A semi-analytical x-space solution for parton evolution

Application to non-singlet and singlet DGLAP equation

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JH, Oliver Schüle, Fabian Wunder, arXiv:2404.18667

Outline

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2. Semi-analytical x-space solution for parton evolution
3. Numerical Analysis
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1. Motivation

Our idea

- x-space ansatz for parton distributions
- Overcomplete family of spanning functions closed under evolution equation $\left(\frac{\ln^m(x)x^n}{m!}\text{ for DGLAP}\right)$
- Grasp analytic behaviour in x generated by evolution equation
- Transforms evolution equation into ODE for scale-dependent coefficients
- Truncate ODE to finite subsystem, solve numerically

Simplest example: Test idea on LO DGLAP equation

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Example: DGLAP equation

- Evolution basis

$$q_{\mathrm{ns},\,ij} \equiv q_i - q_j \qquad \text{and} \qquad q_{\mathrm{s}} \equiv \sum_{i=1}^{n_{\mathrm{f}}} [q_i + \bar{q}_i]$$

- Non-singlet DGLAP equation

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2}\, q_{\mathrm{ns}}\left(\mu^2, x\right) = \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{\mathrm{d}\xi}{\xi}\, P_{qq}(\xi)\, q_{\mathrm{ns}}\left(\mu^2, \frac{x}{\xi}\right)$$

- Singlet DGLAP equation

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu^2} \begin{pmatrix} q_{\mathrm{s}}\left(\mu^2, x\right) \\ g\left(\mu^2, x\right) \end{pmatrix} = \frac{\alpha_s\left(\mu^2\right)}{2\pi} \int_x^1 \frac{\mathrm{d}\xi}{\xi} \begin{pmatrix} P_{qq}(\xi) & 2n_{\mathrm{f}} P_{qg}(\xi) \\ P_{gq}(\xi) & P_{gg}(\xi) \end{pmatrix} \begin{pmatrix} q_{\mathrm{s}}\left(\mu^2, \frac{x}{\xi}\right) \\ g\left(\mu^2, \frac{x}{\xi}\right) \end{pmatrix}$$

Existing methods

Mellin-space methods:

- Analytic solution in Mellin-space, numerical transformation to x-space
  `partonevolution`, `QCD-PEGASUS`, `EKO`

x-space methods:

- Discretization in x with interpolation ansatz to transform evolution equation into ODE
  `QCDnum`, `APFEL`, `HOPPET`, `ChiliPDF`
- Discretization in x and μ, differentiation and integration discretized, solved step-by-step
  `BF1`, `BFP1`
- Expand PDF and evolution kernels in Laguerre polynomials in $\ln(x)$, yields (truncated) sum over Laguerre coefficients, evolve recursively
  `LAG1`, `LAG2NS`

2. Semi-analytical x-space solution for parton evolution

General idea

- General integro-differential equation with integral operator $\mathbf{P}\otimes$ in $\vec{x}$-space

$$\frac{\mathrm{d}}{\mathrm{d}\mu}\mathbf{f}(\mu,\vec{x}) = (\mathbf{P}\otimes\mathbf{f})(\mu,\vec{x}) \tag{1}$$

- Ansatz with suitable set of spanning functions $\mathbf{f}_m$

$$\mathbf{f}(\mu,\vec{x}) = \sum_m a_m(\mu)\,\mathbf{f}_m(\vec{x})$$

- Transform (1) into ODE for coefficients

$$\frac{\mathrm{d}}{\mathrm{d}\mu}a_m(\mu) = \mathcal{P}_{mn}(\mu)\,a_n(\mu)\,.$$

- Truncate to finite system and solve numerically
- Evolution matrix $\mathcal{P}$ can be applied to any initial condition

