] Gluon fusion $2 \to 1$

$$\frac{\partial G(\eta,k_\perp)}{\partial\eta} = \frac{\bar\alpha_s}{\pi}\left[\int \frac{d^2 l_\perp}{l_\perp^2} G(\eta,k_\perp+l_\perp) - \int_0^{k_\perp} \frac{d^2 l_\perp}{l_\perp^2} G(\eta,k_\perp)\right] - g_{\rm TPV}\frac{\alpha_s^2}{S_\perp(8\pi)^2} G^2(\eta,k_\perp)$$

$$g_{\rm TPV} = 8(2\pi)^4$$

the dipole gluon distribution $G(\eta,k_\perp) = \frac{k_\perp^2 N_c}{2\pi^2\alpha} S_\perp \int \frac{d^2 r_\perp}{(2\pi)^2} e^{-ik_\perp\cdot r_\perp} \frac{1}{N_c}\langle {\rm Tr}\left[U^\dagger(0)U(r_\perp)\right]\rangle$

$$N(\eta,k_\perp) = \frac{2\alpha_s\pi^3}{N_c S_\perp} G(\eta,k_\perp)$$

$$\frac{\partial N(\eta,k_\perp)}{\partial\eta} = \frac{\bar\alpha_s}{\pi}\left[\int \frac{d^2 l_\perp}{l_\perp^2} N(\eta,k_\perp+l_\perp) - \int_0^{k_\perp} \frac{d^2 l_\perp}{l_\perp^2} N(\eta,k_\perp)\right] - \bar\alpha_s N^2(\eta,k_\perp)$$

- this form is the same as the BK equation in the momentum space [Balitsky, NPB 96; Kovchegov, PRD 99]

Gluon fusion $2 \to 1, 3 \to 1, 4 \to 1 \ldots$

$$\frac{\partial \mathcal{N}(\eta,k_\perp)}{\partial\eta} = \frac{\bar\alpha_s}{\pi}\left[\int \frac{d^2 l_\perp}{l_\perp^2} \mathcal{N}(\eta,l_\perp+k_\perp) - \int_0^{k_\perp} \frac{d^2 l_\perp}{l_\perp^2} \mathcal{N}(\eta,k_\perp)\right] - \bar\alpha_s \mathcal{N}^2(\eta,k_\perp)$$

WW gluon distribution $\mathcal{N}(\eta,k_\perp) = \int \frac{d^2 r_\perp}{2\pi} \frac{e^{-ik_\perp\cdot r_\perp}}{r_\perp^2}\left[1 - \frac{1}{N_c}\langle U^\dagger(0)U(r_\perp)\rangle\right]$ [Kovchegov, PRD, 00; Marquet, Soyez, NPA, 05]

GLR evolution Equation

- Resolved and unresolved branching

$$\int \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta, k_\perp + l_\perp) \approx \int_\mu \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta, k_\perp + l_\perp) + \int_0^\mu \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta, k_\perp)$$

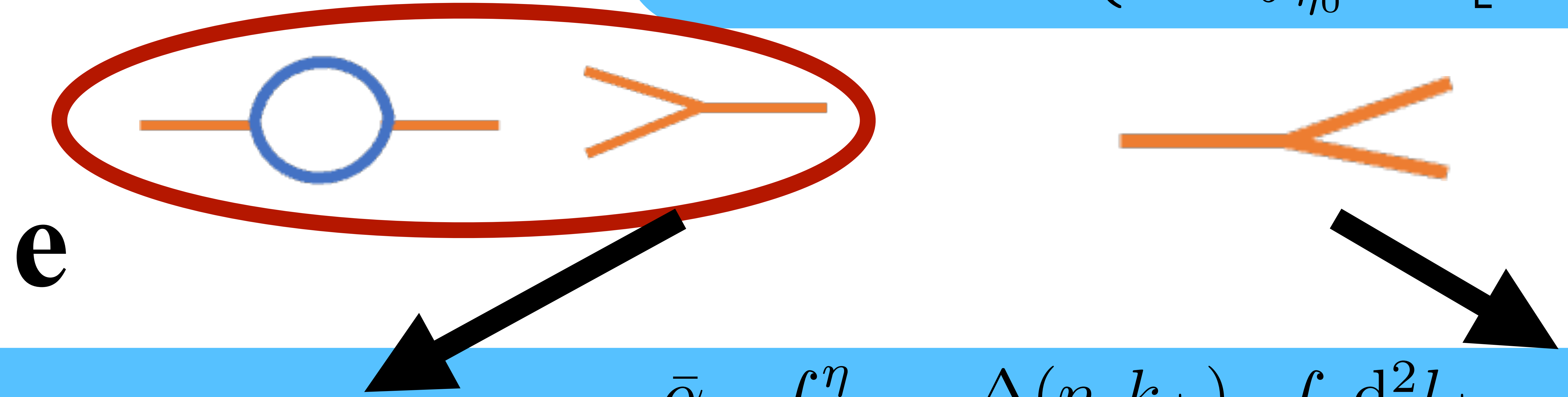
- The GLR equation can be rewrite as (unfolded one)

Independent on the choice of μ

$$\frac{\partial N(\eta, k_\perp)}{\partial \eta} = \frac{\bar{\alpha}_s}{\pi} \int_\mu \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta, l_\perp + k_\perp) - \bar{\alpha}_s \ln \frac{k_\perp^2}{\mu^2} N(\eta, k_\perp) - \bar{\alpha}_s N^2(\eta, k_\perp)$$

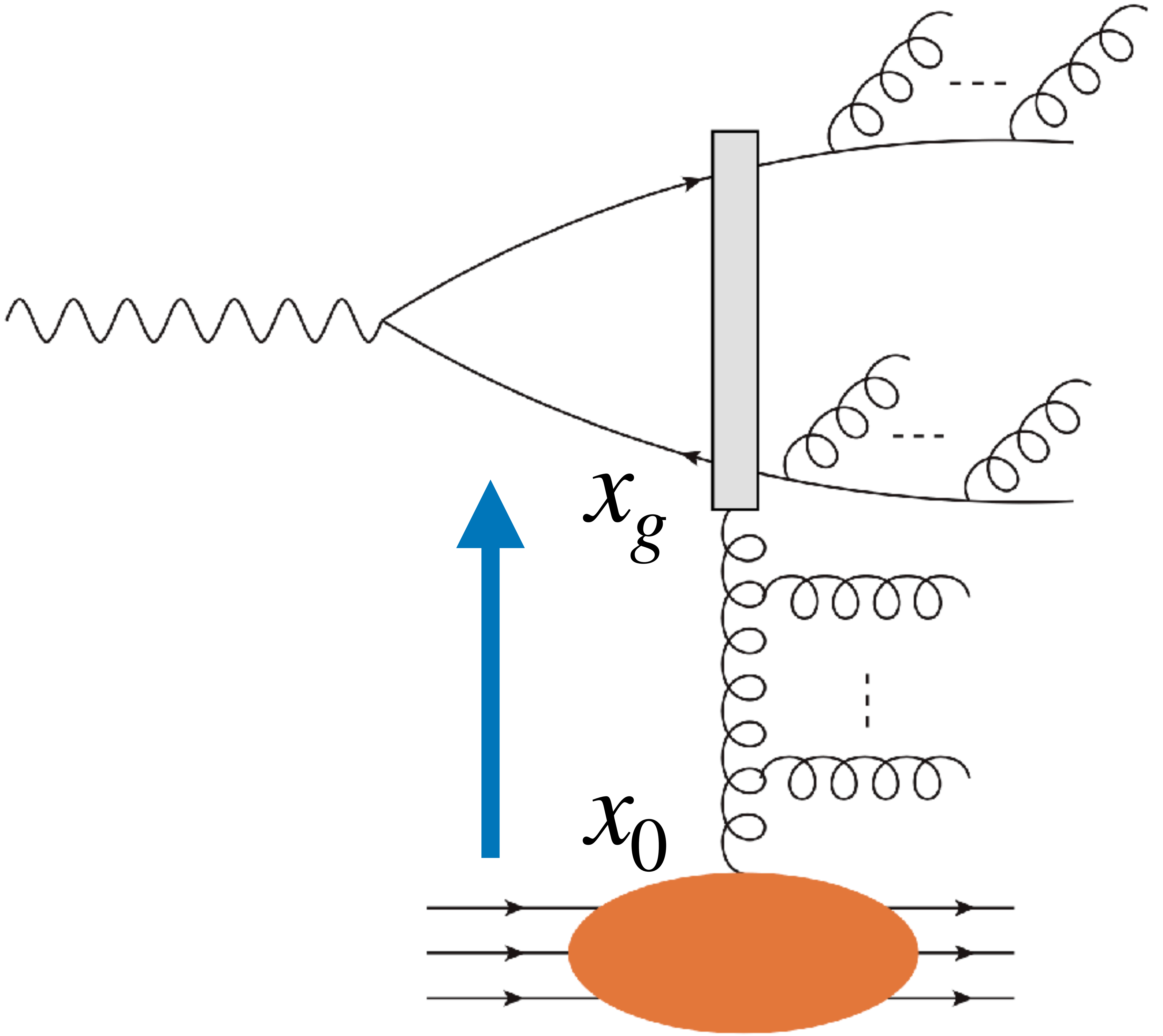
- Non-Sudakov form factor resums the virtual and non-linear term
- The integral GLR equation (folded one)

$$\Delta(\eta, k_\perp) = \exp\left\{ -\bar{\alpha}_s \int_{\eta_0}^{\eta} d\eta' \left[\ln \frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp) \right] \right\}$$



$$N(\eta, k_\perp) = N(\eta_0, k_\perp)\Delta(\eta, k_\perp) + \frac{\bar{\alpha}_s}{\pi} \int_{\eta_0}^{\eta} \mathrm{d}\eta' \frac{\Delta(\eta, k_\perp)}{\Delta(\eta', k_\perp)} \int_\mu \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N(\eta', l_\perp + k_\perp)$$

