Exclusive and semi-inclusive diffractive processes in the shockwave approach

Michael Fucilla

Université Paris-Saclay, CNRS/IN2P3, IJCLab

in collaboration with

R. Boussarie A. V. Grabovsky, E. Li, L. Szymanowski, S. Wallon

based on

[M. F., Grabovsky, Li, Szymanowski, Wallon, JHEP 03 (2024) 159]
[M. F., Grabovsky, Li, Szymanowski, Wallon, JHEP 04 (2023) 137]
[R. Boussarie, M. F., L. Szymanowski, S. Wallon (to appear)]

QCD evolution (2024), Pavia, 27-31 May 2024

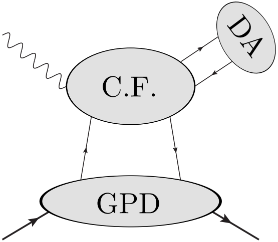
This project is supported by the European Union's Horizon 2020 research and innovation programme under Grant agreement no. 824093

Deeply virtual meson production (DVMP)

- Exclusive ρ-meson leptoproduction

  $\gamma^{(*)}(p_\gamma) + P(p_0) \to \rho(p_\rho) + P(p_0')$

- Extensively studied at HERA



- NLO corrections to the production of a longitudinally polarized ρ-meson at small-x

  [Ivanov, Kotsky, Papa (2004)]

  [Boussarie, Grabovsky, Ivanov, Szymanowski, Wallon (2017)]

  [Mäntysaari, Pentalla (2022)]

- Transversally polarized ρ-meson production start at the **twist-3**

  [Diehl, Gousset, Pire (1999)] **[Collins, Diehl (2020)]**

- Collinear treatment at the twist-3 leads to end point singularities

  [Mankiewicz, Piller (2000)] **[Anikin, Teryaev (2002)]**

Transversely polarized ρ-meson production

- Momentum space impact factor for the exclusive ρ-meson production at the twist-3 in the dilute limit (BFKL scheme) and forward case

  [Anikin, Ivanov, Pire, Szymanowski, Wallon (2009)]

- Phenomenological studies at small-x

  [Besse, Szymanowski, Wallon (2013)]

  [Bolognino, Celiberto, Ivanov, Papa (2018)]

  [Bolognino, Szczurek, Schäfer (2019)]

  i. Restricted to s-channel helicity conserving (SCHC) amplitudes

- Exclusive light-meson production at the **twist-3** within the Shockwave approach

  [Boussarie, M.F. , Szymanowski, Wallon (to appear)]

  i. Saturation corrections to DVMP in the transversely polarized case
  ii. Both forward and non-forward results
  iii. Beyond SCHC
  iiii. Coordinate and momentum space representations
  iiiii. Linearization **[Caron-Huot (2013)]** $\Longrightarrow$ BFKL results

Theoretical framework

- Effective background field operator formalism of small-x physics

[Balitsky (2001)]

$$[\psi_{\text{eff}}\,(z_0)]_{z_0^+<0} = \psi\,(z_0) - \int \mathrm{d}^D z_2 G_0\,(z_{02}) \left(V^\dagger_{\boldsymbol{z}_2} - 1\right) \gamma^+ \psi\,(z_2)\,\delta(z_2^+)$$

$$[\bar{\psi}_{\text{eff}}\,(z_0)]_{z_0^+<0} = \bar{\psi}\,(z_0) + \int \mathrm{d}^D z_1 \bar{\psi}\,(z_1)\,\gamma^+\,(V_{\boldsymbol{z}_1} - 1)\,G_0\,(z_{10})\,\delta(z_1^+)$$

$$[A^{\mu a}_{\text{eff}}\,(z_0)]_{z_0^+<0} = A^{\mu a}\,(z_0) + 2i \int \mathrm{d}^D z_3 \delta(z_3^+) F^b_{-\sigma}\,(z_3)\,G^{\mu\sigma_\perp}\,(z_{30}) \left(U^{ab}_{\boldsymbol{z}_3} - \delta^{ab}\right)$$

- Higher-twist formalisms

  i. **Covariant collinear factorization**

  [Braun, Filyanov (1990)]

  [Ball, Braun, Koike, Tanaka (1998)]

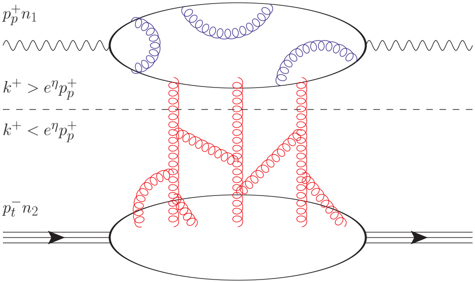
  ii. Light-cone collinear factorization

  [Ellis, Furmanwski, Petronzio (1982)]

  [Anikin, Teryaev (2002)]

Shockwave approach

- High-energy approximation $s = (p_p + p_t)^2 \gg \{Q^2\}$



$$p_p = p_p^+ n_1 - \frac{Q^2}{2p_p^+} n_2$$

$$p_t = \frac{m_t^2}{2p_t^-} n_1 + p_t^- n_2$$

$$p_p^+ \sim p_t^- \sim \sqrt{\frac{s}{2}}$$

$$n_1^2 = n_2^2 = 0 \quad n_1 \cdot n_2 = 1$$

- Separation of the gluonic field into “fast” (quantum) part and “slow” (classical) part through a rapidity parameter $\eta < 0$

[I. Balitsky (1996-2001)]

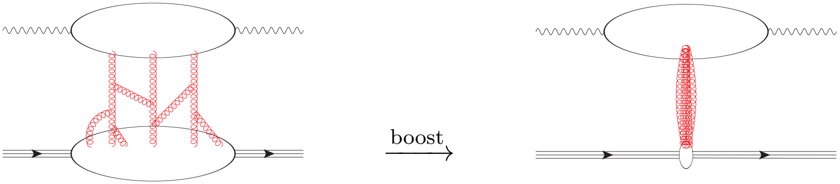
$$\mathcal{A}^\mu(k^+, k^-, \vec{k}) = A^\mu(k^+ > e^\eta p_p^+, k^-, \vec{k}) + b^\mu(k^+ < e^\eta p_p^+, k^-, \vec{k})$$

$e^\eta \ll 1$

Shockwave approach

- Large longitudinal Boost: $\Lambda = \sqrt{\frac{1+\beta}{1-\beta}} \sim \frac{\sqrt{s}}{m_t}$

$$\begin{cases} b^+(x^+,x^-,\vec{x}) & = \Lambda^{-1} b_0^+(\Lambda x^+, \Lambda^{-1}x^-, \vec{x}) \\ b^-(x^+,x^-,\vec{x}) & = \Lambda b_0^-(\Lambda x^+, \Lambda^{-1}x^-, \vec{x}) \\ b^i(x^+,x^-,\vec{x}) & = b_0^i(\Lambda x^+, \Lambda^{-1}x^-, \vec{x}) \end{cases}$$



$b_0^\mu(x)$

$$b^\mu(x^+,x^-,\vec{x}) = \delta(x^+)\mathbf{B}(\vec{x})n_2^\mu + \mathcal{O}\left(\Lambda^{-1}\right)$$