Solving the DGLAP equation

- Ansatz: $$q_{\rm ns}(\mu^2, x) = \sum_{m=0,\, n=0,-1}^{\infty} a_{mn}(\mu^2)\, \frac{\ln^m(x)\, x^n}{m!}$$
- Plug into LO DGLAP equation
$$\sum_{m,n=0}^{\infty} \frac{{\rm d}a_{mn}(\mu^2)}{{\rm d}\ln\mu^2} \frac{\ln^m(x)\, x^n}{m!} = \frac{\alpha_s(\mu^2)}{2\pi} C_{\rm F} \sum_{m,n=0}^{\infty} a_{mn}(\mu^2)$$
$$\times \left[\frac{1}{x} I_1^{m,n+1} + x\, I_1^{m,n-1} + 2\, I_2^{m,n} + 2\, \ln(1-x)\, \frac{\ln^m(x)\, x^n}{m!} + \frac{3}{2} \frac{\ln^m(x)\, x^n}{m!} \right]$$
- Collect wrt spanning functions
$$\frac{{\rm d}a_{MN}(\mu^2)}{{\rm d}\ln\mu^2} = \frac{\alpha_s(\mu^2)}{2\pi} \mathcal{P}^{ij}_{MN}\, a_{ij}(\mu^2)$$
- Truncate infinite system of equations
$$q(\mu^2, x) = \sum_{m=0}^{M_{\rm max}} \sum_{n=0,-1}^{N_{\rm max}(m)} a_{mn}(\mu^2)\, \frac{\ln^m(x)\, x^n}{m!}$$

Solving the DGLAP equation

- Differential equation for the coefficients

$$\frac{\mathrm{d}a_{MN}\left(\mu^2\right)}{\mathrm{d}\ln\mu^2} = \frac{\alpha_s\left(\mu^2\right)}{2\pi}\,\mathcal{P}^{ij}_{MN}\,a_{ij}\left(\mu^2\right)$$

- Solved numerically, e.g. by matrix exponential

$$a_{MN}\left(\mu^2\right) = \prod_{k=0}^{K}\exp\left[\Omega\left(\mu^2_{k+1},\mu^2_k\right)\right]a_{MN}\left(\mu^2_0\right)\,,\quad \mu^2_K = \mu^2$$

- With Magnus expansion, $\Omega = \Omega_1 + \Omega_2 + \ldots,$

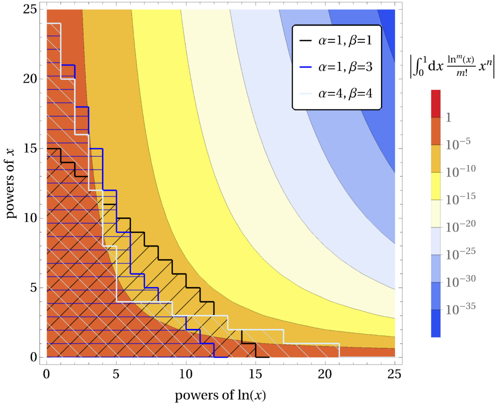
$$\Omega_1\left(\mu^2,\mu^2_0\right) = \int_{\ln\mu^2_0}^{\ln\mu^2}\mathrm{d}\ln\mu^2_1\left(\alpha_s\left(\mu^2_1\right)\mathcal{P}^{(0)} + \alpha^2_s\left(\mu^2_1\right)\mathcal{P}^{(1)} + \alpha^3_s\left(\mu^2_1\right)\mathcal{P}^{(2)} + \mathcal{O}\left(\alpha^4_s\right)\right)$$

$$\Omega_2\left(\mu^2,\mu^2_0\right) = \frac{1}{2}\int_{\ln\mu^2_0}^{\ln\mu^2}\mathrm{d}\ln\mu^2_1\int_{\ln\mu^2_0}^{\ln\mu^2_1}\mathrm{d}\ln\mu^2_2\,\alpha_s\left(\mu^2_1\right)\alpha^2_s\left(\mu^2_2\right)\left[\mathcal{P}^{(0)},\,\mathcal{P}^{(1)}\right] + \mathcal{O}\left(\alpha^4_s\right)$$