The forward evolution algorithm



The forward evolution algorithm

First step: non-Sudakov form factor

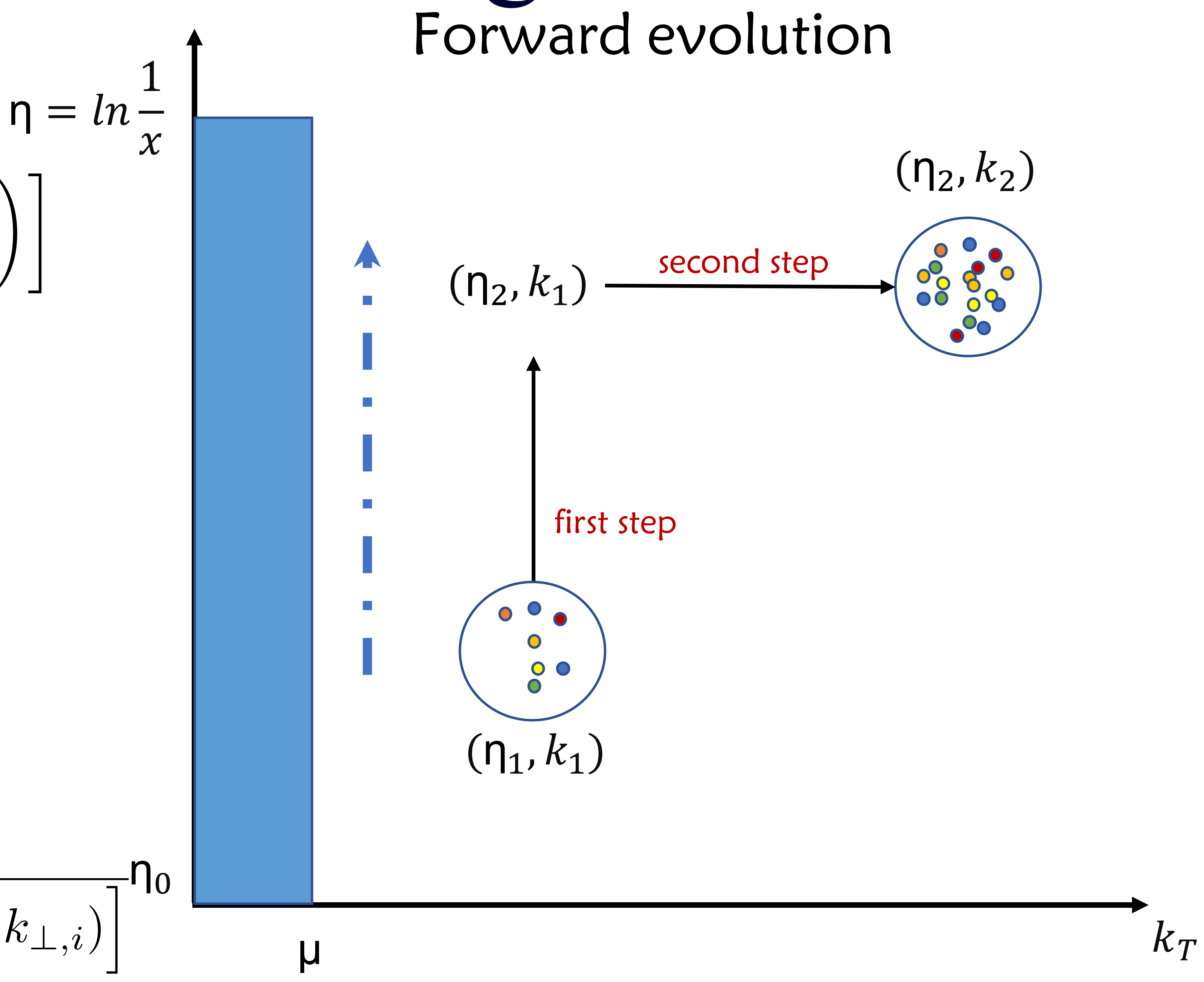
$$\mathcal{R} = \exp\left[-\bar{\alpha}_s \int_{\eta_i}^{\eta_{i+1}} d\eta' \left(\ln \frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp)\right)\right]$$

Second step: Real splitting kernel

$$\mathcal{R}_2 \int_{\mu}^{P_\perp} \frac{d^2 l'_\perp}{l'^2_\perp} = \int_{\mu}^{|l_\perp|} \frac{d^2 l'_\perp}{l'^2_\perp}$$

The generated event has to be re-weighted

$$\mathcal{W}(\eta_i, \eta_{i+1}; k_{\perp,i}) = \frac{\int_{\eta_i}^{\eta_{i+1}} d\eta \ln(P_\perp^2/\mu^2)}{\int_{\eta_i}^{\eta_{i+1}} d\eta \left[\ln(k_{\perp,i}^2/\mu^2) + N(\eta, k_{\perp,i})\right]}$$



The forward evolution algorithm

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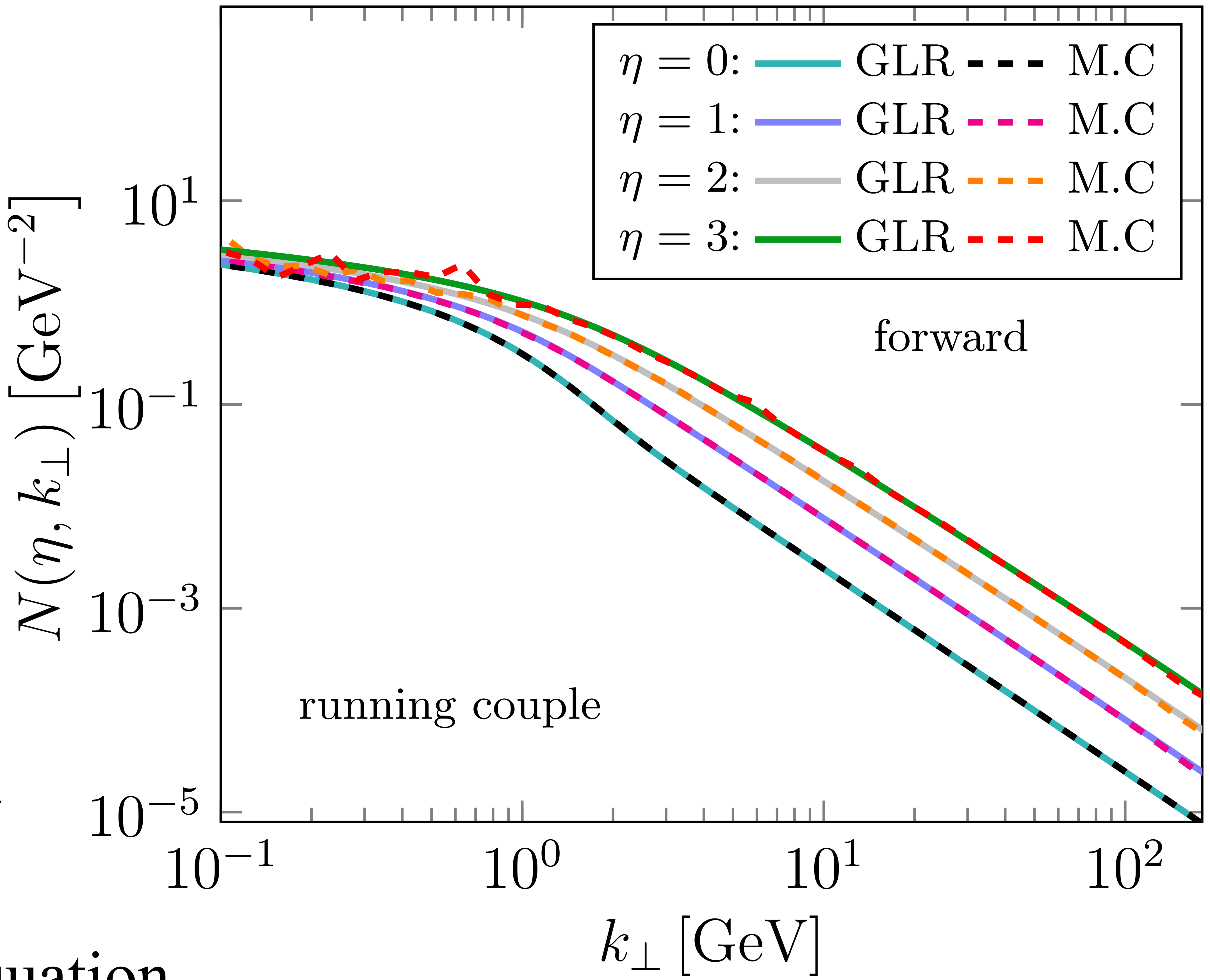
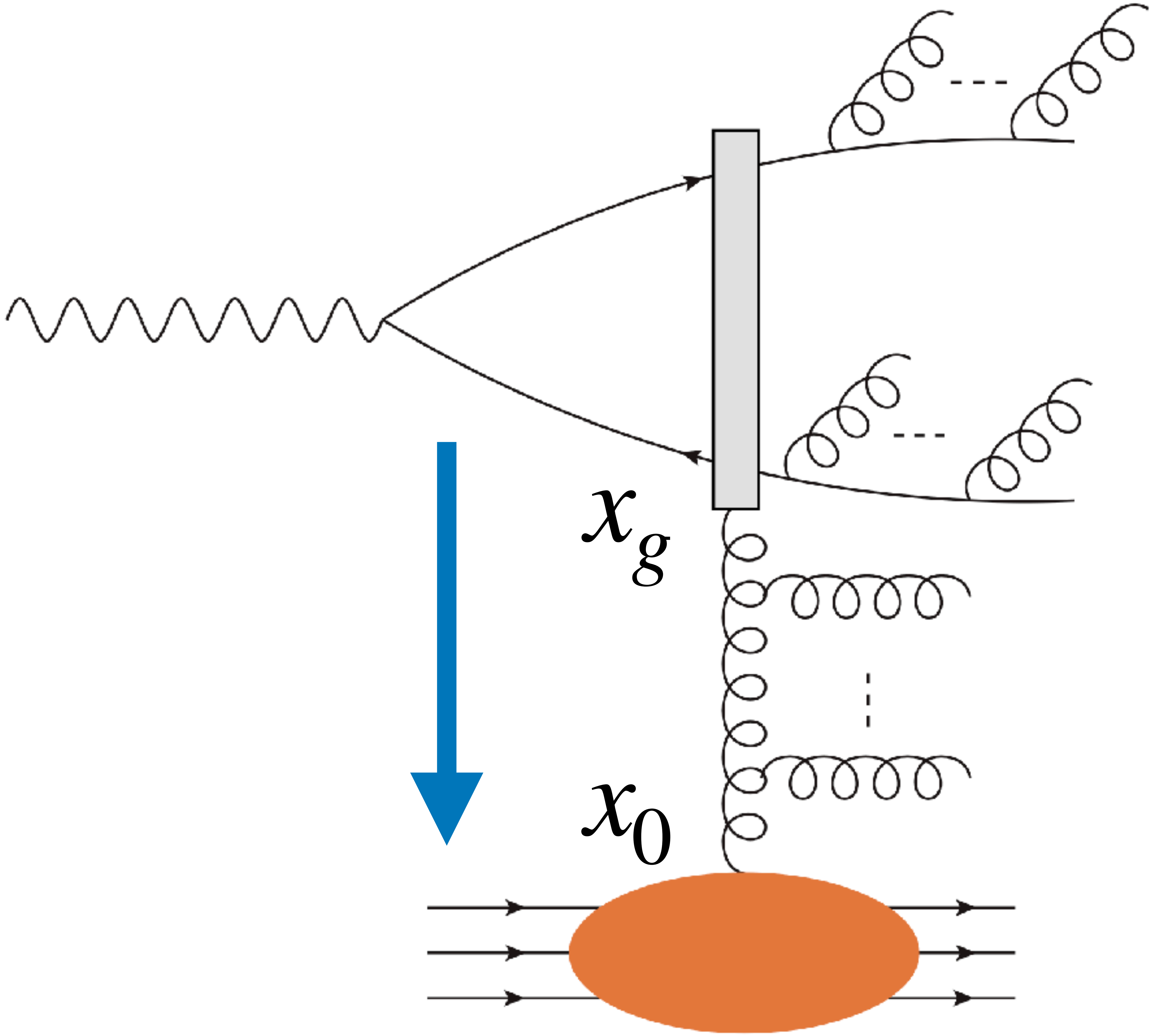
- The initial condition likes

$$N(\eta=0, k_\perp) = \int \frac{d^2 r_\perp}{2\pi} e^{-ik_\perp \cdot r_\perp} \frac{1}{r_\perp^2} \left(1 - \exp\left[-\frac{1}{4} Q_{s0}^2 r_\perp^2 \ln(e + \frac{1}{\Lambda r_\perp})\right]\right)$$