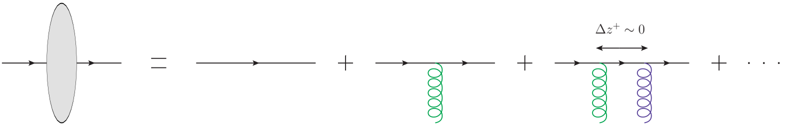
Shockwave approximation

- Light-cone gauge $A \cdot n_2 = 0$

$$A \cdot b = 0 \implies \textit{Simple effective Lagrangian}$$

Shockwave approach

- Interactions with the simple shockwave field
  - *i.* Independence on x^- $\implies$ conservation of p^+ (eikonal approximation)
  - *ii.* $\delta(x^+)$ $\implies$ interactions at a single transverse coordinate.
- Quark line through the shockwave



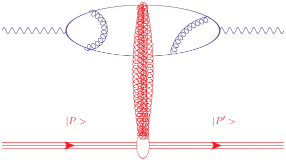
$$V_{\vec{z}_i} = 1 + ig\int_{-\infty}^{+\infty} dz_i^+ b_\eta^-\left(z_i^+, \vec{z}_i\right) + (ig)^2\int_{-\infty}^{+\infty} dz_i^+ dz_j^+ b_\eta^-\left(z_i^+, \vec{z}_i\right) b_\eta^-\left(z_j^+, \vec{z}_i\right)\theta\left(z_{ij}^+\right) + \cdots$$

- Multiple interactions with the target $\rightarrow$ ***path-ordered Wilson lines***

$$V_{\vec{z}}^{\eta} = \mathcal{P}\exp\left[ig\int_{-\infty}^{+\infty} dz_i^+ b_\eta^-\left(z_i^+, \vec{z}\right)\right]$$

Shockwave approach

- Factorization in the Shockwave approximation

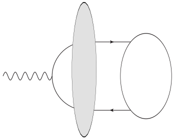


$$\mathcal{M}^\eta = N_c \int d^d \boldsymbol{z}_1 d^d \boldsymbol{z}_2 \, \Phi^\eta(\boldsymbol{z}_1, \boldsymbol{z}_2) \langle P' \left| \mathcal{U}_{12}^\eta(\boldsymbol{z}_1, \boldsymbol{z}_2) \right| P \rangle$$

- **Dipole operator** $\mathcal{U}_{ij}^\eta = 1 - \frac{1}{N_c} \mathrm{Tr}\left(V_{\vec{z}_i}^\eta V_{\vec{z}_j}^{\eta\dagger}\right)$
- Evolution equations
  - **Balitsky-JIMWLK** evolution equations
    [Balitsky (1995)]
    [Jalilian-Marian, Iancu, McLerran, Weigert, Kovner, Leonidov]
  - Large N_c → **Balitky-Kovchegov (BK)** non-linear equation
    [Balitsky (1995)] [Kovchegov (1999)]
  - Evolution at the NLO
    [Balitsky, Chirilli (2007)] [Kovner, Lublinsky, Mulian (2013)]

ρ-meson production: diagrams

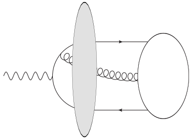
- **2-body contribution**



i. Dependence of the leading Fock state wave function – with a minimal number of (valence) partons – on transverse momentum

$$\mathcal{A}_2 = -ie_f \int \mathrm{d}^D z_0 \theta(-z_0^+) \left\langle P\left(p'\right) M\left(p_M\right) \middle| \overline{\psi}_{\text{eff}}\left(z_0\right) \hat{\varepsilon}_q \mathrm{e}^{-i(q\cdot z_0)} \psi_{\text{eff}}\left(z_0\right) \middle| P\left(p\right) \right\rangle$$

- **3-body contribution**

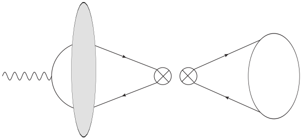


i. Distribution with a non-minimal parton configuration

$$\mathcal{A}_{3,q} = \left(-ie_q\right)\left(ig\right) \int \mathrm{d}^D z_4 \mathrm{d}^D z_0 \theta(-z_4^+)\theta(-z_0^+)$$
$$\times \left\langle P\left(p'\right) M\left(p_M\right) \middle| \overline{\psi}_{\text{eff}}\left(z_4\right) \gamma_\mu A^{\mu a}_{\text{eff}}(z_4) t^a G(z_{40}) \hat{\varepsilon}_q \mathrm{e}^{-i(q\cdot z_0)} \psi_{\text{eff}}\left(z_0\right) \middle| P\left(p\right) \right\rangle$$

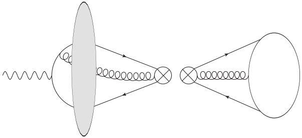
ρ-meson production: factorization

- **2-body contribution**



$$\mathcal{A}_2 = ie_f \int \mathrm{d}^D z_0 \int \mathrm{d}^D z_1 \int \mathrm{d}^D z_2 \theta(-z_0^+)\delta(z_1^+)\delta(z_2^+) \left\langle M(p_M) \left| \mathrm{tr}_c \overline{\psi}(z_1)\Gamma^\lambda \psi(z_2) \right| 0 \right\rangle$$
$$\times \left\langle P(p') \left| 1 - \frac{1}{N_c}\mathrm{tr}\left(V_{\boldsymbol{z}_1} V^\dagger_{\boldsymbol{z}_2}\right) \right| P(p) \right\rangle \frac{1}{4}\mathrm{tr}_D \left[\gamma^+ G_0(z_{10})\hat{\varepsilon}_q \mathrm{e}^{-i(q\cdot z_0)} G_0(z_{02})\gamma^+\Gamma_\lambda\right]$$

- **3-body contribution**



$$\mathcal{A}_{q3} = -ie_q \int \mathrm{d}^D z_4 \mathrm{d}^D z_3 \mathrm{d}^D z_2 \mathrm{d}^D z_1 \mathrm{d}^D z_0 \theta(-z_4^+)\delta(z_3^+)\delta(z_2^+)\delta(z_1^+)\theta\left(-z_0^+\right)\mathrm{e}^{-i(q\cdot z_0)}$$
$$\times \left\langle P(p') \left| \mathrm{tr}\left(V_{\boldsymbol{z}_1} t^a V^\dagger_{\boldsymbol{z}_2} t^b U^{ab}_{\boldsymbol{z}_3}\right) \right| P(p) \right\rangle \left\langle M(p_M) \left| \overline{\psi}(z_1)\Gamma^\lambda g F_{-\sigma}(z_3)\psi(z_2) \right| 0 \right\rangle$$
$$\times \frac{1}{N_c^2 - 1}\mathrm{tr}_D \left[\gamma^+ G_0(z_{14})\gamma_\mu G^{\mu\sigma_\perp}(z_{34}) G_0(z_{40})\hat{\varepsilon}_q G_0(z_{02})\gamma^+\Gamma_\lambda\right] - \text{n.i.}$$

Results: 2-body contribution

- Dipole amplitude

$$\mathcal{A}_2 = \int_0^1 \mathrm{d}x \int \mathrm{d}^2\boldsymbol{r}\Psi\left(x,\boldsymbol{r}\right) \int \mathrm{d}^d\boldsymbol{b}\, \mathrm{e}^{i(\boldsymbol{q}-\boldsymbol{p}_M)\cdot\boldsymbol{b}} \left\langle P\left(p'\right) \middle| 1 - \frac{1}{N_c}\mathrm{tr}\left(V_{\boldsymbol{b}+\bar{x}\boldsymbol{r}} V^\dagger_{\boldsymbol{b}-x\boldsymbol{r}}\right) \middle| P\left(p\right) \right\rangle$$