LO, NLO, NNLO contribution

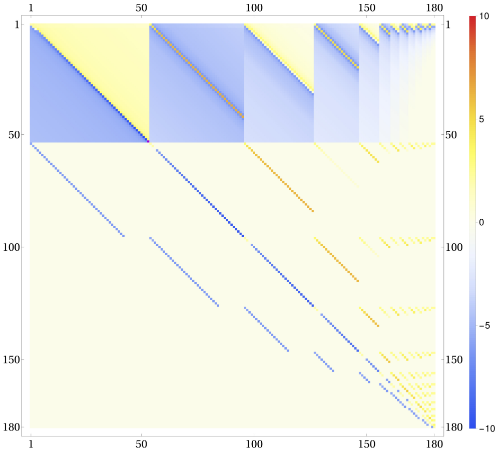
3. Numerical Analysis

Contribution of different basis functions



$$N_{\max}^{\alpha,\beta}(m) = \begin{cases} \left\lfloor \frac{M_{\max} - m}{\alpha} \right\rfloor & \text{if } (\alpha+1)\,m > M_{\max} \\ \left\lfloor -\beta\, m + \frac{1+\beta}{1+\alpha} M_{\max} \right\rfloor & \text{else} \end{cases}$$

Magnitude of entries in the non-singlet evolution matrix



- Ordered by (m, n), i.e. $(1, x, x^2, \ldots, \ln(x), x\ \ln(x), \cdots)$
- Cut-off parameters $(\alpha, \beta) = (2, 11)$

Choice of basis cut-off

Numerical performance may be spoiled by

(1) omitting "large" basis functions

(2) omitting matrix elements strongly coupling to initial condition

(3) large coefficients / very large number of non-negligible coefficients in initial conditions

- Choose suitable basis functions to avoid (3)
- (1) and (2) determine sensible cut-off
- Finite cut-off always leads to (1), precise evolution possible if omitted basis functions do not couple strongly to initial conditions
- Rescaling basis functions shuffles overall factors between (1) - (3)

$\Longrightarrow$ Check numerical performance for sufficiently realistic example

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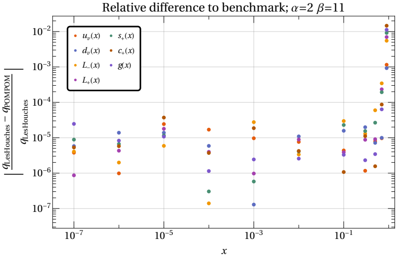
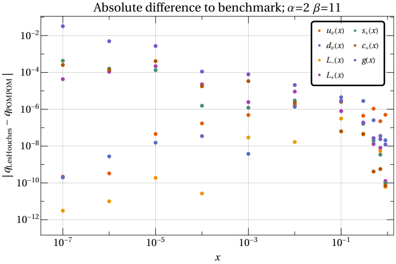
⟹ Check numerical performance for sufficiently realistic example

Comparison to Les Houches benchmark tables

- Benchmark values for checking PDF evolution codes [hep-ph/02043160]
- Parametrize input distributions in terms of spanning functions
- Evolve from $\mu_0^2 = 2\,\text{GeV}^2$ to $\mu^2 = 10^4\,\text{GeV}^2$ using `POMPOM` Mathematica code for LO DGLAP evolution