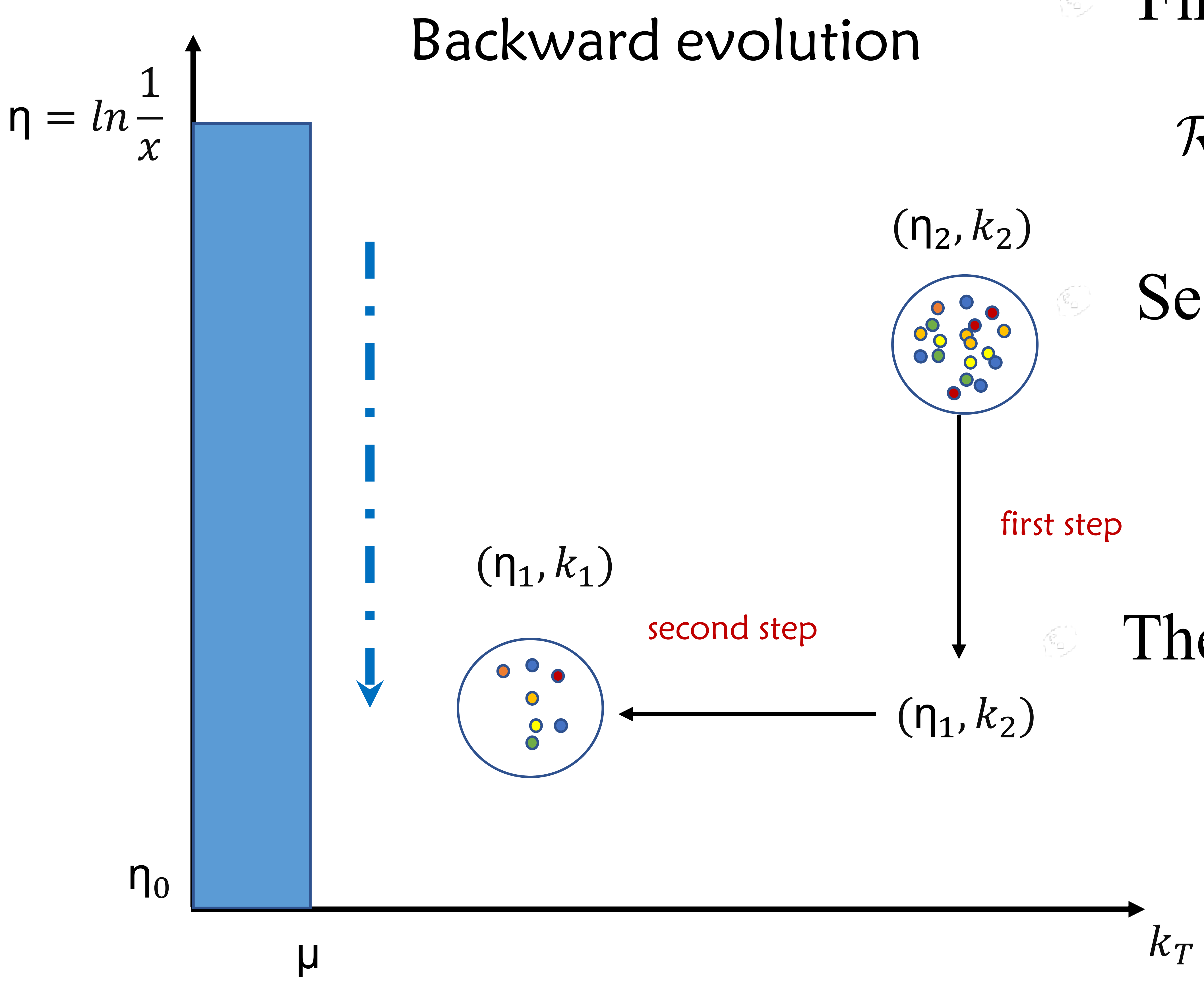
- Agree with the numerical solutions of the GLR equation.

The backward evolution algorithm



- As a more efficient procedure, the backward evolution approach is also presented.
- Using the numerical solution of the GLR equation. $N(\eta, k_\perp)$

The backward evolution algorithm



- First step: backward non-Sudakov form factor

$$\mathcal{R} = \exp\left[-\frac{\bar{\alpha}_s}{\pi}\int_{\eta_i}^{\eta_{i+1}} d\eta \int_\mu \frac{d^2 l_\perp}{l_\perp^2} \frac{N(\eta, k_{\perp,i+1} + l_\perp)}{N(\eta, k_{\perp,i+1})}\right]$$

- Second step: Real splitting

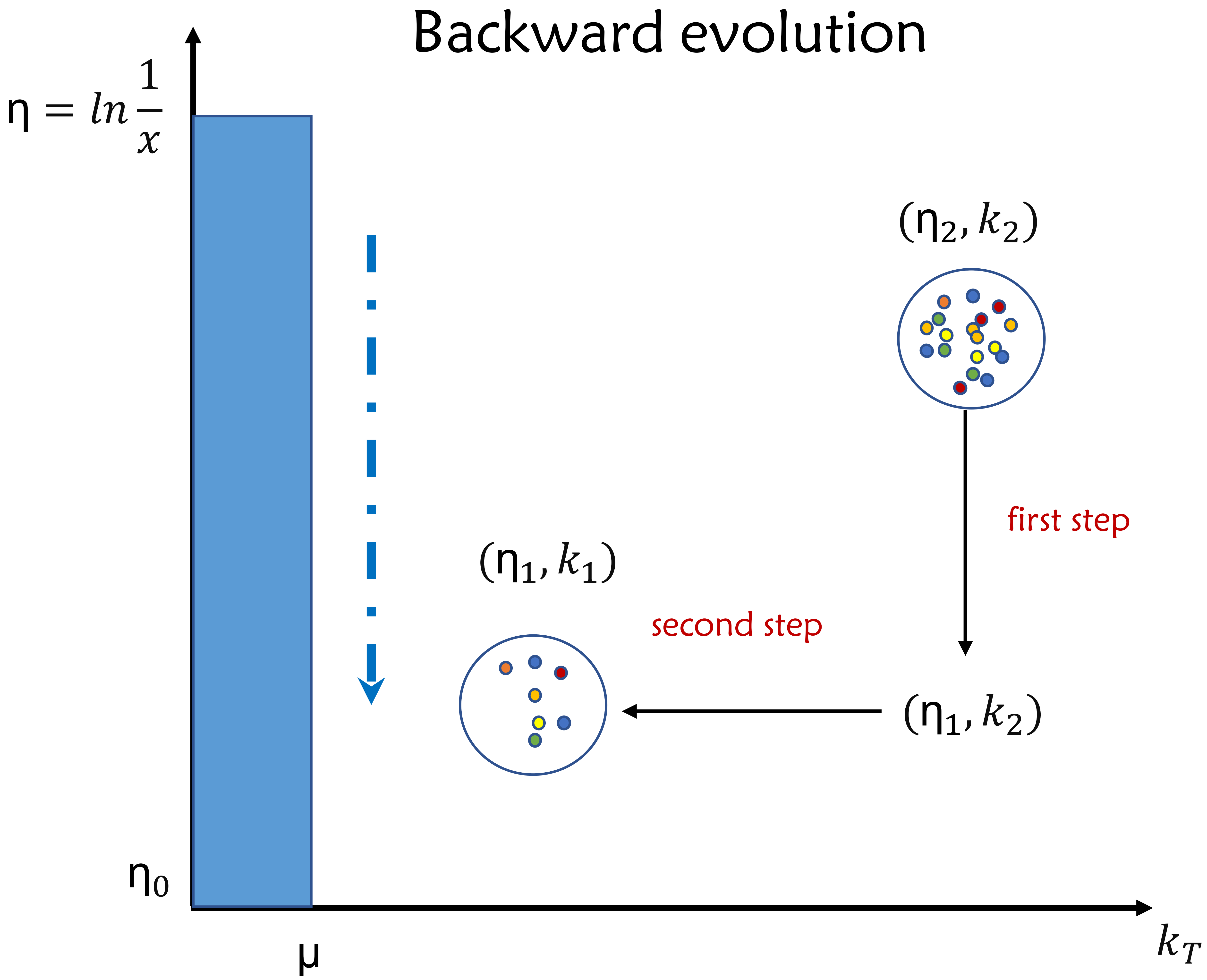
$$\frac{\bar{\alpha}_s}{\pi}\int_\mu^{l_\perp} \frac{d^2 l'_\perp}{l'^2_\perp} N(\eta_i, k_{\perp,i+1} + l'_\perp) = \mathcal{R}_2 \frac{\bar{\alpha}_s}{\pi}\int_\mu^{P_\perp} \frac{d^2 l'_\perp}{l'^2_\perp} N(\eta_i, k_{\perp,i+1} + l'_\perp)$$

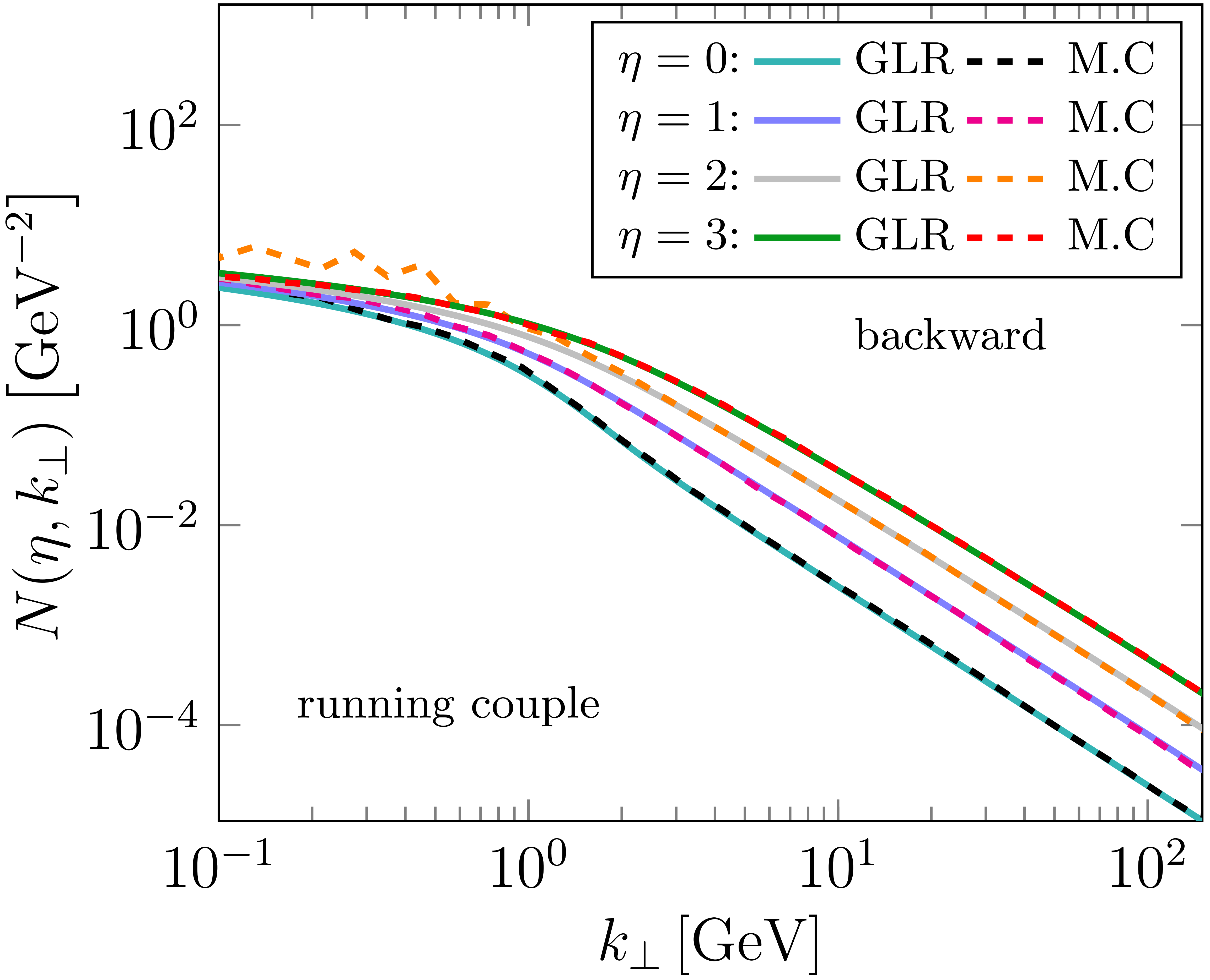
- The generated event has to be re-weighted

$$\mathcal{W}_{\text{backward}} = \frac{1}{\mathcal{W}_{forward}}$$

- As a more efficient procedure, the backward evolution approach is also presented.
- Using the numerical solution of the GLR equation.