- Wavefunction overlap

$$\Psi_2\left(x,\boldsymbol{r}\right) = e_q \delta \left(1 - \frac{p_M^+}{q^+}\right)\left(\varepsilon_{q\mu} - \frac{\varepsilon_q^+}{q^+} q_\mu\right)$$

$$\times \left[\phi_{\gamma^+}(x,\boldsymbol{r})\left(2x\bar{x}q^\mu - i(x-\bar{x})\frac{\partial}{\partial r_{\perp\mu}}\right) + \epsilon^{\mu\nu+-}\phi_{\gamma^+\gamma^5}(x,\boldsymbol{r})\frac{\partial}{\partial r_\perp^\nu}\right] K_0\left(\sqrt{x\bar{x}Q^2\boldsymbol{r}^2}\right)$$

- 2-body matrix elements

$$\phi_{\gamma^+}(x,\boldsymbol{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dr^- e^{ixp_M^+ r^-} \left\langle M\left(p_M\right) \middle| \overline{\psi}\left(r\right)\gamma^+\psi\left(0\right) \middle| 0 \right\rangle_{r^+=0}$$

$$\phi_{\gamma^+\gamma^5}(x,\boldsymbol{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dr^- e^{ixp_M^+ r^-} \left\langle M\left(p_M\right) \middle| \overline{\psi}\left(r\right)\gamma^+\gamma^5\psi\left(0\right) \middle| 0 \right\rangle_{r^+=0}$$

Results: 3-body contribution

- 3-body amplitude

$$\mathcal{A}_3 = \left(\prod_{i=1}^{3}\int dx_i\theta(x_i)\right)\delta(1-x_1-x_2-x_3)\int d^2\boldsymbol{z}_1 d^2\boldsymbol{z}_2 d^2\boldsymbol{z}_3 e^{i\boldsymbol{q}(x_1\boldsymbol{z}_1+x_2\boldsymbol{z}_2+x_3\boldsymbol{z}_3)}$$

$$\times\Psi_3\left(x_1,x_2,x_3,\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{z}_3\right)\left\langle P\left(p'\right)\middle|\mathcal{U}_{\boldsymbol{z}_1\boldsymbol{z}_3}\mathcal{U}_{\boldsymbol{z}_3\boldsymbol{z}_2}-\mathcal{U}_{\boldsymbol{z}_1\boldsymbol{z}_3}-\mathcal{U}_{\boldsymbol{z}_3\boldsymbol{z}_2}+\frac{1}{N_c^2}\mathcal{U}_{\boldsymbol{z}_1\boldsymbol{z}_2}\middle|P\left(p\right)\right\rangle$$

- Wavefunction overlap

$$\Psi_3\left(x_1,x_2,x_3,\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{z}_3\right)=\frac{e_q q^+}{2(4\pi)}\frac{N_c^2}{N_c^2-1}\left(\varepsilon_{q\rho}-\frac{\varepsilon_q^+}{q^+}q_\rho\right)$$

$$\times\left\{\chi_{\gamma^+\sigma}\left[\left(4ig_{\perp\perp}^{\rho\sigma}\frac{x_1x_2}{1-x_2}\frac{Q}{Z}K_1(QZ)+T_1^{\sigma\rho\nu}(x_1,x_2,x_3)\frac{z_{23\perp\nu}}{\boldsymbol{z}_{23}^2}K_0(QZ)\right)-(1\leftrightarrow 2)\right]\right.$$

$$\left.-\chi_{\gamma^+\gamma^5\sigma}\left[\left(4\epsilon^{\sigma\rho+-}\frac{x_1x_2}{1-x_2}\frac{Q}{Z}K_1(QZ)+T_2^{\sigma\rho\nu}(x_1,x_2,x_3)\frac{z_{23\perp\nu}}{\boldsymbol{z}_{23}^2}K_0(QZ)\right)+(1\leftrightarrow 2)\right]\right\}$$

- 3-body matrix elements

$$\chi_{\Gamma^\lambda,\sigma}\equiv\chi_{\Gamma^\lambda,\sigma}(x_1,x_2,x_3,\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{z}_3)=$$

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}z_1^-}{2\pi}\frac{\mathrm{d}z_2^-}{2\pi}\frac{\mathrm{d}z_3^-}{2\pi}\mathrm{e}^{-ix_1q^+z_1^- - ix_2q^+z_2^- - ix_3q^+z_3^-}\left\langle M\left(p_M\right)\middle|\overline{\psi}\left(z_1\right)\Gamma^\lambda gF_{-\sigma}\left(z_3\right)\psi\left(z_2\right)\middle|0\right\rangle_{z_{1,2,3}^+=0}$$

Covariant collinear factorization

- Light-cone collinear factorization

  [Ellis, Furmanwski, Petronzio (1982)] [Anikin, Teryaev (2002)]

  i. Factorization in mom. space around the dominant light-cone direction

  ii. Overcomplete set of distributions → reduced by QCD eqs. of motion and invariance of the amplitude under rotation on the light-cone

  [Anikin, Ivanov, Pire, Szymanowski, Wallon (2009)]

- **Covariant collinear factorization**

  [Ball, Braun, Koike, Tanaka (1998)]

  i. Minimal basis of independent distributions

  ii. Minimal numbers of parameters

  iii. Easy to perform the calculation directly into coordinate space

- 2- and 3-body operators in gauge invariant form

$$\langle M(p_M)|\overline{\psi}(z)\Gamma_\lambda\,[z,0]\,\psi(0)|0\rangle$$

$$\langle M(p_M)|\overline{\psi}(z)\gamma_\lambda\,[z,tz]\,gF^{\mu\nu}(tz)\,[tz,0]\,\psi(0)|0\rangle$$

$$\langle M(p_M)|\overline{\psi}(z)\gamma_\lambda\,[z,tz]\,g\tilde{F}^{\mu\nu}(tz)\,[tz,0]\,\psi(0)|0\rangle$$

where

$$[z,0]=\mathcal{P}_{\exp}\left[ig\int_0^1 dt A^\mu\,(tz)\,z_\mu\right]$$

- Expansion of the matrix elements in powers of deviation from the light-cone $z^2=0$

  [Balitsky, Braun (1989)]

Covariant collinear factorization

- Matrix elements without gauge links can be easily related to the fully gauge invariant one within **twist-3** accuracy
- Three-body matrix element

$$\left\langle M(p)\left|\overline{\psi}\left(z_q\right)\gamma^\lambda gF^{\mu\nu}\left(z_g\right)\psi\left(z_{\overline{q}}\right)\right|0\right\rangle \simeq \left\langle M(p)\left|\overline{\psi}(z_q)[z_q,z_g]\gamma^\lambda gF^{\mu\nu}(z_g)[z_g,z_{\overline{q}}]\psi(z_{\overline{q}})\right|0\right\rangle_{z_i^+=0}$$

- Two-body matrix element

$$\left\langle M\left(p_M\right)\left|\overline{\psi}\left(r\right)\left[r,0\right]\Gamma^\lambda\psi\left(0\right)\right|0\right\rangle_{r^+=0} \longleftrightarrow \left\langle M\left(p_M\right)\left|\overline{\psi}\left(r\right)\Gamma^\lambda\psi\left(0\right)\right|0\right\rangle_{r^+=0}$$

- In the two body case we need to expand the gauge link to relate the two matrix elements

$$[z,0]=\mathcal{P}\exp\left[ig\int_0^1 dtA^\mu(tz)z_\mu\right]\simeq 1+ig\int_0^1 dtA^\mu\left(tz\right)z_\mu+\text{h.t.}$$