Check sum rule violation

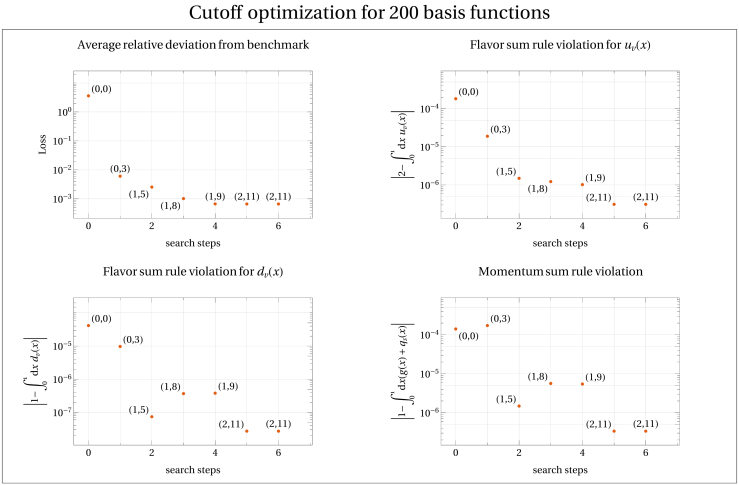
- Self-contained measure for quality of evolution
- Momentum sum rule

$$\int_0^1 \mathrm{d}x\, x\,(q_{\mathrm{s}}(x) + g(x)) = 1$$

- Flavor sum rule

$$\int_0^1 \mathrm{d}x\, q_{v,i}(x) = \int_0^1 \mathrm{d}x\, (q_i(x) - \overline{q}_i(x)) = \text{number of valence quarks } i$$

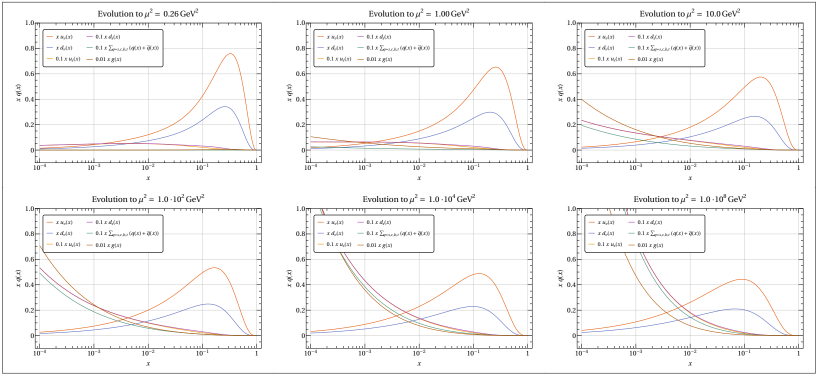
Comparison to Les Houches benchmark tables



- Use gradient descent on cutoff parameters (α, β)
- Tested for 50, 100, 150, 200 basis functions

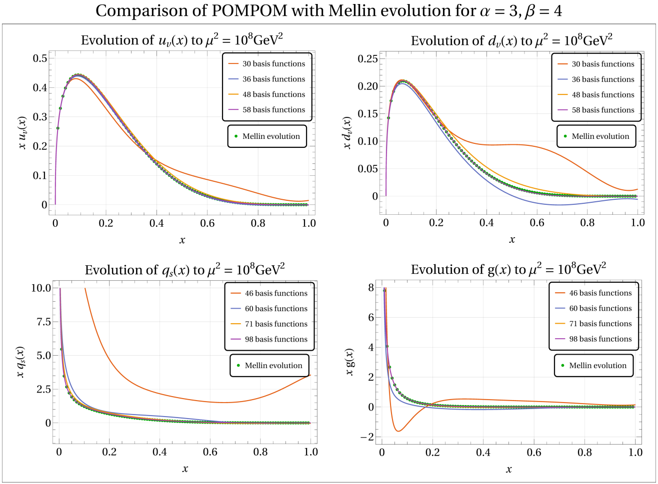
Evolution of a full set of PDFs

DGLAP evolution with POMPOM: 200 basis functions, $\alpha = 4, \beta = 3$



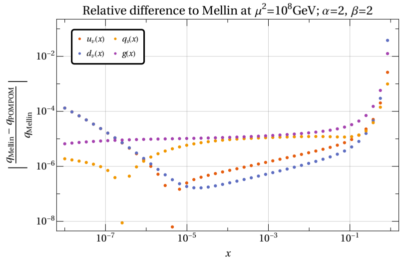
- GRV98 input distributions at $\mu_0^2 = 0.26\,\mathrm{GeV}^2$ [hep-ph/9806404]
- Sum-rule violation as metric to optimize cutoff
- Variable flavor number scheme