The backward evolution algorithm





- As a more efficient procedure, the backward evolution approach is also presented.
- Agree with the numerical solutions of the GLR equation.

Parton shower algorithms

GLR v.s. DGLAP/CCFM

$$\Delta(\eta, k_\perp) = \exp\left\{-\bar{\alpha}_s \int_{\eta_0}^{\eta} d\eta' \left[\ln\frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp)\right]\right\}$$

$$\Delta_a(t, t') = \exp\left\{-\sum_{b\in\{q,g\}} \int_t^{t'} \frac{\mathrm{d}\bar{t}}{\bar{t}} \int_{z_{\min}}^{z_{\max}} \mathrm{d}z\, \frac{\alpha_s}{2\pi}\, \frac{1}{2}\, P_{ab}(z)\right\}$$

GLR	DGLAP/CCFM
gluon splitting gluon fusion	parton splitting

The evolution variable:

GLR	DGLAP/CCFM
$\eta = \ln(1/x)$	Q

The generated event:

GLR	DGLAP/CCFM
reweight	Unitary

Kinematical constraint in the GLR evolution equation

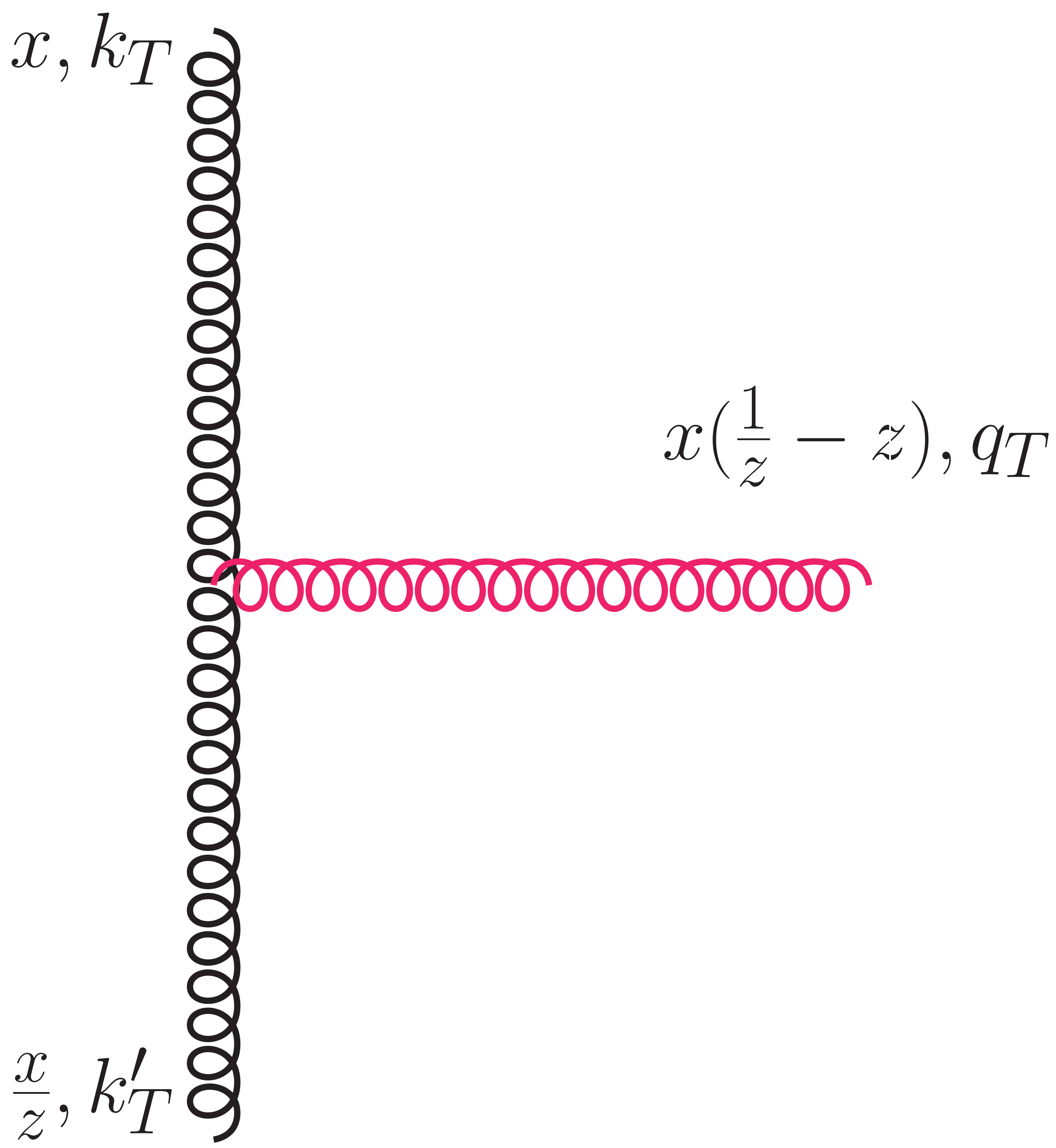
[Kwiecinski, Martin, Sutton, Z. Phys. C, 96;

Deak, Kutak, Li, Stasto, EPJC, 19]

- The key observation is that the virtuality of a gluon should arise mainly from the transverse momentum

$$k_T^2 > |k^+k^-| \qquad k^- = k'^- - q^- \simeq -q^- = -q_T^2/q^+$$

$$k^+k^- \simeq -\frac{k^+}{q^+}q_T^2 = -\frac{k^+}{k'^+ - k^+}q_T^2 = -\frac{z}{1-z}q_T^2$$



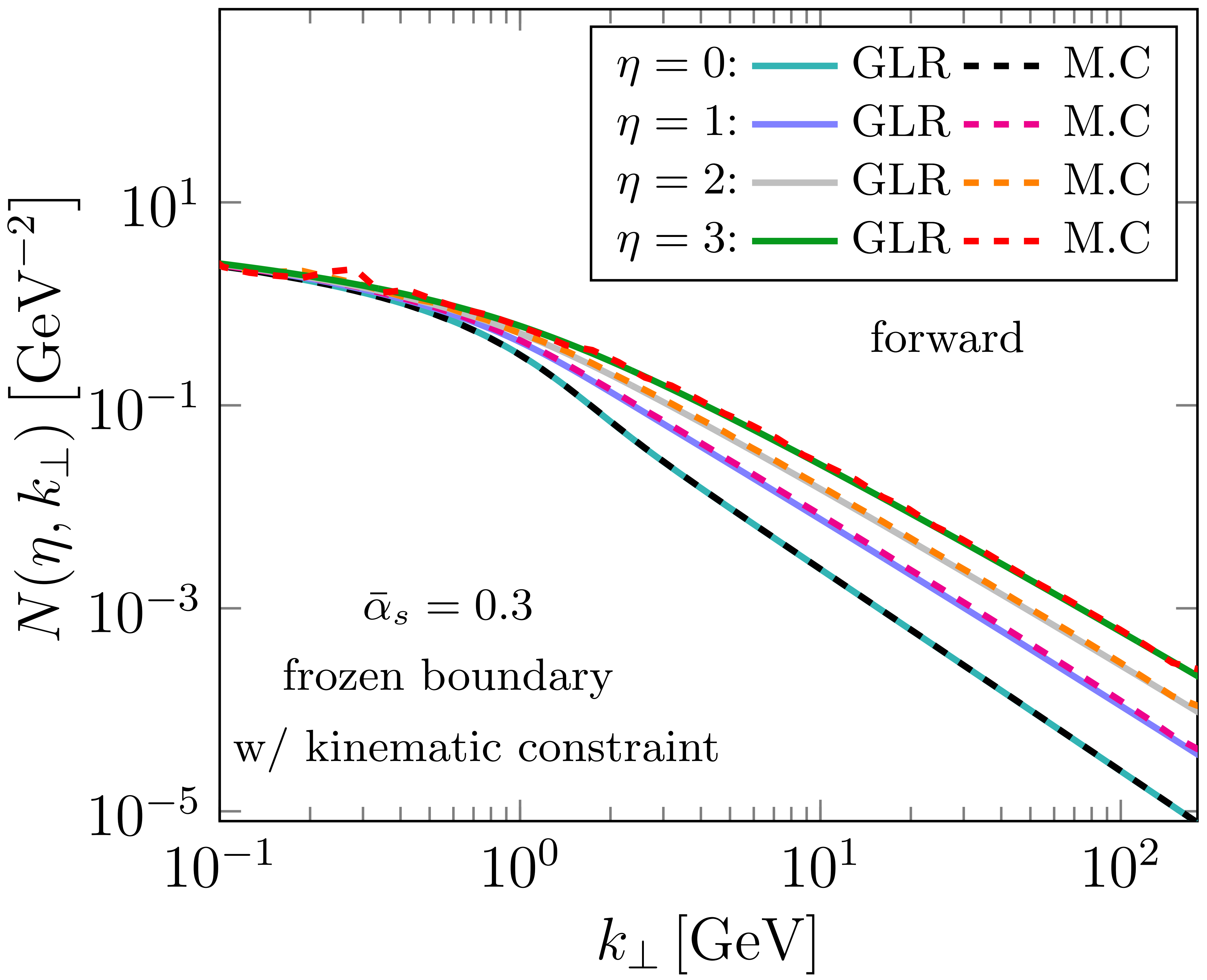
- The on-shell condition give the kinematical constraint