- In a given n light-cone gauge

$$A^\mu(z)=\int_0^\infty \mathrm{d}\sigma\,\mathrm{e}^{-\epsilon\sigma}n_\nu F^{\mu\nu}(z+\sigma n)$$

- Then the difference between two matrix elements is parameterized in terms of a 3-body contribution

Dilute regime: 2-body

- Reggeon definition **[Caron-Huot (2013)]** $R^a(\boldsymbol{z}) \equiv \frac{f^{abc}}{gC_A} \ln\left(U_{\boldsymbol{z}}^{bc}\right)$
- Expansion of the Wilson line in Reggeons

$$V_{\boldsymbol{z}_1} = 1 + ig\boldsymbol{t}^a R^a(\boldsymbol{z}_1) - \frac{1}{2} g^2 \boldsymbol{t}^a \boldsymbol{t}^b R^a(\boldsymbol{z}_1) R^b(\boldsymbol{z}_1) + O\left(g^3\right)$$

- BFKL factorization

$$\mathcal{A}_2^{\text{dilute}} = \frac{g^2}{4N_c}(2\pi)^d \delta^d(\boldsymbol{q} - \boldsymbol{p}_M - \boldsymbol{\Delta}) \int \frac{\mathrm{d}^d \ell}{(2\pi)^d} \mathcal{U}(\ell) \int_0^1 \mathrm{d}x$$
$$\times \underbrace{\left[\Phi_2\left(x, \boldsymbol{\ell} - \frac{x - \bar{x}}{2}\boldsymbol{\Delta}\right) + \Phi_2\left(x, -\boldsymbol{\ell} - \frac{x - \bar{x}}{2}\boldsymbol{\Delta}\right) - \Phi_2(x, \bar{x}\boldsymbol{\Delta}) - \Phi_2(x, -x\boldsymbol{\Delta})\right]}_{\Phi_{2,\text{BFKL}}(x, \boldsymbol{l}, \boldsymbol{\Delta})}$$

- $\mathcal{U}(\boldsymbol{l}) \to k_T$-unintegrated gluon density in the BFKL sense

$$\mathcal{U}(\boldsymbol{\ell}) \equiv \int \mathrm{d}^d \boldsymbol{v} \mathrm{e}^{-i(\boldsymbol{\ell} \cdot \boldsymbol{v})} \left\langle P\left(p'\right) \middle| R^a\left(\frac{\boldsymbol{v}}{2}\right) R^a\left(-\frac{\boldsymbol{v}}{2}\right) \middle| P(p) \right\rangle ,$$

- Three-body case $\to$ combination of more impact factors
- The forward and dilute limit matches the previous results

[Anikin, Ivanov, Pire, Szymanowski, Wallon (2009)]

Diffractive dijet/hadron production

- Precise predictions to detect **saturation** effects at both the EIC or LHC

  [Iancu, Mueller, Triantafyllopoulos (2022)]

- Possibility of studying multi-dimensional **gluon tomography**

  [Feng's talk] [Hatta, Xiao, Yuan (2022)]

  [Hauksson, Iancu, Mueller, Triantafyllopoulos (2024)]

- **Diffractive dijet/hadron(s) production at NLO**

$$\gamma^{(*)}(p_\gamma) + P(p_0) \to j_1(p_{h1}) + j_2(p_{h2}) + P(p_0')$$

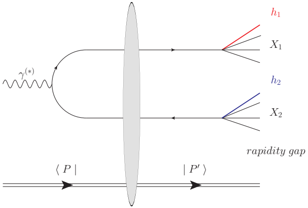
[Boussarie, Grabovsky, Szymanowski, Wallon (2016)]

$$\gamma^{(*)}(p_\gamma) + P(p_0) \to h_1(p_{h1}) + h_2(p_{h2}) + X + P(p_0') \qquad (X = X_1 + X_2)$$

[M. F., Grabovsky, Li, Szymanowski, Wallon (2023)]

$$\gamma^{(*)}(p_\gamma) + P(p_0) \to h_1(p_{h1}) + X + P(p_0')$$

[M. F., Grabovsky, Li, Szymanowski, Wallon (2024)]



i. General kinematics (t, Q^2) and photon polarization

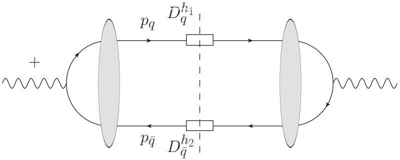
ii. Rapidity gap between $(h_1 h_2 X)$ and P'

iii. $\vec{p}_{h_1}^{\,2}, \vec{p}_{h_2}^{\,2} \gg \Lambda_{\rm QCD}^2$

LO cross-section

- Sudakov decomposition for the momenta: $p_i^\mu = x_i p_\gamma^+ n_1^\mu + \frac{\vec{p}^{\,2}}{2x_i p_\gamma^+} n_2^\mu + p_\perp^\mu$

$$\begin{matrix} & + & - & \perp \\ p_\gamma = & \Big(p_\gamma^+ \; , & -\dfrac{Q^2}{2p_\gamma^+} \; , & \vec{0} \Big) \end{matrix}$$

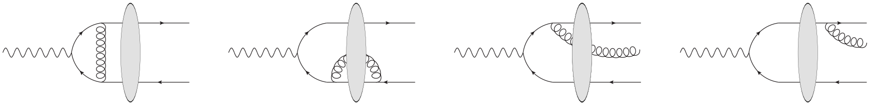


- Collinearity $(p_q^+, \vec{p}_q) = (x_q/x_{h_1})(p_{h_1}^+, \vec{p}_{h_1})$ and $(p_{\bar q}^+, \vec{p}_{\bar q}) = (x_{\bar q}/x_{h_2})(p_{h_2}^+, \vec{p}_{h_2})$

$$\frac{d\sigma_{0JI}^{h_1 h_2}}{dx_{h_1} dx_{h_2} d^d\vec{p}_{h_1} d^d\vec{p}_{h2}} = \sum_q \int_{x_{h_1}}^1 \frac{dx_q}{x_q} \int_{x_{h_2}}^1 \frac{dx_{\bar q}}{x_{\bar q}} \left(\frac{x_q}{x_{h1}}\right)^d \left(\frac{x_{\bar q}}{x_{h2}}\right)^d$$
$$D_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) D_{\bar q}^{h_2}\left(\frac{x_{h_2}}{x_{\bar q}}\right) \frac{d\hat{\sigma}_{JI}}{dx_q dx_{\bar q} d^d\vec{p}_q d^d\vec{p}_{\bar q}} + (h_1 \leftrightarrow h_2)$$

$J, I \rightarrow$ photon polarization for respectively the complex conjugated amplitude and the amplitude.