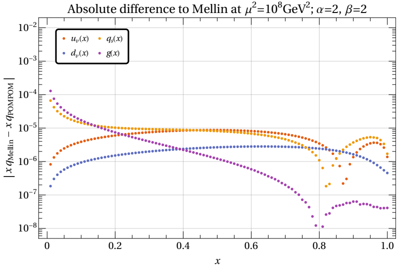
Comparison to Mellin-space evolution



- $\sim$ 50 basis functions valence-like distributions and $\sim$ 100 for sea (additional $1/x$ terms for sea)

Comparison to Mellin-space evolution

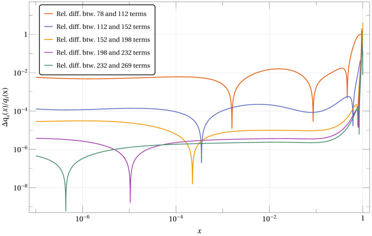


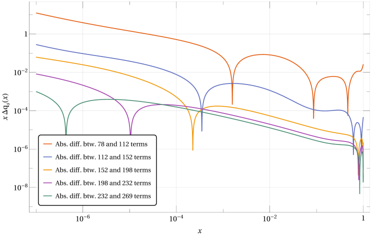


- 200 basis functions

Difference between singlet PDFs $x\, q_s(x)$

- Evolve from scale $\mu_0^2 = 0.26\,\mathrm{GeV}^2$ to $\mu^2 = 10^8\,\mathrm{GeV}^2$
- Cut-off parameters $\alpha = \beta = 2$





4. Conclusion and Outlook

Conclusion and Outlook

- Good agreement with Les Houches benchmarks and Mellin evolution
- Sum rule violations $< 10^{-6}$ for $\mathcal{O}(200)$ basis functions
- Truncation effects under control, can be systematically improved
- `POMPOM`: `Mathematica` and `Python` implementation of LO DGLAP

- Higher order DGLAP evolution
- Apply POMPOM-method to similar evolution equations
- Improve cut-off optimization