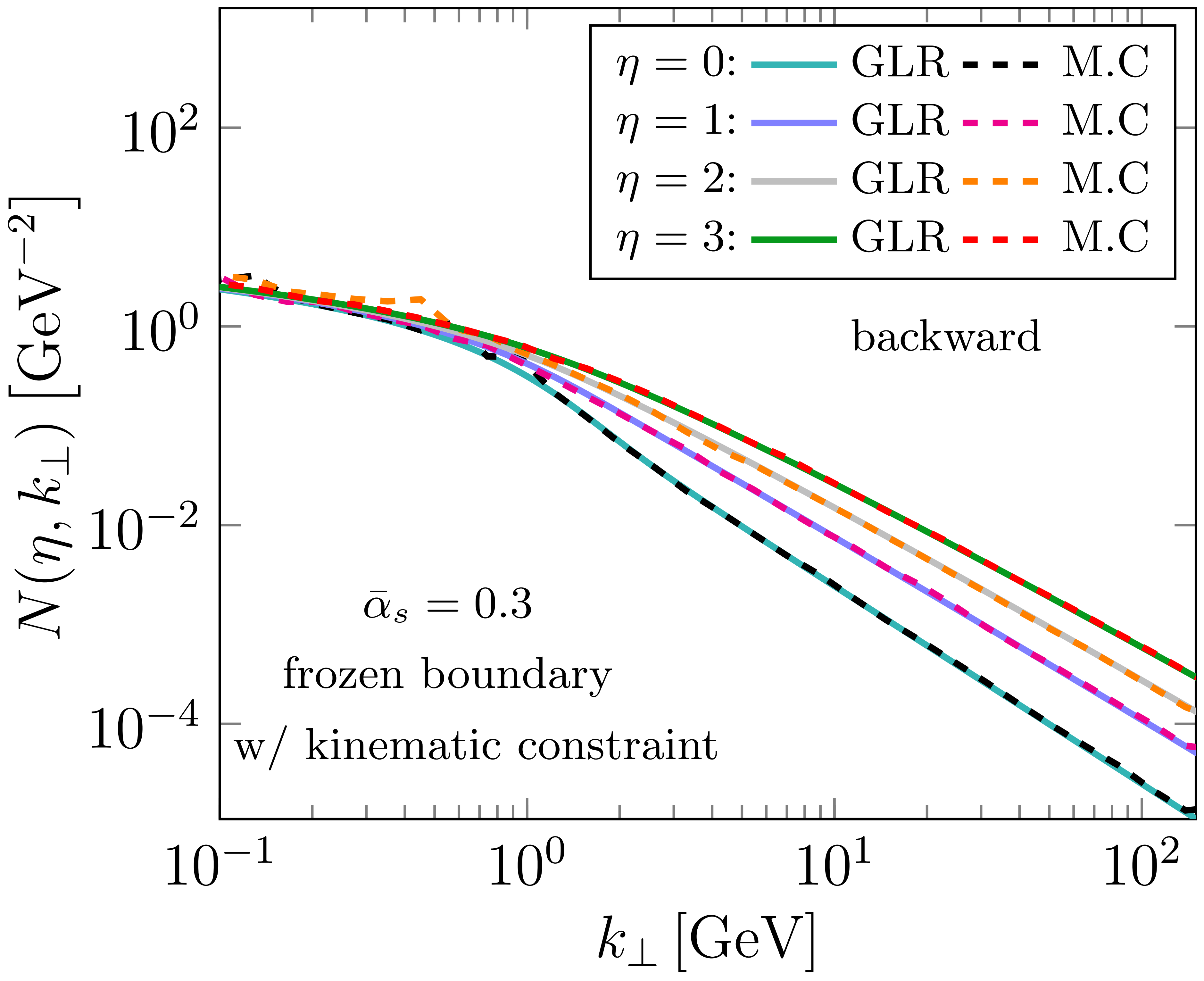
$$q_T^2 < \frac{1-z}{z}k_T^2 \qquad \eta \longrightarrow \eta + \ln\frac{k_\perp^2}{k_\perp^2 + l_\perp^2}$$

- The kinematic constrainted GLR equation can be modified as

$$\frac{\partial N(\eta, k_\perp)}{\partial \eta} = \frac{\bar{\alpha}_s}{\pi}\int \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} N\left(\eta + \ln\frac{k_\perp^2}{k_\perp^2 + l_\perp^2}, l_\perp + k_\perp\right) - \frac{\bar{\alpha}_s}{\pi}\int_0^{k_\perp}\frac{\mathrm{d}^2 l_\perp}{l_\perp^2}N(\eta, k_\perp) - \bar{\alpha}_s N^2(\eta, k_\perp)$$

The forward & backward evolution





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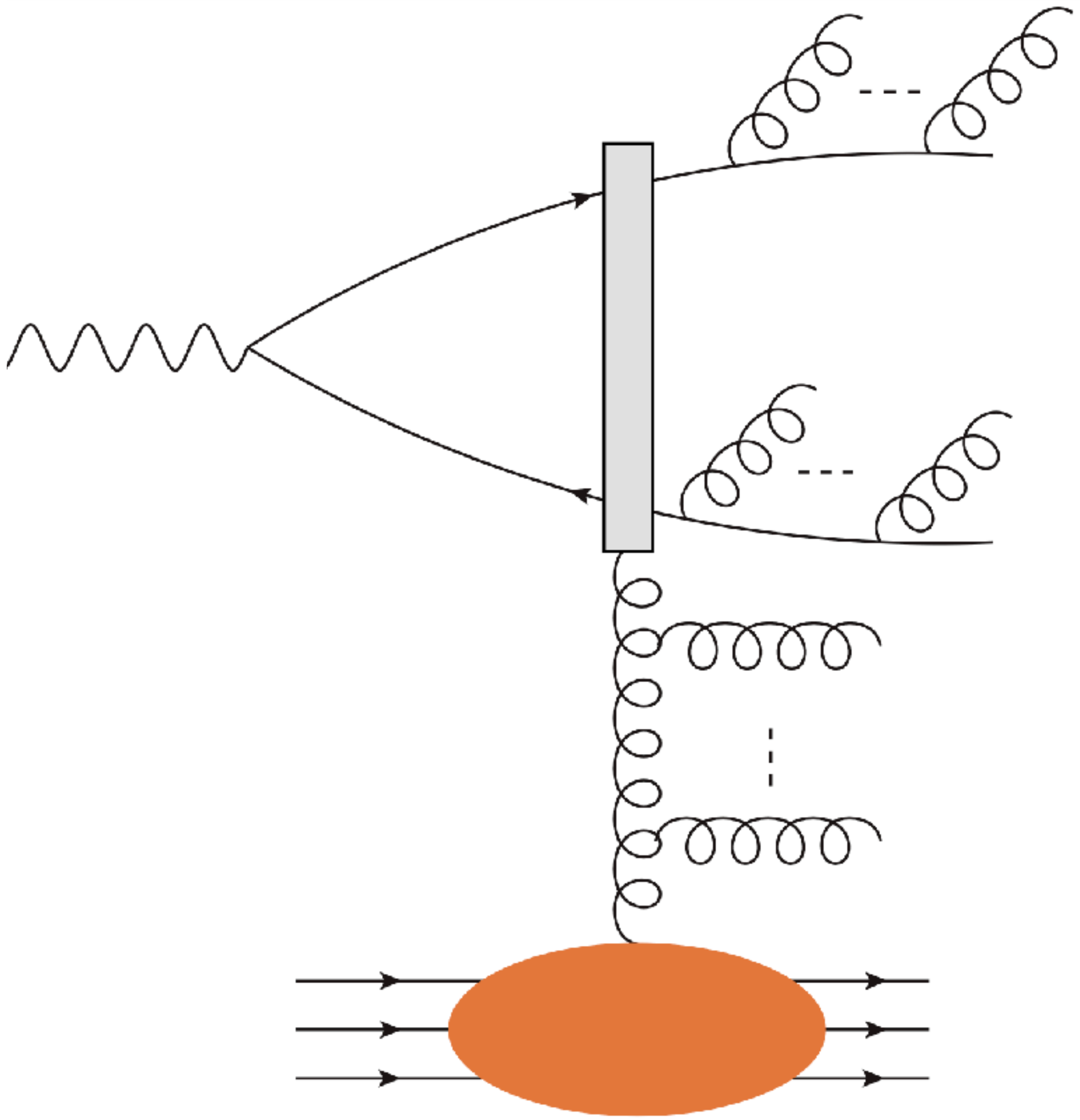
Di-jet/di-hadron production in the DIS



- Q^2 is the invariant mass, and $k_\perp$ is the total transverse momentum of dijet
- When $Q^2 \gg k_\perp^2$, two large logs emerge

$$\ln^2\left(Q^2/k_\perp^2\right)$$
$$\ln\left(Q^2/k_\perp^2\right)$$

- Two contributions: initial and final
- Final radiations can be solved by parton showers based on DGLAP (Pythia…).
- We need to address initial logs in our parton shower.

$$N(Q^2,\eta,k_\perp) = \int \frac{d^2b_\perp}{(2\pi)^2} e^{ik_\perp\cdot b_\perp} e^{-S(\mu_b^2,Q^2)} \int d^2 l_\perp e^{-il_\perp\cdot b_\perp} N(\eta,l_\perp)$$

[Mueller, Xiao, Yuan, PRL, 12; Zheng, Aschenauer, Lee, Xiao, PRD, 14; Xiao, Yuan, Zhou, NPB, 17; Caucal, Salazar, Schenke, Venugopalan, 22-23; Taels, Altinoluk, Beuf, Marquet, JHEP, 22; Mukherjee, Skokov, Tarasov, Tiwari, PRD, 23]

- We can resum both small-x and soft-collinear logarithms at the same time in a consistent way.

$$\ln(1/x) \quad \ln^2\left(Q^2/k_\perp^2\right)$$

The CS+RGE evolution equation

- Collins-Soper evolution equation [Collins, Soper, 81; Collins, Soper, Sterman, 85]

$$\frac{\partial N(\mu^2,\zeta^2,\eta,k_\perp)}{\partial \ln \zeta^2} = \frac{\bar{\alpha}_s}{2\pi}\int_0^{\zeta} \frac{d^2 l_\perp}{l_\perp^2}\left[N(\mu^2,\zeta^2,\eta,k_\perp+l_\perp) - N(\mu^2,\zeta^2,\eta,k_\perp)\right]$$

- renormalization group equation (RGE) [Xiao, Yuan, Zhou, NPB, 17]

$$\frac{\partial N(\mu^2,\zeta^2,\eta,k_\perp)}{\partial \ln \mu^2} = \bar{\alpha}_s\left[\beta_0 - \frac{1}{2}\ln\frac{\zeta^2}{\mu^2}\right] N(\mu^2,\zeta^2,\eta,k_\perp)$$

- Combine CS + RGE

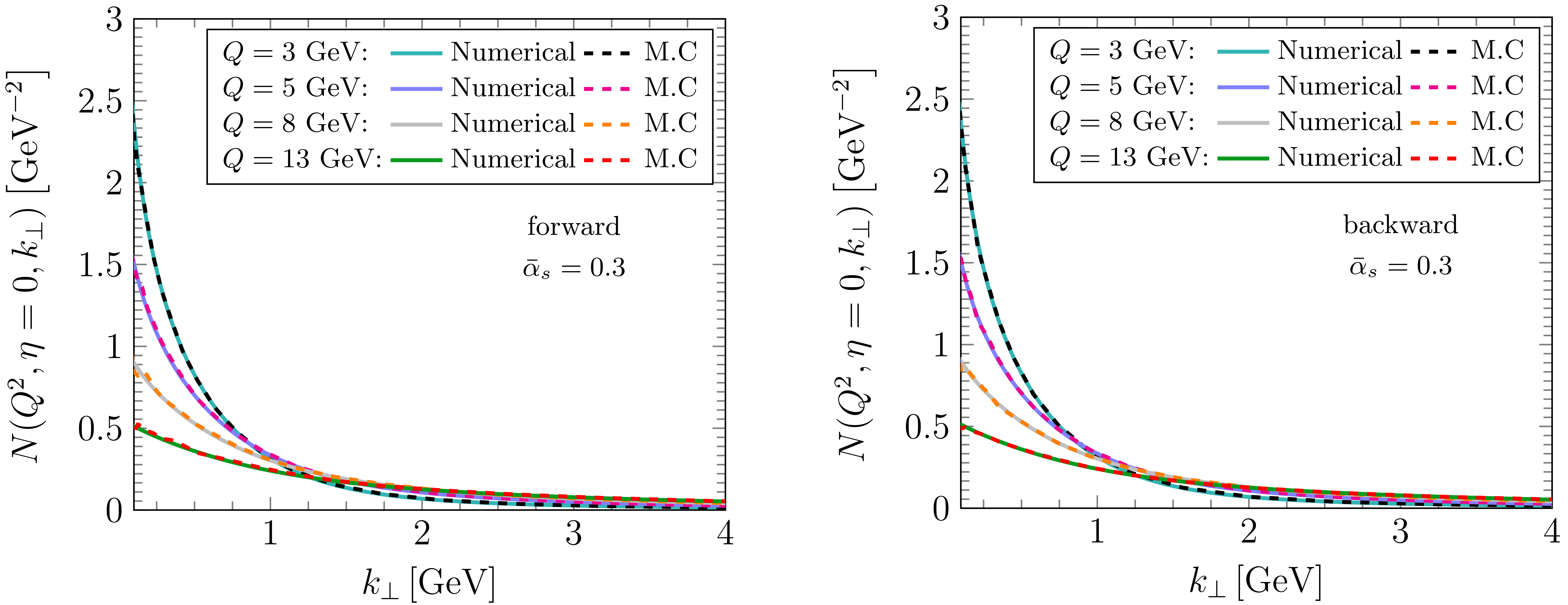
$$\frac{\partial N(Q^2,\eta,k_\perp)}{\partial \ln Q^2} = \frac{\bar{\alpha}_s}{2\pi}\int_0^{Q} \frac{d^2 l_\perp}{l_\perp^2}\left[N(Q^2,\eta,k_\perp+l_\perp) - N(Q^2,\eta,k_\perp)\right] + \bar{\alpha}_s\beta_0 N(Q^2,\eta,k_\perp)$$

where $N(Q^2,\eta,k_\perp) \equiv N(\mu^2 = Q^2,\zeta^2 = Q^2,\eta,k_\perp)$

The forward & backward evolution of CS+RGE

- The initial condition is given as

$$N(Q_0 = 3\ \mathrm{GeV}, \eta = 0, k_\perp) = \int \frac{d^2 r_\perp}{2\pi} e^{ik_\perp \cdot r_\perp} \frac{1}{r_\perp^2} \left[1 - e^{-\frac{Q_s^2 r_\perp^2}{4} \log(\frac{1}{r_\perp \Lambda} + e)}\right]$$



- Agree with the numerical solutions.