Treatment of UV and rapidity divergences



Rapidity divergences $(x_g \to 0)$

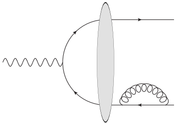
- Coming from Φ_{V_2} (double dipole part of the virtual contribution)
- Regularized by **longitudinal cut-off**: $|p_g^+| = |x_g|p_\gamma^+ > \alpha p_\gamma^+ \implies \ln\alpha$ term
- B-JIMWLK evolution from the non-physical cutoff α to the rapidity e^η

$$U_{\vec{x}}^{\alpha} = U_{\vec{x}}^{e^\eta} + \int_{e^\eta}^{\alpha} d\rho \left(\frac{\partial U_{\vec{x}}^{\rho}}{\partial \rho}\right) \implies \Phi_{V_2} \longrightarrow \tilde{\Phi}_{V_2} = \Phi_{V_2} + 2\ln\left(\frac{e^\eta}{\alpha}\right)\mathcal{K}_{\mathrm{BK}} \otimes \Phi_0 \widetilde{\mathcal{W}}_{123}$$

UV-divergences $(\vec{p}_g^{\,2} \to \infty)$

- Just dressing of the external quark lines

$$\Phi_{\mathrm{dress}} \propto \left(\frac{1}{2\epsilon_{IR}} - \frac{1}{2\epsilon_{UV}}\right)$$

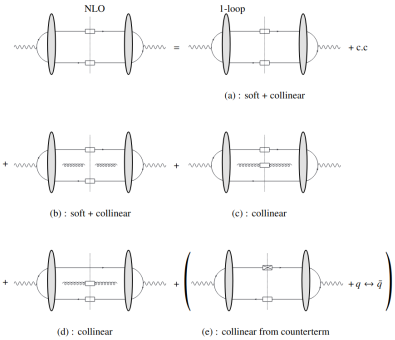


- $\epsilon_{IR} = \epsilon_{UV}$ turns **UV** into **IR** divergences

NLO cross-section in a nutshell

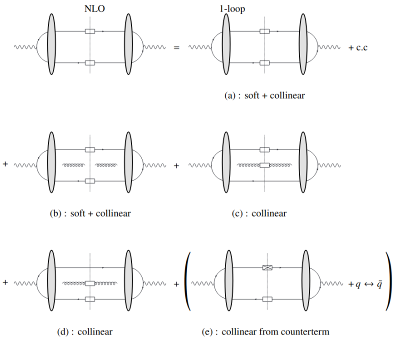
- Different *fragmentation mechanisms*

  i. Quark/anti-quark fragmentation
  ii. Quark/gluon fragmentation
  iii. Anti-quark/gluon fragmentation

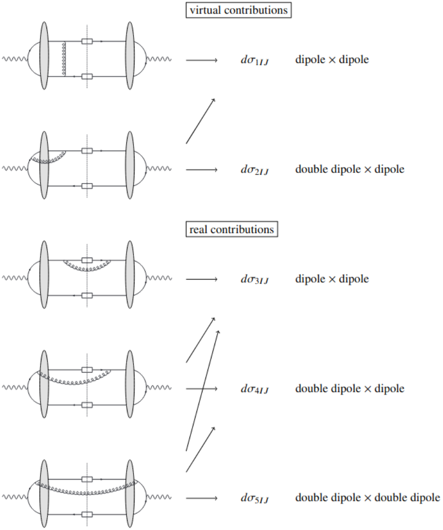


- Different *fragmentation mechanisms*

  i. Quark/anti-quark fragmentation
  ii. Quark/gluon fragmentation
  iii. Anti-quark/gluon fragmentation

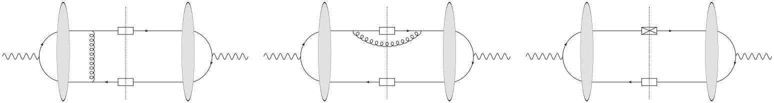


- Operator structure classification



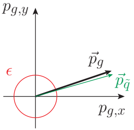
IR singularities: Quark/anti-quark fragmentation

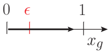
- Divergent contributions



$d\sigma_{1IJ}$ $\qquad$ $d\sigma_{3IJ}$ $\qquad$ $d\sigma_{\text{counter}}$

- Collinear divergence

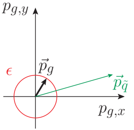


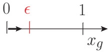


i. $\vec{p}_g \to \vec{p}_{\tilde{q}} = \frac{x_g}{x_q}\vec{p}_q$

ii. x_g generic

- Soft divergence





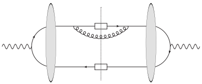
i. $\vec{p}_g \equiv x_g \vec{u}$

ii. $x_g \to 0$ and $\vec{u}$ generic

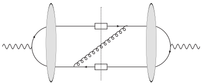
- Soft and collinear divergence ($x_g \to 0$ and $\vec{u} \to \frac{\vec{p}_q}{x_q}$)

IR singularities

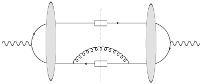
- Divergences: $q\bar{q}$-fragmentation



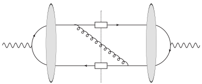
(1): soft + collinear (qg)



(2): soft



(3): soft + collinear ($\bar{q}g$)

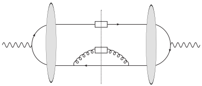


(4): soft

- Treatment of divergences in a nutshell

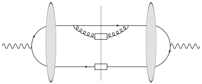
$$d\sigma_1 + d\sigma_{3,\mathrm{soft}} + \underbrace{(d\sigma_3^{(1)} - d\sigma_{3,\mathrm{soft}}^{(1)})}_{d\sigma_{3,\mathrm{collinear}}^{(1)}} + (d\sigma_3^{(2)} - d\sigma_{3,\mathrm{soft}}^{(2)}) + \underbrace{(d\sigma_3^{(3)} - d\sigma_{3,\mathrm{soft}}^{(3)})}_{d\sigma_{3,\mathrm{collinear}}^{(3)}} + ((d\sigma_3^{(4)} - d\sigma_{3,\mathrm{soft}}^{(4)})) + d\sigma_{\mathrm{counter}}$$

- Divergences: qg-fragmentation



(5): collinear $\longrightarrow$ $d\sigma_{3,\mathrm{collinear}}^{\bar{q}g(5)}$

- Divergences: $\bar{q}g$-fragmentation



(6): collinear $\longrightarrow$ $d\sigma_{3,\mathrm{collinear}}^{\bar{q}g(6)}$

Renormalization of FFs and gluon fragmentation

- Renormalized quark FFs (similar for the anti-quark)

$$\tilde{D}_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) = D_q^{h_1}\left(\frac{x_{h_1}}{x_q}, \mu_F\right) - \frac{\alpha_s}{2\pi}\left(\frac{1}{\hat{\epsilon}} + \ln\frac{\mu_F^2}{\mu^2}\right)\left[\left[P_{qq}\otimes D_q^{h_1}\right]\left(\frac{x_{h_1}}{x_q}, \mu_F\right) + \left[P_{gq}\otimes D_g^{h_1}\right]\left(\frac{x_{h_1}}{x_q}, \mu_F\right)\right]$$

Renormalization of FFs and gluon fragmentation

- Renormalized quark FFs (similar for the anti-quark)

$$\tilde{D}_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) = D_q^{h_1}\left(\frac{x_{h_1}}{x_q}, \mu_F\right) - \frac{\alpha_s}{2\pi}\left(\frac{1}{\hat{\epsilon}} + \ln\frac{\mu_F^2}{\mu^2}\right)\left[[P_{qq} \otimes D_q^{h_1}]\left(\frac{x_{h_1}}{x_q}, \mu_F\right) + [P_{gq} \otimes D_g^{h_1}]\left(\frac{x_{h_1}}{x_q}, \mu_F\right)\right]$$

$$\downarrow$$

$$d\sigma_{LL}^{h_1 h_2}\Big|_{\text{ct}} = \frac{4\alpha_{\text{em}} Q^2}{(2\pi)^{4(d-1)} N_c} \sum_q Q_q^2 \int_{x_{h_1}}^1 dx_q \int_{x_{h_2}}^1 dx_{\bar{q}} x_q x_{\bar{q}} \left(\frac{x_q}{x_{h_1}}\right)^d \left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d \delta(1 - x_q - x_{\bar{q}})$$

$$\times \mathcal{F}_{LL}\left(-\frac{\alpha_s}{2\pi}\right)\left(\frac{1}{\hat{\epsilon}} + \ln\frac{\mu_F^2}{\mu^2}\right)\Bigg\{\underbrace{[P_{qq} \otimes D_q^{h_1}]\left(\frac{x_{h_1}}{x_q}, \mu_F\right)}_{(1)} D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right)$$

$$+\underbrace{[P_{gq} \otimes D_g^{h_1}]\left(\frac{x_{h_1}}{x_q}, \mu_F\right)}_{(6)} D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}}, \mu_F\right) + \left[\left(q, x_q, x_{h_1}\right) \leftrightarrow \left(\bar{q}, x_{\bar{q}}, x_{h_2}\right)\right]\Bigg\} + (h_1 \leftrightarrow h_2)$$