5. Backup slides

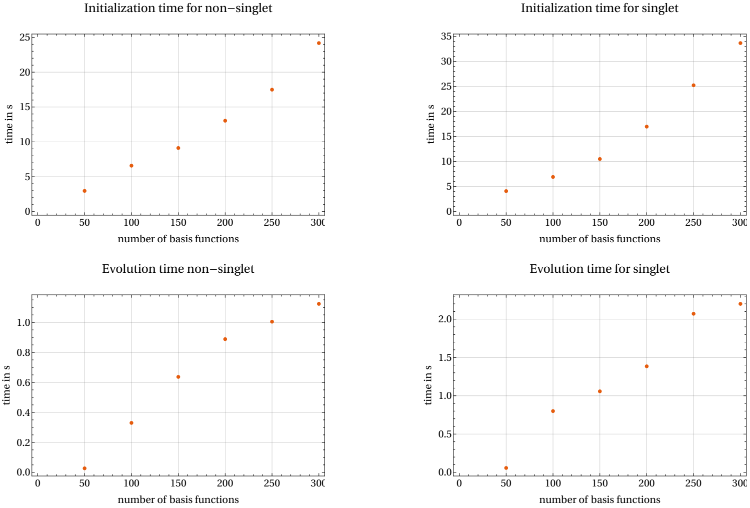
Relation to Mellin-space

- Mellin transform $\mathcal{M}$ of the non-singlet pdf

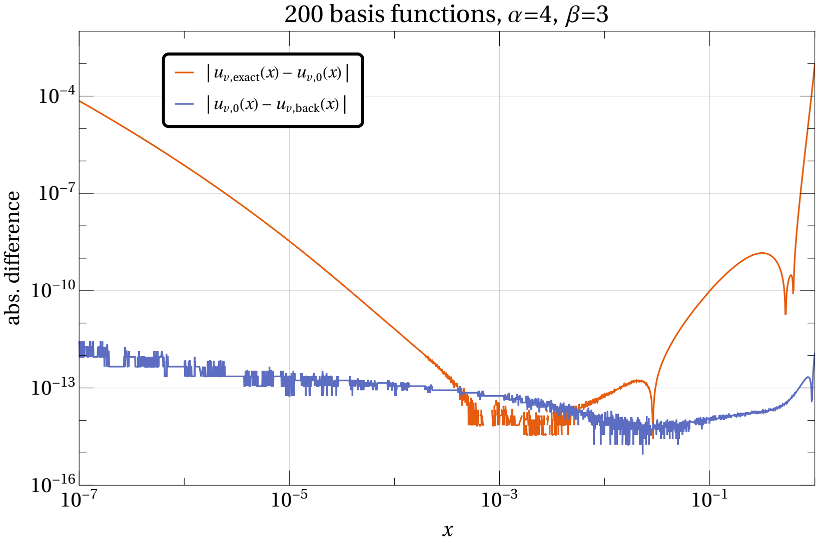
$$\mathcal{M}[q_{\text{ns}}](\mu^2, s) = \int_0^1 \mathrm{d}x\, x^{s-1}\, q_{\text{ns}}(\mu^2, x) = \sum_{m,n=0}^{\infty} a_{mn}(\mu^2)\, \frac{(-1)^m}{(n+s)^{m+1}}$$

- Coefficients a_{mn} in Mellin-space identical to coefficients in x-space
- Scale evolution decouples from switch x-space $\leftrightarrow$ Mellin-space
- Spanning functions $\frac{(-1)^m}{(n+s)^{m+1}}$ are meromorphic functions of s on $\mathbb{C}$, poles on real axis for $s = n$
- Exploit to obtain simple analytic form of Mellin transform of PDF

Timing


Initialization time for non−singlet
time in s
number of basis functions
Initialization time for singlet
time in s
number of basis functions
Evolution time non−singlet
time in s
number of basis functions
Evolution time for singlet
time in s
number of basis functions

Back Evolution



GRV98 input distributions and parameters

- LO input distributions at $\mu_0^2 = 0.26\,\mathrm{GeV}^2$ $(\Delta \equiv \bar{d} - \bar{u})$

$$
\begin{aligned}
x\,u_v(x,\mu_0^2) &= 1.239\,x^{0.48}\,(1-x)^{2.72}\,(1 - 1.8\sqrt{x} + 9.5x) \\
x\,d_v(x,\mu_0^2) &= 0.614\,(1-x)^{0.9}\,x\,u_v(x,\mu_0^2) \\
x\,\Delta(x,\mu_0^2) &= 0.23\,x^{0.48}\,(1-x)^{11.3}\,(1 - 12.0\sqrt{x} + 50.9x) \\
x\,(\bar{u}+\bar{d})(x,\mu_0^2) &= 1.52\,x^{0.15}\,(1-x)^{9.1}\,(1 - 3.6\sqrt{x} + 7.8x) \\
x\,g(x,\mu_0^2) &= 17.47\,x^{1.6}\,(1-x)^{3.8} \\
x\,s(x,\mu_0^2) &= x\,\bar{s}(x,\mu_0^2) = 0
\end{aligned}
$$