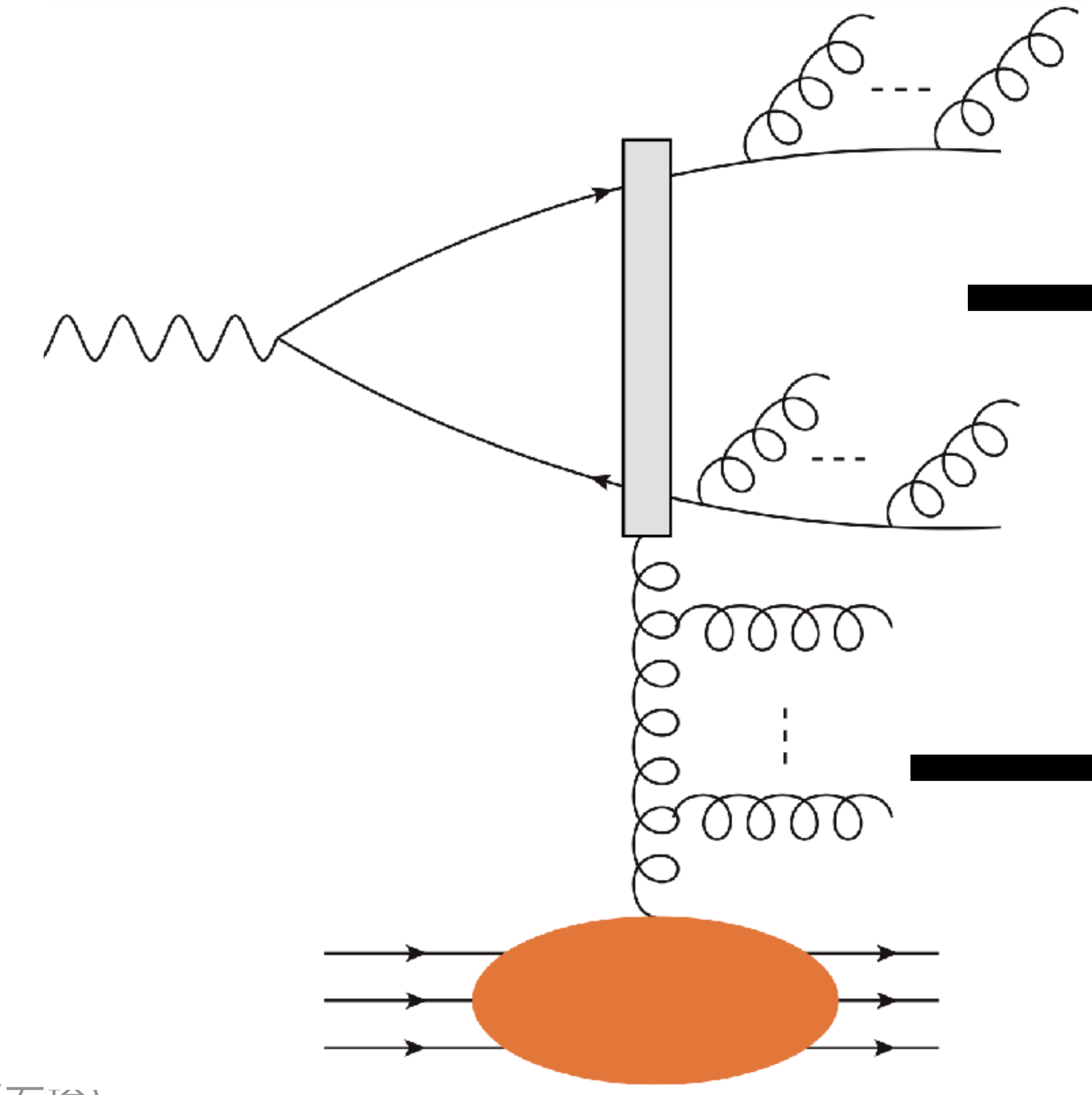
Di-jet/di-hadron production in the DIS

$$N_{\text{event}} = \mathscr{H}_{hard} \otimes \mathcal{N}(k_\perp) \otimes D(z) \otimes S_{\text{ISR}} \otimes S_{\text{FSR}} \otimes P_{\text{MPI}} \otimes P_{\text{decay}} \cdots$$

$$\frac{d\sigma^{\gamma^* A \to q\bar{q}X}}{dy_1 dy_2 d^2P_\perp d^2q_\perp} = \frac{S_\perp N_c \alpha_{\text{em}} e_q^2}{3\pi^2} x_\gamma f_\gamma(x_\gamma, \mu) \frac{z(1-z)}{P_\perp^4} \left(z^2 + (1-z)^2\right) N(x_g, q_\perp)$$

Working in progress

[Dominguez, Marquet, Xiao, Yuan, PRD, 11]



→ Pythia

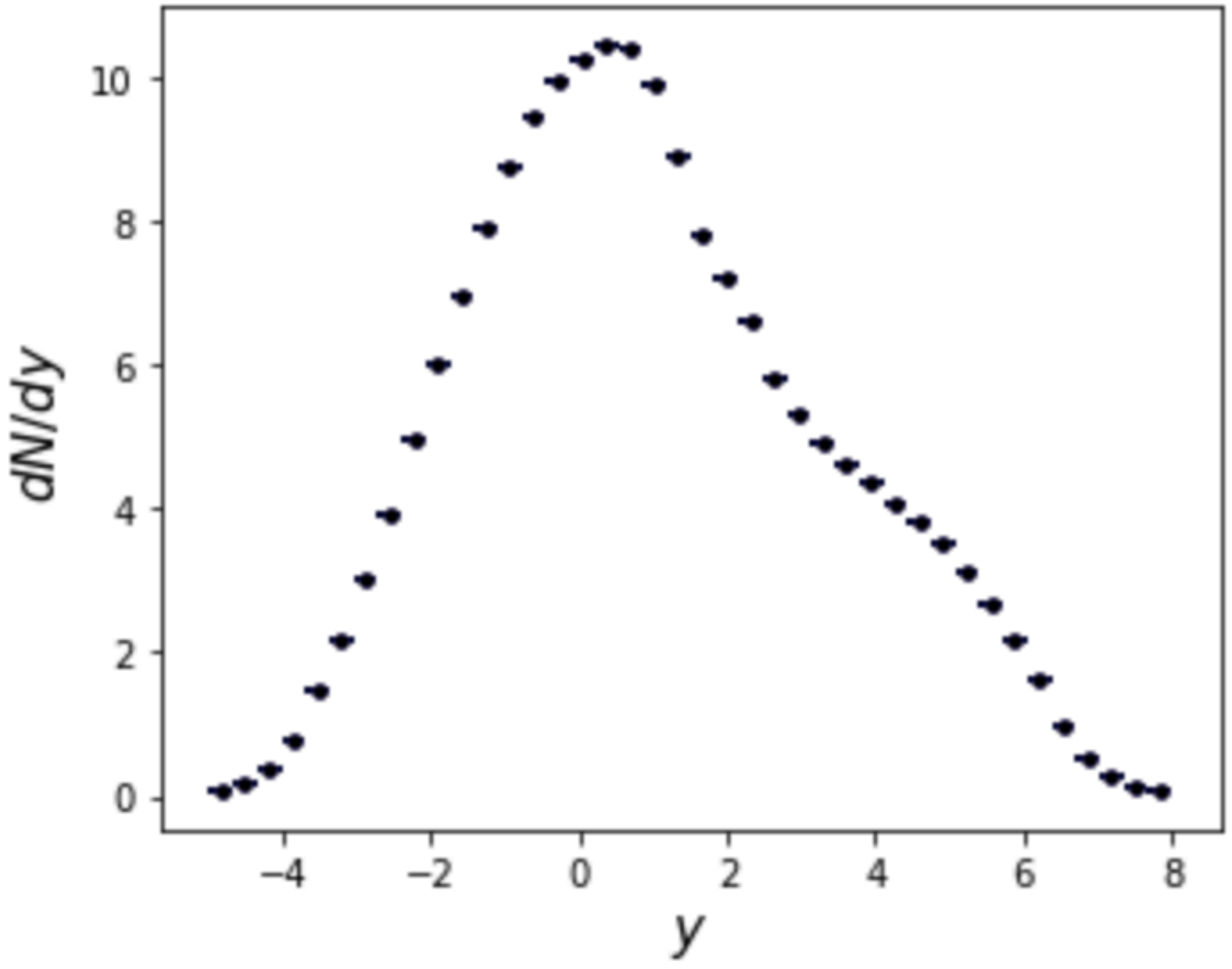
→ Cascade radiation from saturation region → Pythia hadronization