Renormalization of FFs and gluon fragmentation

- Renormalized quark FFs (similar for the anti-quark)

$$\tilde{D}_q^{h_1}\left(\frac{x_{h_1}}{x_q}\right) = D_q^{h_1}\left(\frac{x_{h_1}}{x_q},\mu_F\right) - \frac{\alpha_s}{2\pi}\left(\frac{1}{\hat{\epsilon}} + \ln\frac{\mu_F^2}{\mu^2}\right)\left[\left[P_{qq}\otimes D_q^{h_1}\right]\left(\frac{x_{h_1}}{x_q},\mu_F\right) + \left[P_{gq}\otimes D_g^{h_1}\right]\left(\frac{x_{h_1}}{x_q},\mu_F\right)\right]$$

$$\downarrow$$

$$d\sigma_{LL}^{h_1 h_2}\Big|_{\text{ct}} = \frac{4\alpha_{\text{em}}Q^2}{(2\pi)^{4(d-1)}N_c}\sum_q Q_q^2\int_{x_{h_1}}^1 dx_q\int_{x_{h_2}}^1 dx_{\bar{q}}x_q x_{\bar{q}}\left(\frac{x_q}{x_{h_1}}\right)^d\left(\frac{x_{\bar{q}}}{x_{h_2}}\right)^d\delta(1-x_q-x_{\bar{q}})$$

$$\times\mathcal{F}_{LL}\left(-\frac{\alpha_s}{2\pi}\right)\left(\frac{1}{\hat{\epsilon}}+\ln\frac{\mu_F^2}{\mu^2}\right)\Bigg\{\underbrace{\left[P_{qq}\otimes D_q^{h_1}\right]\left(\frac{x_{h_1}}{x_q},\mu_F\right)}_{(1)}D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right)$$

$$+\underbrace{\left[P_{gq}\otimes D_g^{h_1}\right]\left(\frac{x_{h_1}}{x_q},\mu_F\right)}_{(6)}D_{\bar{q}}^{h_2}\left(\frac{x_{h_2}}{x_{\bar{q}}},\mu_F\right) + \left[\left(q,x_q,x_{h_1}\right)\leftrightarrow\left(\bar{q},x_{\bar{q}},x_{h_2}\right)\right]\Bigg\} + (h_1\leftrightarrow h_2)$$

- Finite part of the cross sections

$$d\sigma_{h_1,h_2} = \sum_{(a,b)} D_a^{h_1}\otimes D_b^{h_2}\otimes d\hat{\sigma}_{ab} \qquad (a,b) = \{(q,\bar{q}),(q,g),(g,\bar{q})\}$$

- Extension to the **semi-inclusive diffractive DIS (SIDDIS)** at the NLO **[M.F., Grabovsky, Li, Szymanowski, Wallon (2024)]**

Summary and outlook

Summary

- **Transversally polarized ρ-meson production**

  [Boussarie, M.F., Szymanowski, Wallon (to appear)]

- **Full NLO computation of diffractive dijet, di-hadron production and SIDDIS**

  [Boussarie, Grabovsky, Szymanowski, Wallon (2016)]

  [M. F., Grabovsky, Li, Szymanowski, Wallon (2023)]

  [M. F., Grabovsky, Li, Szymanowski, Wallon (2024)]

- *General kinematics* (Q^2, t) and *arbitrary photon polarization* means either photo or electro-production

- Detection of saturation and BFKL effects at both the EIC or at LHC via Ultra Peripheral Collisions (UPC)

Outlook

- Extension of the ρ-transverse production at the NLO

- Special kinematic configurations $\rightarrow$ diffractive dijet production in the back-to-back limit, TMD factorization in SIDDIS

[Boussarie, M.F., Yuan, Szymanowski, Wallon (ongoing work)]

Thanks for your attention

Backup

Balitsky-JIMWLK evolution equations

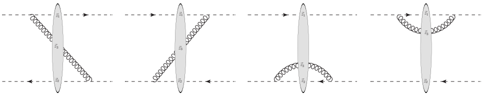
- ***Balitsky-JIMWLK evolution equations*** for the dipole
  [Balitsky — Jalilian-Marian, Iancu, McLerran, Weigert, Kovner, Leonidov]

$$\frac{\partial \mathcal{U}_{12}^{\eta}}{\partial \eta} = \frac{\alpha_s N_c}{2\pi^2} \int d^2 \vec{z}_3 \left(\frac{\vec{z}_{12}^{\,2}}{\vec{z}_{23}^{\,2} \vec{z}_{31}^{\,2}} \right) \Big[\underbrace{\mathcal{U}_{13}^{\eta} + \mathcal{U}_{32}^{\eta} - \mathcal{U}_{12}^{\eta}}_{\text{BFKL}} - \mathcal{U}_{13}^{\eta} \mathcal{U}_{32}^{\eta} \Big]$$

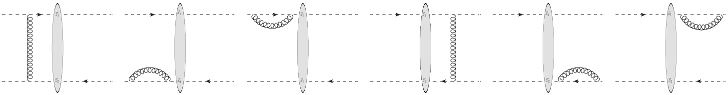
$$\frac{\partial \mathcal{U}_{13}^{\eta} \mathcal{U}_{32}^{\eta}}{\partial \eta} = \cdots \qquad \longleftarrow \quad \textit{Balitsky hierarchy}$$

$$\vdots$$

- Double dipole contribution and Dipole contribution

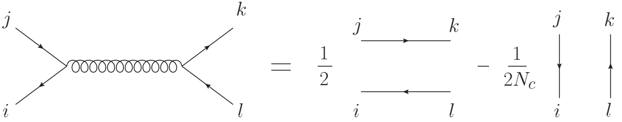


- Dipole contribution



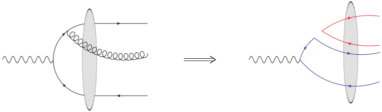
Balitsky-Kovchegov evolution equation

- Large-N_c limit **[G. 't Hooft (1974)]**



$$\boxed{t^a_{ij}t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_c}\delta_{ij}\delta_{kl}\right)}$$

- Double dipole → Dipole × dipole



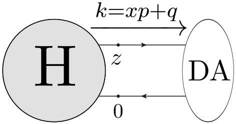
$$\langle \mathcal{U}^\eta_{13}\mathcal{U}^\eta_{32}\rangle \to \langle \mathcal{U}^\eta_{13}\rangle \, \langle \mathcal{U}^\eta_{32}\rangle$$