- Running coupling $\alpha_s(\mu^2) = \frac{4\pi}{\beta_0}\ \ln\left(\frac{\mu^2}{\Lambda^2_{\mathrm{QCD}}}\right)$ with $\beta_0 \equiv 11 - \frac{2}{3} n_f$
- QCD mass scale $\Lambda_{\mathrm{QCD}}^{(n_f=3,4,5,6)} = 204, 175, 132, 66.5\,\mathrm{MeV}$
- Thresholds for $n_f = 4, 5, 6$ at $\mu^2 = 1.96, 20.25, 30625\,\mathrm{GeV}^2$

Master integrals

- $$I_1^{m,n} \equiv x^n \int_x^1 \mathrm{d}\xi \, \frac{\xi^{-n}-1}{1-\xi} \, \frac{\ln^m(x/\xi)}{m!}$$
$$= \begin{cases} \sum_{k=0}^{-n-1} \left[\sum_{j=0}^{m} \frac{1}{(k+1)^{m+1-j}} \frac{\ln^j(x)\, x^n}{j!} - \frac{x^{k+n+1}}{(k+1)^{m+1}} \right] & n \le -2 \\ -\sum_{k=0}^{m} \frac{\ln^k(x)}{k!} + x & n = -1 \\ 0 & n = 0 \\ (-1)^m \sum_{k=1}^{n-1} \frac{x^{-k}}{k^{m+1}} - \sum_{k=0}^{m} (-1)^k H_{n-1,k+1} \frac{\ln^{m-k}(x)}{(m-k)!} - \frac{\ln^{m+1}(x)}{(m+1)!} & n \ge 1 \end{cases}$$

- $$I_2^{m,n} \equiv x^n \int_x^1 \frac{\mathrm{d}\xi}{(1-\xi)} \, \frac{\ln^m(x/\xi) - \ln^m(x)}{m!}$$
$$= \sum_{k=0}^{m-1} \zeta_{m-k+1} \frac{\ln^k(x)}{k!} + \frac{\ln^m(x)}{m!} \sum_{k=0}^{\infty} \frac{x^k}{k} - \sum_{k=1}^{\infty} \frac{x^k}{k^{m+1}}$$

- $$I_3^{m,n} \equiv x^n \int_x^1 \mathrm{d}\xi \, \frac{\ln^m(x/\xi)\, \xi^{-n}}{m!}$$
$$= \begin{cases} \sum_{j=0}^{m} \frac{1}{(1-n)^{m+1-j}} \frac{\ln^j(x)\, x^n}{j!} - \frac{x}{(1-n)^{m+1}} & n \ne 1 \\ -\frac{\ln^{m+1}(x)\, x}{(m+1)!} & n = 1 \end{cases}$$

ODE for LO non-singlet coefficients

- $$\frac{\mathrm{d}\, a_{MN}(\mu^2)}{\mathrm{d}\ln\mu^2} = \frac{\alpha_s(\mu^2)\, C_F}{2\pi}\left[\sum_{m>M}\alpha(N, m-M)a_{mN} + \beta(N)\, a_{MN} + \gamma(N)\left(\delta_{M0}\sum_{\substack{m,n\\ n\neq N}}\frac{a_{mn}}{(N-n)^{m+1}} + a_{M-1,N}\right)\right]$$

- $$\alpha(N,l) \equiv \begin{cases} 2\zeta_{l+1} - 1 & N = 0 \\ 2\zeta_{l+1} - (-1)^l & N = 1 \\ 2\zeta_{l+1} - (-1)^l\,(H_{N,l+1} + H_{N-2,l+1}) & N \geq 2 \end{cases}$$

- $$\beta(N) \equiv \begin{cases} \frac{1}{2} & N = 0,1 \\ \frac{3}{2} - H_N - H_{N-2} & N \geq 2 \end{cases}$$

- $$\gamma(N) \equiv \begin{cases} -1 & N = 0,1 \\ -2 & N \geq 2 \end{cases}$$

- $H_{n,m} \equiv \sum_{k=1}^{n} \frac{1}{k^m}$ and $H_n \equiv H_{n,1} = \sum_{k=1}^{n}\frac{1}{k}$