Di-jet/di-hadron production in the DIS

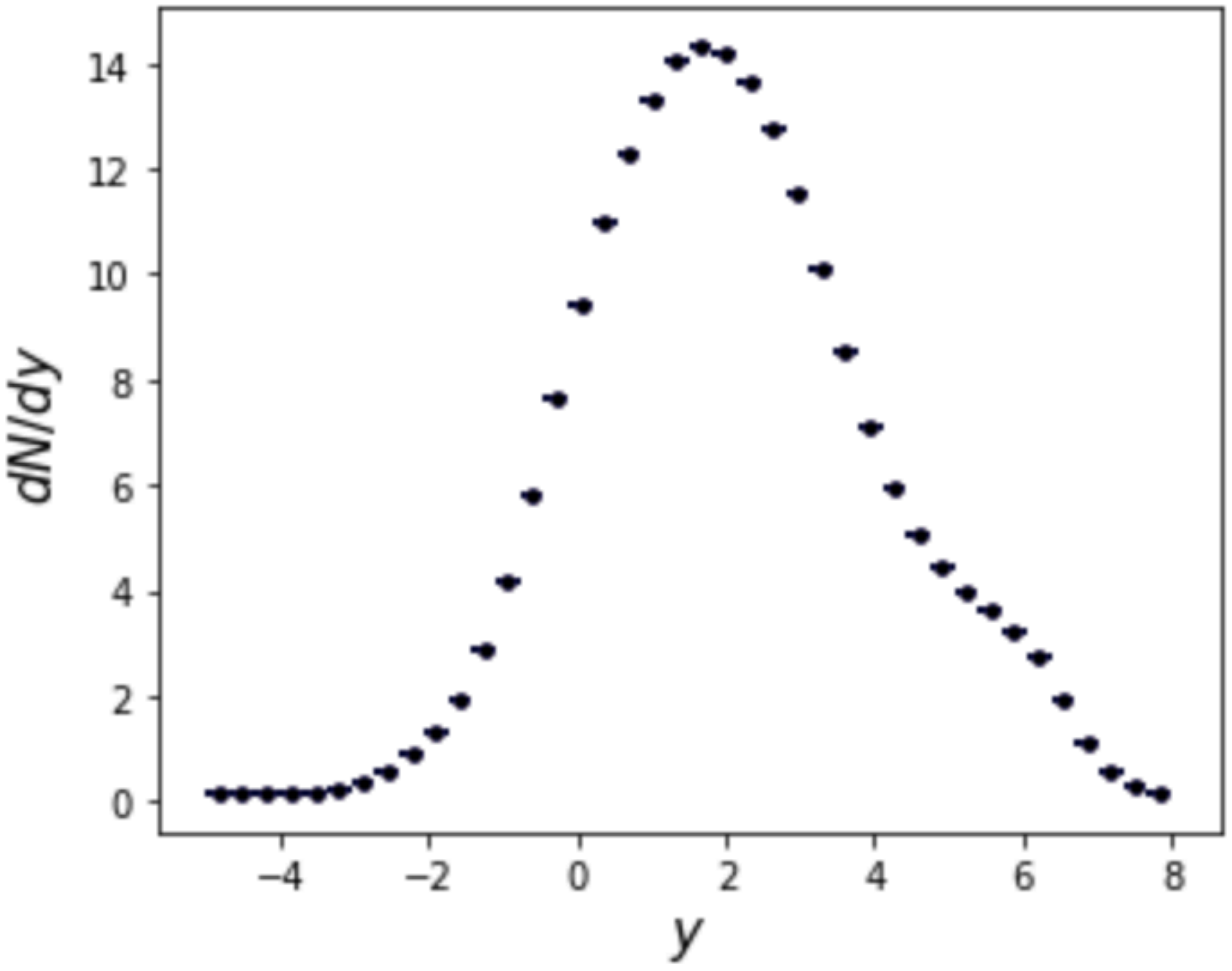
Lepton-proton collider at HERA (Photon is quasi-real photon.)

Working in progress

Preliminary results



Small-x Cascade
Pythia + hadronization



Pythia

Summary and outlook

- The first parton shower algorithm incorporating gluon fusion is based on the GLR evolution equation.
- Our work paves the way for developing an event generator that incorporates the saturation effect.
- Di-jet production in eA collisions is working in progress.
- We also plan to integrate our algorithms into eHIJING.

- How to develop a parton shower algorithm based on BK equation?

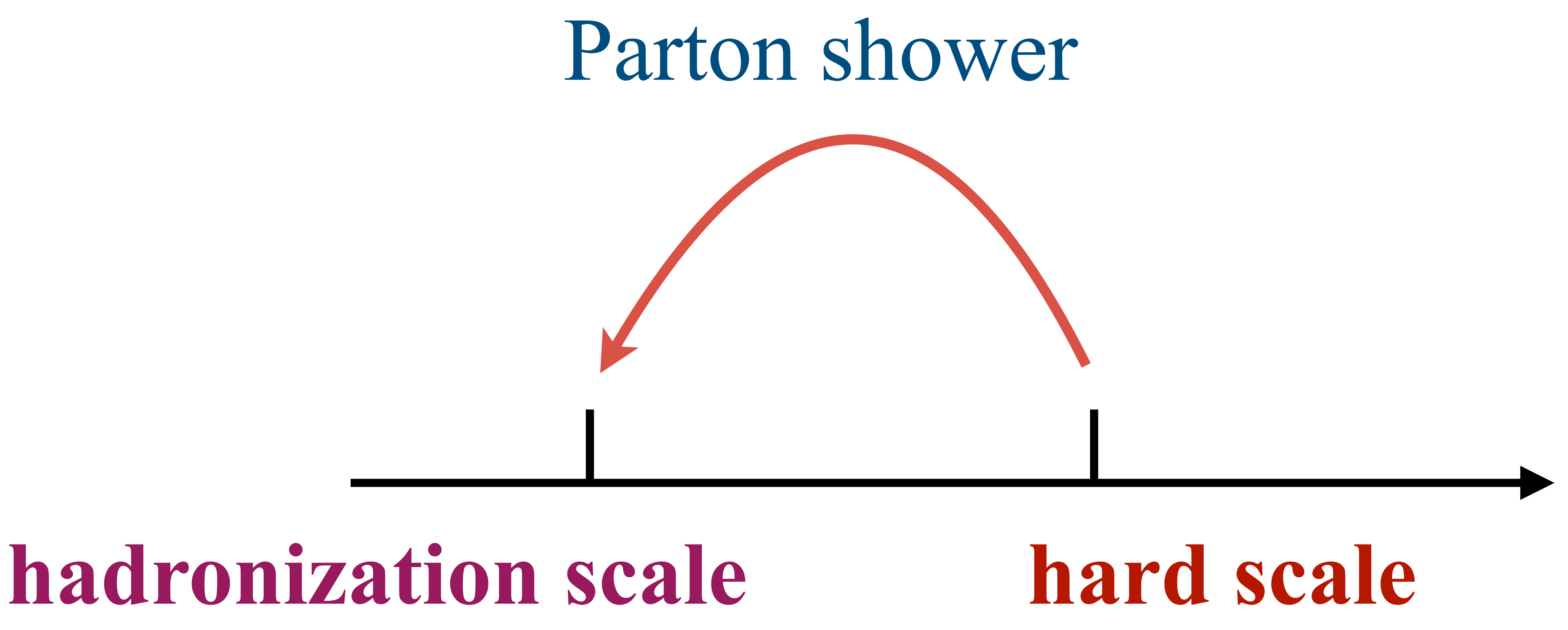
Thank you !

Backups

Parton shower algorithms in M.C. event generator

Sudakov form factor

$$\Delta_a(t,t') = \exp\left\{ - \sum_{b\in\{q,g\}} \int_t^{t'} \frac{\mathrm{d}\bar{t}}{\bar{t}} \int_{z_{\min}}^{z_{\max}} \mathrm{d}z \, \frac{\alpha_s}{2\pi} \frac{1}{2} P_{ab}(z) \right\}$$



The three commonly event generators:

HERWIG	PYTHIA	SHERPA

The corresponding parton shower algorithms:

CAT	PYSHOW	CASCADE

Based on the following evolution equation:

CCFM	DGLAP	CCFM

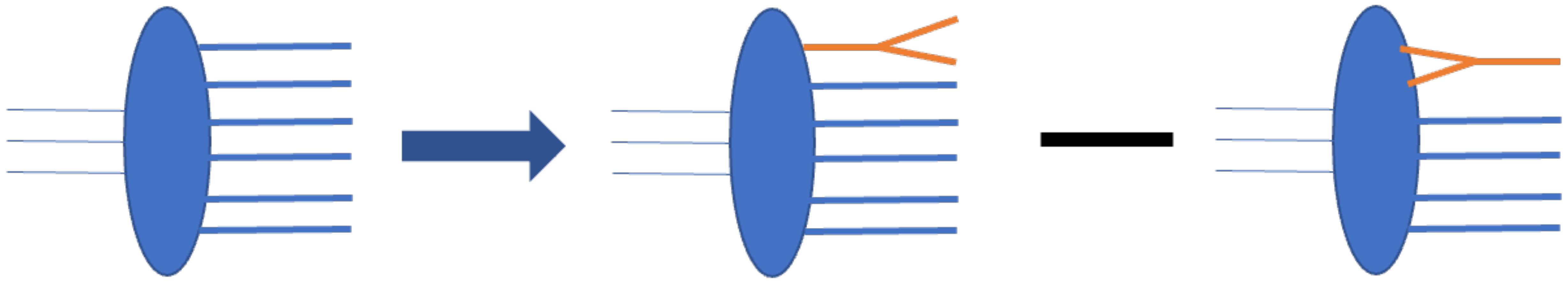
Parton shower algorithms are dedicated to simulating the radiation behavior of quarks and gluons.

Parton shower: a model for the evolution from high scale to hadronization scale based on DGLAP/CCFM.

The same physics as resummation

Non-linear evolution Equation at small-x

$\bar{\alpha}_s = \alpha_s N_c/\pi$



$\eta = (x_0/x)$

Gluon splitting   Gluon fusion $2 \to 1, 3 \to 1, 4 \to 1 \ldots$

- The impact parameter independent BK equation in momentum space is given as [Marquet, Soyez, NPA, 05]

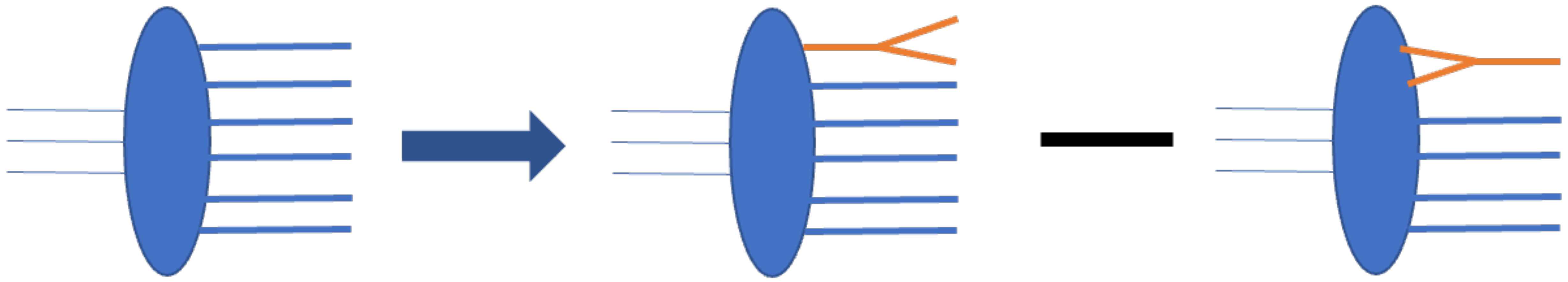
$$\frac{\partial \mathcal{N}(\eta, k_\perp)}{\partial \eta} = \frac{\bar{\alpha}_s}{\pi}\left[\int \frac{\mathrm{d}^2 l_\perp}{l_\perp^2}\mathcal{N}(\eta, l_\perp + k_\perp) - \int_0^{k_\perp} \frac{\mathrm{d}^2 l_\perp}{l_\perp^2}\mathcal{N}(\eta, k_\perp)\right] - \bar{\alpha}_s \mathcal{N}^2(\eta, k_\perp)$$

with Weizsacker-Williams (WW) Dipole distribution

$$\mathcal{N}(\eta, k_\perp) = \int \frac{\mathrm{d}^2 r_\perp}{2\pi}\frac{e^{-ik_\perp \cdot r_\perp}}{r_\perp^2}\left[1 - \frac{1}{N_c}\langle U^\dagger(0) U(r_\perp)\rangle\right]$$

Non-linear evolution Equation at small-x

$\bar{\alpha}_s = \alpha_s N_c / \pi$



$\eta = \ln(x_0/x)$

Gluon splitting   Gluon fusion $2 \to 1, 3 \to 1, 4 \to 1 \ldots$

- The impact parameter independent BK equation in coordinate space is given as [NPB 96; PRD 99]

$$\frac{\partial \mathcal{N}(\eta, r_\perp)}{\partial \ln \eta} = \frac{\bar{\alpha}_s}{2\pi} \int d^2 r_{1_\perp} \frac{r_\perp^2}{r_{1\perp}^2 r_{2\perp}^2} \left[\mathcal{N}(\eta, r_{1\perp}) + \mathcal{N}(\eta, r_{2\perp}) - \mathcal{N}(\eta, r_\perp) - \mathcal{N}(\eta, r_{1\perp})\mathcal{N}(\eta, r_{2\perp})\right]$$

$$\mathcal{N}(\eta, k_\perp) = \int \frac{\mathrm{d}^2 r_\perp}{2\pi} \frac{e^{-ik_\perp \cdot r_\perp}}{r_\perp^2} \left[1 - \frac{1}{N_c} \langle U^\dagger(0) U(r_\perp)\rangle\right] \qquad \mathcal{N}(\eta, r_\perp) = 1 - \frac{1}{N_c} \langle U^\dagger(0) U(r_\perp)\rangle$$

- The impact parameter independent BK equation in coordinate space is given as [Marquet, Soyez, NPA, 05]

$$\frac{\partial \mathcal{N}(\eta, k_\perp)}{\partial \eta} = \frac{\bar{\alpha}_s}{\pi} \left[\int \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} \mathcal{N}(\eta, l_\perp + k_\perp) - \int_0^{k_\perp} \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} \mathcal{N}(\eta, k_\perp)\right] - \bar{\alpha}_s \mathcal{N}^2(\eta, k_\perp)$$

- GLR equation is the non-linear evolution equation that describes the gluon diffusion process.