- Hierarchy of equations broken → closed non-linear **BK-equation**
  [I. I. Balitsky (1995)] [Y. V. Kovchegov (1999)]

$$\frac{\partial\langle \mathcal{U}^\eta_{12}\rangle}{\partial\eta} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_3 \left(\frac{\vec{z}^{\,2}_{12}}{\vec{z}^{\,2}_{23}\vec{z}^{\,2}_{31}}\right)\left[\langle \mathcal{U}^\eta_{13}\rangle + \langle \mathcal{U}^\eta_{32}\rangle - \langle \mathcal{U}^\eta_{12}\rangle - \langle \mathcal{U}^\eta_{13}\rangle\langle \mathcal{U}^\eta_{32}\rangle\right]$$

with $\langle \mathcal{U}^\eta_{12}\rangle \equiv \langle P'|\mathcal{U}^\eta_{12}|P\rangle$

Light-cone collinear factorization



- 2-body amplitude

$$\mathcal{A}_2 = \int \frac{d^4k}{(2\pi)^4} \int d^4 z e^{-ik\cdot z} \langle M(p)|\overline{\psi}^{\,i}_{\alpha}(z)\,\psi^{\,j}_{\beta}(0)\,|0\rangle H^{ij}_{2,\alpha\beta}$$

- 2-body amplitude after Fierz decomposition

$$\mathcal{A}_2 = \frac{1}{4N_c} p^+ \int \frac{\mathrm{d}x}{2\pi} \int \frac{\mathrm{d}q^-}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{q}}{(2\pi)^d} \int \mathrm{d}^D z \; \mathrm{e}^{-ixp^+z^- - iq^-z^+ + i(\boldsymbol{q}\cdot\boldsymbol{z})}$$
$$\times \left\langle M(p) \left| \overline{\psi}(z)\,\Gamma_\lambda \psi(0) \right| 0 \right\rangle \mathrm{tr}\left[H_2(xp+q)\,\Gamma^\lambda \right]$$

- Taylor expansion of the hard part

$$H_2(xp+q) = H_2(xp) + q_{\perp\mu} \left[\frac{\partial}{\partial q_{\perp\mu}} H_2(xp+q) \right]_{k=xp} + \mathrm{h.t.}$$

Light-cone collinear factorization

- 2-body factorized form up to twist-3

$$\mathcal{A}_2 = \frac{1}{4N_c}\int \mathrm{d}x\; p^+ \int \frac{\mathrm{d}z^-}{2\pi}\mathrm{e}^{-ixp^+z^-}$$
$$\times\left\{\left\langle M\left(p\right)\left|\overline{\psi}\left(z^-\right)\Gamma_\lambda\psi\left(0\right)\right|0\right\rangle \mathrm{tr}\left[H_2\left(xp\right)\Gamma^\lambda\right]\right.$$
$$\left.+i\left\langle M\left(p\right)\left|\overline{\psi}\left(z^-\right)\overleftrightarrow{\partial}_{\perp\mu}\Gamma_\lambda\psi\left(0\right)\right|0\right\rangle \mathrm{tr}\left[\partial_\perp^\mu H_2\left(xp\right)\Gamma^\lambda\right]\right\}$$

- 3-body contribution

$$\mathcal{A}_3 = \int\frac{\mathrm{d}^Dk_q}{(2\pi)^D}\frac{\mathrm{d}^Dk_g}{(2\pi)^D}\int \mathrm{d}^Dz_q\mathrm{d}^Dz_g\mathrm{e}^{-i(k_q\cdot z_q)-i(k_g\cdot z_g)}$$
$$\times\left\langle M\left(p\right)\left|\overline{\psi}_\alpha^{\,i}\left(z_q\right)\Gamma_\lambda\; gA_\mu^a\left(z_g\right)\psi_\beta^{\,j}\left(0\right)\right|0\right\rangle \mathrm{tr}\left[H_{3,\alpha\beta}^{ija,\mu}\left(k_q,k_g\right)\Gamma^\lambda\right]$$

- 3-body contribution factorized

$$\mathcal{A}_3 = \frac{1}{2(N_c^2-1)}\int \mathrm{d}x_q\mathrm{d}x_g\left(p^+\right)^2\int\frac{\mathrm{d}z_q^-}{2\pi}\frac{\mathrm{d}z_g^-}{2\pi}\mathrm{e}^{-ix_qp^+z_q^- - ix_gp^+z_g^-}$$
$$\times\left\langle M\left(p\right)\left|\overline{\psi}\left(z_q^-\right)\Gamma_\lambda gA_\mu\left(z_g^-\right)\psi\left(0\right)\right|0\right\rangle \mathrm{tr}\left[t^bH_3^{\mu,b}\left(x_qp,x_gp\right)\Gamma^\lambda\right]\;.$$

- General three-body small-x amplitude

$$\mathcal{A}_3 = \left(\prod_{i=1}^{3}\int dx_i\theta(x_i)\right)\delta(1-x_1-x_2-x_3)\int d^2\boldsymbol{z}_1 d^2\boldsymbol{z}_2 d^2\boldsymbol{z}_3 e^{i\boldsymbol{q}(x_1\boldsymbol{z}_1+x_2\boldsymbol{z}_2+x_3\boldsymbol{z}_3)}$$
$$\times\Psi_3\left(x_1,x_2,x_3,\boldsymbol{z}_1,\boldsymbol{z}_2,\boldsymbol{z}_3\right)\left\langle P\left(p'\right)\left|\mathcal{U}_{\boldsymbol{z}_1\boldsymbol{z}_3}\mathcal{U}_{\boldsymbol{z}_3\boldsymbol{z}_2}-\mathcal{U}_{\boldsymbol{z}_1\boldsymbol{z}_3}-\mathcal{U}_{\boldsymbol{z}_3\boldsymbol{z}_2}+\frac{1}{N_c^2}\mathcal{U}_{\boldsymbol{z}_1\boldsymbol{z}_2}\right|P\left(p\right)\right\rangle$$

- Momentum space impact factor

$$\Phi_3\left(\{x\},\{\boldsymbol{p}\}\right)=\frac{e_q q^+ m_M}{4}c_f\left(\varepsilon_{q\rho}-\frac{\varepsilon_q^+}{q^+}q_\rho\right)\left(\varepsilon_M^{*\beta}-\frac{p_M^\beta}{p_M^+}\varepsilon_M^{*+}\right)\delta(q^+-p_M^+)$$
$$\times\left(\prod_{j=1}^{3}\frac{\theta(1-x_j)\theta(x_j)}{x_j}\right)\frac{(2\pi)^3\delta^{(2)}\left(\sum_{i=1}^3\boldsymbol{p}_i+x_i\boldsymbol{p}_M\right)}{\left[Q^2+\sum_{i=1}^3(\boldsymbol{p}_i+x_i\boldsymbol{p}_M)^2/x_i\right]}\Bigg\{g_{\beta\sigma}f_{3M}^V V(x_1,x_2)\left(4g_{\perp\perp}^{\rho\sigma}\frac{x_1x_2}{1-x_2}\right.$$
$$\left.+\tilde{T}_1^{\sigma\rho\nu}(\{x\})\big|_{\boldsymbol{k}_i=-x_i\boldsymbol{p}_M}\frac{x_1x_2(p_3+x_3p_M)_{\perp\nu}-x_1x_3(p_2+x_2p_M)_{\perp\nu}}{(\boldsymbol{p}_1+x_1\boldsymbol{p}_M)^2+x_1(1-x_1)Q^2}\right)-\epsilon_{-+\sigma\beta}f_{3M}^A A(x_1,x_2)$$
$$\times\left(4\frac{x_1x_2}{1-x_2}\epsilon^{\sigma\rho+-}+i\tilde{T}_2^{\sigma\rho\nu}(\{x\})\big|_{\boldsymbol{k}_i=-x_i\boldsymbol{p}_M}\frac{x_1x_2(p_3+x_3p_M)_{\perp\nu}-x_1x_3(p_2+x_2p_M)_{\perp\nu}}{(\boldsymbol{p}_1+x_1\boldsymbol{p}_M)^2+x_1(1-x_1)Q^2}\right)\Bigg\}$$
$$+(1\leftrightarrow 2)$$