Kinematical constraint in the GLR evolution equation

- **Fixed boundary condition**: By adopting this boundary condition, we set the $k_\perp$ dependent gluon distribution to zero when $x_g > 0.01$ since this region is beyond the applicable window of the CGC calculation. This prescription is equivalent to removing all the events with $x_g > 0.01$ in our calculation.

- **Frozen boundary condition**: In this case, to extend the dipole gluon distribution in the large x_g region, we freeze it at $x_g = 0.01$. That is to say, when $x_g > 0.01$, the input dipole scattering amplitude simply retains its value at the initial condition at $x_g = 0.01$.

Fixed boundary condition: forward

First step: non-Sudakov form factor

$$\mathcal{R} = \exp\left[-\bar{\alpha}_s \int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta' \left(\ln \frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp)\right)\right]$$

Second step: Real splitting kernel

$$\mathcal{R} = \frac{1}{\mathcal{C}} \frac{\bar{\alpha}_s}{\pi} \int_{\Lambda_{\text{cut}}}^{l_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} \exp\left\{-\bar{\alpha}_s \int_{\eta_i}^{\eta_{i+1}+\ln\frac{(k_{\perp,i}-l'_\perp)^2}{(k_{\perp,i}-l'_\perp)^2+l'^2_\perp}} \mathrm{d}\eta \left[\ln \frac{k_{\perp,i}^2}{\Lambda_{\text{cut}}^2} + N(\eta, k_{\perp,i})\right]\right\},$$

$$\mathcal{C} = \frac{\bar{\alpha}_s}{\pi} \int_{\Lambda_{\text{cut}}}^{\min[P_\perp, \sqrt{(k_{\perp,i}-l'_\perp)^2 \frac{1-z}{z}}]} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} \exp\left\{-\bar{\alpha}_s \int_{\eta_i}^{\eta_{i+1}+\ln\frac{(k_{\perp,i}-l'_\perp)^2}{(k_{\perp,i}-l'_\perp)^2+l'^2_\perp}} \mathrm{d}\eta \left[\ln \frac{k_{\perp,i}^2}{\Lambda_{\text{cut}}^2} + N(\eta, k_{\perp,i})\right]\right\},$$

The generated event has to be re-weighted

$$\mathcal{W}_{kc,1}(\eta_i, \eta_{i+1}; k_{\perp,i}) = \frac{(\eta_{i+1}-\eta_i) \int_{\Lambda_{\text{cut}}}^{\min\left[P_\perp, \sqrt{\frac{1-z}{z}(k_{\perp,i}-l_\perp)^2}\right]} \frac{\mathrm{d}^2 l_\perp}{l_\perp^2} e^{-\bar{\alpha}_s \int_{\eta_{i+1}}^{\eta_{i+1}+\ln\frac{(k_{\perp,i}-l_\perp)^2}{(k_{\perp,i}-l_\perp)^2+l_\perp^2}} \mathrm{d}\eta \left[\ln \frac{k_{\perp,i}^2}{\Lambda_{\text{cut}}^2} + N(\eta, k_{\perp,i})\right]}}{(\eta_{i+1}-\eta_i) \ln \frac{k_{\perp,i}^2}{\Lambda_{\text{cut}}^2} + \int_{\eta_i}^{\eta_{i+1}} d\eta N(\eta, k_{\perp,i})}$$

Frozen boundary condition: forward

- First step: non-Sudakov form factor

$$\mathcal{R} = \exp\left[-\bar{\alpha}_s \int_{\eta_i}^{\eta_{i+1}} \mathrm{d}\eta' \left(\ln \frac{k_\perp^2}{\mu^2} + N(\eta', k_\perp)\right)\right]$$

- Second step: Real splitting kernel

$$\mathcal{R}_2 \int_{\mu}^{P_\perp} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp} = \int_{\mu}^{|l_\perp|} \frac{\mathrm{d}^2 l'_\perp}{l'^2_\perp}$$

- The generated event has to be re-weighted

$$\mathcal{W}_{kc,2}(\eta_i, \eta_{i+1}; k_{\perp,i}, k_{\perp,i+1}) = \frac{(\eta_{i+1} - \eta_i) \ln \frac{P_\perp^2}{\Lambda_{\mathrm{cut}}^2}}{(\eta_{i+1} - \eta_i) \ln \frac{k_{\perp,i}^2}{\Lambda_{\mathrm{cut}}^2} + \int_{\eta_i}^{\eta_{i+1}} d\eta N(\eta, k_{\perp,i})} \frac{N\left(\eta_i + \ln\left[\frac{k_{\perp,i+1}^2}{k_{\perp,i+1}^2 + l_\perp^2}\right], k_{\perp,i}\right)}{N(\eta_i, k_{\perp,i})}$$

Frozen boundary condition: backward

- First step: backward non-Sudakov form factor

$$\Pi_{ns}(\eta_{i+1},\eta_i;k_{\perp,i+1}) = \exp\left[-\frac{\bar{\alpha}_s}{\pi}\int_{\eta_i}^{\eta_{i+1}} d\eta \int_{\Lambda_{\rm cut}}^{P_\perp} \frac{d^2 l_\perp}{l_\perp^2} \frac{N\left(\eta+\ln\left[\frac{k^2_{\perp,i+1}}{k^2_{\perp,i+1}+l^2_\perp}\right], k_{\perp,i+1}+l_\perp\right)}{N(\eta, k_{\perp,i+1})}\right]$$

- Second step: Real splitting kernel

$$\mathcal{R} = \frac{1}{\mathcal{C}}\frac{\bar{\alpha}_s}{\pi}\int_{\Lambda_{\rm cut}}^{l_\perp} \frac{d^2 l'_\perp}{l'^2_\perp} N\left(\eta_{i+1}+\ln\left[\frac{k^2_{\perp,i+1}}{k^2_{\perp,i+1}+l'^2_\perp}\right], k_{\perp,i+1}+l'_\perp\right)$$

$$\mathcal{C} = \frac{\bar{\alpha}_s}{\pi}\int_{\Lambda_{\rm cut}}^{P_\perp} \frac{d^2 l'_\perp}{l'^2_\perp} N\left(\eta_{i+1}+\ln\left[\frac{k^2_{\perp,i+1}}{k^2_{\perp,i+1}+l'^2_\perp}\right], k_{\perp,i+1}+l'_\perp\right).$$

- The generated event has to be re-weighted

$$\mathcal{W}_{\rm backward} = \frac{1}{\mathcal{W}_{forward}}$$

The forward evolution of CS+RGE

- The integral equation (folded one)

$$N(Q^2,\eta,k_\perp) = N(Q_0^2,\eta,k_\perp)\Delta_s(Q^2) + \int_{Q_0^2}^{Q^2}\frac{dt}{t}\frac{\Delta_s(Q^2)}{\Delta_s(t)}\frac{\bar\alpha_s(t)}{2\pi}\int_{\Lambda_{\rm cut}}^{Q}\frac{d^2l_\perp}{l_\perp^2}N(t,\eta,k_\perp+l_\perp)$$

With Sudakov form factor $\Delta_s(Q^2) = \exp\left[-\int_{Q_0^2}^{Q^2}\frac{dt}{t}\frac{\bar\alpha_s(t)}{2}\left(\ln\frac{t}{\Lambda_{\rm cut}^2} - 2\beta_0\right)\right]$

First step: Sudakov form factor

$$\mathcal{R} = \exp\left[-\int_{Q_i^2}^{Q_{i+1}^2}\frac{dt}{t}\bar\alpha_s(t)\left(\frac{1}{2}\ln\frac{t}{\Lambda_{\rm cut}^2} - \beta_0\right)\right]$$

Second step: Real splitting kernel

$$\int_{\Lambda_{\rm cut}}^{Q_{i+1}}\frac{d^2l'_\perp}{l'^2_\perp} = \mathcal{R}_2\int_{\Lambda_{\rm cut}}^{|l_\perp|}\frac{d^2l'_\perp}{l'^2_\perp}$$

The generated event has to be re-weighted

$$\mathcal{W}_{\rm CS}(Q_{i+1}^2,Q_i^2) = \frac{\int_{Q_i^2}^{Q_{i+1}^2}\frac{dt}{t}\alpha_s(t)\ln\frac{t}{\Lambda_{\rm cut}^2}}{\int_{Q_i^2}^{Q_{i+1}^2}\frac{dt}{t}\alpha_s(t)\left[\ln\frac{t}{\Lambda_{\rm cut}^2} - 2\beta_0\right]}$$

Ignoring the single log, the event is unitary.

The backward evolution of CS+RGE

- First step: Sudakov form factor

$$\mathcal{R} = \exp\left[-\int_{Q_i^2}^{Q_{i+1}^2} \frac{dt}{t}\frac{\bar{\alpha}_s(t)}{2\pi}\int_{\Lambda_{\rm cut}}^{\sqrt{t}} \frac{d^2 l_\perp}{l_\perp^2}\frac{N(t,\eta,k_{\perp,i+1}+l_\perp)}{N(t,\eta,k_{\perp,i+1})}\right]$$

- Second step: Real splitting

$$\mathcal{R}\int_{\Lambda_{\rm cut}}^{Q_i} \frac{d^2 l'_\perp}{l'^2_\perp} N(Q_i^2,\eta,k_{\perp,i+1}+l'_\perp) = \int_{\Lambda_{\rm cut}}^{l_{\perp,i}} \frac{d^2 l'_\perp}{l'^2_\perp} N(Q_i^2,\eta,k_{\perp,i+1}+l'_\perp)$$

- The generated event has to be re-weighted

$$\mathcal{W}_{\rm CS,back}(Q_{i+1}^2,Q_i^2) = \frac{\int_{Q_i^2}^{Q_{i+1}^2}\frac{dt}{t}\alpha_s(t)\left[\ln\frac{t}{\Lambda_{\rm cut}^2}-2\beta_0\right]}{\int_{Q_i^2}^{Q_{i+1}^2}\frac{dt}{t}\alpha_s(t)\ln\frac{t}{\Lambda_{\rm cut}^2}}$$