Dilute Regime: 3-body

- Linearization in the three-body case → combination of more impact factors

$$\mathcal{A}_3^{\text{dilute}} = \left(\prod_{i=1}^{3}\int dx_i\theta(x_i)\right)\delta(1-x_1-x_2-x_3)\frac{-g^2}{4N_c}\int\frac{d^d\boldsymbol{l}}{(2\pi)^d}\mathcal{U}(\ell)$$

$$\Big\{\Phi_3'\left(\{x\},\bar{x}_1\boldsymbol{\Delta},-x_2\boldsymbol{\Delta},-x_3\boldsymbol{\Delta}\right)-\Phi_3'\left(\{x\},\left(\frac{1-2x_1}{2}\right)\boldsymbol{\Delta}-\boldsymbol{l},-x_2\boldsymbol{\Delta},\left(\frac{1-2x_3}{2}\right)\boldsymbol{\Delta}+\boldsymbol{l}\right)$$

$$-\Phi_3'\left(\{x\},\left(\frac{1-2x_1}{2}\right)\boldsymbol{\Delta}+\boldsymbol{l},-x_2\boldsymbol{\Delta},\left(\frac{1-2x_3}{2}\right)\boldsymbol{\Delta}-\boldsymbol{l}\right)+\Phi_3'\left(\{x\},-x_1\boldsymbol{\Delta},-x_2\boldsymbol{\Delta},\bar{x}_3\boldsymbol{\Delta}\right)$$

$$+\Phi_3'\left(\{x\},-x_1\boldsymbol{\Delta},-x_2\boldsymbol{\Delta},\bar{x}_3\boldsymbol{\Delta}\right)-\Phi_3'\left(\{x\},-x_1\boldsymbol{\Delta},\left(\frac{1-2x_2}{2}\right)\boldsymbol{\Delta}+\boldsymbol{l},\left(\frac{1-2x_3}{2}\right)\boldsymbol{\Delta}-\boldsymbol{l}\right)$$

$$-\Phi_3'\left(\{x\},-x_1\boldsymbol{\Delta},\left(\frac{1-2x_2}{2}\right)\boldsymbol{\Delta}-\boldsymbol{l},\left(\frac{1-2x_3}{2}\right)\boldsymbol{\Delta}+\boldsymbol{l}\right)+\Phi_3'\left(\{x\},-x_1\boldsymbol{\Delta},\bar{x}_2\boldsymbol{\Delta},-x_3\boldsymbol{\Delta}\right)$$

$$-\frac{1}{N_c^2}\Big[\Phi_3'\left(\{x\},\bar{x}_1\boldsymbol{\Delta},-x_2\boldsymbol{\Delta},-x_3\boldsymbol{\Delta}\right)-\Phi_3'\left(\{x\},\left(\frac{1-2x_1}{2}\right)\boldsymbol{\Delta}-\boldsymbol{l},\left(\frac{1-2x_2}{2}\right)\boldsymbol{\Delta}+\boldsymbol{l},-x_3\boldsymbol{\Delta}\right)$$

$$-\Phi_3'\left(\{x\},\left(\frac{1-2x_1}{2}\right)\boldsymbol{\Delta}+\boldsymbol{l},\left(\frac{1-2x_2}{2}\right)\boldsymbol{\Delta}-\boldsymbol{l},-x_3\boldsymbol{\Delta}\right)+\Phi_3'\left(\{x\},-x_1\boldsymbol{\Delta},\bar{x}_2\boldsymbol{\Delta},-x_3\boldsymbol{\Delta}\right)\Big]\Big\}$$

where

$$\Phi_3'(\{x\},\{\boldsymbol{p}+x\boldsymbol{p}_M\})\equiv\Phi_3(\{x\},\{\boldsymbol{p}\})$$

Explicit 3-body in the dilute and $\Delta = 0$ limit

- Forward and dilute limit in momentum space

$$\mathcal{A}^{\text{dilute}}_{3T,\boldsymbol{\Delta}=\mathbf{0}} = e_q m_M \frac{g^2}{N_c}(2\pi)q^+\delta\left(q^+ - p_M^+\right)(2\pi)^2\delta^2\left(\boldsymbol{q}-\boldsymbol{p}_M\right)\int\frac{\mathrm{d}^d\boldsymbol{\ell}}{(2\pi)^d}\mathcal{U}(\boldsymbol{\ell})$$
$$\times\left(\prod_{i=1}^{3}\int_0^1\frac{dx_i}{x_i}\right)\frac{\delta(1-x_1-x_2-x_3)}{x_3}\frac{\boldsymbol{\ell}^2}{Q^2}\Bigg\{T_{\text{f.}}\left[f^V_{3M}V\left(x_1,x_2\right)-f^A_{3M}A\left(x_1,x_2\right)\right]$$
$$\times 2x_1\left(\frac{x_3c_f}{\boldsymbol{\ell}^2+\frac{x_2x_3}{x_2+x_3}Q^2}+\frac{x_3c_f}{\boldsymbol{\ell}^2+\frac{x_1x_3}{x_1+x_3}Q^2}-\frac{\bar{x}_3\left(1-c_f\right)}{\boldsymbol{\ell}^2+\frac{x_1x_2}{x_1+x_2}Q^2}+\frac{x_2-\bar{x}_1c_f}{\boldsymbol{\ell}^2+x_1\bar{x}_1Q^2}+\frac{x_1-\bar{x}_2c_f}{\boldsymbol{\ell}^2+x_2\bar{x}_2Q^2}\right)$$
$$-T_{\text{n.f.}}\left[f^V_{3M}V\left(x_1,x_2\right)+f^A_{3M}A\left(x_1,x_2\right)\right]$$
$$\times\left(\frac{\left(1-c_f\right)x_1\bar{x}_3}{\bar{x}_3\boldsymbol{\ell}^2+x_1x_2Q^2}-\frac{c_fx_3^2}{\bar{x}_1\boldsymbol{\ell}^2+x_2x_3Q^2}-\frac{\left(x_2-\bar{x}_1c_f\right)x_1x_2}{\bar{x}_1\left(\boldsymbol{\ell}^2+x_1\bar{x}_1Q^2\right)}-\frac{\left(x_1-\bar{x}_2c_f\right)\bar{x}_2}{\left(\boldsymbol{\ell}^2+x_2\bar{x}_2Q^2\right)}\right)\Bigg\},$$

- Helicity structures

$$T_{\text{f.}} = \frac{(\boldsymbol{\varepsilon}_q\cdot\boldsymbol{l})(\boldsymbol{\varepsilon}^*_M\cdot\boldsymbol{l})}{\boldsymbol{l}^2}-\frac{\boldsymbol{\varepsilon}_q\cdot\boldsymbol{\varepsilon}^*_M}{2}\qquad T_{\text{n.f.}}=\boldsymbol{\varepsilon}_q\cdot\boldsymbol{\varepsilon}^*_M$$

- The forward and dilute limit matches the previous result

[Anikin, Ivanov, Pire, Szymanowski, Wallon (2009)]